

On the so called Boy or Girl Paradox

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Abstract

A quite old problem has been recently revitalized by Leonard Mlodinow's book *The Drunkard's Walk*, where it is presented in a way that has definitely confused several people, that wonder why the prevalence of the name of one daughter among the population should change the probability that the other child is a girl too. I try here to discuss the problem from scratch, showing that the rarity of the name plays no role, unless the strange assumption of two identical names in the same family is taken into account. But also the name itself does not matter. What is really important is ‘identification’, meant in an acceptation broader than usual, in the sense that a child is characterized by a set of attributes that make him/her uniquely identifiable (‘that one’) inside a family. The important point of how the information is acquired is also commented, suggesting an explanation of why several people tend to consider the informations “at least one boy” and “a well defined boy” (elder/youngest or of a given name) equivalent.

1 Introduction

A classical series of problems in elementary probability theory is about the gender combinations ($m-m$, $m-f$, $f-m$ and $f-f$) in a family of two children. Being this an academic exercise (in the bad sense of the term), usually one does not attempt to assess how much one believes that these combinations happen in a real family. This means that the well known male over female birth asymmetry is neglected, as are neglected gender correlations within a family, like those induced by genetic factors, or by the possibility of monovular twins.

Once the conditions are properly defined, the usual questions, besides the trivial one of male/female, are

Eldest	Youngest		
	m	f	$m \cup f$
m	1/4	1/4	1/2
f	1/4	1/4	1/2
$m \cup f$	1/2	1/2	1

Table 1: Table of equiprobable cases of the four possible sequences of child's gender. The symbol ' \cup ' stands for 'OR'.

- Q₁) What is the probability of two boys?
 Q₂) What is the probability of two boys, if the eldest child is a boy?
 Q₃) What is the probability of two boys, if at least one child is a boy?

These questions can be promptly answered looking at contingency table 1 that lists the space of the four equiprobable elementary cases.

- A₁) The probability of two boys is 1/4, or 25%, since it is just the probability of each elementary event, that all together have to sum up to unity, or 100%.
 A₂) If the eldest child is a boy, the space of possibilities is squeezed to the first row of the table. We remain with two equiprobable cases, each of which gets probability 1/2. In formulae:

$$P(Em \cap Ym | Em, I_0) = \frac{P(Em \cap Ym | I_0)}{P(Em | I_0)} \quad (1)$$

$$= \frac{1/4}{1/2} = \frac{1}{2}. \quad (2)$$

[The symbol ' \cap ' stands for a logical 'AND'; '|' stands for 'given', or 'conditioned by'; ' Em ' and ' Ym ' are short forms for "the eldest is male" and "the youngest is male"; the condition I_0 is the *background status of information* under which the probabilities are evaluated, that includes the simplifying hypotheses stated above; when there is a further condition, like Em and I_0 in the l.h.s. of Eq. (1), they are both indicated after the conditional symbol '|', separated by a comma.]

- A₃) Finally, the information that there is at least one boy in the family reduces the space of possibilities to three equiprobable cases, of which only one is that of our interest, thus getting 1/3 (the symbol ' \cup ' in the following formulae indicated a logical 'OR'). Formally

$$P(Em \cap Ym | Em \cup Ym, I_0) = \frac{P[(Em \cap Ym) \cap (Em \cup Ym) | I_0]}{P(Em \cup Ym | I_0)} \quad (3)$$

$$= \frac{P(Em \cap Ym | I_0)}{P(Em \cup Ym | I_0)} \quad (4)$$

$$= \frac{1/4}{3/4} = \frac{1}{3} \quad (5)$$

Obviously, the problem can be turned into probability of girl-girl by symmetry.

The ‘complication’ (mainly induced confusion) comes when the information about the name of one child is provided (‘Florida’ in *The Drunkard’s Walk* [1]):

Q_4 What is the probability of two boys, if one of the children is called Mark?¹

2 How the child name changes the probabilities

At this point the question is how table 1 is changed by the information that one child is known by gender and name (the latter usually implying the former).

The way the problem is often solved is to assume that the same name can be given to two different children in the same family. Frankly, I never heard of this possibility before. And, anyhow, if such a strange behavior occurs in some very rare cases, it seems to me of an importance much lower than all other questions that have been neglected (male/female asymmetry, genetic biases, etc.). What I found annoying is that this peculiar solution is not reported as a mathematical curiosity (see e.g. the Drunkard’s Walk or the Wikipedia page site dedicated to the so called ‘paradox’ [2] – a puzzle one is unable to solve is not necessarily a paradox), but as it would be ‘the solution’.

