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ELECTRIC FIELD FOR A UNIFORMLY CHARGED CYLINDRICAL BUNCH WITH ELLIPTICAL CROSS SECTION

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Abstract

The focusing properties of different accelerator's devices, as quadrupoles and bending magnets, can change the charge distribution of a bunch from round to elliptical one. Depending on the intensity of this effect, the dynamics properties of the bunch could be affected. For the SPARC project the ratio between the two semi axes can reach the value of 0.2. The consequent increase of the design emittance reduces the performances of the machine. In this note we study the electric field generated by a bunch of finite length and elliptical cross section, travelling with constant velocity *v*. This electric field is then compared to that of a bunch with circular cross section of diameter equal to the sum of the two semi axes. The comparison can be used to verify the validity of the round beam approximation in the study of the beam dynamics.

1 INTRODUCTION

The focusing properties of different accelerator's devices, as quadrupoles and bending magnets, can change the charge distribution of a bunch from round to elliptical one, thus affecting the dynamic properties of the bunch. For the SPARC project the ratio between the two semi axes can reach the value of 0.2 thus increasing the design emittance of the machine.

In order to compare the effect of an elliptical cross section bunch on the beam dynamics with respect to that of a circular cross section bunch, it is important to know the electromagnetic field produced by the bunch itself.

In section 2 we describe how to obtain the electric field of a uniformly charged infinite cylinder, with elliptical cross section.

In section 3 we derive the on axis longitudinal and radial electric field of a uniformly charged cylinder with circular cross section and finite length. The cylinder is moving along the z-axis at velocity v. For the longitudinal electric field an exact solution can be obtained, whilst for the radial electric field only an approximate solution is derived limited to a radial linear dependence.

In section 4 we obtain an integral expression of the on axis longitudinal and radial electric field of a bunch moving along the z-axis at velocity v and shaped like a uniformly charged finite cylinder having an elliptical cross section (fig.1). Furthermore we derive an approximate solution of the longitudinal electric field by a series expansion of the integral expression.

In section 5 we solve numerically the above equations and compare the numeric solutions of the integrals with approximate formulas. Moreover it is well known that for an infinite bunch with elliptical cross section, the radial electric field calculated on the two semi axis a and b, is equal to the radial electric field of an infinite bunch with circular cross section of radius $r = (a+b)/2$.We show that this general rule cannot be applied for finite distributions.

In section 6 we relate the electric field's behaviour of a finite length bunch with the so called aspect ratio defined as $A = R/\gamma L$, where R is the bunch radius, L its length and γ the relativistic parameter.

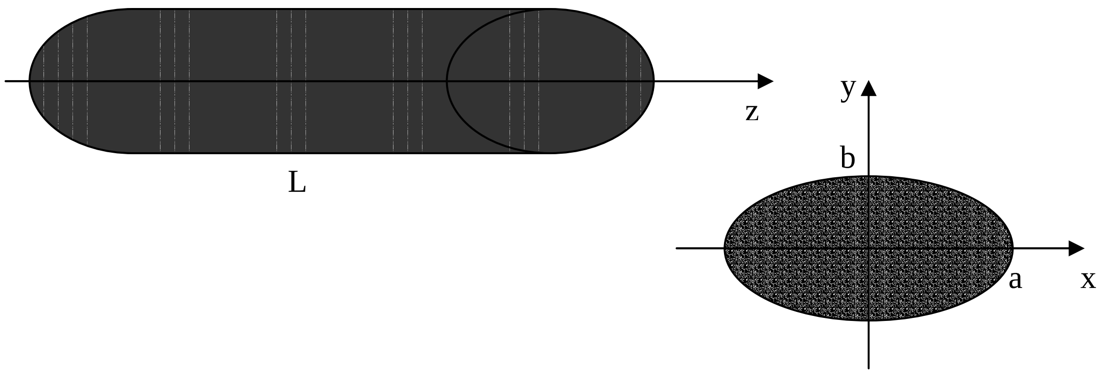


FIG. 1: Uniformly charged bunch of finite length *L*, with elliptical cross section of semi axis *a* and *b*.

2 INFINITE CYLINDER WITH ELLIPTICAL CROSS SECTION

The electric field of an elliptical cross section cylinder with infinite length can be derived following ref. [7]. Starting from the potential of an ellipsoid of semi axis *a*, *b*, *c* and charge *q* with $c\text{->}\infty$, we obtain the potential of an infinite cylinder with elliptical cross section moving at velocity *v*. From the potential we get the radial electric field inside the cylinder [1-6]

$$E_x = \frac{\rho}{\varepsilon_0}\frac{bx}{a+b} \tag{1}$$

$$E_y = \frac{\rho}{\varepsilon_0}\frac{ay}{a+b} \tag{2}$$

where ρ is the charge density in the laboratory frame.

Only the radial electric field exists since the longitudinal electric field is zero for an infinite distribution. It is important to note that the electric fields component on the cylinder axes, in *a* and *b*, are the same, that is $E_{x=a} = E_{y=b}$, and they are equal to the electric field generated by an infinite cylinder with circular cross section

$$E_r = \frac{\rho}{2\varepsilon_0} r \text{ at } r = \frac{a+b}{2}$$

In section 5 we will discuss if the same method can also be used in the case of a finite distribution.

3 FINITE CYLINDER WITH CIRCULAR CROSS SECTION

The longitudinal and radial electric field for a circular cross section finite cylinder are [7]:

$$E_z(z, r=0) = \frac{\rho}{2\varepsilon_0\gamma}\left[\sqrt{R^2+\gamma^2(L-z)^2} - \sqrt{R^2+\gamma^2 z^2} + \gamma|z| - \gamma|z-L|\right] \tag{3}$$

and

$$E_r(r,z) = \frac{\rho\gamma}{4\varepsilon_0} r\left[\frac{(L-z)}{\sqrt{r^2+\gamma^2(L-z)^2}} + \frac{z}{\sqrt{r^2+\gamma^2 z^2}}\right] \tag{4}$$

where *r* is the radial coordinate, *L* the bunch’s length and ρ its charge density.

The longitudinal electric field is calculated on the bunch’s axis z, whilst the radial electric field is obtained from the on axis longitudinal electric field considering linear terms in *r*. As a consequence eq. (4) is an approximation and its accuracy increases with longer bunch. Both equations (3) and (4) are obtained in appendix A.1.

