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Considerations about the use of the analytical formulas for SPARC FEL

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1 Introduction

The aim of this study is to obtain a better understanding of the validity limits of the analytical formulas that are commonly used in order to evaluate the saturation length in SASE-FELS. The following two actions have been done:

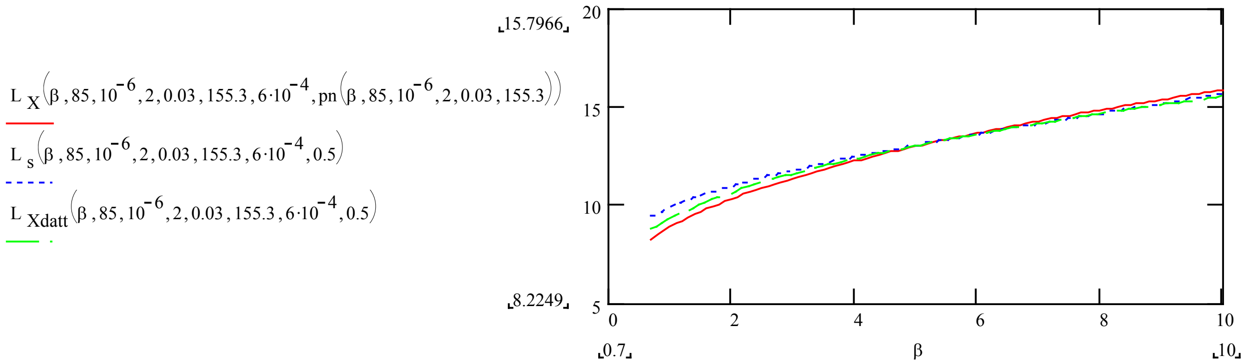
1. Xie [1] and Dattoli [2] formulas have been compared in order to evaluate the differences and both analytical models have been compared with the results of numerical codes (Ginger, Genesis) as they have been retrieved from literature or internal notes.
2. A parametric analysis of the FEL SPARC working area has been done.

Dattoli and Xie formulas are reported in the following table (Dattoli on the left, Xie on the right)

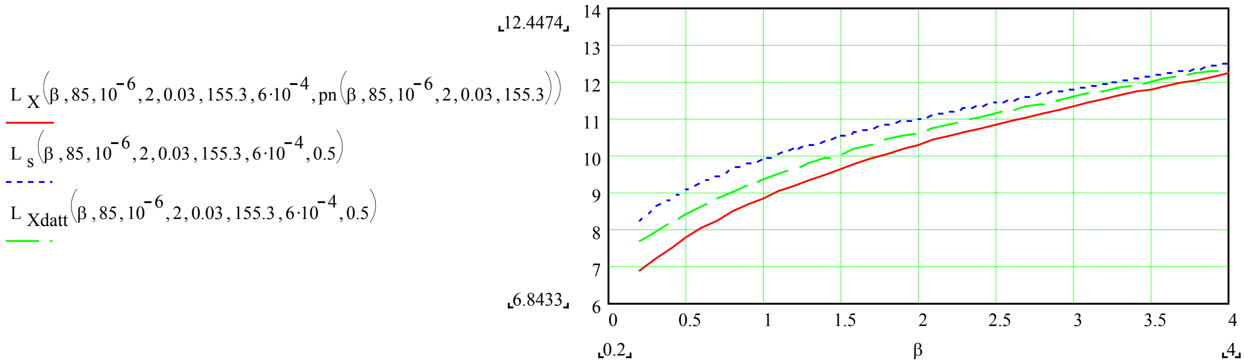
Tabella 1

Dattoli model	Xie model
$L_s = \frac{\lambda_u}{4\pi\sqrt{3}\rho_D\chi}\left[\ln(9\frac{P_F}{P_0})+1\right]$ $P_F = 1.67\eta(\chi)\rho_D P_E$	$L_{sat} = L_g \ln(\frac{P_{sat}}{\alpha P_n})$ $P_{sat} \approx 1.6\rho\left(\frac{L_{1d}}{L_g}\right)^2 P_{beam}.$
With	With
Diffraction effect: $\rho_D = \left[(1+\tilde{\mu}_x^D)(1+\tilde{\mu}_y^D)\right]^{-\frac{1}{6}}\rho$	Combined effect of diffraction, emittance and energy spread $\frac{L_{1d}}{L_g} = F(\eta_d;\eta_\varepsilon;\eta_\gamma)$ $\frac{L_{1d}}{L_g} = \frac{1}{1+\eta}$ $\eta = a_1\eta_d^{a_2} + a_3\eta_\varepsilon^{a_4} + a_5\eta_\gamma^{a_6} + a_7\eta_\varepsilon^{a_8}\eta_\gamma^{a_9} + a_{10}\eta_d^{a_{11}}\eta_\gamma^{a_{12}} + a_{13}\eta_d^{a_{14}}\eta_\varepsilon^{a_{15}} + a_{16}\eta_d^{a_{17}}\eta_\varepsilon^{a_{18}}\eta_\gamma^{a_{19}}$
Emittance and energy spread effect $\chi(\tilde{\mu}_\varepsilon,\tilde{\mu}_x,\tilde{\mu}_{x'},\tilde{\mu}_y,\tilde{\mu}_{y'}) = \frac{\chi(0,\tilde{\mu}_x,\tilde{\mu}_{x'},\tilde{\mu}_y,\tilde{\mu}_{y'})\exp(-c\tilde{\mu}_\varepsilon^2)}{1+0.185\frac{\sqrt{3}}{2}\chi(0,\tilde{\mu}_x,\tilde{\mu}_{x'},\tilde{\mu}_y,\tilde{\mu}_{y'})\tilde{\mu}_\varepsilon^2}$	

In figs. 1 and 2 the behavior of saturation length versus β as derived by Xie and Dattoli formulas are compared (in the case of the typical SPARC parameters, (85 A, 1 mm mrad, λu=3 cm, 155.3 MeV,K=2) according to a Mathcad program, directly supplied by Dattoli, in which, however, the Xie formula was slightly modified adding one gain length to the saturation length and giving to the input power Pnoise the fixed value 0.5 W. The original Xie formula as in tab. 1 with the noise derived from the expression given in ref.1, i.e. $P_n \approx \rho^2 c E_0/\lambda$, was then added to the graphs.

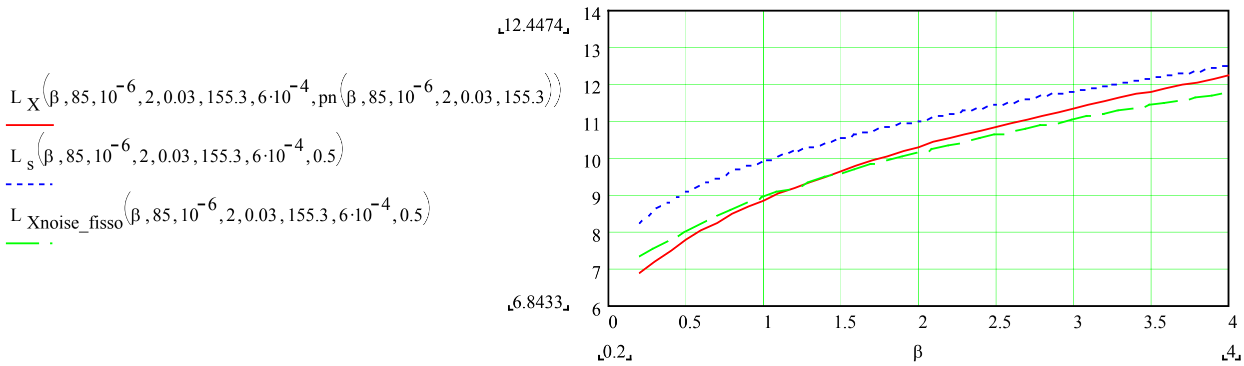


• Fig. 1 Saturation length curves vs β in the following 3 cases: LX = original Xie model (variable noise with β),Ls= Dattoli model (original, noise= 0.5 W),Lxdatt= Xie model as has been written in the Dattoli Mathcad model.