Nevertheless, let us first see what happens when this possibility is allowed. (By the way, one might think of families with children coming from previous marriages, in which case identical names might occur, but this possibility is excluded in the often implicit assumptions of this kind of puzzles, often formulated as “a lady has two children, ...”.)

2.1 Allowing identical names for two children of a family

Just to stay close to the formulation of the problem in the recent disputes that have triggered this paper, let us focus on girl probabilities, assuming we know that one of the child is a girl of a given name. Splitting the female category into ‘female of that given name’ (fN) and ‘female with any other name’ ($f\bar{N}$), there are now three possible cases for each child, $\{m, fN, f\bar{N}\}$, no longer equiprobable.

Calling r the fraction of girls owning that name in the population, we get the following probabilities, under the new background condition I_1 : $P(m | I_1) = 1/2$, $P(fN | I_1) = r \times 1/2 = r/2$ and $P(f\bar{N} | I_1) = (1 - r)/2$. The nine possibilities and their probabilities, calculated using the product rule (justified by elder/youngest name independence), are reported in table 2. From the table we can calculate the probability

¹This question can be turned into “what is the probability that Mark has a brother or a sister?” and any normal and sane person might wonder about the sense of this *madness*, as said a friend of mine with zero math skill, when he saw a first draft of this paper on my desk, because – he explained – “it is absolutely equally likely that he has either a brother or a sister”.

Eldest		Youngest			
		m	f	$m \cup f$	
			fN	$f\bar{N}$	
m		1/4	$r/4$	$(1-r)/4$	1/2
	fN	$r/4$	$r^2/4$	$r(1-r)/4$	$r/2$
f	$f\bar{N}$	$(1-r)/4$	$r(1-r)/4$	$(1-r)^2/4$	$(1-r)/2$
$m \cup f$		1/2	$r/2$	$(1-r)/2$	1
			1/2		

Table 2: Table of probabilities of the possible cases assuming that eldest and youngest children can have the same name (see text).

of both females, if we know one girl by name:

$$P[(Ef \cap Yf) | (EfN \cup YfN), I_1] = \frac{P[(Ef \cap Yf) \cap (EfN \cup YfN) | I_1]}{P[(EfN \cup YfN) | I_1]}. \quad (6)$$

The denominator is given by the five elements emphasized in boldface in table 2, whose probability sum up to $r - r^2/4$. The numerator is given by the three elements that have m neither in the rows nor in the columns, whose probabilities are $r^2/4$, $r(1-r)/4$ and $r(1-r)/4$, adding up to $(2r - r^2)/4$. We get then

$$P[(Ef \cap Yf) | (EfN \cup YfN), I_1] = \frac{(2r - r^2)/4}{r - r^2/4} \quad (7)$$

$$= \frac{1}{2} \left[\frac{1 - r/2}{1 - r/4} \right] \quad (8)$$

$$\approx \frac{1}{2} - \frac{r}{8} \quad (\text{for } r \ll 1) \quad (9)$$

As we can see, the probability does depend on r , but it tends rapidly to 1/2 for small values of r , as also shown in table 3 for some numerical values of this parameter.²

² The value $r = 0.02 = 1/50$ is that used in Ref. [2], for which the probabilities of table 2 acquire the following values

0.2500	0.0050	0.2450	0.5000
0.0050	0.0001	0.0049	0.0100
0.2450	0.0049	0.2401	0.4900
0.5000	0.1000	0.4900	1.0000

from which we get the following table of *expected values* in 10000 families

2500	50	2450	5000
50	1	49	100
2450	49	2401	4900
5000	100	4900	10000

[By the way, I would like to point out that quoting expected values is a way to state ...what we

r	$P(\text{two girls} \mid fN, I_1)$
0.3	0.45946
0.2	0.47368
0.1	0.48718
0.02	0.49749
0.01	0.49875
0.001	0.49988
0.0001	0.49999

Table 3: Probability of two girls in family, if we know by name a daughter, calculated as a function of the prevalence of that name within the girls of that population. [Note, just for mathematical curiosity, that if $r = 1$ (all girls have the same name), Eq. (8) gives a probability of $1/3$, thus recovering Q_3 . In fact, in this case telling the name adds no more information to “at least one is female”.]