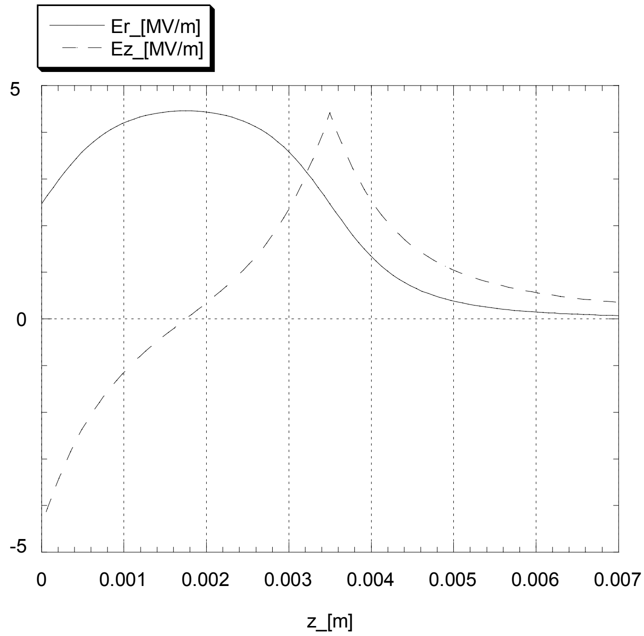


FIG. 2: Longitudinal (dashed line) and radial (continuous line) electric field of a bunch of length *L=3.4mm*, circular cross section of radius *r=1mm*, charge *q=1nC* and relativistic factor γ*=1.*

The longitudinal and radial electric field, generated by the estimated bunch of the SPARC project, is shown in fig.2. The longitudinal electric field grows inside the bunch and its absolute value is at its maximum on the bunch's head and tail, whilst the radial one is at his maximum in the centre of the bunch.

4 FINITE CYLINDER WITH ELLIPTICAL CROSS SECTION SECTION

The procedure used to calculate the longitudinal and radial electric field of a bunch with elliptical cross section is similar to that explained in the previous section. To this purpose we can use eq. (5) from the appendix A.1 to calculate the longitudinal electric field:

$$E_z(z,0)=\frac{\rho}{4\pi\varepsilon_0\gamma}\int_0^{2\pi}\left\{\sqrt{r'^2+\gamma^2(L-z)^2}-\sqrt{r'^2+\gamma^2 z^2}-\gamma|L-z|-\gamma|z|\right\}d\phi \qquad (5)$$

The radius *r'*, in the elliptic case, is not constant but depends on the angle ϕ as:

$$r'^2=\frac{b^2}{1-e^2\cos^2\phi} \qquad (6)$$

where the eccentricity of the ellipse is $e=\sqrt{1-\frac{b^2}{a^2}}$.

It can be easily verified that as *L* approaches infinite the longitudinal electric field goes to zero. To simplify the integral (5) we expand eq. (6) in *e,* up to the fourth order:

$$r'^2 = \frac{b^2}{1-e^2\cos^2\phi} \cong b^2(1+e^2\cos^2\phi + e^4\cos^4\phi + ...) \tag{7}$$

In appendix A.2 the second order approximation is derived; that is we neglect the fourth order term in the expansion (7) and solve the corresponding integral of eq.(5) obtaining

$$E_z = \frac{\rho}{\pi\varepsilon_0\gamma}\left\{\alpha\cdot \mathrm{E}[\Gamma^2] - \beta\cdot \mathrm{E}[\Lambda^2] - \frac{\rho}{2\varepsilon_0}\left[|L-z| - |z|\right]\right\} \tag{8}$$

with

$$\alpha = \sqrt{b^2(1+e^2)+\gamma^2(L-z)^2}$$

$$\beta = \sqrt{b^2(1+e^2)+\gamma^2 z^2}$$

$$\Gamma^2 = \frac{b^2e^2}{b^2(1+e^2)+\gamma^2(L-z)^2}$$

$$\Lambda^2 = \frac{b^2e^2}{b^2(1+e^2)+\gamma^2 z^2}$$

and

$$\mathrm{E}[x^2] = \int_0^{\pi/2}\sqrt{1-x^2\sin^2\varphi}\,d\varphi$$

Moreover if we keep the fourth order term in eq.(7), the square roots of eq.(5) become

$$\sqrt{b^2+k^2}\sqrt{1+\frac{b^2}{b^2+k^2}(e^2\cos^2\phi+e^4\cos^4\phi)} \tag{9}$$

$$\sqrt{b^2+k'^2}\sqrt{1+\frac{b^2}{b^2+k'^2}(e^2\cos^2\phi+e^4\cos^4\phi)} \tag{10}$$

where

$$\begin{aligned} k^2 &= \gamma^2(L-z)^2 \\ k'^2 &= \gamma^2 z^2 \end{aligned} \tag{11}$$

We can expand the square root (9) as

$$\sqrt{b^2+k^2}\begin{Bmatrix} 1+\dfrac{1}{2}\left(\dfrac{b^2}{b^2+k^2}\right)(e^2\cos^2\phi+e^4\cos^4\phi) \\ -\dfrac{1}{8}\left(\dfrac{b^2}{b^2+k^2}\right)^2(e^2\cos^2\phi+e^4\cos^4\phi)^2+ \\ \dfrac{3}{48}\left(\dfrac{b^2}{b^2+k^2}\right)^3(e^2\cos^2\phi+e^4\cos^4\phi)^3 \end{Bmatrix} \tag{12}$$

Expanding the square and the cube in the above expression and keeping the term up to the fourth order we have:

$$\sqrt{b^2+k^2}\left\{1+\frac{1}{2}\left(\frac{b^2}{b^2+k^2}\right)e^2\cos^2\phi+\left(\frac{b^2(3b^2+4k^2)}{8(b^2+k^2)^2}\right)e^4\cos^4\phi\right\} \tag{13}$$

The same can be done for eq.(10) getting eq.(13) with *k* replaced by *k'*

$$\sqrt{b^2+k'^2}\left\{1+\frac{1}{2}\left(\frac{b^2}{b^2+k'^2}\right)e^2\cos^2\phi+\left(\frac{b^2(3b^2+4k'^2)}{8(b^2+k'^2)^2}\right)e^4\cos^4\phi\right\} \tag{14}$$