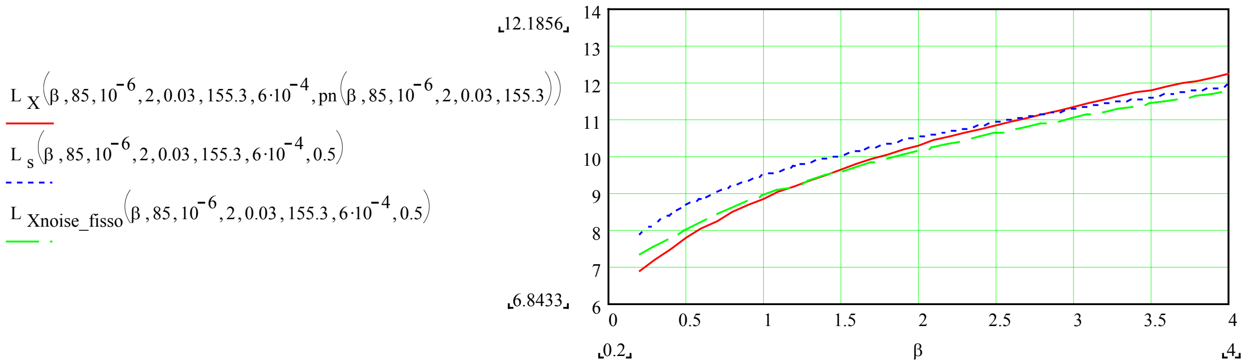
• Fig. 2 the same plot of figure 1 in an expanded scale.

In figure 3 we compare: (a) the original Xie formula with fixed noise, (b) the original Xie formula with variable noise (c) the original Dattoli formula that is always at fixed noise (0.5 W). It can be observed that for β values between 1 and 2 the Xie curve does not depend so much from the fact of having a fixed or variable noise. In addition figure 3 shows that the two formulas give a difference of 1 m in the saturation length.



• Figura 3 – In addition to LX e Ls as from the previous figures the case of the original Xie formula at fixed noise (0.5 W).

Perhaps the difference between the predicted value of the saturation length between the two models could be due to the fact that while in the Dattoli formula the diffraction effect is separated from the effect of emittance and energy spread, in the Xie model the terms that take into account these three effects are mixed.
Still eliminating the "+1" term in the expression of the saturation length given by the Dattoli formula the difference between the values predicted by the two formulas is not completely recovered, expecially for low β values, as it can be seen in fig. 4.



• Figure 4 L_{Xnoise_fisso} e L_X as in figure 3, L_s is the Dattoli formula without adding 1.

Therefore, to avoid hardly manageable models modifications, that, anyway do not vanish the differences between the predictions between the models, **in the following only the original Xie and Dattoli formulas will be used, as they appear in literature**.

In Xie and Dattoli models the saturation length, that of course is the parameters of greatest interest for the device realization, is determined by the combination of two parameters that are actually computed: the effective gain length Lg and the saturation power or maximum power P_{sat} o P_F (table 1).
So in order to do a comparison between the two theories it is necessary to compare independently the values of these two parameters (paragraph 2), and, next (paragraph 3), the theoretic saturation length will be compared with numerical codes results.
The comparison will be done for the cases listed in table 2.

Tabella 2

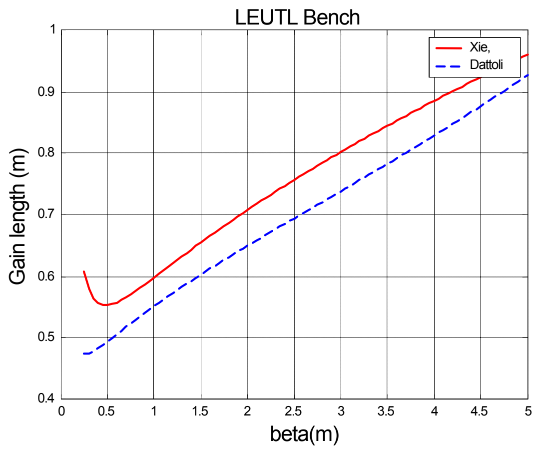
LEUTL bench [3]	SPARC [2]	SPARC (start to end) [5]	VISA1 [4]

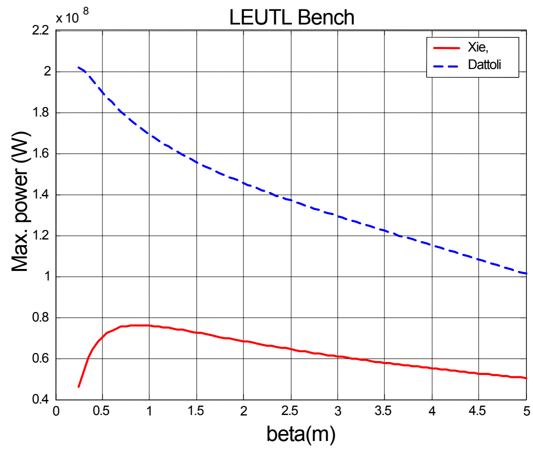
lamfel=516.751e-9;
lamu=0.033;
betax=1.46;
betay=1.46;
alfax=0;
alfay=0;
gammas=1+220/.511;
emittxns=5;
emittyns=5;
currslice=0.15;
derms=1e-3;
	lamfel=487e-9;
	lamu=0.03;
	betax=1.55;
	betay=1.55;
	alfax=0;
	alfay=0;
	gammas=1+155.3/.511;
	emittxns=1;
	emittyns=1;
	currslice=0.085;
	derms=6e-4;
		lamfel=485.5e-9;
		lamu=0.03;
		betax=1.6;
		betay=1.6;
		alfax=0;
		alfay=0;
		gammas=1+155.3/.511;
		emittxns=0.5;
		emittyns=0.5;
		currslice=0.089 –
		0.08- 0.061;
		derms=4e-4;
			lamfel=800e-9;
			lamu=0.018;
			betax=0.2707;
			betay=0.2707;
			alfax=0;
			alfay=0;
			gammas=142.1;
			emittxns=2;
			emittyns=2;
			currslice=0.2;
			derms=0.176e-2;

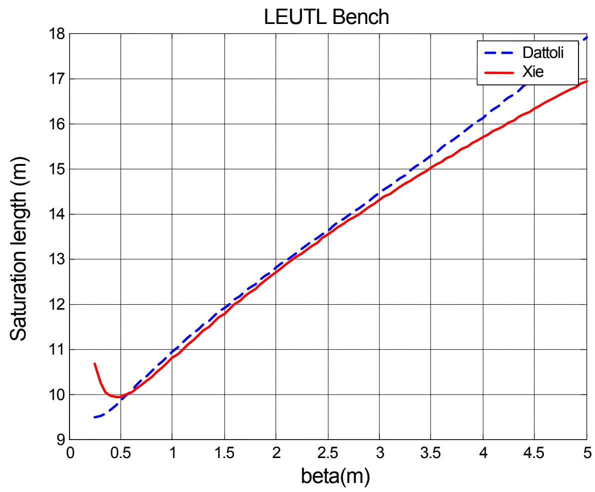
2. Comparison between Dattoli and Xie models

In this paragraph the values of the **gain length** and of the **saturation power** given by Xie and Dattoli formulas will be compared. The saturation power will be also compared with the results of numerical simulations.
The saturation length that, as it has been pointed above, is derived from the gain length and saturation power, will be compared with the results of the numerical codes in the following paragraph.

