A REVIEW ON BELL INEQUALITY TESTS WITH NEUTRAL KAONS

A. Bramon, R. Escribano

Grup de Física Teòrica, Universitat Autònoma de Barcelona, E–08193 Bellaterra, Spain

G. Garbarino

Dipartimento di Fisica Teorica, Università di Torino and INFN, Sezione di Torino, I-10125 Torino, Italy

Dedicated to the memory of R.H.Dalitz
from whom we learnt the 'strangeness' of kaon physics.

Abstract

Recent proposals aiming to confront Local Realistic theories with Quantum Mechanics by performing Bell tests with entangled neutral kaons, such as those produced by ϕ decays at Daphne, are reviewed. Some difficulties appear because of the reduced number of useful, non–commuting kaonic observables and the low efficiency of the strangeness measurements. The possibilities to overcome this and other loopholes are analyzed.

1 Introduction

A classical book by R. H. Dalitz [1] offers an accurate description of the development of the 'strange' particle physics since its origin in the 1950s. The 'strangeness' of their behavior was associated with the fact that these particles were copiously produced in ordinary, non–strange particle reactions always in pairs. Present day examples of such 'associated productions' are the electron–positron and the s–wave proton–antiproton annihilations into the state

$$\frac{1}{\sqrt{2}}\left[|K^0\rangle_l|\bar{K}^0\rangle_r - |\bar{K}^0\rangle_l|K^0\rangle_r\right] \qquad (1)$$

consisting of two strange, neutral kaons which, after collimation, form a left– and a right–moving beam as indicated by the subindexes. Independently, another classical book by D. Bohm, 'Quantum Theory', appeared in 1951 [2]. The nowadays famous gedanken experiment by Einstein, Podolsky and Rosen [3] was discussed there in its simplest form, *i. e.*, in terms of the singlet state formed by two spin–1/2 objects which is quite similar to the two–kaon state (1). In the Bohm singlet state, each spin–1/2 points both into any given spatial direction and its opposite one; similarly, each particle in (1) is both a kaon and an antikaon at the very same time. According to quantum mechanics, each separate spin–1/2 particle or kaon in the two–particle states just considered cannot be represented by a wave function or state vector; only the global system, such as that in Eq. (1), has a definite state vector and is thus the single, indivisible quantum. In both considered cases, apparently one has to deal with a rather simple two–particle (bipartite) quantum state, but the *entanglement* or quantum correlations between its two partners adds to the 'strangeness' of kaon physics the weirdness of quantum mechanics.

Indeed, one of the most counterintuitive and subtle aspects of quantum mechanics refers to the correlations shown by the distant parts of composite systems like the above mentioned two. This became evident in 1935, when Einstein, Podolsky and Rosen (EPR) [3], discussing a gedanken experiment with entangled states, arrived at the conclusion that the description of physical reality given by the quantum wave function cannot be complete. Bohr, in his famous response [4], noted that EPR's criterion of physical reality contained an ambiguity if applied to quantum phenomena and gave rise to one of the most important and long standing debates in physics. According to Bohr, EPR's assumption that a quantum system has real and well defined properties also when does not interact with other systems (including measuring apparata) is contradicted by the basic axioms of quantum mechanics.

For about 30 years the debate triggered by EPR and Bohr remained basically a matter of philosophical belief. Then, in 1964, Bell [5] interpreted EPR's argument as the need for the introduction of additional, unobservable variables aiming to restore *completeness*, *relativistic causality* (or *locality*) and *realism* in quantum theory. He established a theorem which proved that any *local hidden–variable* (*i. e.*, *local realistic* [6]) theory is incompatible with some statistical predictions of quantum mechanics. Since then, various forms of Bell inequalities [7]- [11] have been the tool for an experimental discrimination between local realism (LR) and quantum mechanics (QM).

Such a discrimination is possible only if the predictions coming from QM cannot be reproduced with LR models. These models allow the derivation of Bell inequalities which necessarily relate the statistical results one has to expect from a given entangled system when its two members are potentially subjected to alternative joint measurements chosen by the experimenters. If such a choice among experiments exists, we refer to them as *active* measurements. Each one of these experiments projects then each measured kaon into one of the two states of the chosen measurement basis. This is a common feature in Refs. [7]- [11] but care has to be taken when extrapolating these considerations to unstable systems such as neutral kaons [12]. Admittedly, this instability allows for different decay modes, which effectively correspond to different quantum measurements. But the inequalities involving these *passive* measurements, with no choice on the experimenter part, are not *Bell* inequalities since they cannot discriminate LR from QM, as we will discuss later on.

Many experiments confronting QM versus LR have been performed, mainly with entangled optical photons [10, 13, 14, 15, 16] and, more recently, with entangled ions [17]. All these tests obtained results in good agreement with QM but, according to several authors, they do not represent a conclusive proof against LR. The tests are affected by another type of criticisms, which are certainly less severe than that mentioned in the preceding paragraph but have been discussed for many years [18]. These tests only seem to show the violation of the so called *non–genuine* Bell inequalities. Indeed, because of non–idealities of the apparata and other technical problems, *supplementary assumptions* not implicit in LR were needed in the interpretation of the experiments. Consequently, no one of these tests has been strictly *loophole free* [10, 18, 19], *i. e.*, able to test a *genuine* Bell inequality, which has to be a necessary consequence of LR alone.

One of these criticisms, frequently referred to as the *detection* or *efficiency loophole*, is particularly relevant for kaons. It has been proven [9, 10, 20] that for any bipartite and entangled state one can derive Bell inequalities without the introduction of (plausible but not testable) supplementary assumptions concerning undetected events. In particular, the most appropriate inequality for confronting LR vs QM has been derived long ago by Clauser and Horne [9]. For maximally entangled (non–maximally entangled) states, if one assumes that all detectors have the same overall detection efficiency η, these genuine Clauser–Horne inequalities are violated by QM only if $\eta > 0.83$ [21] ($\eta > 0.67$ [22]). Since such detection thresholds cannot be presently achieved in photon experiments, only non–genuine inequalities have been tested experimentally.

Several of these photonic tests violated non–genuine inequalities by the amount predicted by QM but they could not overcome the detection loophole. Indeed, local realistic models exploiting the detector inefficiencies and reproducing the experimental results can be contrived [9, 23] for these tests. Only the recent test with entangled beryllium ions of Ref. [17], for which $\eta \simeq 0.97$, did close the detection loophole. On the other hand, an experiment with entangled photons [14] closed the other main existing loophole, the *locality loophole*. In this test, the measurements on the two photons were carried out under strict space–like separation conditions, thus avoiding any possible exchange of subluminal signals between the two measurement event regions. But this is not the case for the high efficiency experiment [17] with two beryllium ions separated only by a few microns. In other words, no experiment closing simultaneously both the detection *and* locality loopholes has been performed till now.

Extensions to other kind of entangled systems are thus important. Over the past ten years or so there has been an increased interest on the possibility to test LR vs QM in particle physics,*e. g.*, by using entangled neutral kaons [24]–[41]. This is also a manifestation of the desire to go beyond the usually considered spin–singlet case and to have new entangled systems made of massive particles with peculiar quantum–mechanical properties (apart from the classical book by Dalitz [1], other detailed reviews of neutral kaons are [29, 42, 43]). Entangled $K^0\bar{K}^0$ states (1) are copiously produced in the decay of the $\phi(1020)$ resonance [44] and in proton–antiproton annihilation processes at rest [45, 46]. For kaons, the strong nature of hadronic interactions should contribute to close the detection loophole, since it enhances the efficiencies to detect the products of kaon decays and kaon interactions with ordinary matter (pions, kaons, nucleons, hyperons,...). Moreover, the two kaons produced in ϕ decays or $p\bar{p}$ annihilations at rest fly apart from each other at relativistic velocities and can fulfill the condition of space–like separation. Therefore, contrary to the experiment with ion pairs of Ref. [17], the locality loophole could be closed with kaon pairs by using equipments able to prepare, very rapidly, the alternative kaon measurement settings.

In this contribution, our purpose is to review the Bell inequalities proposed to test LR vs QM using $K^0\bar{K}^0$ entangled pairs. The proposals are discussed in the light of the basic requirements —specified in Section 2— necessary to establish genuine Bell inequalities. Each measurement is associated to a specific basis and the bases relevant for our discussion are studied in Section 3. The alternative measurements one can perform on each neutral kaon at a given time are rather reduced, as we show in Section 4. The preparation of the two–kaon entangled state is discussed in Section 5 and can be performed in many different ways; a given, fixed state, however, has to be used for all the alternative measurements contemplated in a given Bell inequality. The various forms of inequalities are derived and related in Section 6. In Section 7 the different proposals with neutral kaons are discussed.

2 Requirements for a genuine Bell inequality

The requirements for deriving a Bell inequality from LR can be summarized as follows:

(1) A non–factorisable or entangled state must be used. Here, as in most cases, a two–particle (bipartite) state is considered. The simplest example is the state (1);

(2) Alternative and mutually exclusive measurements, corresponding to non–commuting observables, must be chosen at will and performed on both members of that state;

(3) Each one of the different single measurements has to have dichotomic outcomes. However, if the possibility of undetected events is considered, they can count as a third outcome;

(4) The measurement process on each member of the two–particle state must be space–like separated from the measurement on the other member.