Eqs.(13) and (14) can be inserted in the integral (5) and solved easily with respect to the variable ϕ thus obtaining the fourth order approximation for the longitudinal electric field

$$E_z(z,0)=\frac{\rho}{4\pi\varepsilon_0\gamma}\begin{Bmatrix}\sqrt{b^2+k^2}\left[2\pi+\dfrac{b^2\pi}{2(b^2+k^2)}e^2+\dfrac{3b^2\pi(3b^2+4k^2)}{32(b^2+k^2)}e^4\right]- \\ \sqrt{b^2+k'^2}\left[2\pi+\dfrac{b^2\pi}{2(b^2+k'^2)}e^2+\dfrac{3b^2\pi(3b^2+4k'^2)}{32(b^2+k'^2)}e^4\right]\end{Bmatrix} \tag{15}$$

The radial electric field on the contrary can't be obtained from the longitudinal one as in the previous case. For an elliptical cross section, in fact, it does not exist any ϕ angle symmetry (see appendix A.1). For this reason we calculate the radial electric field referring to fig.3 and eq. (A.1):

$$\underline{E}=\frac{q(1-\beta^2)}{4\pi\varepsilon_0 r^2(1-\beta^2\sin^2\theta)^{\frac{3}{2}}}\hat{r} \tag{16}$$

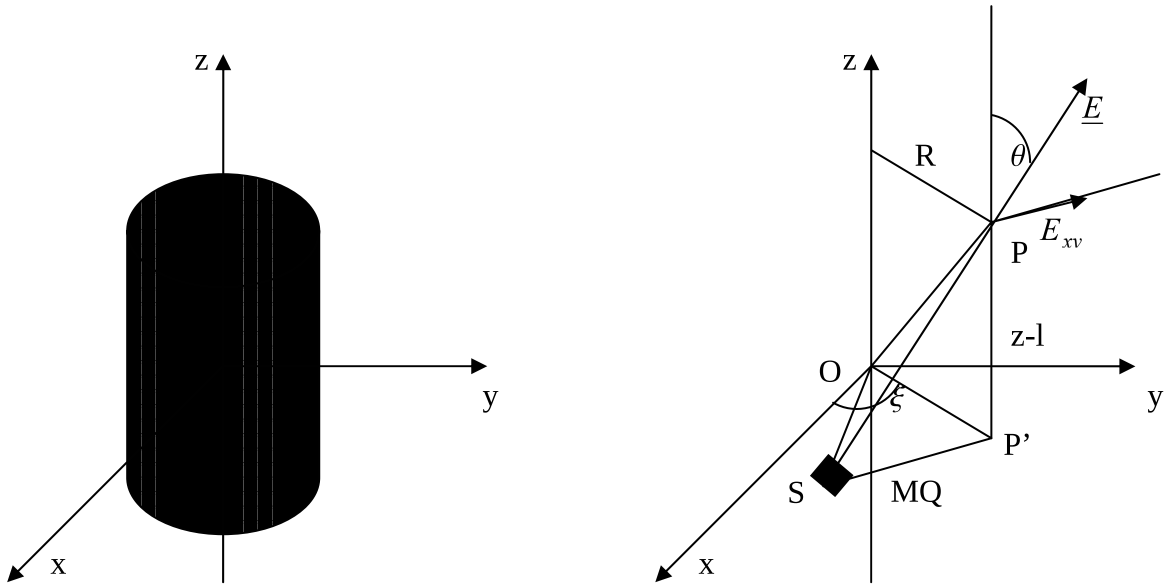


FIG. 3: 3D-coordinate system used to calculate the radial electric field of a uniformly charged cylinder with elliptical cross section.

The box on the right hand side of fig.3 represents the elementary charge density to be integrated over the bunch's volume. The observer's position is at a distance $\overline{SP}$ from the elementary charge and it is placed in the bunch's volume. Eq. (16), with *r* replaced by $\overline{SP}$, represents the electric field $\underline{E}$. Since we are interested to the radial electric field E_r, we project it on the *x-y* plane: the elementary electric field component belonging to the plane *x-y* is

$$dE_{xy} = \frac{\rho(1-\beta^2)\sin\theta}{4\pi\varepsilon_0 \overline{SP}^2 (1-\beta^2 \sin^2\theta)^{3/2}} d\tau \qquad (17)$$

where ρ is the charge density and θ represents the angle between the axis z and the observer vector. Referring to fig.4 we obtain the elementary radial electric field

$$dE_r = \frac{\rho(1-\beta^2)\sin\theta\cos\alpha}{4\pi\varepsilon_0 \overline{SP}^2 (1-\beta^2 \sin^2\theta)^{3/2}} d\tau \qquad (18)$$

where α is the angle between the *x-y* electric field component and the radial vector $\hat{r}$.

In eq. (17) and (18) $\rho d\tau = \sigma d\sigma d\phi dl$ represents the elementary density charge and σ is the distance $\overline{OS}$

From fig.4 we can easily write:

$$\overline{SP}\sin\theta = \overline{SP'}$$

$$\overline{SP} = \sqrt{(z-l)^2 + \overline{SP'}^2} \qquad (19)$$

where *P'* is the projection of the observer *P* on the x-y plane and

$$\overline{SP'} = \sqrt{\sigma^2 + R^2 - 2\sigma R \cos(\xi - \phi)} \tag{20}$$

R is the observer's distance from the bunch center and ξ the angle between the observer and the x axis.

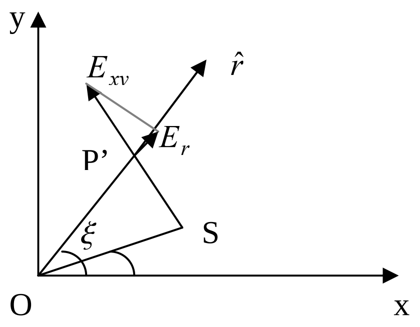


FIG. 4: 2D-coordinate system.

Inserting eq.(19) and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

in eq.(18) we have:

$$E_r = \int\int\int \frac{\rho\gamma}{4\pi\varepsilon_0} \frac{\overline{SP'}\cos\alpha}{\left(\gamma^2(z-l)^2 + \overline{SP'}^2\right)^{3/2}} \sigma d\sigma d\phi dl \tag{21}$$

with

$$0 < l < L$$

$$0 < \sigma < \sqrt{\frac{b^2}{1-e^2\cos^2\phi}}$$

$$0 < \phi < 2\pi$$

The integral (21) can be solved with respect to the coordinate l, thus obtaning:

$$dE_r = \frac{\rho\gamma}{4\pi\varepsilon_0} \frac{1}{\overline{SP'}} \left[\frac{z}{\sqrt{\gamma^2 z^2 + \overline{SP'}^2}} - \frac{z-L}{\sqrt{\gamma^2(z-L)^2 + \overline{SP'}^2}} \right] \cos\alpha\sigma d\sigma d\phi$$

