RADIATION DAMPING

- Synchrotron radiation
- RF cavity and synchrotron oscillations
- Calculation of Damping times
- Wigglers
- Quantum excitation and beam sizes
- Emittance control

BEAM LIFETIME

- Quantum fluctuations
- Beam-gas interaction
- Touschek effect

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SYNCHROTRON RADIATION

Emission of Synchrotron Radiation (S.R.) exerts a strong influence on electron beam dynamics in storage rings.

The average rate of emission of synchrotron radiation leads to damping of synchrotron and betatron oscillations.

The quantum fluctuations in the emission give rise to a growth of the oscillation amplitudes.

The beam sizes are determined by the equilibrium of the two processes.

At present energies, these effects strongly affect the design of electron machines, while are negligible for proton machines.

For the next proton collider LHC synchrotron radiation effects have to be taken into account.

In the following we will refer to electrons.

For a charged particle in circular motion the instantaneous rate of power emitted by S.R. is:

$$P = \frac{2}{3}\frac{e^2 c}{4\pi\varepsilon_0}\frac{\beta^4\gamma^4}{\rho^2} \qquad E^2 B^2$$

For ultrarelativistic particles (β~1) **the radius of curvature is related to the field of the bending magnets by:**

***E* [GeV] = .3 *B*ρ [Tm]**

The energy loss per turn is:

$$U_0 = \oint \frac{P}{\beta c} ds = \frac{2}{3}\frac{e^2}{4\pi\varepsilon_0}\beta^4\gamma^4 \oint \frac{ds}{\rho^2}$$

For e^- (β~1) **it can be written in a practical form:**

$$U_0 = \frac{C_\gamma}{2\pi} E^4 \oint \frac{ds}{\rho^2} \quad ; \quad C_\gamma = 8.85\ 10^{-5}\ \mathrm{GeV}^{-3}\ \mathrm{m}$$

for isomagnetic lattice (uniform bending radius):

$$U_0[eV] = 8.85\ 10^4 \frac{E^4[GeV]}{\rho[m]}$$

The total beam power is given by the total circulating current $I_b = Ne/T_0$:

P_b [W] = U_0 [eV] I_b [A]

Energy loss per turn and related parameters for various electron storage rings

	E (GeV)	ρ (m)	L (m)	T_0 (μs)	$U_{0,dip}$* (MeV)
Adone	.51	5	105	.35	.001
DAΦNE	.51	1.4	98	.31	.004
PEP B LE	3.1	30.5	2200	13.6	.27
PEP B HE	9.0	165	2200	13.6	3.5
LEP	100.	3100	$3\ 10^4$	89	2855

The same quantities for the next proton storage ring.

	E (GeV)	ρ (m)	L (m)	T_0 (s)	$U_{0,dip}$ (MeV)
LHC	7700	2568	$3\ 10^4$	89	.011

* dip = from dipoles, excluding contributions from wigglers.

CHARACTERISTICS OF SYNCHROTRON RADIATION

For ultrarelativistic electrons the emitted radiation is confined within a narrow cone around the direction of motion ($\theta_{rms} = 1/\gamma$)**.**

The effect of the radiation is to decrease the energy of the electron without changing its direction of motion.

The critical frequency ω_C is defined as the frequency for which half of the power is emitted above ω_C and half below.

$$\omega_c = \frac{3}{2}\frac{c\gamma^3}{\rho}$$

The spectral distribution, normalized to the critical frequency, does not depend on the particle energy and can therefore be represented by a universal distribution function $S(\omega/\omega_c)$.