2.1 LEUTL Benchmark







In the case of LEUTL the saturation length values given by the two models give very similar results in the range of interest of β values, also if the single parameters that combined together give the saturation length do not agree very well. In particular it seems that in the case of the Dattoli model the shorter gain length that should give a reduction of the saturation length is compensated by the larger saturation power which produces an increase of the saturation length.

- Figure 5 Gain length, maximum power and saturation length with LEUTL_benchmark data

For these parameters, the GENESIS calculation results are reported in ref. [3],. The table 2 of ref 3, reproduced below, shows the value of the saturation power (about 70 MW) for β=1.46 as it is predicted by GENESIS, TDA3D and "theory", ("theory" means the Xie model). For this value of β the Dattoli model gives 156.7 MW.

Table 2: Saturation Power Predicted by Simulation Programs and Theory

Method	Saturation Power
TDA3D Simulation	73.20 MW
GENESIS Simulation	70.74 MW
Theory in reference [7]	72.80 MW

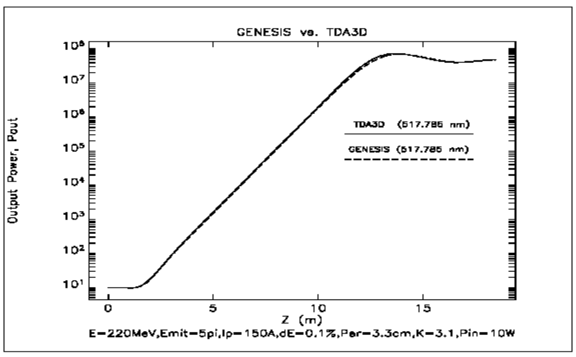
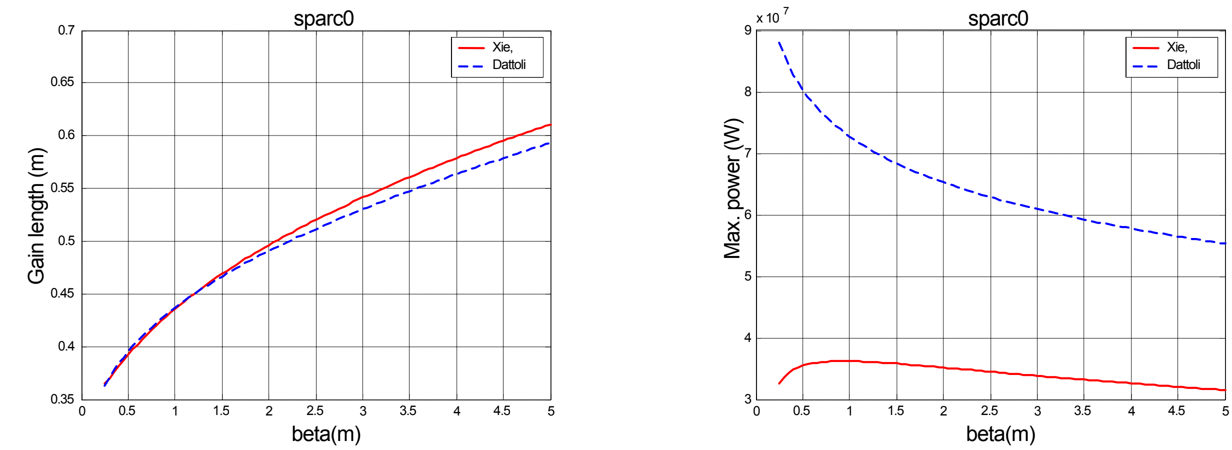
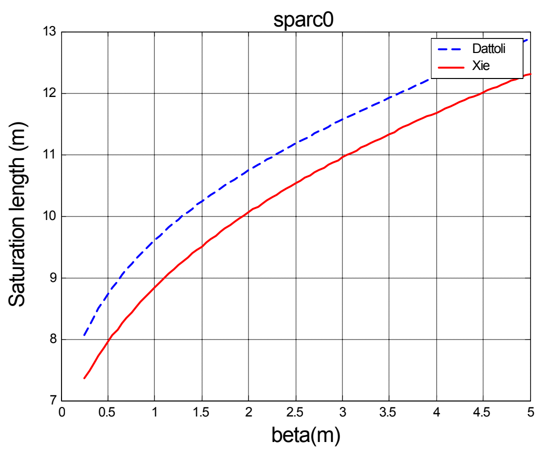


Figure 11: Input power amplification along the long undulator (solid line for TDA3D and dashed for GENESIS).

- Figure 5a: Tab. 32 and figure 11 from ref 3

2.2 SPARC



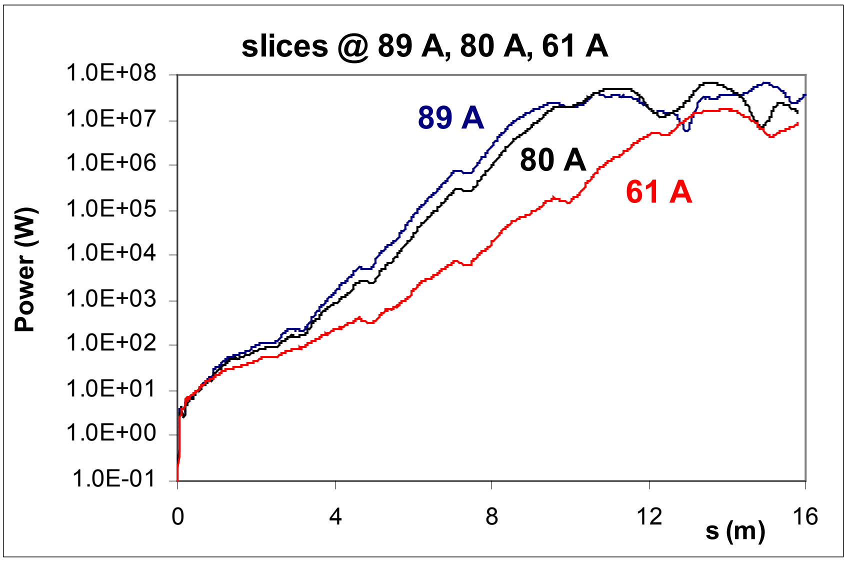


In this case as it results from the plots of figure 6 the gain length values that are very similar for both models do not compensate the difference in the calculation of the saturation power. In ref. 2 the GINGER calculation (fig. 2 in ref. 2) shows a maximum power of about 30 MW that is in good agreement with Xie formula.