Using eq. (20) and

$$\cos\alpha = \frac{-\sigma^2 + \overline{SP'}^2 + R^2}{2\overline{SP'}R} = \frac{R - \sigma\cos(\xi-\phi)}{\sqrt{R^2 + \sigma^2 - 2R\sigma\cos(\xi-\phi)}}$$

eq.(21) becomes:

$$E_r(R,\xi,z) = \frac{\rho\gamma}{4\pi\varepsilon_0}\left\{zI_1 - (z-L)I_2\right\} \tag{22}$$

with

$$I_1 = \int\int \frac{\sigma\left(R - \sigma\cos(\xi-\phi)\right)}{\left(R^2 + \sigma^2 - 2R\sigma\cos(\xi-\phi)\right)\sqrt{\gamma^2 z^2 + R^2 + \sigma^2 - 2R\sigma\cos(\xi-\phi)}}\, d\sigma d\phi$$

$$I_2 = \int\int \frac{\sigma\left(R - \sigma\cos(\xi-\phi)\right)}{\left(R^2 + \sigma^2 - 2R\sigma\cos(\xi-\phi)\right)\sqrt{\gamma^2 (z-L)^2 + R^2 + \sigma^2 - 2R\sigma\cos(\xi-\phi)}}\, d\sigma d\phi$$

and

$$0 < \sigma < \sqrt{\frac{b^2}{1-e^2\cos\phi}}$$

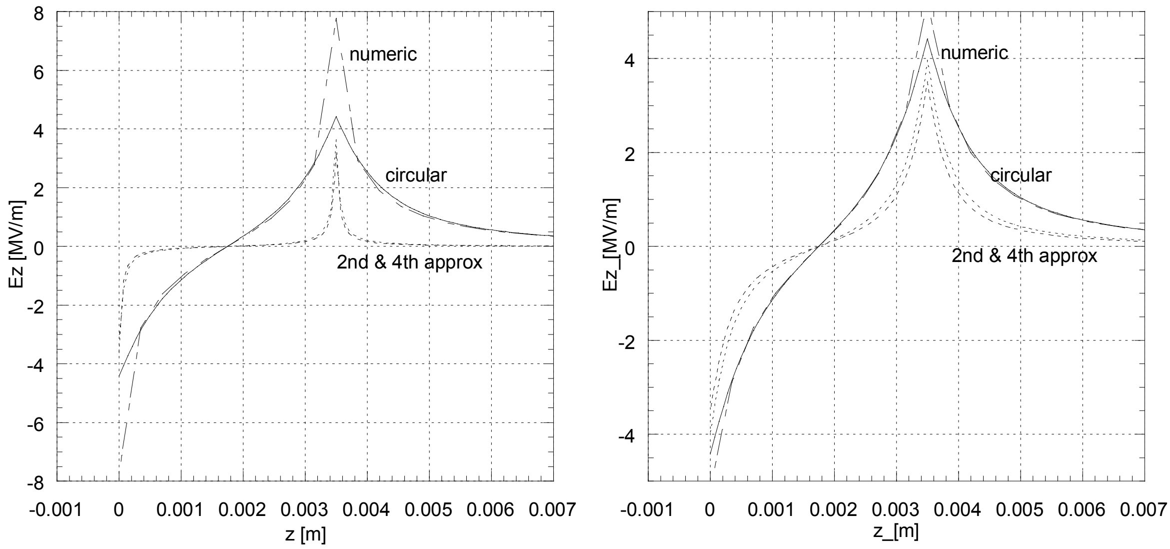
$$0 < \phi < 2\pi$$

Eq.(22) represents the radial electric field an observer sees if he is placed at a distance R from the bunch center with an angle ξ respect to the x axis coordinate and at a distance z form the bunch's tail.

It has been verified numerically that as L approaches infinite the integral (22) approaches the electric fields of an infinite elliptical cylinder (eq. (1) and (2)).

5 COMPARISON BETWEEN NUMERICAL AND ANALYTICAL FIELD

The two approximations (8) and (15) for the longitudinal electric field can be compared with the numerical solution of the integral (5) and with the longitudinal electric field of the circular cross section cylinder with $r = \frac{a+b}{2}$. Fig.5 shows the comparison for different values of the eccentricity:



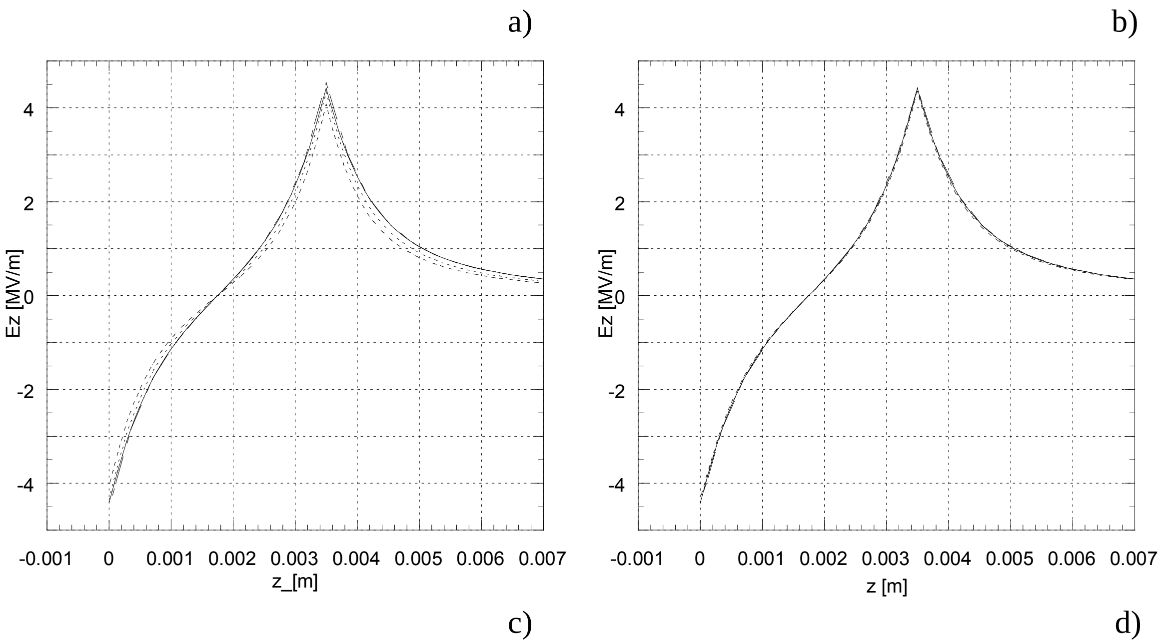
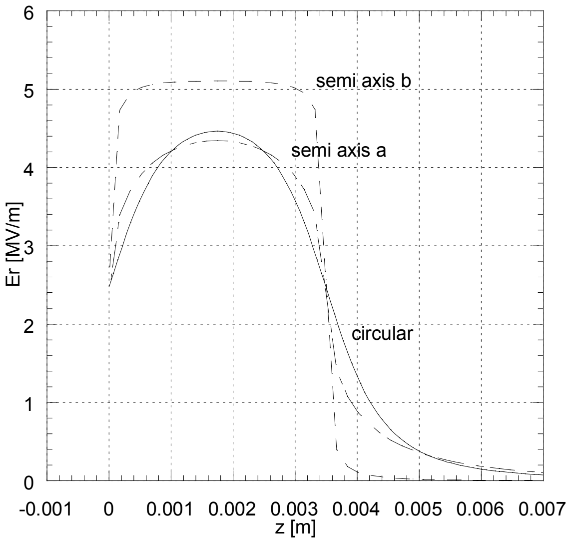
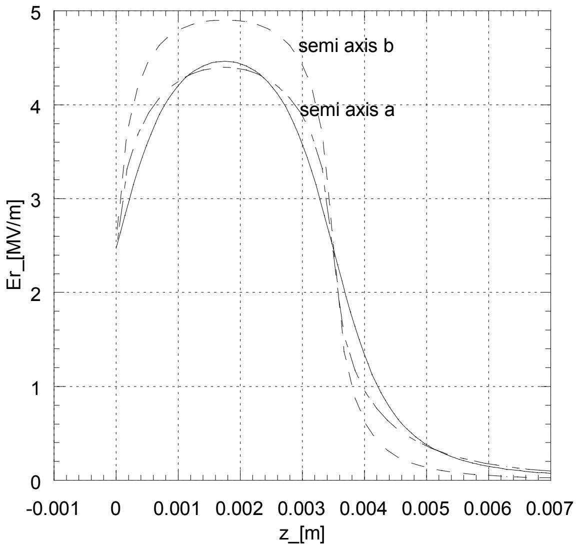