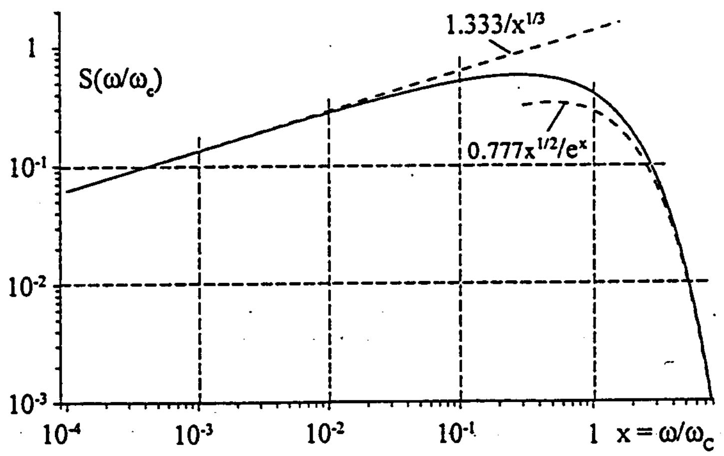
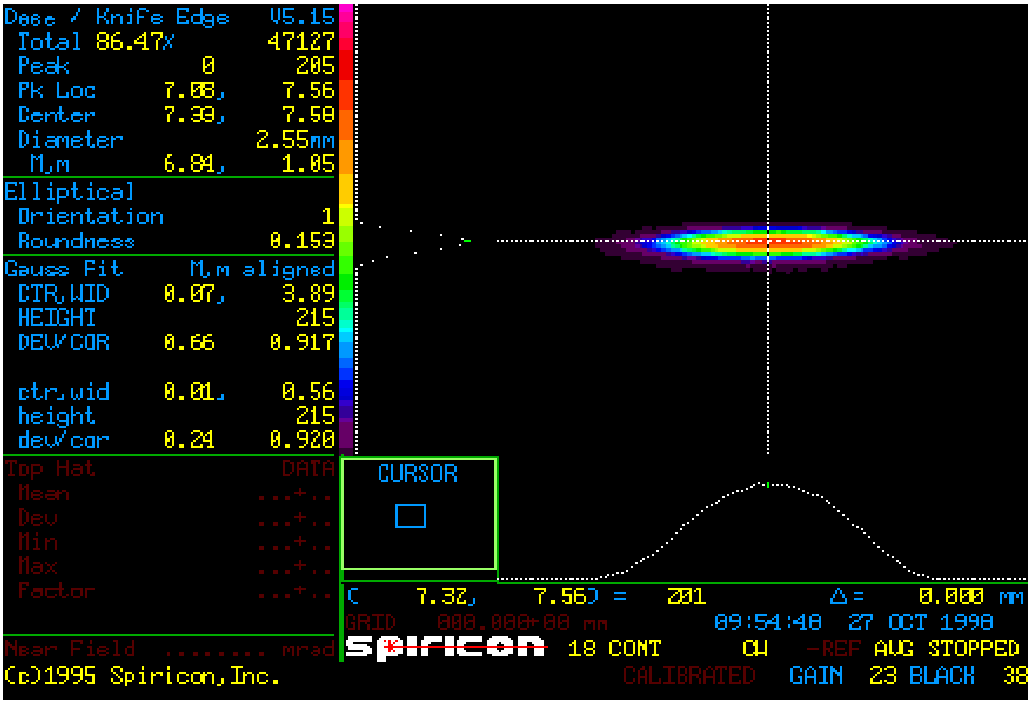
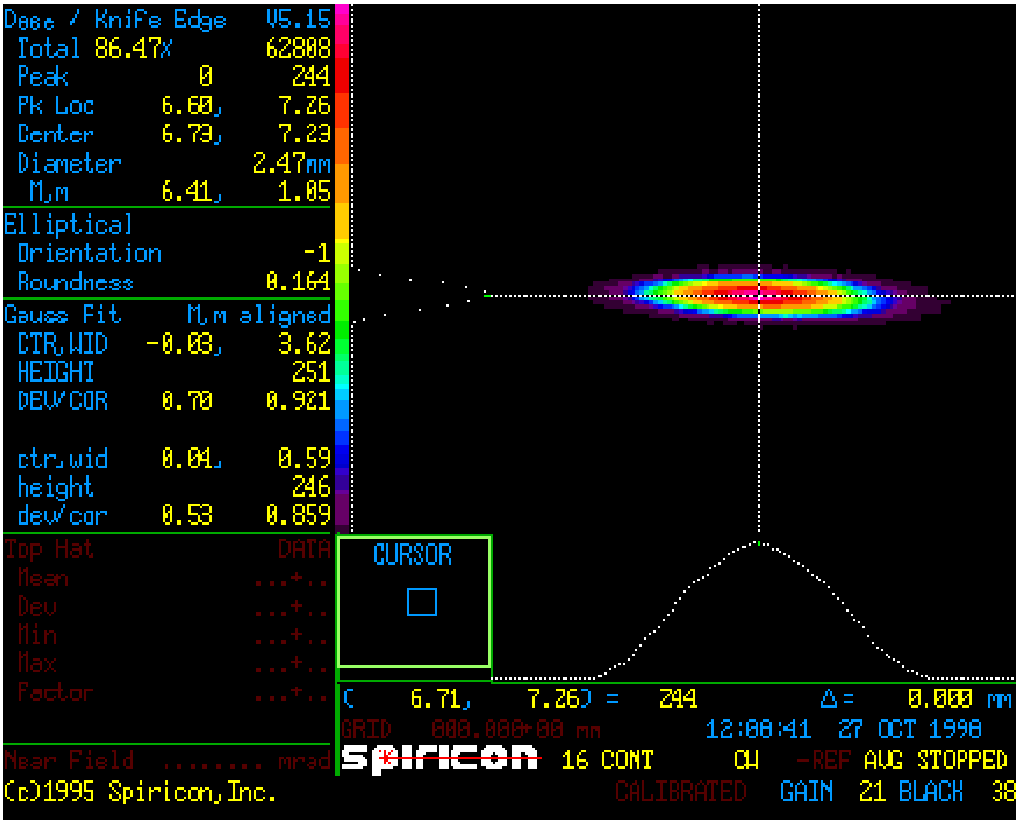


Fig. 9.5. Universal function of the synchrotron radiation spectrum $S(\omega/\omega_c)$

DENSITÀ TRASVERSALE MISURATA CON LUCE DI SINCROTRONE



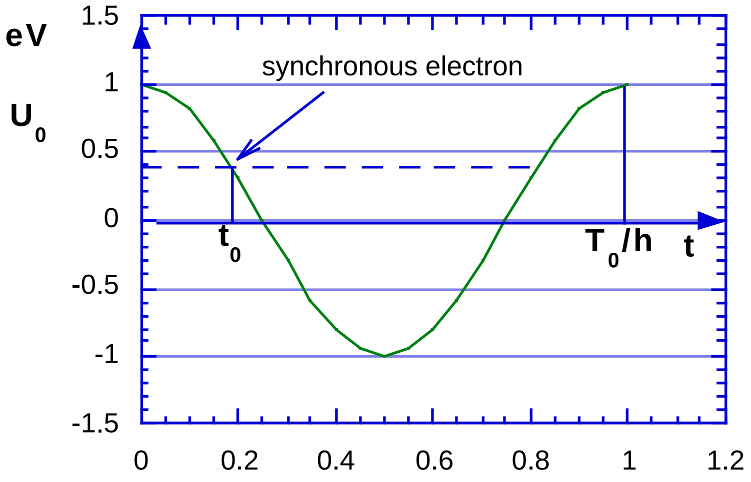
e+



e−

ENERGY OSCILLATIONS

The energy lost by S.R. has to be replaced by means of the electric field in the Radio Frequency (RF) cavities.