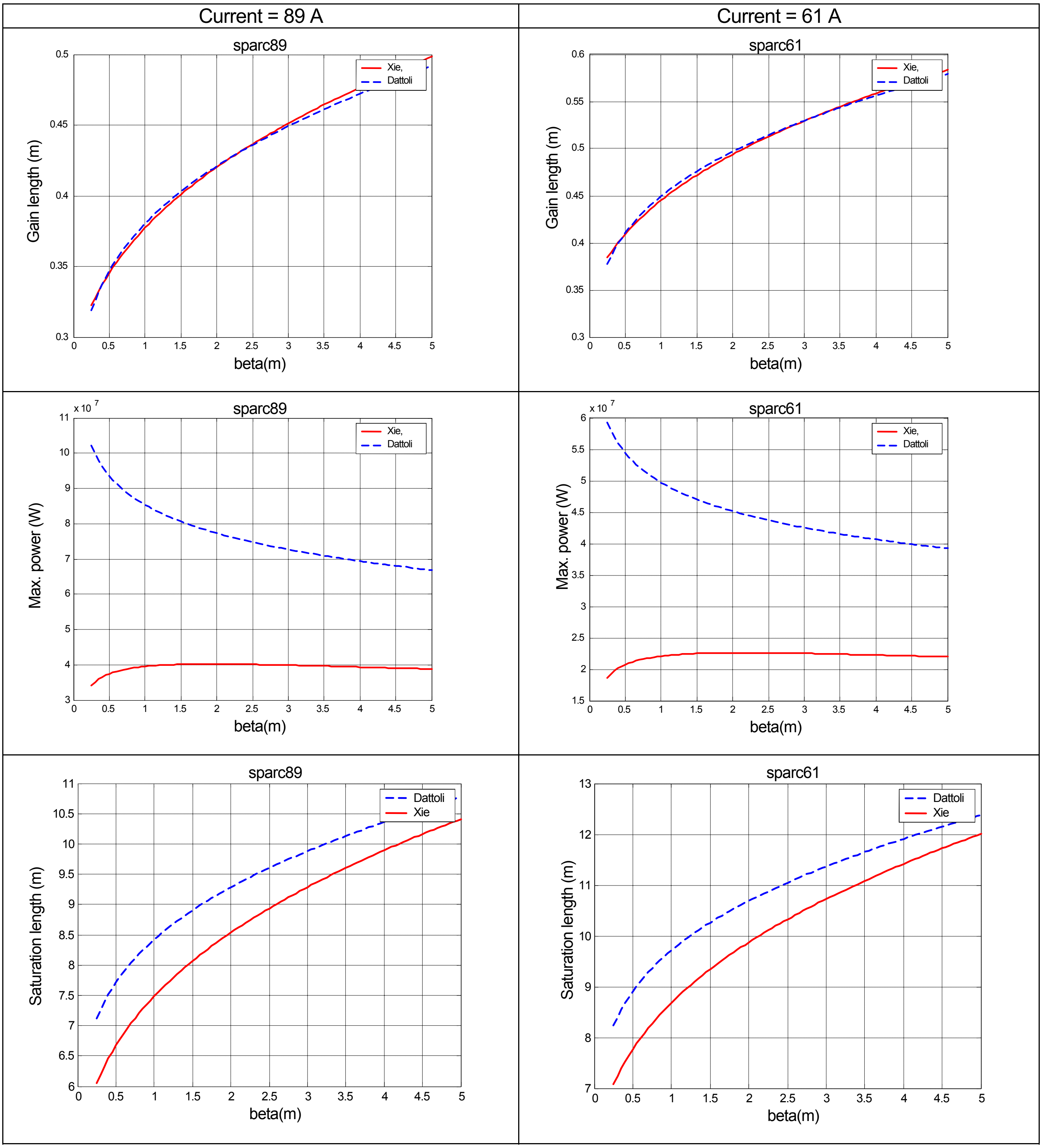
• Fig.6: Gain length, maximum power and saturation length with SPARC data

2.3 SPARC (start_to_end: PARMELA+GENESIS)

We refer to the calculation done by Bartolini with GENESIS (fig.7) starting from PARMELA output data and published in various Conferences papers and in the draft version of the SPARC Technical Design Report [5]. The calculation has been done for single slices.



• Fig. 7 GENESIS simulations on 3 representative slices of the bunch

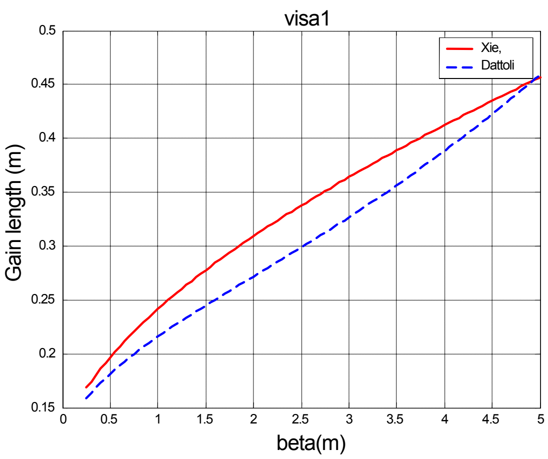


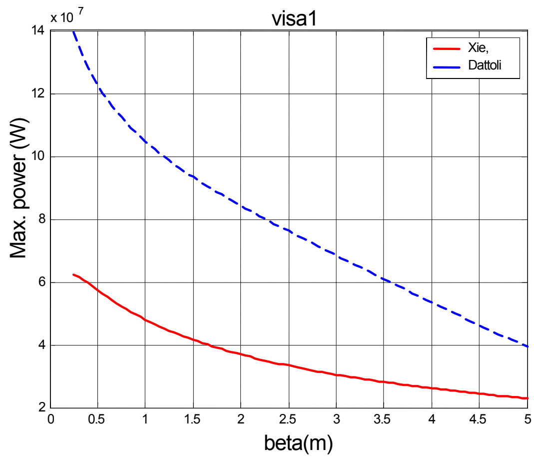
• Fig. 8 Gain length, maximum power and saturation length with SPARC start to end data for I=89 A and I= 61 A

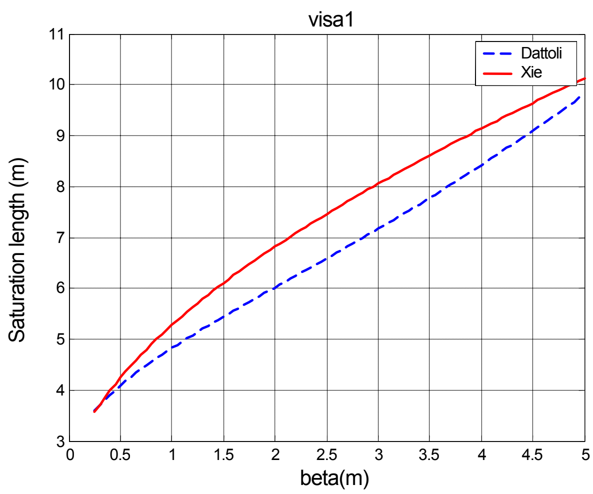
Also in this case Xie formula predicts a value of the maximum power in good agreement with the value predicted by the code, while the value predicted by the Dattoli model is about twice. The gain length given by the two models agree very well, while the larger maximum power predicted by the Dattoli model produces a higher value of the saturation length.

2.4 VISA

We refer to the VISA parameters list that is available on the experiment web page web [4]. In this case the analytical formulas give the results of figure 9.







Also in this case the larger difference is given by the value of the maximum power. About the saturation length the two models agree well in the low β region that is also the VISA operating region.

• Fig. 9 Gain length, maximum power and saturation length with VISA data

In the following table the values predicted by the analytical formulas and the values reported in the VISA parameters list are shown for β=0.27 m.

