Normal distribution

Some technical remarks

- Standard normal: $Z \sim \mathcal{N}(0,1)$.
- Interesting properties of the log of the pdf of the normal (not to be confused with the 'lognormal'!):

$$\begin{aligned}
\varphi(x) \equiv -\ln f(x) &= \frac{(x-\mu)^2}{2\,\sigma^2} + k \\
\varphi_{min} &= k \\
\text{`}\Delta\varphi^{(\frac{1}{2})}\text{'} \; = \; \varphi(\mu \pm \sigma) - \varphi_{min} &= \frac{1}{2} \\
\frac{d\varphi}{dx} &= \frac{x-\mu}{\sigma^2} \quad \xrightarrow[==0]{} \quad x_m = \mu \\
\frac{d^2\varphi}{d^2x} &= \frac{1}{\sigma^2}
\end{aligned}$$

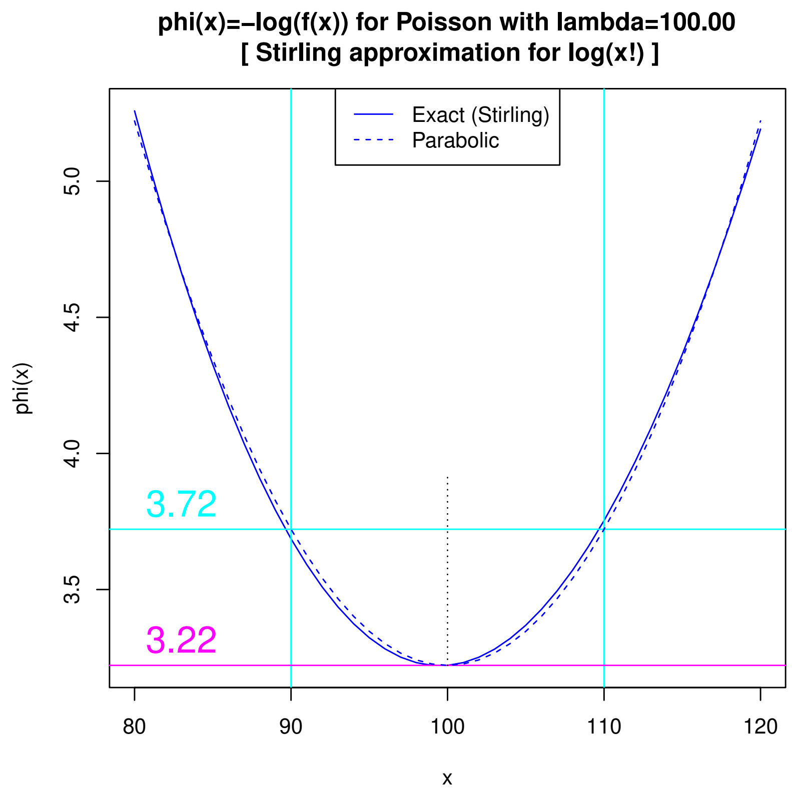
$\Rightarrow$ Useful 'tricks' to evaluate mean and variance of distributions "assumed to be almost normal".

- Beware when they become **rules**! (Or **prescriptions**, or even **'principles'**. . .)

(Quite academic) example of the 'Gaussian tricks'

In the case of a Poisson distribution, we can calculate $\varphi(x)$, making x continuous and using Stirling's approximation:

$$\begin{aligned}
\varphi(x) &= -\ln\left(\frac{e^{-\lambda}\lambda^x}{x!}\right) \\
&= +\lambda - x\ln\lambda + \ln(x!) \\
&\approx \lambda - x\ln\lambda + x\ln x - x + \frac{1}{2}\ln x + \frac{1}{2}\ln 2\pi
\end{aligned}$$



A numerical example ($\lambda = 100$), using $\mathrm{E}[X] = \sigma^2 = \lambda$, and comparing with a parabolic function around the minimum.

Exercise on the 'Gaussian trick' applied to the Poisson distribution

Evaluation of E[X] and $\sigma^2(X)$, for large λ and treating x as continuous

$$\varphi(x) = \lambda - x\ln\lambda + x\ln x - x + \frac{1}{2}\ln x + \frac{1}{2}\ln 2\pi \qquad (1)$$

$$\frac{d\varphi}{dx} = \qquad (2)$$

$$\frac{d^2\varphi}{d^2x} = \qquad (3)$$

$\lambda = 100$:

1. 'estimate' E[X] mimizing (1);
2. 'estimate' E[X] from the root of (2), i.e. from $d\varphi/dx = 0$;
3. then 'estimate' $\sigma^2(X)$ from (3), after having 'got' E[X] from the previous items.
4. Finally 'estimate' $\sigma(X)$ using the '$\Delta\varphi$ trick'.