At a ϕ–factory, or in proton–antiproton annihilations at rest, the first requirement poses no serious problems. Indeed, entanglement has been confirmed experimentally, over macroscopic distances, for $K^0\bar{K}^0$ pairs at CPLEAR [45] using active strangeness measurements and can be demonstrated at the DaΦne ϕ–factory as well [47]. However, care has to be taken to define the state at a specific (proper) time τ, or specific times τ_l and τ_r if these are different for the left– and right–moving members of the entangled state. Indeed, contrary to what happens in photonic experiments, neutral kaons decay and oscillate in time. Only when these times are fixed we have a well defined state to perform Bell–tests.

Difficulties appear with requirement number (2). Indeed, among the differences between the spin–singlet state of entangled photons and the $K^0\bar{K}^0$ entangled state (1), the most important one is that while for photons one can measure the linear polarization along *any* space direction chosen at will, measurements on neutral kaons essentially reduce to only *two* kinds: one can chose to detect either the strangeness or the lifetime of each kaon. These are then two useful and direct measurement choices which can be somehow enlarged by kaon regeneration effects before the final detection (see Section 3.3). But the problem essentially remains and complicates considerably the possibilities of Bell–tests with neutral kaons.

Another property of neutral kaons, not shared by photons, is that the former are unstable and decay via different modes. Each one of these modes is associated with a specific kaon basis and the observation of a kaon decay into a given mode represents a *passive* measurement [12]. Indeed, the experimenter has no control on when the kaon decays nor into which of the various channels it decays. In general, the information thus obtained does not refer to the specific state under consideration (because of kaon time evolution), nor to a desired basis *actively* chosen by the experimenter. As a result, the inequalities that some authors have proposed, which make use uniquely of decay–mode observations, cannot discriminate between LR and QM and, in this sense, are not Bell inequalities. The reason is quite obvious: since the experimenter is not allowed to exert his/her free will, a LR model can immediately be constructed which always gives the same predictions as QM *and* violates the proposed inequality. But this is an absurdity since, by definition, a Bell inequality has to contradict some QM prediction. Since there are no active changes of measurements, the LR model is constructed by just adopting the set of decay distributions predicted by QM as the complete set of hidden–variables. This point was first discussed in a related context by Kasday time ago [48] but has been ignored by many authors when deriving Bell inequalities in the domain of particle physics. In the case of entangled $B^0\bar{B}^0$ pairs, for which only decay mode measurements can be performed, the situation is then more unfortunate than with kaon pairs. Neutral kaons are then unique among pseudo–scalar mesons: the lack of active measurement procedures for B–mesons makes impossible the derivation of relevant Bell inequalities [12, 49]. In this review, centered in discriminating QM from LR, we do not discuss these Bell–tests based on passive measurements, although most of them are of clear interest showing, among other things, the entanglement between pairs of separated particles.

The requirement (4) on locality deserve also some comments. Kaons move at relativistic velocities and can travel macroscopic distances away from the production point before decaying. These distances are certainly much shorter than those involved in photonic experiments (a recent one has shown two–photon entanglement over 144 km [50]) but much larger than those for ions (separated only some microns in Ref. [17]). During these survival distances each kaon has to be submitted to either one or another measurement and this implies changing the experimental setup, typically, by placing or removing material pieces (kaon regenerators). Actively changing from one setup to another in such a way that the two (left and right) distant measurement events are space–like separated could imply serious technical difficulties. For this reason, some authors [51] prefer to consider static measurements setups (fixed pieces acting as different absorbers) which, even if they will not be able to close the locality loophole, look more feasible and still of interest.

Finally, in order to establish the feasibility of the real test, one has to derive the detection efficiencies necessary for a meaningful quantum mechanical violation of the considered Bell inequality. With all this in mind and in the light of the basic requirements (1)–(4), in Section 7 we proceed to analyse various proposals of Bell–tests with entangled kaon–antikaon pairs. Before, we present a general discussion on measurement bases, quantum states and genuine and non–genuine Bell inequalities for neutral kaons.

3 Bases in 'quasi–spin' space

Thanks to the analogy with spin–1/2 particles, neutral kaon states can be conveniently described with the formalism of 'quasi–spin'. The strangeness eigenstates K^0 and $\bar{K}^0$ (specified in subsection 3.1) are considered as members of a quasi–spin doublet, with $|K^0\rangle \equiv \binom{1}{0}$ having 'spin up' and $|\bar{K}^0\rangle \equiv \binom{0}{1}$ having 'spin down'. A particular superposition, with unitary norm, of these strangeness eigenstates together with the corresponding orthogonal state:

$$|K_\alpha\rangle = \alpha|K^0\rangle + \bar{\alpha}|\bar{K}^0\rangle,$$
$$|K_\alpha^\perp\rangle = -\bar{\alpha}^*|K^0\rangle + \alpha^*|\bar{K}^0\rangle, \tag{2}$$

with $\langle K_\alpha|K_\alpha\rangle = \langle K_\alpha^\perp|K_\alpha^\perp\rangle = |\alpha|^2 + |\bar{\alpha}|^2 = 1$ and $\langle K_\alpha|K_\alpha^\perp\rangle = 0$, define the generic basis $\{K_\alpha, K_\alpha^\perp\}$ along the quasi–spin axis α. Any operator acting on the quasi–spin space can be expressed in terms of the Pauli matrices, σ_x, σ_y and σ_z. The formalism is appropriate for all two–level quantum systems or 'qubits' in the novel language of quantum information.

3.1 Strangeness basis: $\{K^0, \bar{K}^0\}$

Neutral kaons are spinless and s–wave quark–antiquark bound meson states, $K^0 \sim d\bar{s}$ and $\bar{K}^0 \sim s\bar{d}$. They define the 'strangeness' or 'strong–interaction' basis which consists of the two eigenstates $|K^0\rangle$ and $|\bar{K}^0\rangle$ with strangeness $S = +1$ and $S = -1$, respectively. This is the suitable basis to analyze S–conserving electromagnetic and strong interaction processes, such as the creation of $K^0\bar{K}^0$ systems from non–strange initial states (*e. g.*., $e^+e^- \to \phi(1020) \to K^0\bar{K}^0$, $p\bar{p} \to K^0\bar{K}^0$), and the detection of neutral kaons via strong kaon–nucleon interactions. This 'strangeness' basis is orthonormal, $\langle K^0|\bar{K}^0\rangle = 0$. In the quasi–spin picture, the strangeness operator evidently corresponds to σ_z:

$$\sigma_z|K^0\rangle = +|K^0\rangle, \quad \sigma_z|\bar{K}^0\rangle = -|\bar{K}^0\rangle.$$

Weak interaction phenomena —such as K^0–$\bar{K}^0$ mixing, K^0–$\bar{K}^0$ oscillations and neutral kaon evolution in time—, as well as kaon propagation in a medium —with the associated regeneration effects— introduce other relevant bases.

3.2 Free–space basis: $\{K_S, K_L\}$

The so called K–short and K–long states, $|K_S\rangle$ and $|K_L\rangle$, are the normalized eigenvectors of the effective weak Hamiltonian $H_{\rm free}$ governing neutral kaon time evolution in free–space:

$$i\frac{d}{d\tau}|K_{S,L}(\tau)\rangle = H_{\rm free}|K_{S,L}(\tau)\rangle, \quad H_{\rm free} = \begin{pmatrix} \lambda_+ & \lambda_-/r \\ r\lambda_- & \lambda_+ \end{pmatrix}, \tag{3}$$

where $r \equiv (1-\epsilon)/(1+\epsilon)$, ϵ is the CP–violation parameter [42, 52] and τ is the kaon proper time.

The (complex) eigenvalues of the previous (non–hermitian) Hamiltonian are

$$\begin{aligned} \lambda_S &= \lambda_+ + \lambda_- = m_S - \frac{i}{2}\Gamma_S, \\ \lambda_L &= \lambda_+ - \lambda_- = m_L - \frac{i}{2}\Gamma_L, \end{aligned} \tag{4}$$

where $m_{S,L}$ are the $K_{S,L}$ masses and $\Gamma_{S,L} \equiv 1/\tau_{S,L}$ their decay widths, with lifetimes $\tau_S = (0.8953 \pm 0.0005) \times 10^{-10}$ s and $\tau_L = (5.18 \pm 0.04) \times 10^{-8}$ s [52]. We also introduce $\Delta m \equiv m_L - m_S \simeq 0.475\,\Gamma_S$ and $\Delta\Gamma \equiv \Gamma_L - \Gamma_S \simeq -\Gamma_S$, to be used later on.