	VISA table	Xie	Dattoli
Saturation length	3.4 m	3.6 m	3.48
Maximum power	62 MW	61.9 MW	137.5 MW

3. SATUIRATION LENGTHS : comparison with numerical codes

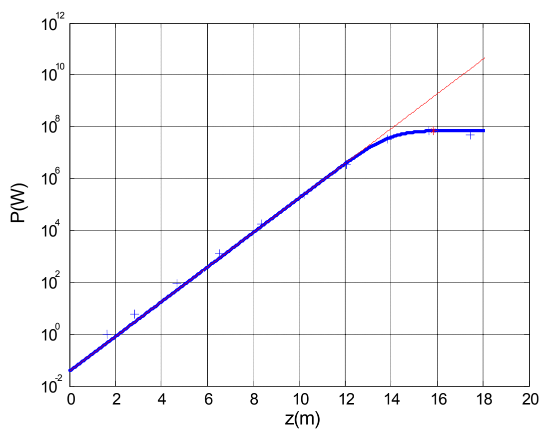
In order to compare the saturation length predicted by the codes with the values given by the analytical models it is convenient to take into account that the signal growth along the ondulator can be expressed by [7]

$$P(z) = \frac{\frac{P0}{9} \cdot e^{z/Lg}}{1 + \frac{P0}{9Psat} \cdot (e^{z/Lg} - 1)} \qquad (3.1)$$

The comparison has been done accordingly to the following procedure

- The gain length Lg and the maximum power Psat have been determined by Xie formulas, which give a value for the maximum power that agrees better with the code value in the examined cases
- The initial power is determined by Xie noise
- The power as a function of z is given by the formula (3.1)

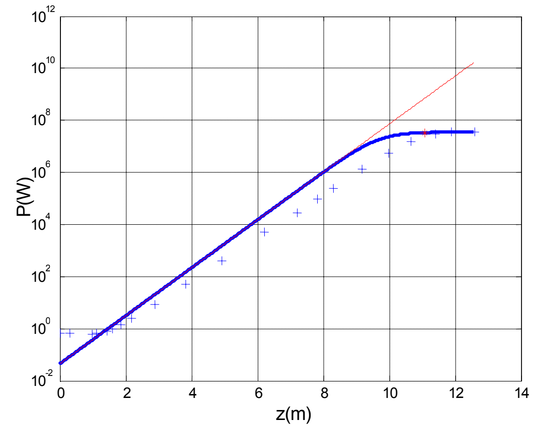
The formula 3.1 shows that the saturation power is reached asintotically along z. Instead the saturation length predicted by the Xie and Dattoli models, is in practice the intercept of the horizontal line, at constant power equal to Psat, with the rectilinear line (in a semilogaritmic scale) which represents the logarithmic evolution along z. So the effective saturation length is larger than the saturation length that is computed by the formulas.
From the following figures that refer respectively to the case of LEUTL (GENESIS calculation), the case of SPARC (GINGER calculation) and the case of SPARC start to end (GENESIS calculation), it can be noted that this approximation and the GENESIS simulation agree very well, while the agreement with GINGER is less good. From an accurate analysis of the data we have estimated an additional length equal to 3 gain lengths. In each figure the point Lsat (Xie) + 3*Lg(Xie) is indicated by a red asterisk and when the agreement is good it corresponds to the saturation length derived by the numerical code.



LEUTL (GENESIS)

The comparison has been done with the evolution along z computed by GENESIS (ref. 8), represented by the crosses.
The dashed red line is the evolution accordingly with the Xie formula
The continuous blue line is the formula 3.1.
The red point is the effective saturation length.
The agreement is good.

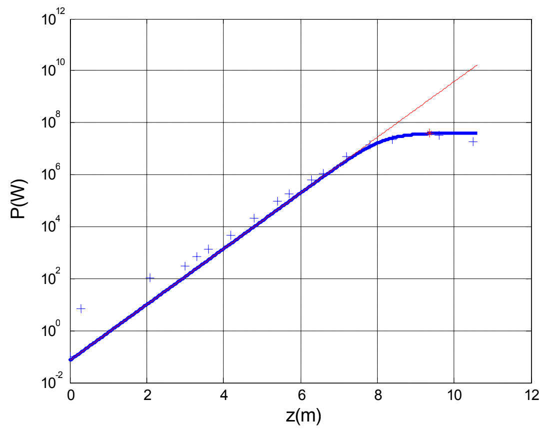
• Fig. 10 Comparison between formulas and codes in the LEUTL case



SPARC (GINGER)

The comparison has been done with the power evolution in function of z computed by GINGER (ref. 2), represented by crosses.
The dashed red line is the evolution accordingly with Xie formula
The continuous blue line is the formula 3.1.
The red point is the effective saturation length.

• Fig. 11 Comparison between formulas and codes in the case of Sparc



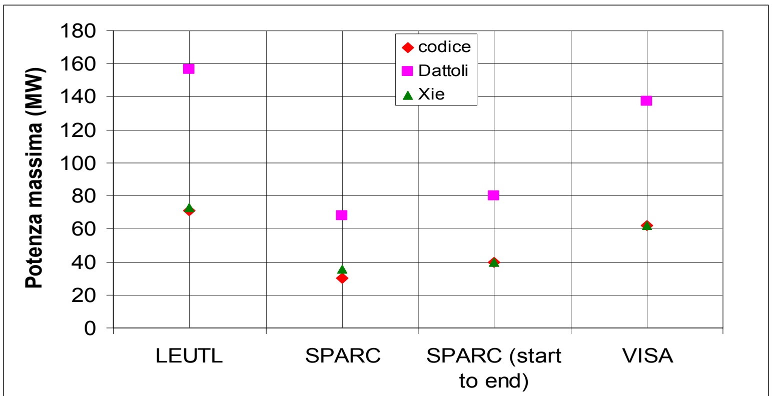
SPARC (start to end, PARMELA + GENESIS)

The comparison has been done with the power evolution in function of z computed by GENESIS (ref. 8), represented by crosses.
The dashed red line is the evolution accordingly with Xie formula
The continuous blue line is the formula 3.1.
The red point is the effective saturation length.
The agreement is good.

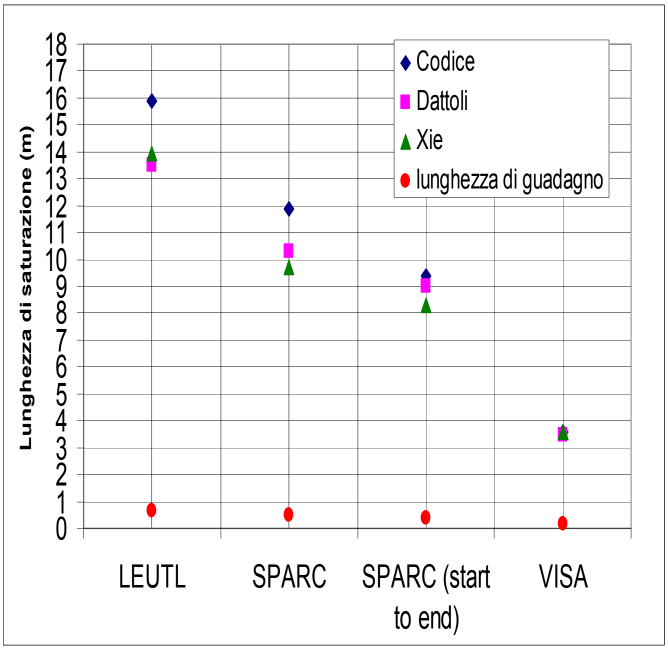
• Fig. 12 Comparison between formulas and codes in the case of Sparc (start to end)