FIG. 5: Comparisons between the longitudinal electric field obtained numerically (dashed line), the longitudinal electric field of the circular cross section case with *r=(a+b)/2* (continuous line) and the longitudinal electric field approximated by the 2^{nd} and the 4^{th} order terms (dotted line). The length, the charge and the relativistic factor are the same of fig. 2; the eccentricity is *(a) e=0.999671 (b/a=0.026)* , *(b) e=0.979055 (b/a=0.2*), *(c) e=0.830901(b/a=0.556)* , *(d) e=0.56296 (b/a=0.826).*

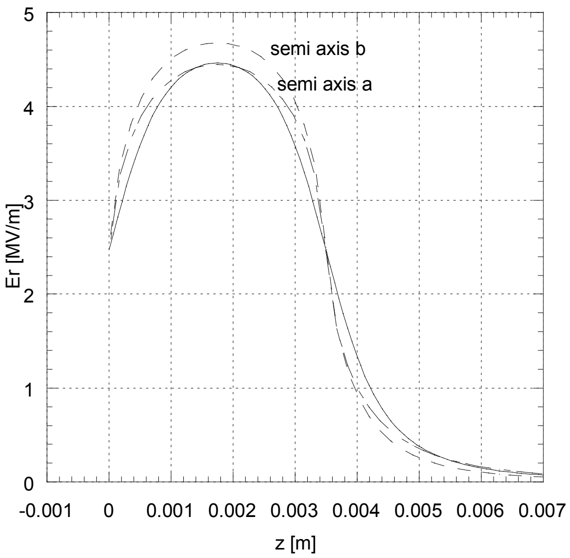
When the eccentricity is zero, that is the bunch has a circular cross section, all the graphs coincide; more the eccentricity approaches to unity, more the graphs draw away. For all the different eccentricity’s values, the circular cross section bunch electric field (continuous line) is the closest to the numerical solution, as can be seen form fig.5.
Fig.6 shows a comparison between the radial electric field obtained numerically from the above eq. (22), on the semi axis *a* (ξ=0) and *b* ($\xi = \frac{\pi}{2}$), and eq. (4), that is the radial electric field for a circular cross section finite bunch whose radius is $r = \frac{a+b}{2}$.



a)



b)



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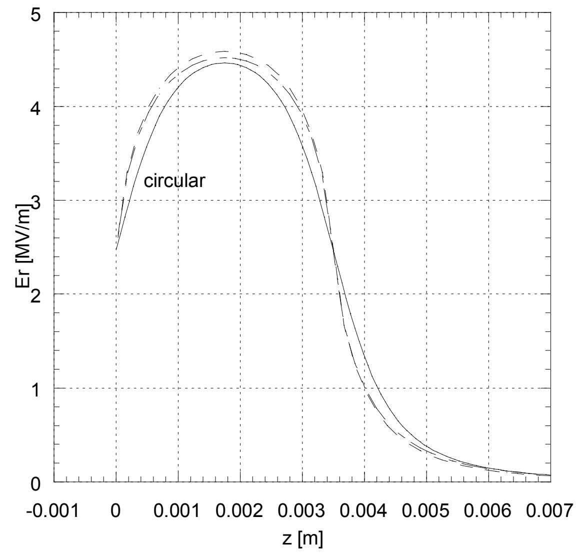

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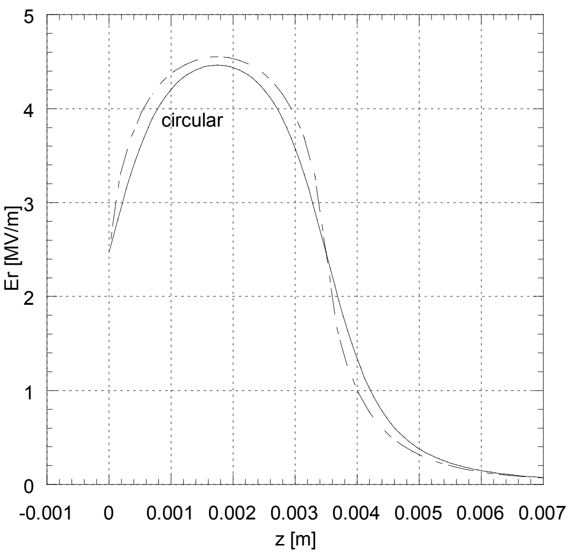


e)