eV accelerating voltage

t time of arrival at the RF Cavity

The sinchronous particle travels on the reference trajectory (closed orbit) with energy E_0 and revolution period $T_0 = 1/f_0 = L/c$.

$$E_0[GeV] = \frac{.3}{2\pi}\oint Bds[Tm]$$

The RF frequency must be a multiple of the revolution frequency $f_{RF} = hf_0$.

The synchronous particle arrives at the RF cavity at time t_0 so that the energy gained from the RF is equal to the energy lost by S.R. in one turn U_0.

For γ >>1

ε = E - E_0 > 0 L > L_0 t > t_0 eV < U_0

ε = E - E_0 < 0 L < L_0 t < t_0 eV > U_0

$$\frac{\Delta T}{T_0} = \frac{\Delta L}{L_0} = \alpha \frac{\varepsilon}{E_0}$$

with α the momentum compaction factor.

This produces stable energy and phase oscillations around E_0, t_0.

The synchronous particle is in the bunch center; τ = Δs/c > 0 is the time distance for an e^- ahead of the center of the bunch.

Assuming that changes in ε and τ occur slowly with respect to T_0:

$$\frac{d\tau}{dt} = -\alpha \frac{\varepsilon}{E_0}$$

on average in one turn:

$$\frac{d\varepsilon}{dt} = \frac{eV(\tau) - U(\varepsilon)}{T_0}$$

Consider small oscillation and linear voltage approximation:

$$eV(\tau) = U_0 + e\dot{V}_0\tau$$

We obtain the usual equation of harmonic motion for the energy oscillations with an additional damping term.

$$\frac{d^2\varepsilon}{dt^2} + 2\alpha_\varepsilon \frac{d\varepsilon}{dt} + \Omega^2\varepsilon = 0$$

$$\alpha_\varepsilon = \frac{1}{2T_0}\frac{dU}{d\varepsilon}$$

$$\Omega^2 = \frac{e}{T_0}\dot{V}_0\frac{\alpha}{E_0}$$

The solution is:

$$\varepsilon(t) = Ae^{-\alpha_e t}\cos(\Omega t - \phi)$$

$$\tau(t) = \frac{-\alpha}{E_0\Omega}Ae^{-\alpha_e t}\sin(\Omega t - \phi)$$

Calculation of damping rate

The rate of energy loss changes with energy because

- **it is itself a function of energy**
- **the orbit deviates from the reference orbit and there may be a change in path length.**

$$U(\varepsilon) = \frac{1}{c}\oint P dl$$

P **is a function of** E^2 **and** B^2**.**

$$P = P_0 + \frac{2P_0}{E_0}\varepsilon$$

and

$$\frac{dU(\varepsilon)}{d\varepsilon} = \frac{1}{c}\oint \frac{2P_0}{E_0} ds = \frac{2U_0}{E_0}$$

Taking into account also the path lengthening the damping coefficient is:

$$\alpha_\varepsilon = \frac{1}{2T_0}\frac{U_0}{E_0}(2 + D) \quad ; \quad D = \frac{\oint D(1-2n)/\rho^3 ds}{\oint 1/\rho^2 ds}$$

Generally for separated functions lattice D **<< 1**

$$\tau_\varepsilon = \frac{1}{\alpha_\varepsilon} \approx \frac{T_0 E_0}{U_0}$$

Damping of Betatron Oscillations

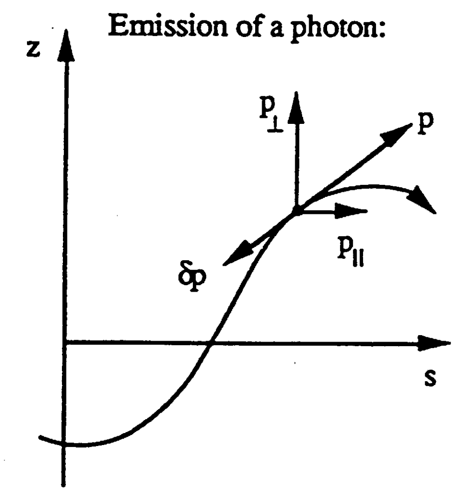
VERTICAL PLANE

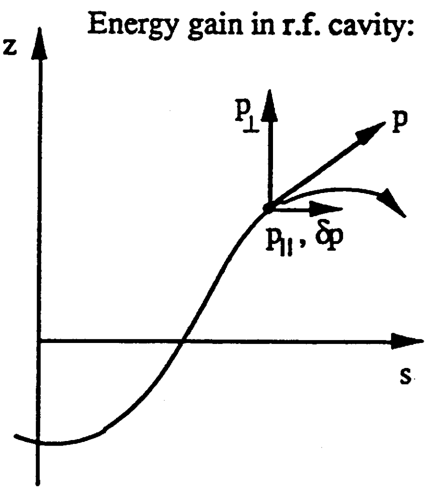
$$z = A\sqrt{\beta}\cos(\phi(s)+\phi_0) \qquad z' = \frac{A}{\sqrt{\beta}}\sin(\phi(s)+\phi_0)$$

A **is the amplitude of the oscillation**

$$A^2 = \gamma z^2 + 2\alpha z z' + \beta z'^2$$

Effect of energy loss due to S.R. and energy gain in the RF cavity.