The corresponding K–short and K–long eigenstates are:

$$\begin{aligned} |K_S\rangle &= \frac{1}{\sqrt{2(1+|\epsilon|^2)}}\left[(1+\epsilon)|K^0\rangle + (1-\epsilon)|\bar{K}^0\rangle\right], \\ |K_L\rangle &= \frac{1}{\sqrt{2(1+|\epsilon|^2)}}\left[(1+\epsilon)|K^0\rangle - (1-\epsilon)|\bar{K}^0\rangle\right], \end{aligned} \tag{5}$$

or, ignoring an irrelevant global phase:

$$|K_{S,L}\rangle = \frac{1}{\sqrt{1+|r|^2}}\left[|K^0\rangle \pm r|\bar{K}^0\rangle\right]. \tag{6}$$

The proper time propagation of the short– and long–lived states, having well–defined masses and decay widths, shows no oscillation between these two states and, according to Eqs. (3), is simply given by

$$|K_{S,L}(\tau)\rangle = e^{-im_{S,L}\tau} e^{-\frac{1}{2}\Gamma_{S,L}\tau}|K_{S,L}(\tau=0)\rangle \equiv e^{-i\lambda_{S,L}\tau}|K_{S,L}\rangle. \tag{7}$$

The $\tau = 0$ states $|K_{S,L}\rangle$ define a quasi–orthonormal basis with $\langle K_S|K_S\rangle = \langle K_L|K_L\rangle = 1$ and

$$\langle K_S|K_L\rangle = \langle K_L|K_S\rangle = \frac{1-|r|^2}{1+|r|^2} = \frac{\epsilon+\epsilon^*}{1+|\epsilon|^2} \simeq 0,$$

due to the smallness of the CP–violation parameter ϵ with modulus $|\epsilon| \simeq (2.284 \pm 0.014) \times 10^{-3}$ and phase $\phi \simeq 43.5^\circ$ [52].

In the quasi–spin space, the weak interaction eigenstates are indeed very 'similar' to the CP eigenstates $|K_1\rangle$ ($CP=+1$) and $|K_2\rangle$ ($CP=-1$):

$$|K_S\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}\left[|K_1\rangle + \epsilon|K_2\rangle\right], \tag{8}$$
$$|K_L\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}\left[|K_2\rangle + \epsilon|K_1\rangle\right].$$

But, while the $K_{S,L}$ basis is useful to discuss free–space propagation, the CP–basis describes weak kaon decays either into two or three final pions from the K_1 and K_2 states, respectively. These two $CP=\pm$ states are the eigenstates of σ_x, $\sigma_x|K_1\rangle = +|K_1\rangle$ and $\sigma_x|K_2\rangle = -|K_2\rangle$. Thus, the limit of CP–conservation corresponds to the invariance under quasi–spin rotations around the x axis. In this limit one has

$$|K_S\rangle \to |K_1\rangle = \frac{1}{\sqrt{2}}\left[|K^0\rangle + |\bar{K^0}\rangle\right], \tag{9}$$
$$|K_L\rangle \to |K_2\rangle = \frac{1}{\sqrt{2}}\left[|K^0\rangle - |\bar{K^0}\rangle\right],$$

and strict orthogonality between the K_S and K_L states is recovered.

3.3 Inside–matter basis: $\{K'_S, K'_L\}$

The dynamics of neutral kaons propagating inside a homogeneous medium of nucleonic matter, which we can consider as a 'regenerator' and/or an 'absorber', is governed by the Hamiltonian

$$H_{\rm medium} = H_{\rm free} - \frac{2\pi\nu}{m_K}\begin{pmatrix} f_0 & 0 \\ 0 & \bar{f}_0 \end{pmatrix}, \tag{10}$$

showing an additional, strong interaction term where m_K is the mean $K_{S,L}$ mass, f_0 and $\bar{f}_0$ are the forward scattering amplitudes for K^0 and $\bar{K^0}$ on nucleons and ν is the nucleonic density of the homogeneous medium.

The eigenvalues of $H_{\rm medium}$ are

$$\lambda'_S = \lambda_+ - \frac{\pi\nu}{m_K}(f_0+\bar{f}_0) + \lambda_-\sqrt{1+4\rho^2}, \tag{11}$$
$$\lambda'_L = \lambda_+ - \frac{\pi\nu}{m_K}(f_0+\bar{f}_0) - \lambda_-\sqrt{1+4\rho^2},$$

and the corresponding eigenstates

$$|K'_S\rangle = \frac{1}{\sqrt{1+|r\bar{\rho}|^2}}\left[|K^0\rangle + r\bar{\rho}|\bar{K}^0\rangle\right], \qquad (12)$$
$$|K'_L\rangle = \frac{1}{\sqrt{1+|r(\bar{\rho})^{-1}|^2}}\left[|K^0\rangle - r(\bar{\rho})^{-1}|\bar{K}^0\rangle\right],$$

where we have introduced the dimensionless regenerator parameter ρ, as well as the auxiliary parameter $\bar{\rho}$ and its inverse $(\bar{\rho})^{-1}$,

$$\rho \equiv \frac{\pi\nu}{m_K}\frac{f_0-\bar{f}_0}{\lambda_S-\lambda_L}, \qquad (13)$$
$$\bar{\rho} \equiv \sqrt{1+4\rho^2}+2\rho \ , \quad (\bar{\rho})^{-1} = \sqrt{1+4\rho^2}-2\rho. \qquad (14)$$

The proper time propagation of these $K'_{S,L}$ states inside matter is given by

$$|K'_{S,L}(\tau)\rangle = e^{-i\lambda'_{S,L}\tau}|K'_{S,L}\rangle, \qquad (15)$$

and shows no K'_S–K'_L oscillations but a decreasing intensity in time given by the imaginary part of $\lambda'_{L,S}$. The latter comes from weak decays, essentially as in free–space propagation, plus absorption via strong kaon interactions with the medium driven by the imaginary part of $f_0+\bar{f}_0$. In this sense, the medium acts as an 'absorber'. The difference between f_0 and $\bar{f}_0$ appearing in ρ is responsible for 'rotations' in the quasi–spin space and transitions between K_S and K_L states. For surviving kaons, the medium acts as a 'regenerator', giving rise to the well known ρ–dependent regeneration effects.

Again, the $\tau = 0$ states form a quasi–orthonormal basis,

$$\langle K'_S|K'_L\rangle = \langle K'_L|K'_S\rangle^* = \frac{1-|r|^2(\bar{\rho}^*/\bar{\rho})}{\sqrt{1+|r\bar{\rho}|^2}\sqrt{1+|r/\bar{\rho}|^2}}, \qquad (16)$$

due to the smallness of ϵ ($r \simeq 1$), as before, and to the low efficiency of usual regenerators ($\rho \simeq \mathcal{R}e\,\rho \sim 10^{-2}$ and $\bar{\rho} \simeq \bar{\rho}^*$ [26, 42]), in spite of the strong character of the induced kaon–nucleon interactions.

Two limiting cases illustrate the relationships among the three bases we have considered: i) for a very low density medium: $\nu, \rho \to 0$ and $\bar{\rho} \to 1$ imply $|K'_{S,L}\rangle \to |K_{S,L}\rangle$, thus recovering the states in Eq. (6) and ii) for extremely high density media (absorbers): $\nu, |\rho|, |\bar{\rho}| \to \infty$ implies $|K'_S\rangle \to |\bar{K}^0\rangle$ and $|K'_L\rangle \to |K^0\rangle$.

4 Dichotomic measurements on neutral kaon states

4.1 Strangeness measurements

When a kaon–nucleon reaction occurs at a given place of a medium, the distinct strong interactions of the $S = +1$ and $S = -1$ neutral kaons on the bound nucleons inside the medium project the arbitrary state of an incoming kaon into one of the two orthogonal members of the strangeness basis $\{K^0, \bar{K}^0\}$ [1, 29, 42]. The quantum number S of the incoming kaon state is determined by identifying the products (usually pions, kaons, nucleons and hyperons) of the strangeness conserving kaon–nucleon strong interaction. Simple examples of $\bar{K}^0$ identifying reactions at low energies are $\bar{K}^0 p \to \Lambda \pi^+$, $\bar{K}^0 p \to \Sigma \pi$ and $\bar{K}^0 n \to \Lambda \pi^0$, while the lowest threshold reaction $K^0 p \to K^+ n$ identifies incoming states as K^0's. This strangeness measurement is then analogous to the projective von Neumann measurements with two–channel analyzers for polarized photons or Stern–Gerlach setups for spin–1/2 particles.

Unfortunately, the efficiency for such strangeness measurements at moderate kaon energies as in $\phi \to K^0 \bar{K}^0$ and $p\bar{p} \to K^0 \bar{K}^0$ is certainly less than what people have been naively expecting from the strong nature of these interactions [45]. The reason, rather than being the difficulty in detecting the final state particles (for which one can have rather high efficiencies), stems from the low probability in initiating the strong reaction. Indeed, the efficiency to *induce* either a $\bar{K}^0$–nucleon or a K^0–nucleon interaction at a given time τ turns out to be close to 1 only for infinitely dense absorber materials or for ultrarelativistic kaons, where, by Lorentz contraction, the absorber is seen by the incoming kaon as extremely thin and dense ($\nu \to \infty$). In this case, kaon–nucleon strong interactions occur and the incident kaons are immediately projected into one member of the inside–matter or the strangeness basis which are coincident in the present limit; *i. e.*, the incident kaon is projected into either $|K'_S\rangle \to |\bar{K}^0\rangle$ or $|K'_L\rangle \to |K^0\rangle$. But in ordinary cases, when a thin absorber is placed to measure strangeness, the incident kaon likely fails to interact with a nucleon and, although the kaon can be efficiently detected beyond the absorber, the desired strangeness measurement has not been performed. Ordinary matter is too transparent for kaons. This contrasts with the polarization measurements where the low efficiency corresponds to the final detection of photons having passed through the efficient analyzers. It would be highly desirable to identify very efficient kaon absorbers. Since this does not seem to be viable at present, one has to play with small strangeness detection efficiencies, which originate both conceptual and practical difficulties when discussing Bell–type tests for entangled kaons [37, 38, 39, 41].