FIG. 6: Comparisons between the radial electric field obtained numerically on the semi axis *a* and *b* (θ=0 and $\theta=\pi/2$ respectively) (dashed lines) , the radial electric field of the circular cross

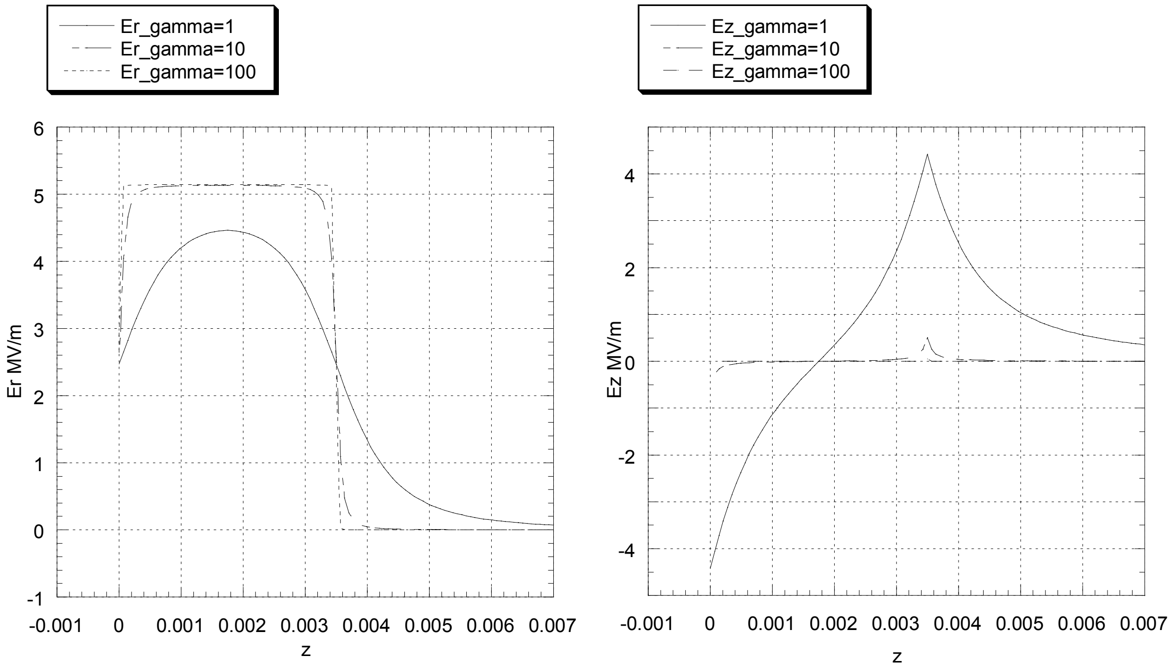
section case with *r=(a+b)/2* (continuous line). The length, the charge and the relativistic factor are the same of fig. 2; the eccentricity is *(a) e=0.999671 (b/a=0.026)* , *(b) e=0.979055 (b/a=0.2)*, *(c) e=0.830901(b/a=0.556)* , *(d) e=0.56296 (b/a=0.826), (e) e=0 (b/a=1)* .

In the case of an elliptical cross section finite bunch, as can be seen from the figures, the radial electric field calculated on the major semi axis, $x=a$, is different from the one calculated on the minor semi axis, $y=b$. The radial electric field in $x=a$ is close to the circular cross section one even for high eccentricity; meanwhile the one calculated in $y=b$ is still far from the approximation for lower eccentricity. Of course as the eccentricity approaches to zero all the plots become closer; when the eccentricity is zero the fields, calculated on the two semi axis, are of course the same. It is important to note that, since the radial electric field of eq. (4) is an approximation, the fields of fig.6e do not coincide perfectly.

6 ROLE OF THE ASPECT RATIO PARAMETER IN THE FIELD FORM FACTOR

We can explain the radial fields' different behaviour on the two semiaxis, with respect to the infinite length case, observing that in this case the length of the bunch plays an important role. When the eccentricity is close to unity then $b<<L$ and as a consequence the field's behaviour is similar to that of an infinite bunch, that is it is constant with the longitudinal coordinate z and shows a discontinuity as it reaches the edges of the bunch. On the contrary since a is comparable to the bunch length L its behaviour is different from an infinite bunch.
As L approaches zero we obtain the radial electric field of an elliptical disk uniformly charged. Using eq.(22) we can numerically verify that the radial electric field in $x=a$ and $y=b$ are still different.

Besides the bunch is moving with a velocity v along the z axis, so when its velocity increases, or the relativistic parameter γ increases, the radial electric field becomes more and more squared. Of course the same behaviour can be obtained for a circular cross section bunch as well, just increasing its length or increasing its relativistic parameter γ as shown in fig.7. Note that the longitudinal electric field approaches zero as for an infinite bunch.



a) b)

FIG. 7: Radial (a) and longitudinal (b) electric field for a circular cross section bunch. The length and the charge of the bunch are the same of fig.2

It follows from the above explanation that the field's behaviour in x=a and y=b depends on the size of the ellipse's semi axis with respect to the bunch's length or its relativistic parameter γ, that is on the so called aspect ratio (ref. [8]):

$$A = \frac{R}{\gamma L}$$

where R is the major semi axis a or the minor semi axis b. We deduce that as the aspect ratio decreases the electric field becomes more and more squared.

In appendix A.1 the longitudinal and radial electric field, for a circular cross section bunch, as a function of the aspect ratio, are obtained. For an elliptical cross section cylinder the fields can be written as a function of the aspect ratio as well:

$$E_r(A,\xi,z) = \frac{\rho\gamma L}{4\pi\varepsilon_0}\left\{\frac{z}{L}I_1{}' - (\frac{z}{L}-1)I_2{}'\right\}$$

$$E_z(z,0) = \frac{\rho L}{4\pi\varepsilon_0}\int_0^{2\pi}\left\{\sqrt{u^2+(1-\frac{z}{L})^2} - \sqrt{u^2+\left(\frac{z}{L}\right)^2} - \left|1-\frac{z}{L}\right| - \left|\frac{z}{L}\right|\right\}d\phi$$

where

$$I_1' = \int\int \frac{s\left(A - s\cos(\xi-\phi)\right)}{\left(A^2 + s^2 - 2As\cos(\xi-\phi)\right)\sqrt{\left(\frac{z}{L}\right)^2 + A^2 + s^2 - 2As\cos(\xi-\phi)}}\, ds d\phi$$