After the RF cavity, since $z' = p_\perp / p_{\parallel}$, **we have:**

$$\delta z' = -z'\frac{\delta\varepsilon}{E_0}$$

$$\frac{\langle \delta A\rangle}{A} = -\frac{1}{2}\frac{\delta\varepsilon}{E_0} \qquad \text{average over one turn} \qquad \frac{\Delta A}{A} = -\frac{U_0}{2E_0}$$

$$\alpha_z = \frac{U_0}{2E_0T_0}$$

HORIZONTAL PLANE

In the horizontal plane the damping coefficient has an additional term which accounts for the path length variation:

$$\alpha_x = \frac{U_0}{2E_0T_0}(1-D)$$

Damping Partition

$$\alpha_i = \frac{J_i U_0}{2E_0T_0}$$

$i = x,z$ **or** ε **and** J_i **are the damping partition numbers:**

$$J_x = 1 - D \quad ; \quad J_z = 1 \quad ; \quad J_\varepsilon = 2 + D$$

The sum of the damping rates for the three planes is a costant:

$$J_x + J_z + J_\varepsilon = 4$$

For damping in all planes simultaneously all J_i **> 0 and hence** $-2 < D < 1$

RADIATION DAMPING EFFECTS

equilibrium beam sizes
multicycle injection
damping rings
counteracts the beam instabilities

INFLUENCE OF S.R. EMISSION ON MACHINE DESIGN

RF system
vacuum system: heating and gas desorption
radiation damage
radiation background in collider experiments.

HIGH ENERGY STORAGE RINGS

The radius of curvature is increased to keep the power of the emitted radiation below an acceptable level $\rho \approx E^2$.

LOW ENERGY STORAGE RINGS

It is often useful to insert in the ring special devices, wiggler magnets, in order to increase the power emitted by S.R. and reduce the damping times.

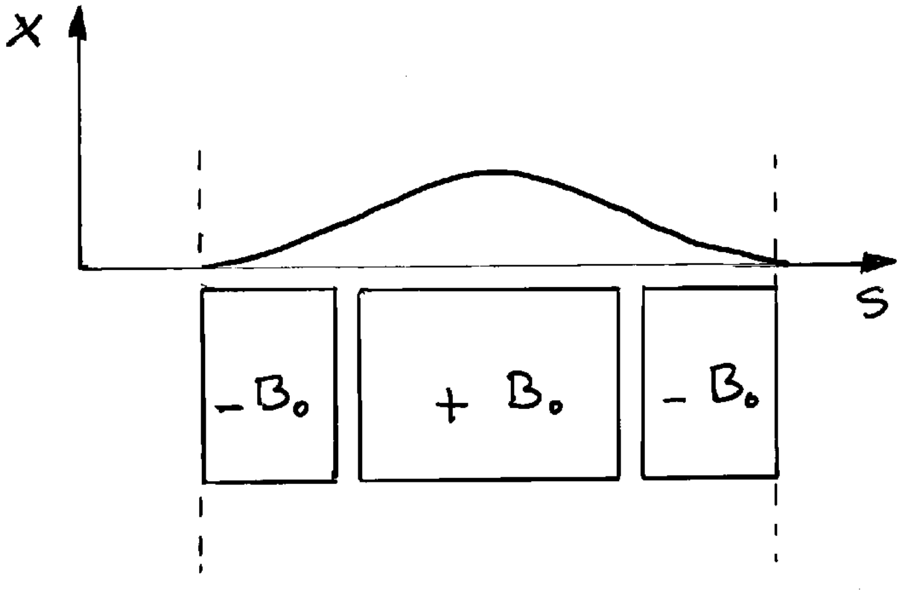
WIGGLER MAGNETS

A wiggler magnet is made of a series of dipole magnets with alternating polarity so that the total bending angle (i.e. the field integral along the trajectory) is zero.

This device can be inserted in a straight section of the ring with minor adjustments of the optical functions.
The damping time becomes faster because U_0 **increases.**

The trajectory in a wiggler period is shown below.

Any number of periods can be added in order to get the desired damping time.



Wigglers are also used in synchrotron light sources to increase the critical frequency and the emitted power with respect to that of the bending magnets.