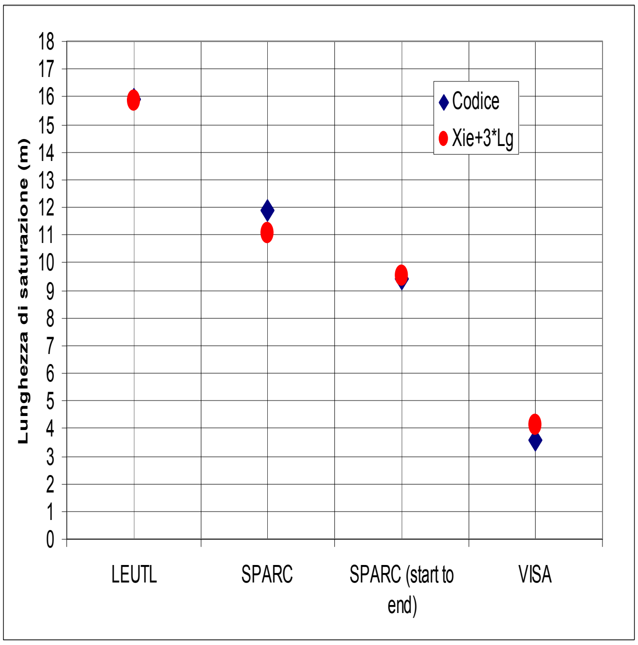
4. ANALYSIS OF THE RESULTS OF THE COMPARISON FORMULAS-CODES

The results of this analysis are summarized in the plots of figure 13. The plot in fig. 13a shows that in all the cases that have been analyzed the value of the maximum power predicted by the Dattoli model results lager than the Xie model one, that shows a reasonable agreement with the codes. The plot of figure 13b shows the values of the saturation length in function of the experiment, as predicted by the codes and by the analytical models. In figure 13b the value of the gain length is also indicated. In figure 13c, accordingly with the previous paragraphs, we added 3Lg to the saturation length predicted by the Xie model and this quantity has been compared with the value predicted by the codes.



• Fig. 13a Maximum power: comparison between codes nad analytical modelsi





- Fig. 13b Saturation length in function of the experiment Fig. 13c Saturation length in function of the experiment : comparison between the values predicted by the codes and Lsat(Xie)+3Lg

In the following paragraphs for the evaluation of the saturation length we will use the saturation length given by the Xie formula + 3 gain lengths in order to take into account the behaviour of the power near saturation.

5. PARAMETRIC STUDY VS GAP AND UNDULATOR PERIOD

A parametric study of the saturation length in function of the main geometrical parameters of the undulator (period and gap) has been done starting from the following relations that connect these parameters to the radiation wavelength and to the undulator K :

$$\lambda = \lambda_u \cdot (1 + K^2 / 2) / \gamma^2$$

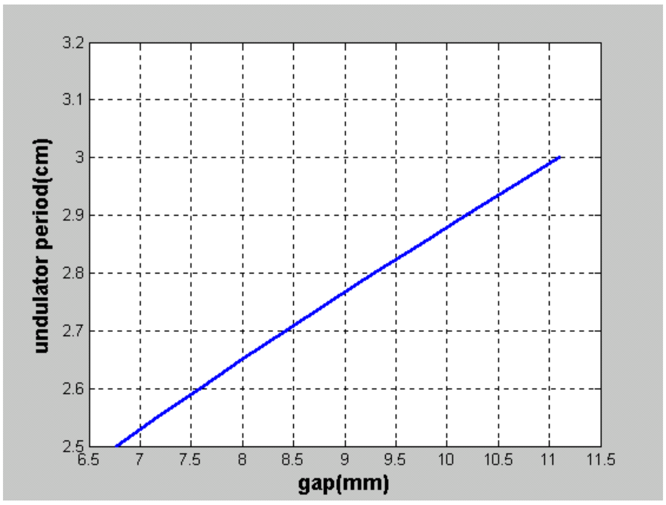
with

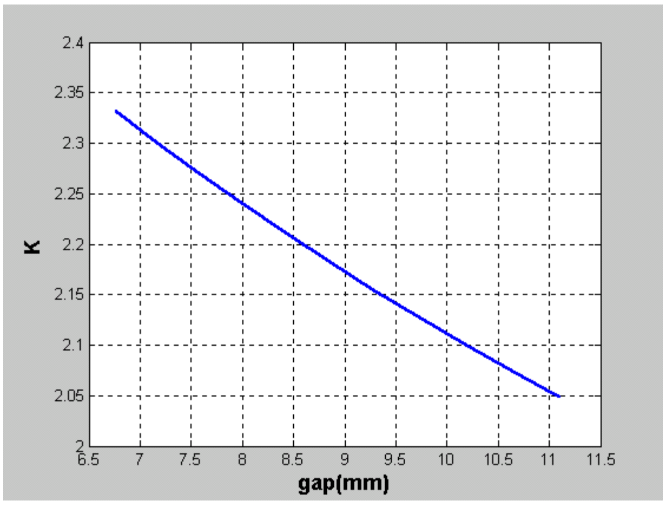
$$K = 0.66 \cdot B(T) \cdot \sqrt{2} \cdot \lambda_u (cm)$$

where from [2]

$$B = 2 \cdot B_r \cdot (1 - \exp(-h \cdot 2 \cdot \pi / \lambda_u)) \cdot \exp(-\pi g / \lambda u) \cdot \sin(\pi M / M)$$

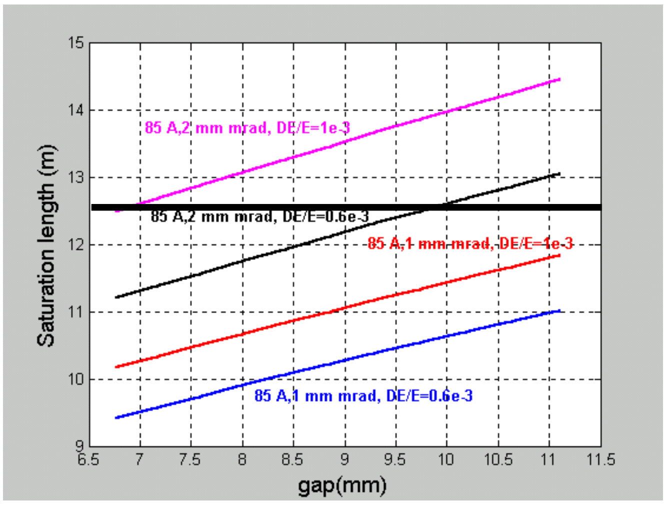
These relations for M=4 (number of magnetic blocks), *Br* =1.3 T (remanent field), h=3.5 cm (magnetic block height) give the plots of figures 14 and 15 giving respectively the value of the undulator period and of undulator K in function of the gap, having fixed the beam energy to 155.3 MeV and the radiation wavelength to 500 nm.