$$I_2' = \int\int \frac{s\left(A - s\cos(\xi-\phi)\right)}{\left(A^2 + s^2 - 2As\cos(\xi-\phi)\right)\sqrt{\left(\frac{z}{L}-1\right)^2 + A^2 + s^2 - 2As\cos(\xi-\phi)}}\, ds d\phi$$

and

$$u^2 = \frac{\left(\frac{b}{\gamma L}\right)^2}{1 - e^2\cos^2\phi}$$

$$s = \frac{\sigma}{\gamma L}$$

$$0 < s < \sqrt{\frac{\left(\frac{b}{\gamma L}\right)^2}{1 - e^2\cos^2\phi}}$$

$$0 < \phi < 2\pi$$

From the previous considerations we deduce that the longitudinal electric field of a finite length bunch with elliptical cross section is well approximated by the longitudinal electric field of a finite length bunch with circular cross section and radius $r = \frac{a+b}{2}$. On the contrary the radial electric fields on the envelope $x=a$ and $y=b$ become more and more different as the aspect ratio A increases and the circular approximation can't be used anymore. Fig.8 shows the ratio between the two radial fields when the aspect ratio grows for different eccentricity's value. The aspect ratio in fig.8 is $A = \frac{a}{\gamma L}$

Note that when the eccentricity decreases the ratio between the two fields becomes closer to unity even for a higher aspect ratio.

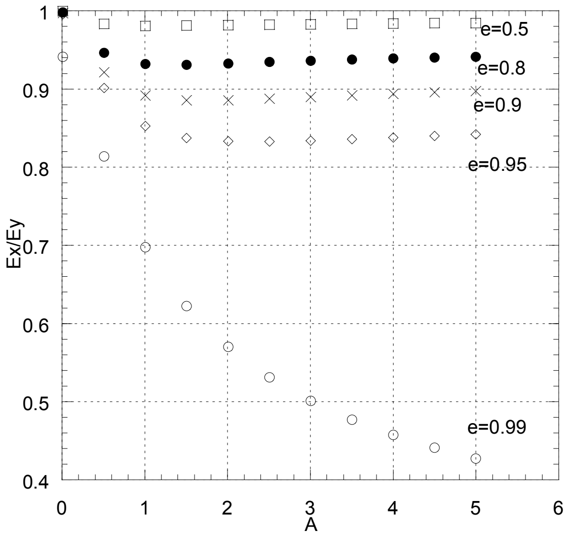


FIG. 8: Ratio between the radial electric field in *x=a* and *y=b* in *z=L/2* as a function of the aspect ratio *A* for different eccentricity *e=0.999671* (*b/a=0.026*), *e=0.95 (b/a=0.31), e=0.9 (b/a=0.43), e=0.8(b/a=0.6)* and *e=0.5* (*b/a=0.866*).

7 CONCLUSIONS

In this paper we reviewed the analytical formulas for the electric field generated by a cylindrical bunch either of infinite length with elliptical cross section and of finite length with circular cross section. Moreover, we obtained the electric field of a finite length bunch with elliptical cross section and compared its numerical solution with the electric field of a circular cross section bunch of finite length whose radius is $r = \frac{a+b}{2}$. Even if this rule fits the infinite cylinder case, it is not appropriate for the finite cylinder case, unless the eccentricity is lower than 0.8 for $\gamma = 1$ (fig.6d and fig.8). The different behaviour of these two cases can be explained by the aspect ratio $A = \frac{R}{\gamma L}$. In particular the radial electric field of a finite bunch becomes that of an infinite bunch when *A* becomes small, that is when the length or □ increases. For example, for a very high eccentricity (*e=0.99*) but small aspect ratio (*A=0.01*) the circular cross section approximation can still be used (fig.8).

8 ACKNOWLEDGEMENTS

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10 APPENDIX

A.1 REFERENCES LONGITUDINAL AND RADIAL ELECTRIC FIELD FOR A CIRCULAR CROSS S ECTION CYLINDRICAL BUNCH

We derive the longitudinal and radial electric field of a uniformly charged finite cylinder with circular cross section (ref.[8]).

The bunch is moving along the z axis at velocity $v=\beta c$. The electric field produced by a charge q, moving at velocity βc and located at the origin of a polar coordinate system, fig.1, in the observation point $P(r,\theta,\phi)$, is:

$$\underline{E} = \frac{q(1-\beta^2)}{4\pi\varepsilon_0 r^2 (1-\beta^2 \sin^2\theta)^{\frac{3}{2}}}\hat{r} \tag{23}$$

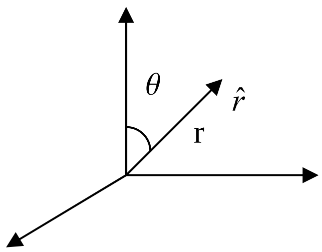

FIG. 1: Spherical coordinate system.

Using eq.(23) and referring to fig.2 we obtain the elementary longitudinal electric field calculated on the bunch's axis:

$$dE_z(z, r=0) = \frac{\rho(1-\beta^2)\cos\theta\, r dr d\phi dl}{4\pi\varepsilon_0 P^2 (1-\beta^2 \sin^2\theta)^{\frac{3}{2}}} \tag{24}$$

the total longitudinal electric field is that

$$E_z(z,0) = \int_0^{2\pi}\int_0^{R}\int_0^{L} \frac{\rho(1-\beta^2)\cos\theta\, r dr d\phi dl}{4\pi\varepsilon_0 P^2 (1-\beta^2 \sin^2\theta)^{\frac{3}{2}}}$$

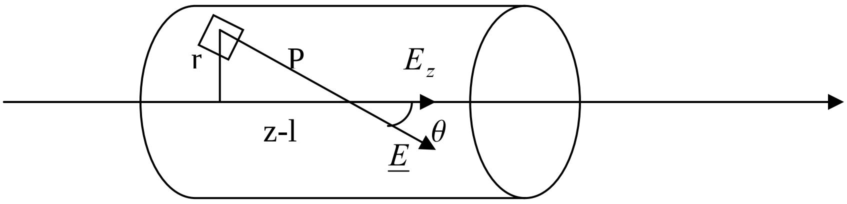

FIG. 2: Longitudinal electric field calculated on the bunch axis.