2.2 Unique names of children within a family

Let us now see what happens if we require that, as it normally happens, children names are unique. The central element of table 2 goes to zero, but the sums along rows and columns have to be preserved³ [for example the probability of $EfN \cap Yf\bar{N}$ becomes $r(1-r)/4 + r^2/4$, that is the same as $r/2 - r/4$, i.e. $r/4$]. The result is shown in table 4 (we label the central value of the table by ‘-’ to remark that this case is impossible by assumption). [Note that the impossibility of identical children names constrains r to be smaller than $1/2$, well above any reasonable value. Remember also that the probabilities of table 4 reflect the several simplifying assumptions of the problem.]

Eldest		Youngest			
		m	f	$f\bar{N}$	$m \cup f$
m		$1/4$	$r/4$	$(1-r)/4$	$1/2$
f	fN	$r/4$	-	$r/4$	$r/2$
	$f\bar{N}$	$(1-r)/4$	$r/4$	$(1-2r)/4$	$(1-r)/2$ $1/2$
$m \cup f$		$1/2$	$r/2$	$(1-r)/2$	1
			$1/2$		

Table 4: Same as table 2, but not allowing the identical names of the children.

expect in a probabilistic sense – and probability theory teaches how to calculate standard expectation uncertainties (the σ 's) - and has little to do with ‘frequentistic approach’, since the probabilities have not been evaluated by past frequencies (statistical data).]

³If no further information is provided, the probability that any child is a female with the special name N has to be $r/2$, no matter if the child in question is the eldest or the youngest. Similarly, the probability of girl with a name different from N has to be $(1-r)/2$.

Contrary to table 2, the four cases that involve fN are now *equiprobable* (each with probability $r/4$). It follows that the probability that the other child is a boy or a girl is 50%, *independently* of the rarity of the name. In formulae (note the new background condition I_2):

$$P[(Ef \cap Yf) | (EfN \cup YfN), I_2] = \frac{P[(Ef \cap Yf) \cap (EfN \cup YfN) | I_2]}{P(EfN \cup YfN | I_2)} \quad (10)$$

$$= \frac{2 \times r/4}{4 \times r/4} \quad (11)$$

$$= \frac{1}{2}. \quad (12)$$

3 Does the name really matter?

At this point it is easy to understand that we could replace fN in the table by fID , where ‘ ID ’ stands now for ‘uniquely identified within the family’, thus getting table 5. Think, for example to the following statements

- “the secretary of the department X of hospital Y in Rome is daughter of my aunt B who has also another child”;
- “the parents of the actress starring in the last movie I have seen have two children”;
- “the mother of that lady has got two children”;
- and so on...

In all these cases the probability that the female in question has a sister is 50%, as everybody that is not fooled by probability theory will promptly tell us (see footnote 1). It is not just a question of knowing her name, or knowing that she is the eldest or the youngest (that’s the reason we recover the answer to $Q_2!$). *What matters is that this person is somehow uniquely ‘identified’ in the family*, where ‘identified’ is within

Eldest		Youngest			
		m	f	$m \cup f$	
m		$1/4$	$r/4$	$(1-r)/4$	$1/2$
f	fID	$r/4$	-	$r/4$	$r/2$
	$f\overline{ID}$	$(1-r)/4$	$r/4$	$(1-2r)/4$	$(1-r)/2$
$m \cup f$		$1/2$	$r/2$	$(1-r)/2$	1
			$1/2$		

Table 5: Same as table 4, but based on ‘identification’ of a girl.

Eldest		Youngest			
		m	f	$m \cup f$	
m	$f\mathcal{ID}$	$1/4$	$\mathbf{r}/4$	$(1-r)/4$	$1/2$
	$f\overline{\mathcal{ID}}$	$\mathbf{r}/4$	$\mathbf{r}^2/4$	$\mathbf{r}(1-r)/4$	$r/2$
f	$f\mathcal{ID}$	$(1-r)/4$	$\mathbf{r}(1-r)/4$	$(1-r)^2/4$	$(1-r)/2$
$m \cup f$	$f\overline{\mathcal{ID}}$	$1/2$	$r/2$	$(1-r)/2$	1
			$1/2$		

Table 6: Same as table 2, but with reference to non unique identification (note the symbol ‘ \mathcal{ID} ’ instead of ‘ ID ’ of table 5) rather than name.

quote marks because it is not requested we know her passport number, but just that we are able to point to her as *that one*.