4.2 Lifetime measurements

To measure if a kaon is propagating in free–space as a K_S or K_L at a given time τ, one has to allow for further propagation in free–space and then detect at which time it subsequently decays. Kaons which show a decay vertex between times τ and $\tau+\Delta\tau$ have to be identified as K_S's, while those decaying later than $\tau+\Delta\tau$ have to be identified as K_L's. Since there are no K_S–K_L oscillations, such subsequent decays do really identify the state at the desired previous time τ. The probabilities for wrong K_S and K_L identification are then given by $\exp(-\Gamma_S\,\Delta\tau)$ and $1-\exp(-\Gamma_L\,\Delta\tau)$, respectively. By choosing $\Delta\tau = 4.8\,\tau_S$, both K_S and K_L misidentification probabilities reduce to $\simeq 0.8\%$, which can be further reduced if the decay mode is also identified (see appendix of Ref. [41]).

Recall that the K_S and K_L states are not strictly orthogonal to each other, $\langle K_S|K_L\rangle = 2\,\mathcal{R}e\,\epsilon/(1+|\epsilon|^2) \neq 0$; thus their identification cannot be exact even in principle. However, ϵ is so small [$|\epsilon| \simeq (2.284 \pm 0.014)\times 10^{-3}$ [52]] and the decay probabilities of the two components so different ($\Gamma_S \simeq 579\,\Gamma_L$) that the K_S vs K_L identification can effectively work [37]. Note also that, contrary to strangeness measurements, K_S vs K_L identifications are not affected by the previous low inefficiencies: by using detectors with very large solid angles, one can play with rather high efficiencies for the detection and proper identification of the kaon decay products.

4.3 Active vs passive measurements

The methods described in the last two subsections to discriminate K^0 vs $\bar{K}^0$ and K_S vs K_L are fully appropriate to establish Bell inequalities and tests. On the one hand, the two measurements correspond to complementary and non–commuting observables with dichotomic outcomes in both cases (essentially, σ_z and σ_x in quasi–spin space). On the other hand, they are clearly *active* measurement procedures since they are performed by exerting the free will of the experimenter, another crucial ingredient to establish genuine Bell inequalities. Indeed, at the chosen measurement time τ, either one places a dense slab of matter or allows for free–space propagation. Needless to say, one measurement excludes the other. In the former case, strangeness is measured and no information is obtained on the lifetime of the observed kaon. Conversely, if free propagation is allowed, one identifies K_S vs K_L but nothing is learned on the strangeness quantum number. As previously discussed, the active strangeness measurement is monitored by strangeness conservation, while the active lifetime measurement is possible thanks to the smallness of Γ_L/Γ_S.

Contrary to what happens with other two–level quantum systems such as spin–1/2 particles or photons, *passive* measurements of lifetime and strangeness for neutral kaons are also possible [12], by randomly exploiting the quantum–mechanical dynamics of kaon decays. To this aim, one has to allow for complete free–space propagation and observe the various kaon decay modes. By neglecting the (small) CP–violation effects ($\epsilon \to 0$), non–leptonic kaon decays into two and three pions permit the identification of K_S's and K_L's, respectively. Alternatively, the strangeness of a given neutral kaon state is measured by observing their semileptonic decays. These decays obey the well tested $\Delta Q = \Delta S$ rule, which allows the modes $K^0 \to \pi^- l^+ \nu_l$ and $\bar{K^0} \to \pi^+ l^- \bar{\nu}_l$, with $l = e, \mu$, but forbids decays into the respective charge–conjugate final states. These procedures for the passive K_S vs K_L and K^0 vs $\bar{K^0}$ discriminations are unambiguous in the approximations given by CP–conservation and the $\Delta Q = \Delta S$ rule, respectively.

However, in passive measurement procedures the experimenter has no control on the time when the lifetime or the strangeness measurement occurs, nor on the basis in which the measurement is performed, in contrast with the previous active, von Neumann projection measurements requiring the intervention of the experimenter, who is free to chose between the two complementary measurements. As discussed in sect.2, for experiments performed with passive measurements only, Kasday construction [48, 49] is therefore possible, thus preventing the derivation of Bell inequalities.

5 Entangled states of neutral kaon pairs

5.1 Maximally entangled states

The simplest and most often discussed two–party or bipartite states are the spin–singlet states consisting of two spin–1/2 particles, as first proposed by

D. Bohm [2]. Let us then first consider the two–kaon entangled state which is the analogous [32, 34, 40, 53] to this standard Bohm state. From both ϕ–resonance decays [44] or s–wave proton–antiproton annihilation [45], one starts at time $\tau = 0$ with an initial state $|\phi(0)\rangle$ with global spin, charge conjugation and parity $J^{PC} = 1^{--}$:

$$
\begin{aligned}
|\phi(0)\rangle &= \frac{1}{\sqrt{2}} \left[|K^0\rangle_l |\bar{K}^0\rangle_r - |\bar{K}^0\rangle_l |K^0\rangle_r \right] \\
&= \frac{1}{\sqrt{2}} \frac{1+|\epsilon|^2}{|1-\epsilon^2|} \left[|K_L\rangle_l |K_S\rangle_r - |K_S\rangle_l |K_L\rangle_r \right],
\end{aligned}
\tag{17}
$$

where l and r denote the 'left' and 'right' directions of motion of the two separating kaons. The weak, CP–violating effects enter only in the last equality. Note that this state is antisymmetric and maximally entangled in the two observable bases. The corresponding measurements will always lead to left–right anticorrelated results.

After production, the left and right moving kaons evolve according to Eq. (7) up to (proper) times τ_l and τ_r, respectively. Formally, this leads to the 'two–times' state

$$
\begin{aligned}
|\phi(\tau_l, \tau_r)\rangle &= \frac{e^{-(\Gamma_L \tau_l + \Gamma_S \tau_r)/2}}{\sqrt{2}} \\
&\quad \times \left\{ |K_L\rangle_l |K_S\rangle_r - e^{i\Delta m(\tau_l - \tau_r)} e^{\Delta\Gamma(\tau_l - \tau_r)/2} |K_S\rangle_l |K_L\rangle_r \right\}
\end{aligned}
\tag{18}
$$

in the lifetime basis, with $\epsilon \to 0$. Equivalently,

$$
\begin{aligned}
|\phi(\tau_l, \tau_r)\rangle &= \frac{1}{2\sqrt{2}} e^{-(\Gamma_L \tau_l + \Gamma_S \tau_r)/2} \\
&\quad \times \left\{ \left(1 - e^{i\Delta m(\tau_l - \tau_r)} e^{\Delta\Gamma(\tau_l - \tau_r)/2}\right) \left[|K^0\rangle_l |K^0\rangle_r - |\bar{K}^0\rangle_l |\bar{K}^0\rangle_r \right] \right. \\
&\quad \left. + \left(1 + e^{i\Delta m(\tau_l - \tau_r)} e^{\Delta\Gamma(\tau_l - \tau_r)/2}\right) \left[|K^0\rangle_l |\bar{K}^0\rangle_r - |\bar{K}^0\rangle_l |K^0\rangle_r \right] \right\}
\end{aligned}
\tag{19}
$$

in the strangeness basis.

Most usually, one considers two–kaon states at a unique time $\tau_l = \tau_r \equiv \tau$. One then has

$$
\begin{aligned}
|\phi(\tau, \tau)\rangle &= \frac{1}{\sqrt{2}} e^{-(\Gamma_L + \Gamma_S)\tau/2} \left[|K^0\rangle_l |\bar{K}^0\rangle_r - |\bar{K}^0\rangle_l |K^0\rangle_r \right] \\
&= \frac{1}{\sqrt{2}} e^{-(\Gamma_L + \Gamma_S)\tau/2} \left[|K_L\rangle_l |K_S\rangle_r - |K_S\rangle_l |K_L\rangle_r \right],
\end{aligned}
\tag{20}
$$

showing the same maximal entanglement and anticorreations at any time τ.

5.2 Non–maximally entangled states

Apart from the previous maximally entangled state of kaons, other non–maximally entangled states are of interest for testing LR vs QM. To prepare these states we start with the initial state (17). A thin, homogeneous regenerator is fixed on the right beam (say), as close as possible to the point where the two–kaon state originates. If the regenerator is very close to this origin and the proper time $\Delta\tau$ required by the right moving neutral kaon to cross the regenerator is short enough, $\Delta\tau << \tau_S$, weak decays can be ignored and the state leaving the thin regenerator is

$$|\phi(\Delta t)\rangle = \frac{1}{\sqrt{2}}\left[|K_S\rangle|K_L\rangle - |K_L\rangle|K_S\rangle + \eta|K_S\rangle|K_S\rangle - \eta|K_L\rangle|K_L\rangle\right], \qquad (21)$$

where the indexes l and r referring to the kaon propagation directions are omitted from now on. The complex parameter η characterizes the regeneration effects and is defined by [42]:

$$\eta \equiv i\rho(\lambda_S - \lambda_L)\Delta\tau = i\frac{\pi\nu}{m_K}(f_0 - \bar{f}_0)\Delta\tau = i\frac{\pi\nu}{p_K}(f_0 - \bar{f}_0)d, \qquad (22)$$

where m_K is the average neutral kaon mass, p_K the kaon momentum and d the total length of the regenerator.