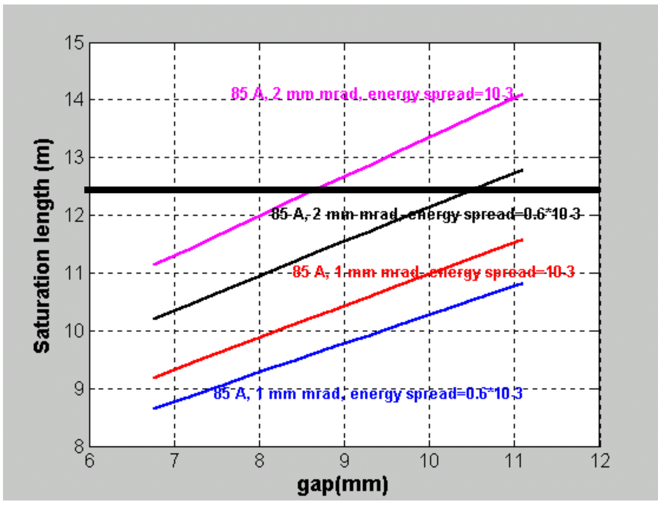




- Fig.14 Undulator period versus gap for 155.3 MeV and 500 nm. Fig. 15 undulator K versis gap for 155.3 MeV and 500 nm

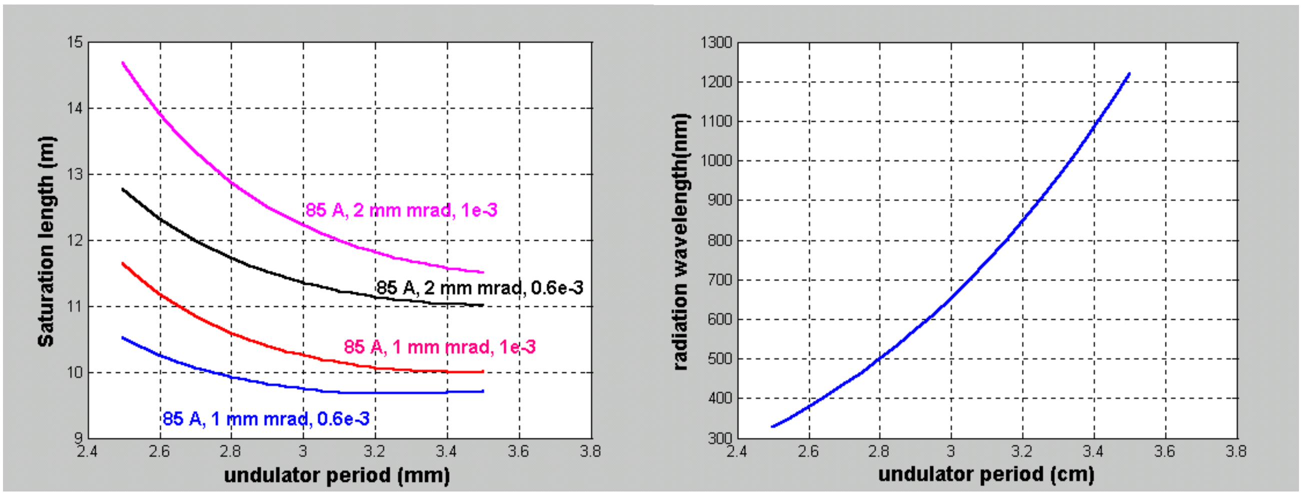
The plots of figure 16 shows the saturation length for a beam current of 85 A at different beam emittance and energy spread. **The saturation length is computed by summing to the Xie saturation length 3 gain lengths.**
In particular in figure 16a the β value is kept constant and equal to 1.55 m, while in figure 16b it is equal to the natural β of the undulator $\gamma\lambda/(\pi K)$. The horizontal black line corresponds to a lenght of 12.5 m that is about the total undulator length minus the drifts: this is the limit length with which the results of the analytical formulas must been compared, because the analytical formulas do not take into account the drifts between the undulator sections.





- Fig. 16a Saturation length in function of the gap for β=1.55 m, I=85 A. Fig. 16b Saturation length in function of the gap for β= natural β at I=85 A

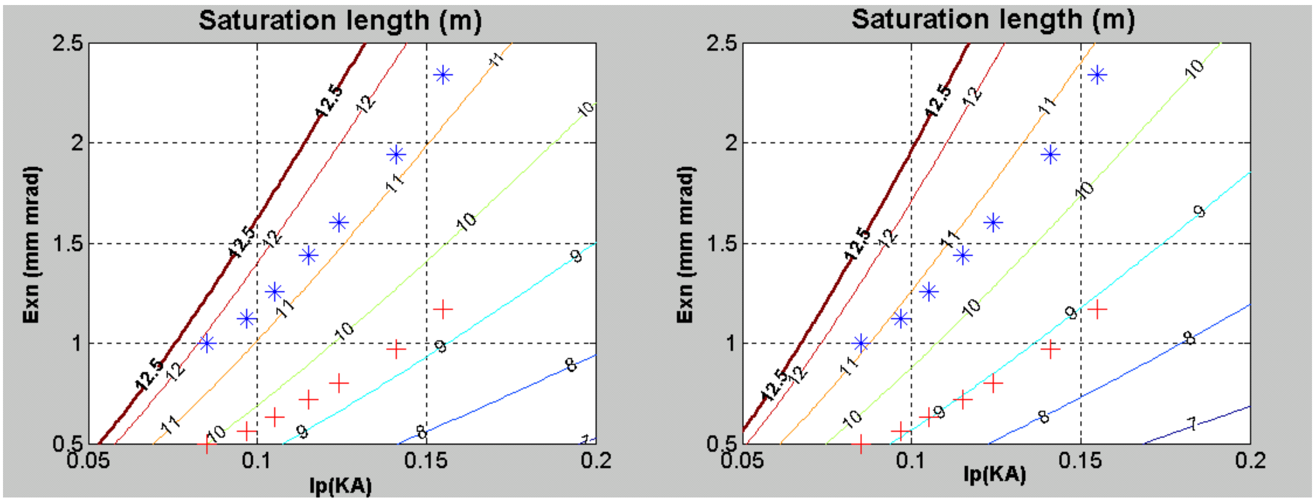
The previous formulas have been used also to investigate the possibility of working at a wavelength different from 500 nm fixing the values of the beam energy to 155.3 MeV and of the gap to 9.3 mm. In this case working to a β value equal to the natural undulator β we obtain for the saturation length and the radiation wavelength the curves of figures 17a and 17b . In this case the saturation length should be minimized for a period of 3-3.2 cm and in all the 4 considered cases of beam emittance and energy spread it could be maintained below the limit length of 12.5 m, but the price to be paid should be an operation at 650 - 700 nm, so this does not seem a good choice.



- Fig. 17a Saturation length in function of the undulator period for gap=9.3 mm, β=natural β, I=85 A. Fig. 17b Saturation length in function of the undulator period for gap=9.3 mm, β= natural β, I=85 A