If this is not the case, and two children could correspond to the same description, then table 2 holds, assuming no correlation between the descriptions (if one is blond, there is high change that the other is blond too, and so on). Therefore we recover it as table 6, but in terms of *non unique identification* (‘ \mathcal{ID} ’) rather than of name. Now it makes sense. In fact, since we are referring here to ‘identification’ in a loose sense, it might really occur that two daughters correspond to the same description (‘goes to college’, or ‘play tennis’, and so on). Finally, the name can be considered a generic identification, in order to include the possibility of identical names in a family (for example in the cases of second marriages).

4 Some Bayesian flavor

Someone asks me about *the* Bayesian solution of the problem (because I am supposed to be *a Bayesian*). But, besides the clarification that “I am not a Bayesian” [3], such a kind of ‘alternative’ solution of the problem does not exist. The solution is already that provided by Eq. (10), because ‘Bayesians’ just make use of probability theory to state the relative beliefs of several hypotheses given some well stated assumptions. In particular, the so called Bayes’ rule for this problem is essentially Eq. (10), that can be possibly written in other convenient forms using the rules of probability.

4.1 Reconditioning the probability of an hypothesis on the light of a new status of information

To make the point clearer, and calling $A = Ef \cap Yf$ (“both children are female”) and $B = EfN \cup YfN$ (“one child is a female of a given particular name”) to simplify the

notation, we can rewrite Eq. (10) as

$$P(A|B, I_2) = \frac{P(A \cap B | I_2)}{P(B | I_2)} \quad (13)$$

$$= \frac{P(B | A, I_2) P(A | I_2)}{P(B | I_2)}. \quad (14)$$

The latter expression shows explicitly how the probability of A is updated, by the extra condition B , via the factor $P(B | A, I_2)/P(B | I_2)$, i.e.

$$P(A|B, I_2) = \frac{P(B | A, I_2)}{P(B | I_2)} \times P(A | I_2). \quad (15)$$

The three ingredients we need to evaluate $P(A|B, I_2)$ can be easily read from table 4,

$$P(A | I_2) = \frac{1}{4} \quad (16)$$

$$P(B | I_2) = r \quad (17)$$

$$P(B | A, I_2) = 2r, \quad (18)$$

from which we get

$$P(A|B, I_2) = \frac{2r}{r} \times \frac{1}{4} = 2 \times \frac{1}{4} = \frac{1}{2}, \quad (19)$$

recovering the result of section 2.2 (note that it must be so because we are strictly using the probabilities of table 4).

4.2 Updating the odds

We can do it in a different way, comparing the probability of “two girls” (A) with that of “only one girl” [let us indicate the latter hypothesis as $C = (Ef \cap Ym) \cup (Em \cap Yf)$]. The probability of C conditioned by B , i.e. $P(C|B, I_2)$, could be obtained in analogy to Eq. (13), reading $P(C \cap B | I_2)$ from table 4. But it can be more instructive to get it *the Bayesian way*, using the formula that shows how relative probabilities are updated by the *Bayes factor* to take into account the new piece of information (this second approach has also the advantage of getting rid of r since the very beginning):

$$\underbrace{\frac{P(A|B, I_2)}{P(C|B, I_2)}}_{\text{‘updated odds’}} = \underbrace{\frac{P(B|A, I_2)}{P(B|C, I_2)}}_{\substack{\text{‘updating factor’} \\ \text{(Bayes factor)}}} \times \underbrace{\frac{P(A|I_2)}{P(C|I_2)}}_{\text{‘initial odds’}}. \quad (20)$$

The initial probability of two girls is one half that of a single girl, i.e.