The states (20) at $\tau = 0$ and (21) only differ in the terms linear in the small parameter η. Indeed, for typical regenerators and at DaΦne energies one has $|\eta| = \mathcal{O}(10^{-3})$ when $d = 1$ mm [26, 42], thus allowing to neglect higher order terms in the state (21). To enhance that difference, we now allow the state (21) to propagate in free space up to a proper time T in the wide range $\tau_S << T << \tau_L \simeq 579\,\tau_S$. One thus obtains the state:

$$\begin{aligned}|\phi(T)\rangle \;=\; & \frac{e^{-(\Gamma_L\,\tau_l + \Gamma_S\,\tau_r)/2}}{\sqrt{2}}\Big[|K_S\rangle|K_L\rangle - |K_L\rangle|K_S\rangle \qquad (23)\\ & -\eta e^{-i\Delta m T} e^{\frac{1}{2}(\Gamma_S - \Gamma_L)T}|K_L\rangle|K_L\rangle + \eta e^{i\Delta m T} e^{\frac{1}{2}(\Gamma_L - \Gamma_S)T}|K_S\rangle|K_S\rangle\Big],\end{aligned}$$

where the $K_L K_L$ component has survived against weak decays much better than the accompanying terms $K_S K_L$ and $K_L K_S$ and has thus been enhanced. On the contrary, the $K_S K_S$ component has been strongly suppressed and can be neglected if $T/\tau_S >> 1$.

The normalization of state (23) to the surviving pairs leads then to:

$$|\Phi\rangle \;=\; \frac{1}{\sqrt{2 + |R_L|^2 + |R_S|^2}} \qquad (24)$$

$$[|K_S\rangle|K_L\rangle - |K_L\rangle|K_S\rangle + R_L|K_L\rangle|K_L\rangle + R_S|K_S\rangle|K_S\rangle] \, ,$$

where:

$$R_L \equiv -re^{\left[-i\Delta m+\frac{1}{2}(\Gamma_S-\Gamma_L)\right]T}, \;\; R_S \equiv re^{\left[i\Delta m-\frac{1}{2}(\Gamma_S-\Gamma_L)\right]T}. \tag{25}$$

Note that the quantity $|R_L| \simeq |r|e^{\frac{1}{2}\Gamma_S T}$ is not necessarily small with an exponential factor compensating the smallness of $|r|$, but we take $R_S \to 0$ from now on. The non–maximally entangled state Φ describes all kaon pairs with both left and right partners surviving up to the common proper time T. Because of the particular normalization of Φ, kaon pairs showing the decay of one (or both) member(s) before time T have to be detected and excluded. Since this occurs prior to any measurement eventually used for a Bell–type test, ours is a 'pre–selection' (as opposed to 'post–selection') procedure which poses no problem when confronting LR with QM.

Once the state is prepared as in Eq. (24), alternative joint measurements on each one of the corresponding kaon pairs have to be considered for a Bell–type test. In Section 7 we will see how one can utilize this state for such tests.

6 Bell inequalities with neutral kaon pairs

In the present Section, our first aim is to show how the Clauser–Horne inequality can be derived from LR and adapted for a generic entangled state of kaon pairs. The obtained CH inequality, equivalent to an inequality which Eberhard proved in a different way, is a *genuine* Bell inequality in the sense that it follows from LR with no need of extra assumptions. When supplementary assumptions, not implicit in LR, are introduced, other, *non–genuine* Bell inequalities can be derived which allow to design more feasible experimental tests of LR vs QM. The potentialities of a Bell inequality derived by Clauser–Horne–Shimony–Holt and of another simple one, due by Wigner, are thus illustrated.

6.1 Clauser–Horne inequalities

In the interpretation with *hidden–variables*, a generic two–kaon entangled state corresponds to a statistical ensemble of kaon pairs specified by different values of these additional, unobservable, deterministic or stochastic variables, which are here globally denoted by the symbol λ. In principle, λ can contain the same information of the quantum mechanical two–kaon state or wave–function but can be further completed, for instance to restore classical determinism in measurement processes. Moreover, in λ one could also include apparatus random hidden–variables, which can influence, locally, the outcomes of measurements. In a general hidden–variable (*i. e.*, realistic) theory, the joint probability to observe particular kaon quasi–spin states $|K_\alpha\rangle = \alpha|K^0\rangle + \bar{\alpha}|\bar{K}^0\rangle$ and $|K_\beta\rangle = \beta|K^0\rangle + \bar{\beta}|\bar{K}^0\rangle$ along the 'left' and 'right' beams, respectively —when measurements along the α and β quasi–spin axes are performed— is given by:

$$P(K_\alpha, K_\beta) = \int d\lambda \, \rho(\lambda) \, p(K_\alpha, K_\beta|\lambda), \tag{26}$$

where $\rho(\lambda)$ is the hidden–variable probability distribution (normalized to unity, $\int d\lambda \, \rho(\lambda) = 1$) and $p(K_\alpha, K_\beta|\lambda)$ the conditional probability that a joint measurement produces the outcome (K_α, K_β) when the kaon pair (and eventually the measuring devices) is in the *state* specified by λ. Note that, since each kaon pair is assumed to be emitted by the source in a way which is independent of the 'adjustable parameters' α and β characterizing the chosen measurement axes, the hidden–variable distribution function $\rho(\lambda)$ is independent of α and β.

By enforcing the *locality* condition, the previous conditional probability $p(K_\alpha, K_\beta|\lambda)$ can be written in the following factorized form:

$$p(K_\alpha, K_\beta|\lambda) = p(K_\alpha, *|\lambda) \, p(*, K_\beta|\lambda), \tag{27}$$

where, for instance:

$$p(K_\alpha, *|\lambda) \equiv p(K_\alpha, K_\gamma|\lambda) + p(K_\alpha, K_\gamma^\perp|\lambda) + p(K_\alpha, U_\gamma|\lambda), \tag{28}$$

taking values between 0 and 1, is independent of the choice of the states $|K_\gamma\rangle$ and $|K_\gamma^\perp\rangle$ forming an orthogonal basis in the quasi–spin space, $\langle K_\gamma|K_\gamma^\perp\rangle = 0$. Note that, in particular $p(*, *|\lambda) = 1$ for any λ. Note also that the last term of the previous equation takes into account eventual undetected events or events for which the proposed measurement failed, denoted by the U_γ argument, due to the various non–perfect efficiencies in measurements along the 'axis' γ in the quasi–spin space. It is important to remark that in the present scheme, the measurement fails or not depending on the values of the hidden–variables. In other words, the possibility of performing or not the desired measurement is correlated with the values of λ. It is also important to emphasize that for fixed

λ, the single–side probabilities $p(K_\alpha, *|\lambda)$ and $p(*, K_\beta|\lambda)$ entering equation (27) are independent of the measurement that one chooses to perform on the other member of the pair: the kaon quasi–spin outcome K_α (K_β) observed along the left (right) beam when measuring along the quasi–spin axis α (β) is independent of the quasi–spin axis β (α) employed to detect the right (left) going kaon.

To derive the Clauser–Horne (CH) inequality, the mathematical lemma of Ref. [9] can be used. It asserts that for any value between 0 and 1 of the real numbers x_1, x_2, x_3 and x_4, the inequality $x_1x_2 - x_1x_4 + x_2x_3 + x_3x_4 \leq x_3 + x_2$ holds. By assigning $x_1 = p(K_\alpha, *|\lambda)$, $x_2 = p(*, K_\beta|\lambda)$, $x_3 = p(K_{\alpha'}, *|\lambda)$ and $x_4 = p(*, K_{\beta'}|\lambda)$, using the factorisable (locality) condition (27) and integrating over the hidden–variable λ as in Eq. (26), one easily obtains the CH inequality:

$$\begin{aligned} -1 \leq S \quad \equiv \quad & P(K_\alpha, K_\beta) - P(K_\alpha, K_{\beta'}) + P(K_{\alpha'}, K_\beta) + P(K_{\alpha'}, K_{\beta'}) \\ & -P(K_{\alpha'}, *) - P(*, K_\beta) \leq 0, \end{aligned} \tag{29}$$

with single–side probabilities given by:

$$\begin{aligned} P(K_{\alpha'}, *) &= P(K_{\alpha'}, K_\gamma) + P(K_{\alpha'}, K_\gamma^\perp) + P(K_{\alpha'}, U_\gamma), \\ P(*, K_\beta) &= P(K_\delta, K_\beta) + P(K_\delta^\perp, K_\beta) + P(U_\delta, K_\beta), \end{aligned} \tag{30}$$

for an arbitrary choice of the quasi–spin axes γ and δ. As noted by Clauser and Horne in Ref. [9], the right–hand side of the CH inequality, $S \leq 0$, has the advantage of being independent of the hidden–variable normalization condition, $\int d\lambda \rho(\lambda) = 1$, thus canceling the influence of the size of the ensemble of detected events.