The energy radiated per turn U_0 **can be written:**

$$U_0 = U_{0,dip} + U_{0,wig} = C_\gamma E^4 \left(\frac{1}{\rho_d} + \frac{1}{2\pi} \int_0^{L_w} \frac{ds}{\rho_w^2} \right)$$

$$C_\gamma = 8.85\ 10^{-5}\ \text{m GeV}^{-3}$$

$$\tau_\varepsilon = \frac{T_0 E_0}{U_0}$$

DAΦNE WIGGLERS

	dipoles	wigglers
B (T)	**1.2**	**1.8**
ρ **(m)**	**1.4**	**.94**
L (m)	**8.8**	**8**
U_0 **(KeV)**	**4.3**	**5.0**

$$U_0 = 9.3\ \text{KeV}\ ; \qquad \tau_\varepsilon = 18\text{ms}$$

Energy oscillation parameters for various electron storage rings

	E (GeV)	$U_{0,dip}$ (MeV)	U_0 (MeV)	T_0 (μs)	T_{synch} (ms)	τ_ε (ms)	τ_ε / T_0
Adone	.51	.001	.001	.35	.05	180	$5\ 10^5$
DAΦNE*	.51	.004	.009	.31	.03	18	$6\ 10^4$
PEP B LE*	3.1	.27	1.24	7.3	.15	18	$2.5\ 10^3$
PEP B HE	9.0	3.5	3.5	7.3	.14	19	$2.6\ 10^3$
LEP	100.	2855	2855	89	1	3.1	35

* Effect at wigglers on U_0

QUANTUM EXCITATION

Radiation damping is related to the *continuous* loss and replacement of energy.

Since the radiation is *quantized,* the statistical fluctuations in the energy radiated per turn cause a growth of the oscillation amplitudes.

The equilibrium distribution of the particles results from the combined effect of quantum excitation and radiation damping.

MEAN SQUARE ENERGY DEVIATION

$$\varepsilon(t) = A\cos(\Omega t - \phi)$$

$$\tau(t) = \frac{-\alpha}{E_0\Omega} A\sin(\Omega t - \phi)$$

The invariate oscillation amplitude is

$$A^2 = \varepsilon^2(t) + \frac{E_0\Omega}{\alpha}\tau^2(t)$$

When a photon of energy u is emitted the change in A^2 **is:**

$$\delta A^2 = -2\varepsilon u + u^2$$

and the total rate of change of A^2**:**

$$\frac{dA^2}{dt} = -\frac{2A^2}{\tau_\varepsilon} + \left\langle N \left\langle u^2 \right\rangle \right\rangle$$

The equilibrium is reached for $d^2A/dt = 0$ **and the mean-square energy deviation is:**

$$\left\langle \varepsilon^2 \right\rangle = \frac{\left\langle A^2 \right\rangle}{2} = \frac{\tau_\varepsilon}{4}\left\langle N \left\langle u^2 \right\rangle \right\rangle$$

The radiation is emitted in photons with energy

$$u = \hbar\omega$$

The total number of photons emitted per electron per second is:

$$N = \int n(u)du = \frac{15\sqrt{3}}{8}\frac{P}{u_c} \quad ; \quad u_c = \hbar\omega_c$$

$$n(u) = \frac{P}{u_c^2}\frac{S(u/u_c)}{(u/u_c)}$$

The mean photon energy is:

$$\langle u \rangle = \frac{\int u \;\; n(u)du}{N} = \frac{P}{N} = \frac{8}{15\sqrt{3}}u_c$$

and the mean square energy:

$$\langle u^2 \rangle = \frac{\int u^2 \;\; n(u)du}{N} = \frac{11}{27}u_c^2$$

DISTRIBUTION OF ENERGY DEVIATION

The energy deviation at a given time can be considered as the sum of all the previous photon emissions, compensated by all the energy gains in the passages through the RF cavities.

Since the typical energy deviation exceeds the typical photon energy the sum contains a large number of statistically independent small terms.

Therefore, for the Central Limit Theorem, the distribution of the energy deviation is Gaussian with standard deviation σ_ε.

$$\sigma_\varepsilon^2 = \langle \varepsilon^2 \rangle = \frac{55}{32\sqrt{3}} \hbar c \gamma^3 \frac{\langle 1/\rho^3 \rangle}{\langle 1/\rho^2 \rangle} \frac{E_0}{J_\varepsilon}$$

and the relative deviation

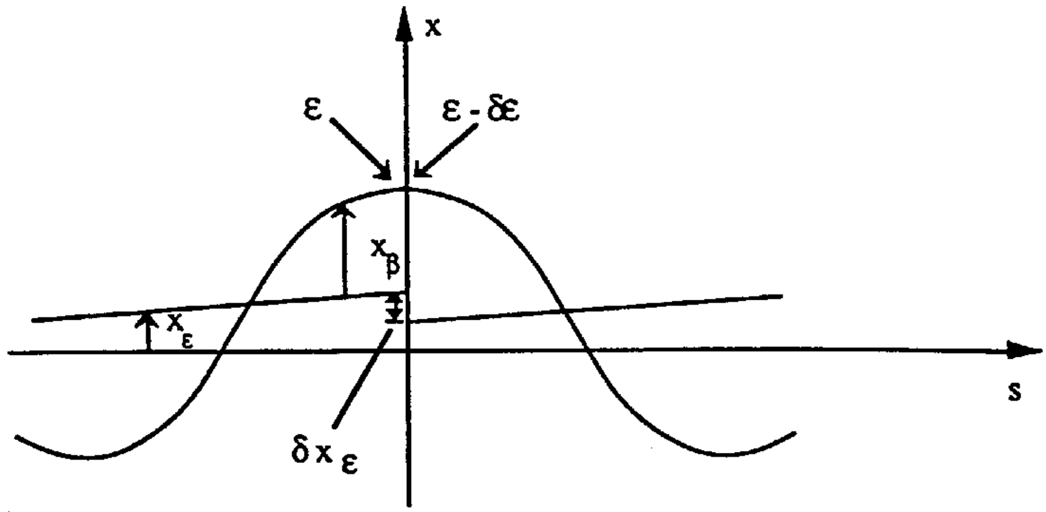
$$\left(\frac{\sigma_\varepsilon}{E_0}\right)^2 = C_q \frac{\gamma^2}{J_\varepsilon} \frac{\langle 1/\rho^3 \rangle}{\langle 1/\rho^2 \rangle} \quad ; \quad C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} = 3.84 \ 10^{-13} \, m$$