$$\frac{P(A|I_2)}{P(C|I_2)} = \frac{1}{2}, \quad (21)$$

while the probability that there is a girl with a precise name is proportional to the number of girls in the family (remember that the condition ‘ I_2 ’ does not allow the same name), namely

$$\frac{P(B|A, I_2)}{P(B|C, I_2)} = 2. \quad (22)$$

It follows

$$\frac{P(A|B, I_2)}{P(C|B, I_2)} = 2 \times \frac{1}{2} = 1 : \quad (23)$$

the girl of which we know the name has equal probability to have a sister (A) or a brother (C), that is

$$P(A|B, I_2) = P(C|B, I_2) = \frac{1}{2}. \quad (24)$$

5 Conclusions

The probability that, knowing the name of one child in a family of two, the other one child is of the same gender has nothing to do with the rarity of the name, unless the crazy possibility of identical names in a family is assumed (and if somebody insists that this can happen, he/she is invited to calculate more realistic probabilities that take into account male/female asymmetry and genetic correlations; also the possibility of identical names of children coming from previous marriages are implicitly excluded in this kind of puzzles, that usually talk of “a lady having two children...”).

Moreover, what matters is not the knowledge of the name, but rather something that allows us to point to him/her as ‘that one’. For this reason Q_2 and Q_4 have the same solution.

I would like to end with some comments on the last of the three text book questions reminded in the introduction. It seems to me that the reason there is quite a broad tendency to confuse Q_3 with Q_4 (or similar questions involving the child identification, including Q_2), is that *in normal life the information about boy/girl is acquired simultaneously with other attributes that make the identification unique* (“my daughter Claudia”). People do not express themselves as in math textbooks, stating that “I have two children, and at least one of them is a boy”, or “my children are not both boys”. We usually gain this information in an indirect way. For this reason several people have some initial difficulty to grasp that “that lady has Claudia and another child” is not the same as “that lady has two children, at least one being a girl”.

Moreover, even if a mother says “if I had two boys”, we may understand from the context (already knowing she has two children) that she has two girls, because we perceived that she emphasized ‘boys’ instead of ‘two’ (in the latter case we could think she has already a boy). Instead, if she said “if my children were both boys”, we usually understand that she is expressing this way because she has a boy and a girl. Therefore,

besides stereotyped recreational puzzles, the evaluation of probabilities, in the sense of how much we have to rationally believe the several hypotheses, can be not trivial. We need to take properly into account all contextual information “when the bare facts wont’do” [4]. Indeed, in probability evaluations not only the ‘facts’ play a role, but also the words, their sound and the expression of the person who says them, and (too often ignored) the question to which they reply [4].

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References

- [1] L. Mlodinow, *The Drunkard’s Walk. How Randomness Rules Our Lives*, Vintage Books, New York, 2008.
- [2] http://en.wikipedia.org/wiki/Boy_or_Girl_paradox.
- [3] G. D’Agostini, *Bayesian reasoning versus conventional statistics in high energy physics*, Proc. XVIII International Workshop on Maximum Entropy and Bayesian Methods, Garching (Germany), July 1998, V. Dose et al. eds., Kluwer Academic Publishers, Dordrecht, 1999.
<http://xxx.lanl.gov/abs/physics/9811046>.
- [4] J. Pearl, *Probabilistic reasoning in intelligent systems: Networks of Plausible Inference*, Morgan Kaufmann Publishers, San Mateo, 1988 (in particular Section 2.3.2).

Appendix — On the direct calculation of the elements of table 4

The elements of table 4 have been evaluated from the condition that the ‘central’ one vanishes and that the marginal probabilities have to be preserved (this means that, for example, the probability that the younger is female with the special name N is $r/2$ if no other information is provided, because $r/2$ is the assumed probability that an individual of that population carries that name). Nevertheless, one might be interested to calculate the eight non vanishing terms in a direct way. But this calculation might reserve surprises, as we shall see.

First row and first column of the table (at least one boy)

Although the elements that contain at least one boy are the easiest ones to be evaluated, let us get them with some pedantic detail, for didascalical purposes and in preparation of the less obvious cases. In particular, we shall rewrite $Em \cap YfN$ as $Em \cap Yf \cap YN$ to remember that we require the eldest child to be a boy, the youngest to be a girl and the name of the girl to be the particular one in which we are interested (YN). Applying the ‘chain rule’ we get