6.2 Eberhard inequalities

If we chose $\gamma = \beta$ and $\delta = \alpha$ in Eqs. (29), (30), the right–hand side of the CH inequality (29) can be rewritten in the form of the following Bell–like inequality first derived by Eberhard [22]:

$$\begin{aligned} P(K_{\alpha'}, K_{\beta'}) \quad \leq \quad & P(K_{\alpha'}, K_\beta^\perp) + P(K_\alpha^\perp, K_\beta) \\ & +P(K_\alpha, K_{\beta'}) + P(K_{\alpha'}, U_\beta) + P(U_\alpha, K_\beta). \end{aligned} \tag{31}$$

The right–hand side CH and Eberhard inequalities just introduced are both genuine Bell inequalities in the sense discussed in Section 2. Unfortunately, due to their specific form, they hardly provide feasible experimental tests able to discriminate between LR and QM. As explained in Refs. [21, 22], the main shortcoming is originated by the existence of thresholds for the relevant measurement efficiencies which have to be overcome in order to attain violations of the considered inequalities by QM. Since in both performed (photon) and proposed (kaon) experiments such thresholds turn out to be hardly reachable within the current experimental capabilities, additional hypotheses beyond realism and locality must be made in order to obtain testable, but non–genuine, inequalities.

6.3 Clauser–Horne–Shimony–Holt inequalities

We then come to an important example of experimentally testable but non–genuine inequality which has been widely adopted in photon experiments, thus allowing the refutation of a restricted class of LR models. Here we will not repeat its derivation, which is due to Clauser, Horne, Shimony and Holt (CHSH) and can be found in Refs. [8, 10]. Our interest is to illustrate the differences with respect to the previously discussed CH inequality, especially concerning the role played by the supplementary assumptions when testing LR.

First, the CHSH inequality refers to expectation values instead of probabilities. In a local hidden–variable theory, the expectation value for a joint kaon measurement along the quasi–spin axes α and β is defined as:

$$E(\alpha,\beta) = \int d\lambda \rho(\lambda) A(\alpha|\lambda) B(\beta|\lambda), \tag{32}$$

where the locality requirement is evident in the factorized form of the left and right beam outcomes $A(\alpha|\lambda)$ and $B(\beta|\lambda)$, for a given state λ, which do not depend on the other–side measurement axis β and α, respectively.

The result of each single–side measurement can take one of the three possible outcomes, 0 or ± 1. For deterministic hidden–variables, we assume that $A(\alpha|\lambda)$ takes the following values: $+1$ when the measurement outcome is K_α, -1 when the outcome is the state $K_\alpha^\perp$ orthogonal to K_α and 0 when the particle is not projected into the $\{|K_\alpha\rangle, |K_\alpha^\perp\rangle\}$ measurement basis. This third outcome 0 presupposes the ability of the experimenter to know when a particular kaon fails to be measured in the chosen basis. In photon experiments the outcome 0 correspond to undetected photons. Since in this case their number is unknown, in 1971 Bell proposed (see Ref. [10]) to use what he called 'event–ready' detectors in order to enumerate the photon pairs emitted by the source which really reach the regions where they are then subject to Bell–measurements. Only if these undetected or 'unmeasured' pairs are included in the ensemble which defines the hidden–variable distribution $\rho(\lambda)$, one is sure that $\rho(\lambda)$ does not depend on the measurement parameters α and β [10, 20]. In practical cases with photons, any conceivable 'event–ready' detector fatally disturb if not destroy the particles. For kaons, good 'event–ready' detectors seems to be at our disposal: one has to detect kaon decays occurring along the beams prior to the measurement times used in the Bell–test and all the initial two–kaon pairs showing (at least one of) such decays have to be excluded from the sample. This amounts to the previously discussed renormalization of the states. Also, kaons which fail to initiate a kaon–nucleon interaction in a thin absorber when trying to measure their strangeness —this is the low efficiency measurement at our energies— can be further detected to decay as a K_S or K_L state and be properly included in the analysis as 'unmeasured', outcome 0 events.

In the lack of 'event–ready' detectors, as in photon experiments, to be sure that $\rho(\lambda)$ does not depend on α and β, one can follow another approach and introduce an *additional assumption* [8], not inferable in the hypothesis of realism and locality, which is plausible but untestable. For photons, this hypothesis amounts to require that 'if a particle passes through a spin analyser, its probability of detection is independent of the analyser's orientation' [8]. For kaons, with essentially only two possible 'orientations' (strangeness or lifetime), the situation is quite different: it is the low probability of inducing the initial kaon–nucleon interaction in strangeness measurements, rather than the detection of the final state products, what contrasts with highly efficient lifetime measurements. Rather than adapting the previous CHSH additional assumption to the kaon case, one can resort to the so–called 'fair sampling' hypothesis [10]. It amounts to assume that the set of effectively measured events represent a fair or undistorted sample of the whole set of states emitted from the source; in other words, the kaon hidden–variables are not correlated with the efficiencies of the measuring apparata. Under this assumption, the efficiency factors in the Bell inequalities are assumed to be 1 and thus no undetected or unmeasured event appears.

Coming back to Eq. (32), in the most general case of stochastic hidden–variables, for the functions $A(\alpha|\lambda)$ and $B(\beta|\lambda)$ we have the obvious constraints:

$|A(\alpha|\lambda)| \leq 1$ and $|B(\beta|\lambda)| \leq 1$. These last expectation values can be seen as averages of the previous deterministic expectation values. One can thus obtain the CHSH inequality in the form:

$$|E(\alpha,\beta) - E(\alpha,\beta') + E(\alpha',\beta) + E(\alpha',\beta')| \leq 2, \tag{33}$$

where, unlike the case of the right–hand side CH inequality, the hidden–variable normalization condition, $\int d\lambda \rho(\lambda) = 1$, has been employed in the derivation.

In terms of joint probabilities, each expectation values can be expressed as:

$$E(\alpha,\beta) = P(Y_\alpha, Y_\beta) + P(N_\alpha, N_\beta) - P(Y_\alpha, N_\beta) - P(N_\alpha, Y_\beta), \tag{34}$$

both in QM and LR, where $Y_{\alpha(\beta)}$ (Yes) and $N_{\alpha(\beta)}$ (No) answer to the question whether the incoming kaon projects into the state K_α (K_β) or otherwise when measuring along the quasi–spin axis α (β). To establish a complete link with the joint probabilities entering the CH inequality (29), or the Eberhard inequality (31), we have to specify the following relations:

$$\begin{aligned} P(Y_\alpha, Y_\beta) &= P(K_\alpha, K_\beta), \qquad (35)\\ P(N_\alpha, N_\beta) &= P(K_\alpha^\perp, K_\beta^\perp) + P(K_\alpha^\perp, U_\beta) + P(U_\alpha, K_\beta^\perp) + P(U_\alpha, U_\beta), \\ P(Y_\alpha, N_\beta) &= P(K_\alpha, K_\beta^\perp) + P(K_\alpha, U_\beta), \\ P(N_\alpha, Y_\beta) &= P(K_\alpha^\perp, K_\beta) + P(U_\alpha, K_\beta), \end{aligned}$$

which hold when undetected or unmeasured events, denoted by $U_{\alpha,\beta}$, can really be identified by efficient 'event–ready' detectors. In such cases, the CHSH inequality (33) is a genuine Bell inequality and is equivalent to the CH inequality (29). In the most common case in which 'event–ready' detectors are not available, and one has to resort to the fair sampling hypothesis, all the detection efficiencies are assumed to be 1, the right–hand sides of relations (35) do not contain probabilities for undetected or unmeasured events and the corresponding CHSH inequality (33) is of the non–genuine type.

6.4 Wigner inequalities

Let us now see how introducing supplementary hypotheses one can obtain testable inequalities from the right–hand side CH inequality (29), or equivalently the Eberhard inequality (31). The first is the fair sampling hypothesis, for which the efficiency factors in the inequalities are assumed to be 1. The corresponding inequality takes the form of what we may call a Wigner–like inequality, but with four instead of three terms and measurement settings:

$$P(K_{\alpha'}, K_{\beta'}) \leq P(K_{\alpha'}, K_{\beta}^{\perp}) + P(K_{\alpha}^{\perp}, K_{\beta}) + P(K_{\alpha}, K_{\beta'}). \qquad (36)$$

This Bell inequality is equivalent to the CHSH inequality (33) when the latter is also considered together with the assumption of fair sampling.

If in addition one demands $P(K_{\alpha}^{\perp}, K_{\beta}) = 0$, a standard (*i. e.*, with three terms and three different measurement settings) Wigner inequality [7] is obtained. Note that this requirement restricts the derivability of the standard Wigner inequality to deterministic local realistic theories only. On the contrary, the four–term Wigner inequality (36) is valid for both deterministic and stochastic hidden–variable theories. For maximally entangled and perfectly–anticorrelated states, $P(K_{\alpha}^{\perp}, K_{\beta}) = 0$ is achieved when $K_{\alpha} \equiv K_{\beta}^{\perp}$, which corresponds to require perfect anticorrelation for joint measurements along the same generic quasi–spin axis α, thus obtaining the standard Wigner inequality:

$$P(K_{\alpha'}, K_{\beta'}) \leq P(K_{\alpha'}, K_{\alpha}) + P(K_{\alpha}, K_{\beta'}). \qquad (37)$$

7 A review of the proposals

7.1 Assuming fair sampling and perfect anticorrelation

We start by reviewing those proposals of Bell–type tests using maximally entangled two–kaon states and based on the Wigner inequality (37), which needs three different quasi–spin measurement axes. These are non–genuine inequalities since to be derived they require, in addition to the hypotheses of realism and locality, the assumption of fair sampling and the condition of perfect anticorrelation, as explained in Section 6.4.