From fig.2 we have:

$$\begin{aligned} \cos\theta &= \frac{z-l}{P} \\ \sin\theta &= \frac{r}{P} \\ P^2 &= r^2 + (z-l)^2 \\ \gamma &= \frac{1}{\sqrt{1-\beta^2}} \end{aligned} \qquad (25)$$

Using eq. (25), eq. (24) becomes

$$E_z(z,0) = \int_0^{2\pi}\int_0^R\int_0^L \frac{\rho(1-\beta^2)\frac{z-l}{P}\,rdrd\phi dl}{4\pi\varepsilon_0 P^2(1-\beta^2\frac{r}{P})^{\frac{3}{2}}} = \frac{\rho\gamma}{2\varepsilon_0}\int_0^{2\pi}\int_0^R\int_0^L \frac{(z-l)rdrd\phi dl}{(r^2+\gamma^2(z-l))^{\frac{3}{2}}} \qquad (26)$$

Making the substitution

$$z - l = z'$$

the integral can be first solved with respect to *l*

$$E_z(z,0) = \frac{\rho}{4\pi\varepsilon_0\gamma}\left[\int_0^R \frac{rdr}{\sqrt{r^2+\gamma^2(L-z)^2}} - \int_0^R \frac{rdr}{\sqrt{r^2+\gamma^2 z^2}}\right]$$

and then with respect to *r*:

$$E_z(z,0) = \frac{\rho}{4\pi\varepsilon_0\gamma}\left[\sqrt{r^2+\gamma^2(L-z)^2} - \sqrt{r^2+\gamma^2 z^2}\right]_0^R$$

thus obtaining

$$E_z(z,0) = \frac{\rho}{4\pi\varepsilon_0\gamma}\int_0^{2\pi}\left[\sqrt{R^2+\gamma^2(L-z)^2}-\sqrt{R^2+\gamma^2 z^2}+\gamma|z|-\gamma|z-L|\right]d\phi \qquad (27)$$

Eq.(27) can be easily integrated with respect to the coordinate ϕ. We obtain the on axis longitudinal electric field of a circular cross section cylindrical bunch of length L:

$$E_z(z,0) = \frac{\rho}{2\varepsilon_0\gamma}\left[\sqrt{R^2+\gamma^2(L-z)^2}-\sqrt{R^2+\gamma^2 z^2}+\gamma|z|-\gamma|z-L|\right] \qquad (28)$$

The radial electric field for a point inside the bunch is calculated in ref. [8] from the relation:

$$E_r(r,z) \cong \frac{r}{2}\left[\frac{\rho}{\varepsilon_0}-\frac{\partial}{\partial z}E_z(z,0)\right] \qquad (29)$$

On the bunch's envelope eq.(29) becomes

$$E_r \cong \frac{R}{2}\left[\frac{\rho}{\varepsilon_0}-\frac{\partial}{\partial z}E_z(z,0)\right] \qquad (30)$$

The above equations are approximated sice they are obtained keeping only linear terms with r. From eq. (30) and the longitudinal electric field obtained in eq.(28) we get:

$$E_r(R,z) = \frac{\rho}{4\varepsilon_0}R\left[\frac{\gamma(L-z)}{\sqrt{R^2+\gamma^2(L-z)^2}}+\frac{\gamma z}{\sqrt{R^2+\gamma^2 z^2}}\right] \qquad (31)$$

Both expressions (28) and (31) can be written in terms of the aspect ratio

$$A = \frac{R}{\gamma L}$$

as in ref.[8]

$$E_z = \frac{q}{2\pi\varepsilon_0 R^2}\left\{\sqrt{A^2+(1-\frac{z}{L})^2}-\sqrt{A^2+z^2}+\left|\frac{z}{L}\right|-\left|1-\frac{z}{L}\right|\right\}$$

or

$$E_z = \frac{\rho L}{2\varepsilon_0}\left\{\sqrt{A^2+(1-\frac{z}{L})^2}-\sqrt{A^2+z^2}+\left|\frac{z}{L}\right|-\left|1-\frac{z}{L}\right|\right\}$$

and

$$E_r(R,z)=\frac{q}{4\pi\varepsilon_0 RL}\left\{\frac{(1-\frac{z}{L})}{\sqrt{A^{\,2}+(1-z/L)^2}}+\frac{\frac{z}{L}}{\sqrt{A^{\,2}+\left(\frac{z}{L}\right)^2}}\right\}$$

or

$$E_r(R,z)=\frac{\rho R}{4\varepsilon_0}\left\{\frac{(1-\frac{z}{L})}{\sqrt{A^2+(1-\frac{z}{L})^2}}+\frac{\frac{z}{L}}{\sqrt{A^2+\left(\frac{z}{L}\right)^2}}\right\}$$

being $\rho=\frac{q}{\pi R^2 L}$.

A.2 SECOND ORDER APPROXIMATION FOR THE ON AXIS LONGITUDINAL ELECTRIC FIELD IN AN ELLIPTICAL CROSS SECTION BUNCH

To simplify the integral

$$E_z(z,0)=\frac{\rho}{4\pi\varepsilon_0\gamma}\int_0^{2\pi}\left\{\sqrt{r'^2+\gamma^2(L-z)^2}-\sqrt{r'^2+\gamma^2z^2}-\gamma|L-z|-\gamma|z|\right\}d\phi \quad (32)$$

we expand the *r’* term as follow

$$r'^2=\frac{b^2}{1-e^2\cos^2\phi}\cong b^2(1+e^2\cos^2\phi+...) \quad (33)$$

In eq. (33) we neglect the term to the forth, obtaining a second order approximation. Inserting (33) in eq. (32), we can write eq.(32) as

$$E_z(z,0)=\frac{\rho}{\pi\varepsilon_0\gamma}\left\{\alpha\int_0^{\frac{\pi}{2}}\sqrt{1-\Gamma^2\sin^2\phi}\,d\phi-\beta\int_0^{\frac{\pi}{2}}\sqrt{1-\Lambda^2\sin^2\phi}\,d\phi\right\}$$

where

$$\alpha=\sqrt{b^2(1+e^2)+\gamma^2(L-z)^2}$$

$$\beta=\sqrt{b^2(1+e^2)+\gamma^2z^2}$$

$$\Lambda^2=\frac{b^2e^2}{b^2(1+e^2)+\gamma^2z^2}$$

or

$$E_z = \frac{\rho}{\pi\varepsilon_0\gamma}\left\{\alpha \cdot \text{EllipticE}[\Gamma^2] - \beta \cdot \text{EllpticE}[\Lambda^2] - \frac{\rho}{2\varepsilon_0}\left[|L-z| - |z|\right]\right\}$$

where

$$\text{EllipticE}[x^2] = \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2\varphi}\, d\varphi$$