$$\begin{aligned}
 P(Em \cap Ym | I_2) &= P(Em | I_2) \times P(Ym | Em, I_2) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \\
 P(Em \cap YfN | I_2) &= P(Em \cap Yf \cap YN | I_2) \\
 &= P(Em | I_2) \times P(Yf | Em, I_2) \times P(YN | Yf, Em, I_2) \\
 &= \frac{1}{2} \times \frac{1}{2} \times r = \frac{r}{4} \\
 P(Em \cap Yf\bar{N} | I_2) &= P(Em \cap Yf \cap Y\bar{N} | I_2) \\
 &= P(Em | I_2) \times P(Yf | Em, I_2) \times P(Y\bar{N} | Yf, Em, I_2) \\
 &= \frac{1}{2} \times \frac{1}{2} \times (1 - r) = \frac{1 - r}{4},
 \end{aligned}$$

recovering the first row of table 4. (Note that \bar{N} stands for ‘a feminine name different from N ’ and not ‘any name but N ’!)

Similarly, the second and third elements of the first column are

$$\begin{aligned}
 P(EfN \cap Ym | I_2) &= P(Ef \cap EN \cap Ym | I_2) \\
 &= P(Ef | I_2) \times P(EN | Ef, I_2) \times P(Ym | Ef, EN, I_2) \\
 &= \frac{1}{2} \times r \times \frac{1}{2} = \frac{r}{4} \\
 P(Ef\bar{N} \cap Ym | I_2) &= P(Ef \cap E\bar{N} \cap Ym | I_2) \\
 &= P(Ef | I_2) \times P(E\bar{N} | Ef, I_2) \times P(Ym | Ef, E\bar{N}, I_2) \\
 &= \frac{1}{2} \times (1 - r) \times \frac{1}{2} = \frac{1 - r}{4}.
 \end{aligned}$$

The first column of table 4 is also recovered.

Probabilities of both females

The probability that two girls have the same name is zero, that is because

$$\begin{aligned}
 P(EfN \cap YfN | I_2) &= P(Ef \cap EN \cap Yf \cap YN | I_2) \\
 &= P(Ef | I_2) \times P(EN | Ef, I_2) \times P(Yf | Ef, EN, I_2) \\
 &\quad \times P(YN | Yf, Ef, EN, I_2) \\
 &= \frac{1}{2} \times r \times \frac{1}{2} \times 0 = 0.
 \end{aligned}$$

The third element of the second row is given by

$$\begin{aligned}
 P(EfN \cap Yf\bar{N} | I_2) &= P(Ef \cap EN \cap Yf \cap Y\bar{N} | I_2) \\
 &= P(Ef | I_2) \times P(EN | Ef, I_2) \times P(Yf | Ef, EN, I_2) \\
 &\quad \times P(Y\bar{N} | Yf, Ef, EN, I_2) \\
 &= \frac{1}{2} \times r \times \frac{1}{2} \times 1 = \frac{r}{4}.
 \end{aligned}$$

(The fourth factor of the r.h.s. of the last equation is 1 because, once we know the eldest child has the name N , the youngest cannot have that name.) Also the second row of table 4 is recovered.

The missing elements of the third row might present some pitfalls. Let us start from $Ef\bar{N} \cap YfN$, that is $Ef \cap E\bar{N} \cap Yf \cap YN$. It can be calculated as

$$\begin{aligned}
P(Ef\bar{N} \cap YfN | I_2) &= P(Ef \cap E\bar{N} \cap Yf \cap YN | I_2) \\
&= P(Yf \cap YN \cap Ef \cap E\bar{N} | I_2) \\
&= P(Yf | I_2) \times P(YN | Yf, I_2) \times P(Ef | Yf, YN, I_2) \\
&\quad \times P(E\bar{N} | Ef, Yf, YN, I_2) \\
&= \frac{1}{2} \times r \times \frac{1}{2} \times 1 = \frac{r}{4}.
\end{aligned}$$

thus obtaining the same value of the third element of the second row, that is $P(EfN \cap Yf\bar{N} | I_2)$, as we expect by symmetry and as it was in table 4.

A pitfall

But one would like to calculate $P(Ef\bar{N} \cap YfN | I_2)$ ‘the other way around’, i.e. applying the chain rule starting from $P(Ef\bar{N} | I_2)$. If one tries to proceed this way, there is *high chance to arrive to the following result*

$$\begin{aligned}
P(Ef\bar{N} \cap YfN | I_2) &= P(Ef \cap E\bar{N} \cap Yf \cap YN | I_2) \\
&= P(Ef | I_2) \times P(E\bar{N} | Ef, I_2) \times P(Yf | Ef, E\bar{N}, I_2) \\
&\quad \times P(YN | Ef, E\bar{N}, Yf, I_2) \\
&= \frac{1}{2} \times (1 - r) \times \frac{1}{2} \times r \\
&= \frac{\mathbf{r(1 - r)}}{4} = \frac{\mathbf{r}}{4} - \frac{\mathbf{r^2}}{4},
\end{aligned}$$

that differs from the value $r/4$ got previously.