7.1.1 A first proposal [27, 31]

For the maximally entangled kaon state (17), Uchiyama [27] derived the inequality:

$$P(K_S, K^0) \leq P(K_S, K_1) + P(K_1, K^0), \qquad (38)$$

which has been rediscussed in detail by Bertlmann and Hiesmayr [31]. Here, the joint probabilities are assumed to be measured at a proper time $\tau = \tau_l = \tau_r$ very close to the instant of the pair creation, $\tau \to 0$. For this reason, the above inequality would eventually test noncontextuality rather than locality. Inserting the quantum–mechanical probabilities into Eq. (38), one obtains the constraint $\mathcal{R}e\,\epsilon \leq |\epsilon|^2$, which is violated by the presently accepted value of the CP–violation parameter ϵ.

Note that the proposed inequality involves passive measurements along the basis consisting of the two unphysical CP eigenstates. Moreover, the smallness of the parameter ϵ and Eq. (8) preclude any realistic attempt of discriminating between lifetime (K_S vs K_L) and CP (K_1 vs K_2) eigenstates. In this sense, the interest of inequality (38) reduces to that of a clear and well defined gedanken experiment.

7.1.2 Improved proposal with strangeness detection and thin regenerators [32, 33]

The authors of Refs. [32, 33] based their study on the K^0 vs $\bar{K}^0$ identification and exploited the phenomenon of kaon regeneration to obtain the three different quasi–spin measurement axes needed to establish a Wigner inequality. The weak interaction eigenstates which emerge after crossing a thin, homogeneous regenerator during a time interval $\Delta\tau$ turn out to be [32, 33]:

$$|K'_S\rangle = |K_S\rangle + \eta|K_L\rangle, \;\; |K'_L\rangle = |K_L\rangle + \eta|K_S\rangle, \tag{39}$$

where

$$\eta \equiv i\,(\lambda_S - \lambda_L)\,\rho\Delta\tau = -\left(i\Delta m + \frac{1}{2}\Delta\Gamma\right)\rho\Delta\tau, \tag{40}$$

and ρ, given by Eq. (13), accounts for the (small) regeneration effects. Eqs. (39) are valid at lowest order in the regeneration parameter ρ.

The maximally entangled two–kaon state adopted in Refs. [32, 33] is the one of Eq. (20), where a single time τ is considered for simultaneous left and right measurements. At this time τ, one kaon enters a regenerator placed on the left hand side and the other kaon enters a right hand side regenerator. The proper time spent by each kaon to cross the corresponding regenerator is $\Delta\tau$. The regenerator parameters are chosen at will by the experimenters: in order to derive a Wigner inequality such as (37), a regenerator is chosen between two available options, both along the left and right beams. The Wigner inequalities thus derived are the following ones:

$$P(K^0, 0; \bar{K}^0, \rho) \leq P(K^0, 0; \bar{K}^0, 2\rho) + P(\bar{K}^0, 2\rho; \bar{K}^0, \rho), \tag{41}$$

$$P(K^0, 0; \bar{K}^0, \rho) \leq P(K^0, 0; K^0, 2\rho) + P(K^0, 2\rho; \bar{K}^0, \rho), \tag{42}$$

where, *e. g.*, $P(K^0, \rho; \bar{K}^0, 2\rho)$ is the probability to detect, at time $\tau + \Delta\tau$, a K^0 on the left after a thin regenerator with parameter ρ and a $\bar{K}^0$ on the right after a double density (2ρ) regenerator. The absence of a regenerator is denoted with a 0. Note that in the present case, each regenerator is considered as part of the corresponding measurement process.

Quantum mechanics predicts a violation of one of the two previous inequalities. In fact, Eq. (41) and (42) imply:

$$\mathcal{R}e\,[i\rho(\lambda_L - \lambda_S)] \geq 0 \iff \mathcal{R}e\,\eta \leq 0, \quad (43)$$

$$\mathcal{R}e\,[i\rho(\lambda_L - \lambda_S)] \leq 0 \iff \mathcal{R}e\,\eta \geq 0. \quad (44)$$

Therefore, inequality (41) [(42)] has to be used for an actual test if the experimental value of $\mathcal{R}e\,\eta$ is positive (negative).

The problem using thin regenerators is that the parameter η is small, typically $|\eta| \simeq 10^{-3} \div 10^{-2}$ [26]. This lowers the level of violation, by quantum mechanics, of the inequalities to some %. Thick regenerators (say larger than a few millimeters) worsen the detector performances and the event reconstruction becomes more difficult because of multiple scatterings. Moreover, one is forced to use thin regenerators since otherwise saturation effects in the regeneration process occurs, due to inevitable K_S–decays (note that the K_S lifetime τ_S corresponds to a distance covered by a K_S coming from the decay $\phi \to K^0\bar{K}^0$ of about 0.6 cm).

7.2 Assuming fair sampling

We now proceed to analyse those proposals based on the fair sampling hypothesis, that is the CHSH inequality (33) or the Wigner inequality (36). Even if these inequalities are non–genuine Bell inequalities, their use opens up the possibility to test the family of both deterministic and stochastic local realistic theories based on the fair sampling.

7.2.1 Proposal with strangeness detection [24]

The analogy between strangeness and linear polarization measurements has been exploited by many authors. In the analysis by Ghirardi et al. [24] one considers the state (19) and performs joint strangeness measurements at two different times on the left beam (τ_1 and τ_2) and at other two different times on the right beam (τ_3 and τ_4). The detection times can be chosen at will and, at least ideally, in accordance with the locality requirement. The proposed inequality, incorporating the fair sampling hypothesis, is in the CHSH form:

$$|E(\tau_1,\tau_3)-E(\tau_1,\tau_4)+E(\tau_2,\tau_3)+E(\tau_2,\tau_4)|\leq 2, \tag{45}$$

where the expectation value $E(\tau_r,\tau_r)$ takes the value $+1$ when either two $\bar{K}^0$'s or no $\bar{K}^0$'s are found in the left (τ_l) and right (τ_r) measurements, and -1 otherwise:

$$E(\tau_l,\tau_r)=P(Y,\tau_l;Y,\tau_r)+P(N,\tau_l;N,\tau_r)-P(Y,\tau_l;N,\tau_r)-P(N,\tau_l;Y,\tau_r). \tag{46}$$

The probabilities entering this correlation function, where Y (Yes) and N (No) answer the question whether a $\bar{K}^0$ is detected at the considered time, can be easily obtained in QM and one gets:

$$E_{\rm QM}(\tau_l,\tau_r)=-\exp\{-(\Gamma_L+\Gamma_S)(\tau_l+\tau_r)/2\}\cos[\Delta m\,(\tau_l-\tau_r)], \tag{47}$$

where, having assumed fair sampling, the inefficiencies in strangeness detection can be ignored.

Because of strangeness oscillations in free–space along both kaon paths, choosing among four different times corresponds to four different choices of measurement directions in the photon case. Unfortunately, the above CHSH inequality is never violated by QM because strangeness oscillations proceed too slowly and cannot compete with the more rapid kaon weak decays.

7.2.2 Proposals with lifetime detection and regenerators [25, 51]

An alternative option is based on K_S vs K_L identification and has been first proposed by Eberhard in Ref. [25]. Here, the two–kaon state of Eq. (20) is considered. To observe if a neutral kaon in a beam is K_S or K_L at a given point (*i. e.*, instant), a kaon detector is located far enough downstream from this point so that the number of undecayed K_S's reaching the detector is negligible. Since $\Gamma_L << \Gamma_S$, almost all K_L's can reach the detector, where they manifest by strong nuclear interactions. In a complementary way, K_S's are identified by their decays (mainly into two–pions) not far from that point of interest. Misidentifications and ambiguous events will certainly appear, but at an acceptably low level, as explained in Ref. [25].

Measurements of K_S vs K_L are thus performed for each one of four experimental setups. In a first setup, the two–kaon state is allowed to propagate in free–space; its normalization is lost because of weak decays, but its perfect antisymmetry is maintained. In the other three setups, regenerators —one thin (4 mm), the other thick (5 cm)— are asymmetrically and alternatively located along one beam, or along the other, or along both. The following interesting inequality relating the number of K_L's and K_S's detected in each experimental setup is then derived from LR:

$$P(K_L, \rho; K_L, \rho') \leq P(K_L, \rho; K_L, 0) + P(K_S, 0; K_S, 0) + P(K_L, 0; K_L, \rho'). \quad (48)$$

Again, ρ and ρ' denote the regenerator parameters and 0 stands for the absence of regenerators. Note that the above Bell inequality is a particular case of the Wigner inequality (36). Due to a constructive interference effect between the two regeneration processes, this Bell inequality turns out to be significantly violated by QM predictions even if the above mentioned misidentifications and ambiguous detection events are taken into account. These successful predictions have some limitations, as already discussed by the author. In particular, they are valid for asymmetric ϕ–factories (where the two neutral kaon beams form a small angle and have a velocity larger than for kaons pairs from symmetric ϕ–factories), whose construction is not foreseen.