EQUILIBRIUM BUNCH LENGTH

A Gaussian distribution in energy results in a similar distribution in the time deviation τ with standard deviation:

$$\sigma_\tau = \frac{\alpha}{\Omega E_0} \sigma_\varepsilon \qquad ; \qquad \Omega^2 = \frac{\alpha}{T_0 E_0} e\dot{V}_0$$

BEAM EMITTANCE - HORIZONTAL PLANE



Effect of energy loss on the off-energy orbit and betatron motion in the horizontal plane.

$$\delta x_\beta = -D(s)\frac{u}{E_0} \quad ; \quad \delta x'_\beta = -D'(s)\frac{u}{E_0}$$

The betatron oscillation invariant is:

$$A^2 = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

and the change due to photon emission:

$$\delta A^2 = \left(\gamma D^2 + 2\alpha D D' + \beta D'^2\right)\frac{u^2}{E_0^2} = \mathcal{H}(s)\frac{u^2}{E_0^2}$$

The average rate of increase of A^2 **is:**

$$\left\langle \frac{dA^2}{dt} \right\rangle = \frac{\left\langle N \left\langle u^2 \right\rangle \mathcal{H} \right\rangle}{E_0^2}$$

Equating to radiation damping the equilibrium mean square value is obtained.

This defines the beam emittance ε_x:

$$\varepsilon_x = \frac{\langle A^2 \rangle}{2} = C_q \frac{\gamma^2}{J_x} \frac{\langle \boldsymbol{H} / \rho^3 \rangle}{\langle 1/\rho^2 \rangle}$$

The emittance is constant for a given lattice and energy.

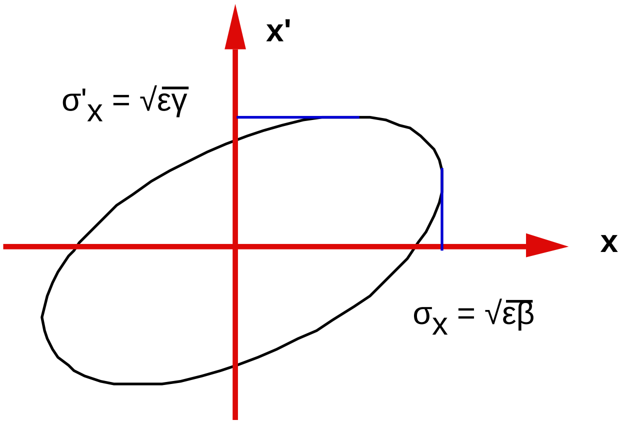
The projection of the distribution on the x,x' axis respectively is Gaussian and the rms beam size and divergence are:

$$\sigma_x = \sqrt{\varepsilon\beta(s)} \qquad \sigma'_x = \sqrt{\varepsilon\gamma(s)}$$

At points in the lattice where dispersion is non zero there is a contribution of the synchrotron motion:

$$\sigma_{x\varepsilon} = D\frac{\sigma_\varepsilon}{E_0} \qquad \sigma'_{x\varepsilon} = D'\frac{\sigma_\varepsilon}{E_0}$$

The two contributions add quadratically.



The area of the ellipse is $\pi\varepsilon$

VERTICAL PLANE

Generally storage rings lie in the horizontal plane and have no bending and no dispersion in the vertical plane.

A very small vertical emittance arises from the fact that the photons are emitted at a small angle ($\theta_{rms} \approx 1/\ \gamma$) with respect to the direction of motion.

The resulting vertical equilibrium emittance is:

$$\varepsilon_z = \frac{C_q}{2J_z} \frac{\langle \beta_z / \rho^3 \rangle}{\langle 1/\ \rho^2 \rangle}$$

Since C_q is very small this vertical emittance can be generally neglected.

In practice the vertical emittance comes from:

- coupling of horizontal and vertical betatron oscillations due to
  - skew quadrupole field errors (angular errors in the quadrupole alignment and vertical orbit in the sextupoles)
  - errors in the compensation of detector solenoids
- vertical dispersion errors
  - angular errors in the dipole alignment and vertical orbit in the quadrupoles.

If the vertical emittance depends only on a large number of small errors randomly distributed along the ring, it can be described in terms of a coupling coefficient κ such that the sum of the horizontal and vertical emittances is constant.

$$\varepsilon_x = \frac{1}{1+\kappa}\varepsilon_{x0} \quad ; \qquad \varepsilon_x = \frac{\kappa}{1+\kappa}\varepsilon_{x0} \qquad\qquad 0 < \kappa < 1$$

ε_{x0} **is often called "natural beam emittance".**

The coupling can be reduced by adjusting the strengths of some skew quadrupoles properly placed in the lattice.