In a similar way, one could be tempted to evaluate the probability that there are two girls, none of them carrying the name N , as

$$\begin{aligned}
P(Ef\bar{N} \cap Yf\bar{N} | I_2) &= P(Ef \cap E\bar{N} \cap Yf \cap Y\bar{N} | I_2) \\
&= P(Ef | I_2) \times P(E\bar{N} | Ef, I_2) \times P(Yf | Ef, E\bar{N}, I_2) \\
&\quad \times P(Y\bar{N} | Ef, E\bar{N}, Yf, I_2) \\
&= \frac{1}{2} \times (1 - r) \times \frac{1}{2} \times (1 - r) \\
&= \frac{(\mathbf{1 - r})^2}{4} = \frac{(\mathbf{1 - 2r})}{4} + \frac{\mathbf{r^2}}{4},
\end{aligned}$$

thus obtaining table 7, that differs from table 4. Namely, in the case of two females, it is now less probable that the youngest girl has the particular name N . The probabilities differ by $r^2/4$, thus being negligible for small r .

One might think it is right so, because it reflects the order of naming the children (“since the name N cannot be given twice, the the eldest girl has a kind of first choice”). But on the other hand, we are dealing here with knowledge (or ignorance), and therefore $P(EfN | I_2)$

Eldest		Youngest			
		m	fN	f	$f\bar{N}$
m		1/4	$r/4$	$(1-r)/4$	1/2
f	fN	$r/4$	-	$r/4$	$r/2$
	$f\bar{N}$	$(1-r)/4$	$r(1-r)/4$	$(1-r)^2/4$	$(1-r)/2$ 1/2
$m \cup f$		1/2	$r/2 - r^2/4$	$(1-r)/2 + r^2/4$	1
			1/2		

Table 7: Same as table 4, but obtained by a **wrong** reasoning that implicitly assumes that the condition $Ef\bar{N}$ does not change the probabilities of YfN and of $Yf\bar{N}$.

and $P(YfN | I_2)$ must be absolutely equal. It is just a question of symmetry in reasoning in conditions of uncertainty. It doesn't matter if we start thinking from the eldest or from the youngest child. Stated in different words, from a probabilistic point of view 'eldest' and 'youngest' are mere labels. The probability 'matrix' must be symmetric.

Moreover, it is curious to realize that table 7 produces a probability of two females that depends on r , as we saw in section 2.1. We get in fact

$$\begin{aligned}
 P[(Ef \cap Yf) | (EfN \cup YfN), I_2] &= \frac{r/4 + r(1-r)/4}{3 \times r/4 + r(1-r)/4} \\
 &= \frac{1}{2} \left[\frac{1-r/2}{1-r/4} \right],
 \end{aligned}$$

exactly the same result of section 2.1 [see Eq. (8)]. Therefore those who maintain that the probability of two girls, provided we know that one child is girl known by name, does depend on the rarity of the name either assume that identical names are possible inside the same family (a bizarre assumption), or have been caught by this pitfall (a mistake in reasoning).

Conditional probabilities of female names

The weak points of the previous evaluations, that lead to table 7, with all its consequences, are the conditional probabilities $P(YN | Ef, E\bar{N}, Yf, I_2)$ and $P(Y\bar{N} | Ef, E\bar{N}, Yf, I_2)$, for which we assumed *intuitively* the values r and $(1-r)$, as if the information that the eldest child is a girl with a name different from N did not change the probability of the name of the other girl. This intuition, *roughly but not exactly correct*, is due to the fact that we tend to consider r small (any modern population has a large amount of possible feminine names) such that the assumption that a girl has any name but the particular one (N) does not change sizably the probability of the name of the other girl. This is the reason why the correct results are recovered for $r \rightarrow 0$, that was the hidden initial assumption!