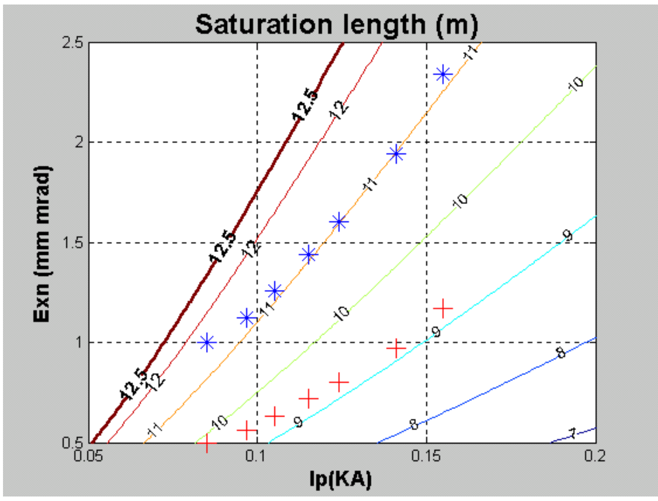
6. PARAMETERS AREA

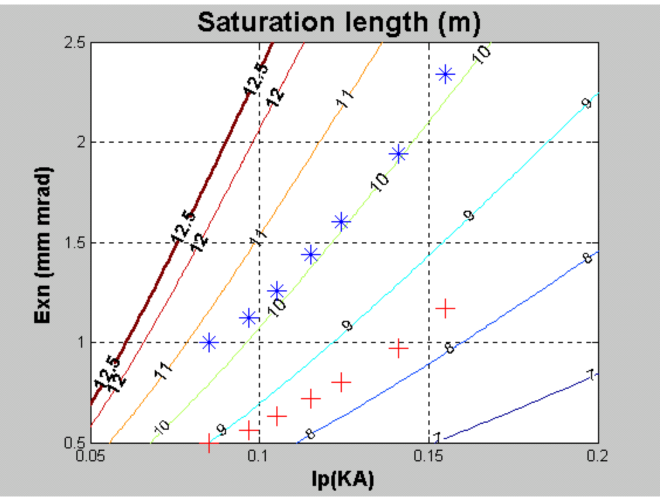
The analytical formulas can be used to individuate the parameters space of FEL-SPARC. As already anticipated in the previous paragraph for an undulator composed by 6 modules 2.13 m long with a period of 3 cm and drift spaces between the modules 36 cm long, **the limit length L* with which the analytical formulas must be compared is the total length of the undulator minus the drifts that corresponds to about 12.5 m.**
The figures of this paragraph show the working area (emittance, peak current) where the constant saturation length curves are shown for an energy spread of $1*10^{-3}$**. The reported saturation length is the sum of the value calculated by the Xie formula and 3 gain lengths in order to take into account the behaviour of the signal growth near the saturation**. The useful working area is the area below the curve corresponding to L*=12.5 m.
We added to this area the couples of the accelerator output beam current and emittance parameters (blue asterisks) resulting from a parametric study done by INFN [6]. As the values in ref. [6] are given without tolerances, while to the 85 A working point an emittance of 1 mm mrad (the computed slice emittance multiplied by 2) has been assigned, including the computed tolerances and the confidence degree, the same points with a doubled emittance are shown.
In this last case for an undulator with a period of 2.8 cm the saturation length is around 11 m. It can be observed that the gain given by the beam current increase is about 0.5 m and not 1.3 m as indicated in ref. [6].



- Fig. 18a Parameters space for SPARC. Energy spread=10^{-3}. Undulator period=3 cm. Gap=11 mm, average β =1.55 m. The constant saturation length curves are derived from the Xie formula + 3Lg

- Fig. 18b Parameters space for SPARC. Energy spread=10^{-3}. Undulator period=2.8 cm. Gap=9.3 mm, average β =1.55 m. The red + points are the values of current and emittance furnished in ref. [6]. The same points with doubled emittance are indicated by blue *. The constant saturation length curves are derived from the Xie formula + 3Lg



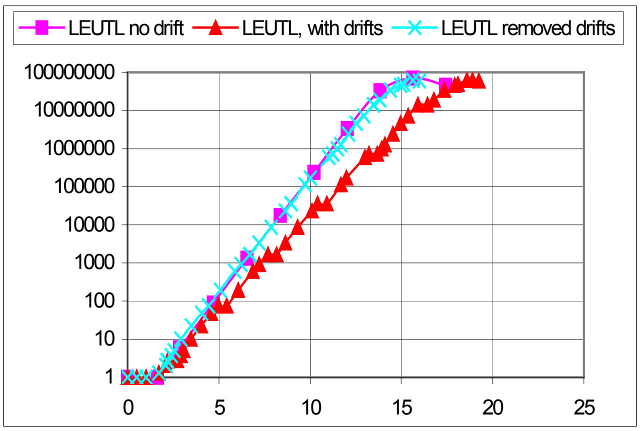


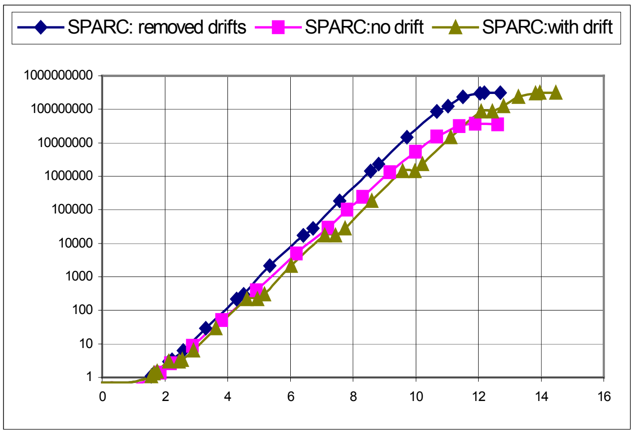
- Fig. 19a Parameters space for SPARC. Energy spread=10^{-3}. Undulator period=3 cm. Gap=11 mm, average β = natural β =1.42. The red + points are the values of current and emittance furnished in ref. [6]. The same points with doubled emittance are indicated by blue *. The constant saturation length curves are derived from the Xie formula + 3Lg

- Fig. 19b Parameters space for SPARC.. Energy spread=10^{-3} Undulator period=2.8 cm, β=natural β =1.26. The red + points are the values of current and emittance furnished in ref. [6]. The same points with doubled emittance are indicated by blue *. The constant saturation length curves are derived from the Xie formula + 3Lg

7. SEGMENTED UNDULATOR

The incidence of the drifts between the undulators has been evaluated for the power growth in function of z. In figure 20a the evolution of the power along z as it has been computer by GENESIS (ref.8) for LEUTL parameters in 3 cases: a) calculation with no drift spaces b) calculation with the undulator including the drifts c) calculations with an undulator including the drifts minus the spaces in which the exponential growth does not occour.





- Fig. 20a LEUTL: GENESIS calculation (data from ref. 8).

Fig. 20 b SPARC:GINGER calculation (data from ref.2)

One can see immediately that in GENESIS calculation the slope of the exponential growth is the same with and without drifts that means that the drifts do not modificate the value of Lg and that the value of the saturation power does not change. Viceversa, GINGER in the case of SPARC (fig. 20b and data from ref.2) predicts a value of the saturation power that is lager when the drift spaces are included and the slope of the curve increases (Lg decreases) so that the saturation length is approximately the same. The GINGER behaviour is not clear, so we have considered as a reference only GENESIS calculations.

About the effective value that must be added to the theoretical saturation length in order to take into account the drift spaces, from the comparison between the curves of the exponential growth it results that it is between a value that is equal to the exact sum of the drifts (case SPARC start-to end, 36 cm multiplied for the number of the drifts covered by the beam) and about 1.2 x this value (LEUTL).

CONCLUSIONS

From this analysis it results that:

1. in all analysed cases the value of the maximum power is overestimated by the Dattoli model, while the Xie model and the codes agree very well.
2. about the gain length the agreement bewteen the two models is good with the exception of the LEUTL case (where the emittance is larger) also if in this case the discrepancies between the saturation power and gain length values give a sort of compensation resulting in a good agreement for the values of the saturation length
3. about the saturation length from the comparison with the codes it results that a good approximation could be to consider that it is equal to the saturation length predicted by the Xie formula + 3 gain lengths. For this approximation the best agreement is found with GENESIS code.
4. from this analysis it seems that it is not necessary to multiply for 2 the average beta in order to find a good agreement between the analytical formulas and the codes results as it has been suggested in ref. [2]
5. The analytical formulas allow to define a beam parameters (current, emittance, energy spread) area and to perform a parametric study concerning the main geometrical parameters of the undulator. A study based on the use of the analytical formulas gives that working with an undulator with a period of 2.8 cm and a gap around 9 mm and a β= the natural undulator β (1.26) it is possible for beam currents $\geq$ 100 A to obtain a tolerance margin of 2.5 m on the undulator length for saturation. This margin is about 1.6 – 1.7 m if the focalization in the undulator does not allow to go below a β value of 1.55m.

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