EFFECT OF THE WIGGLERS ON THE EMITTANCE

At a given energy (assume $J_x=1$) the emittance is proportional to the ratio I_5/I_2 . I_5 and I_2 are the S.R. integrals:

$$I_5 = \oint \frac{\gamma D^2 + 2\alpha D D' + \beta D'^2}{\left|\rho^3\right|} ds$$

$$I_2 = \oint \frac{1}{\rho^2} ds$$

We have already seen that the insertion of wigglers in a machine increases I_2 and therefore the energy radiated per turn.

The emittance depends on the value of I_5 which is determined by the dispersion and optical functions in the wiggler straigth sections.

So the emittance of a given machine can be changed by varying the optical functions only in the wiggler section.

Two examples of the extreme cases are wigglers in a vanishing dispersion or in a high dispersion section.

Inserting the wiggler in a vanishing dispersion section gives a reduction of the emittance.
In fact the self-dispersion in the wiggler is always very small compared with the dispersion in the dipoles.

DAΦNE WIGGLERS

In DAΦNE the wigglers are inserted in a high dispersion section in order to obtain a high emittance value.

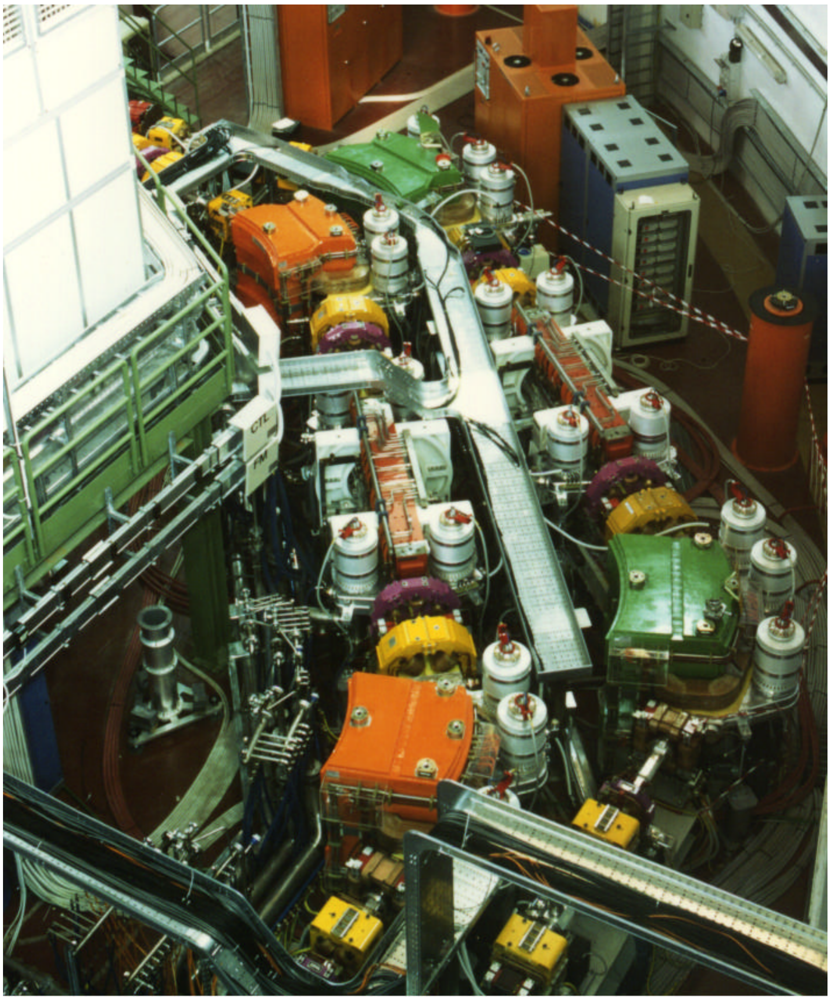
Moreover changing the dispersion and β-function in the wigglers the emittance can be tuned in a wide range.

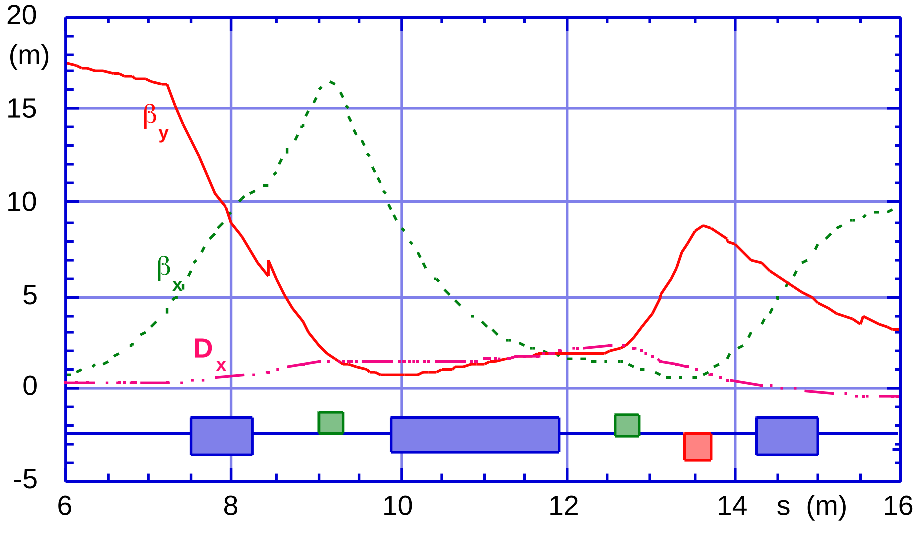
ARC CELL

BWB Bending Wiggler Bending

The wiggler doubles the energy radiated in the bending magnets.