But, strictly speaking, the $Ef\bar{N}$ and YfN are not independent (in probability, or 'stochastically'), as are not independent $Ef\bar{N}$ and $Yf\bar{N}$: the information that the eldest girl has a name different from N has to increase the probability that the youngest girl is called N (and has to decrease the probability that also the youngest girl has a name different from N). Comparing tables 4 and 7 we see that the effect goes this direction and has a size that decreases rapidly with r , going as r^2 .

From this reasoning we get the following qualitative results:

$$\begin{aligned} P(YN | Ef, E\bar{N}, Yf, I_2) &> P(YN | Yf, I_2) \\ P(Y\bar{N} | Ef, E\bar{N}, Yf, I_2) &< P(Y\bar{N} | Yf, I_2). \end{aligned}$$

The only problem is that it is not easy to evaluate these probabilities. But, fortunately, they can be calculated from the general rules of probability, remembering that, as discussed above, the probabilities of $Ef\bar{N} \cap YfN$ and of $Ef\bar{N} \cap Yf\bar{N}$ can be obtained in different ways. More precisely, the probability of $Ef\bar{N} \cap YfN$ can be calculated either directly, from the easy $P(Y\bar{N} | Yf, Ef, EN, I_2)$, as we have done just above in the appendix, or indirectly, requiring $P(YfN | I_2) = r/2$, as it was done building up table 4 in section 2.2. Instead, the last element of the table, $P(Ef\bar{N} \cap Yf\bar{N} | I_2)$, can be only calculated indirectly, either requiring $P(Yf\bar{N} | I_2) = (1 - r)/2$, or from the normalization rule, i.e. from the constraint that all elements of the table have to sum up to one.

Therefore, the conditional probabilities of interest can be finally evaluated from the joint probabilities as

$$\begin{aligned} P(YN | Ef, E\bar{N}, Yf, I_2) &= \frac{P(YN \cap Ef \cap E\bar{N} \cap Yf | I_2)}{P(Ef \cap E\bar{N} \cap Yf | I_2)} \\ &= \frac{P(YN \cap Ef \cap E\bar{N} \cap Yf | I_2)}{P(Ef \cap E\bar{N} | I_2) \cdot P(Yf | Ef \cap E\bar{N}, I_2)} \\ &= \frac{r/4}{(1 - r)/2 \times 1/2} = \frac{r}{1 - r} \\ &\approx r \quad (\text{for } r \ll 1) \\ P(Y\bar{N} | Ef, E\bar{N}, Yf, I_2) &= \frac{P(Y\bar{N} \cap Ef \cap E\bar{N} \cap Yf | I_2)}{P(Ef \cap E\bar{N} \cap Yf | I_2)} \\ &= \frac{P(Y\bar{N} \cap Ef \cap E\bar{N} \cap Yf | I_2)}{P(Ef \cap E\bar{N} | I_2) \cdot P(Yf | Ef \cap E\bar{N}, I_2)} \\ &= \frac{(1 - 2r)/4}{(1 - r)/2 \times 1/2} = \frac{1 - 2r}{1 - r} \\ &\approx 1 - r \quad (\text{for } r \ll 1). \end{aligned}$$

[Obviously, the latter probability could have been calculated easier as $P(Y\bar{N} | Ef, E\bar{N}, Yf, I_2) = 1 - P(YN | Ef, E\bar{N}, Yf, I_2) = 1 - r/(1 - r)$, getting the same result.]

Note that for very small values of r we recover $P(YN | Yf, I_2) = r$ and $P(Y\bar{N} | Yf, I_2) = 1 - r$, respectively. That is, in this limit $Ef\bar{N}$ and YfN , as well as $Ef\bar{N}$ and $Yf\bar{N}$, are approximately independent, in agreement with our initial intuition.

Finally, we remind that the probabilities of the eldest girl name, conditioned by $Yf\bar{N}$, can be obtained by symmetry, i.e. $P(EN | Yf, Y\bar{N}, Ef, I_2) = P(YN | Ef, E\bar{N}, Yf, I_2) = r/(1 - r)$ and $P(E\bar{N} | Yf, Y\bar{N}, Ef, I_2) = P(Y\bar{N} | Ef, E\bar{N}, Yf, I_2) = (1 - 2r)/(1 - r)$, and that all these expressions depend on the simplifying assumptions embedded in this kind of recreational puzzle.