Emittance can be tuned by varing the dispersion in the wiggler.





SINGLE BEAM LIFETIME

Physical processes which affect the single beam lifetime:

- **Quantum lifetime**
- **Beam gas interaction (elastic, inelastic scattering)**
- **Scattering within the bunch (Intrabeam scattering, Touschek effect)**

Beam lifetime: the time interval in which the beam intensity is reduced to a fraction (typically $1/e$) of its initial value.

Aperture limitation

The particle losses are produced by different physical processes but, in all cases, the particles get lost because their betatron or synchrotron oscillation amplitude exceeds an aperture limitation.

TRANSVERSE PLANE

Physical aperture of the vacuum chamber.

LONGITUDINAL PLANE

Energy acceptance of the RF cavity.

DYNAMIC APERTURE

In presence of nonlinearities of the magnetic fields, the region in the sixdimensional phase space where the beam trajectories are stable can be reduced with respect to the physical aperture and to the RF acceptance.

In this case, for beam lifetime calculations, the contour of the stable region in phase space has to be used as aperture limitation.

In most storage rings the dynamic aperture is determined by the sextupoles used to correct the chromaticity.

The sextupole scheme is adjusted by analitycal calculations and tracking simulations in order to have a dynamic aperture larger than the physical aperture.

QUANTUM LIFETIME

The Gaussian distribution of the beam intensity in the transverse plane extends to infinity. Therefore the aperture limitation of the vacuum chamber will cause a constant loss of particles in the tails of the distribution.

The loss is exponential: $N = N_0 e^{-t/\tau_q}$

with $\tau_q = \frac{\tau_x}{2} \frac{e^{\xi}}{\xi}$ **;** $\xi = \frac{x_{\max}^2}{2\sigma_x^2}$

A golden rule to make negligible this cause of beam loss is to choose:

$$x_{max}/\sigma_x \geq 6.5 \text{ .}$$

Analogously for the energy oscillations

$$\varepsilon_{max}/\sigma_\varepsilon \geq 6.5 \text{ .}$$

BEAM-GAS SCATTERING

Elastic scattering

The stored particle is transversally deflected by scattering at the residual gas molecules and increases its betatron oscillation amplitude.

The particle is lost either at the physical or dynamic aperture.

Inelastic scattering

During the scattering process a ligth quantum is emitted and the particle changes its energy or the particle transfers energy to the atom of the gas.

elastic $$\sigma = \frac{2\pi Z^2 r_0^2}{\gamma^2} \frac{\langle\beta\rangle \hat{\beta}}{b^2}$$

inelastic $$\sigma = f\left(Z, \ln(\varepsilon/\varepsilon_m)\right)$$

$$\frac{1}{\tau} = \frac{c\,\sigma}{kT} \sum_{ij} k_{i,j} P_i$$

TOUSCHEK SCATTERING

Touschek scattering is an elastic Coulomb scattering between pairs of particles within a bunch. It results in a change of the longitudinal momentum of the two particles: one loses and the other gains the same fraction of momentum. If the momentum deviation exceeds the acceptance of the ring the particle is lost.

The beam lifetime due to single Touschek scattering is proportional to the third power of the energy, and therefore it is the main limitation for low energy storage rings like DAΦNE.
The Touschek half-lifetime is calculated according to the formula given by H. Bruck:

$$\frac{1}{\tau} = \frac{\sqrt{\pi}\, r_0^2 c N}{\gamma^3 \sigma'_x \varepsilon^2 (4\pi)^{\frac{3}{2}} \sigma_l \sigma_x \sigma_y} C(u_{min})$$

where:

r_0 = classical electron radius
c = velocity of light
γ = electron energy in units of rest mass
N = number of electrons per bunch
σ_x'= angular divergence of the beam
$(4\pi)^{3/2}\ \sigma_l \sigma_x\ \sigma_y$ = beam volume

$$C(u_{min}) = \int_{u_{min}}^{\infty} \frac{1}{u^2}\left[u - u_{min} - \frac{1}{2}\ln\left(\frac{u}{u_{min}}\right)\right] e^{-u}\, du$$

$$u_{min} = \left(\frac{\varepsilon}{\gamma \sigma'_x}\right)^2$$

Physical aperture limitation for the momentum acceptance

ε is the limiting acceptance for the relative momentum deviation of a particle which undergoes a large angle Touschek scattering.

It is the smaller between the RF acceptance, and the momentum acceptance due to the transverse aperture, either physical or dynamic.

At each azimuth s_i along the ring the following quantity is calculated (only for the horizontal plane assuming vanishing vertical dispersion):

$$H(s_i) = \gamma_i D_i + 2\alpha_i D_i D + \beta_i D'^2.$$

The maximum horizontal displacement in a position s_j for a particle which has undergone a relative momentum deviation change ε_i in σ_i is:

$$x_j = \varepsilon_i \left[\sqrt{H_i \beta_j} + |D_j|\right]$$

The limiting value for ε_i is obtained by equating x_j to the physical half-aperture in that position A_x^j and taking the minimum all over the ring:

$$\varepsilon_i = \min_j \left\{ \frac{A_x^j}{\sqrt{H_i \beta_j} + |D_j|} \right\}$$