Fortunately, it has been recently shown that a measurable QM violation of the previous Wigner inequality can also be achieved when the experiment is performed at a symmetric e^+e^- machine. A proposal for such a test at the Frascati ϕ–factory has been indeed put forward [51].

7.2.3 Proposal with both lifetime and strangeness detection [37]

We now discuss a way to use the non–maximally entangled state of Eqs. (24) and (25), which is prepared with the help of a kaon regenerator and corresponds to a proper time T along both kaon beams. Following the approach of Ref. [37], for each kaon on each beam at time T we consider either a strangeness or a lifetime measurement.

With the strategy illustrated in Section 4.2 for lifetime measurements, requiring an extra interval time $\Delta T = 4.8\,\tau_S$ after T, care has to be taken to choose T large enough to guarantee the space–like separation between left and right measurements. Locality then excludes any influence from the experimental setup encountered by one member of the kaon pair at time T on the behaviour of its other–side partner between T and $T+\Delta T$. For kaon pairs from ϕ decays, moving with velocity $\beta \simeq 0.22$, this implies $T > (1/\beta - 1)\Delta T/2 = 1.77\,\Delta T$, with a considerable reduction of the total kaon sample. Indeed, for $\Delta T = 4.8\,\tau_S$ one can choose $T = 2\,\Delta T \simeq 9.6\,\tau_S$, and only 1 in 15000 initial events can be used, having both kaons surviving up to time T. For faster kaons, as in CPLEAR, the situation improves considerably.

In Ref. [37] the following CH inequalities have been derived under the assumption of perfectly efficient experimental apparata:

$$\begin{aligned} &-1 \leq P(\bar{K^0}, K_L) - P(\bar{K^0}, \bar{K^0}) + P(K_S, \bar{K^0}) \\ &+P(K_S, K_L) - P(K_S, *) - P(*, K_L) \leq 0, \end{aligned} \tag{49}$$

$$\begin{aligned} &-1 \leq P(\bar{K^0}, K_S) - P(\bar{K^0}, \bar{K^0}) + P(K_L, \bar{K^0}) \\ &+P(K_L, K_S) - P(*, K_S) - P(K_L, *) \leq 0, \end{aligned}$$

where, for instance:

$$P(K_S, *) \equiv P(K_S, K^0) + P(K_S, \bar{K}^0). \tag{50}$$

In this and the other one–side probabilities, the joint probabilities for the two possible outcomes on the other side are added to guarantee that both kaons have survived up to time T. This respects the particular normalization of the state (24). Note that each one of the two previous inequalities follows from the other by just inverting left and right measurements on the left–right asymmetric state (24). The right–hand side CH inequalities can be rewritten as:

$$\begin{aligned} &\frac{P(\bar{K^0}, K_L) - P(\bar{K^0}, \bar{K^0}) + P(K_S, \bar{K^0}) + P(K_S, K_L)}{P(K_S, *) + P(*, K_L)} \leq 1, \\ &\frac{P(\bar{K^0}, K_S) - P(\bar{K^0}, \bar{K^0}) + P(K_L, \bar{K^0}) + P(K_L, K_S)}{P(*, K_S) - P(K_L, *)} \leq 1. \end{aligned} \tag{51}$$

The CH–like inequalities (49) and (51), actually incorporating the fair sampling hypothesis, can be easily and equivalently rewritten as four–term Wigner inequalities. By properly writing the single–side probabilities in (51), the result can be put in the following form:

$$\begin{aligned} &P(K_S, \bar{K}^0) \leq P(K_S, K_S) + P(K^0, K_L) + P(\bar{K}^0, \bar{K}^0), \\ &P(\bar{K}^0, K_S) \leq P(\bar{K}^0, \bar{K}^0) + P(K_L, K^0) + P(K_S, K_S). \end{aligned} \tag{52}$$

By choosing $T \geq 9.6\tau_S$, as required by locality, the complex parameter R_S of Eq. (25) turns out to be negligible, while $|R_L| = \mathcal{O}(1)$. By substituting the QM predictions in the left–hand side CH inequalities (51), one easily finds:

$$\frac{2 - \mathcal{R}e\, R_L + \frac{1}{4}|R_L|^2}{2 + |R_L|^2} \leq 1, \quad \frac{2 + \mathcal{R}e\, R_L + \frac{1}{4}|R_L|^2}{2 + |R_L|^2} \leq 1, \tag{53}$$

whose only difference is the sign affecting the linear term in $\mathcal{R}e\, R_L$. According to the sign of $\mathcal{R}e\, R_L$, one of these two inequalities is violated if $|\mathcal{R}e\, R_L| \geq 3|R_L|^2/4$. The greatest violation occurs for a purely real value of R_L, $|R_L| \simeq 0.56$, for which one of the two ratios in Eq. (53) reaches the value 1.14. This 14 % violating effect predicted by QM opens up the possibility for a meaningful Bell–type test with neutral kaons which could refute those LR models based on the fair sampling hypothesis.

Values for the parameter R_L satisfying $\mathcal{I}m\, R_L = 0$ and $|\mathcal{R}e\, R_L| = 0.56$, as required, are not difficult to obtain. Indeed, for kaon pairs from ϕ decays and according to the values of the regeneration parameters [26], one can use a thin beryllium regenerator 1.55 mm thick to prepare the state (21), which then converts into the state (24) with the desired value of R_L by propagating in free–space up to $T \simeq 11.1\, \tau_S$.

7.3 An attempt of genuine test [38, 41]

We conclude our review by discussing a proposal which does not assume hypotheses which go beyond the reality and locality requirements. In our opinion, it represent an interesting attempt for a loophole–free test of LR vs QM with neutral kaons.

Hardy's proof without inequalities of Bell's theorem [55] has been applied in Ref. [38] to the non–maximally entangled state of Eqs. (24) and (25). This considerably improved the analysis of Ref. [37]. In such an approach, alternative measurements of strangeness or lifetime are considered, at time T, on each one of the kaon pairs, according to the strategies for active measurement procedures illustrated in Section 4. Hardy's proof is then translated into an Eberhard inequality [41], which could discriminate between LR and QM conditionally on the detection efficiencies for strangeness and lifetime measurements at disposal in the actual test.

Let us first concentrate on the 'non–locality without inequalities' proof of Ref. [38]. Neglecting CP–violation and K_L–K_S misidentification effects, from state (24) with $R_S = 0$ and $R_L = -1$ (called Hardy's state) one obtains the following QM predictions:

$$
\begin{aligned}
P_{\rm QM}(K^0, \bar{K}^0) &= \eta\,\bar{\eta}/12, &(54)\\
P_{\rm QM}(K^0, K_L) &= 0, &(55)\\
P_{\rm QM}(K_L, \bar{K}^0) &= 0, &(56)\\
P_{\rm QM}(K_S, K_S) &= 0, &(57)
\end{aligned}
$$

where η ($\bar{\eta}$) is the overall efficiency for K^0 ($\bar{K}^0$) detection. We note that the values $R_S = 0$ and $R_L = -1$ can be obtained by using, for instance, a beryllium (carbon) regenerator with thickness $d = 2.83$ mm ($d = 0.78$ mm), a detection time $T = 11.1\,\tau_S$ ($T = 11.3\,\tau_S$) and kaon pairs created at a ϕ–factory (proton–antiproton machine). It is found that the necessity to reproduce, under LR, equalities (54)–(56) requires:

$$
P_{\rm LR}(K_S, K_S) \geq P_{\rm LR}(K^0, \bar{K}^0) = \eta\,\bar{\eta}/12 > 0, \qquad (58)
$$

which contradicts Eq. (57). In principle, this allows for an 'all–or–nothing', Hardy–like test of LR vs QM. In Ref. [38] it was concluded that, by requiring a perfect discrimination between K_S and K_L states, an experiment measuring the joint probabilities of Eqs. (54)–(57) closes the efficiency loophole even for infinitesimal values of the strangeness detection efficiencies η and $\bar{\eta}$. However, since K_L and K_S misidentifications (due to the finite value of $\Gamma_S/\Gamma_L \simeq 579$) do not permit an ideal lifetime measurement even when the detection efficiency η_τ for the kaon decay products is 100%, the original proposal must be reanalysed paying attention to the inefficiencies involved in the real test [41].

Retaining the effects due to K_S–K_L misidentifications, for Hardy's state one obtains (see the Appendix of Ref. [41] for details):

$$
\begin{aligned}
P_{\rm QM}(K^0, \bar{K}^0) &= \eta\bar{\eta}/12, &(59)\\
P_{\rm QM}(K^0, K_L) &= 6.77 \times 10^{-4}\,\eta\,\eta_\tau, &(60)\\
P_{\rm QM}(K_L, \bar{K}^0) &= 6.77 \times 10^{-4}\,\bar{\eta}\,\eta_\tau, &(61)\\
P_{\rm QM}(K_S, K_S) &= 1.19 \times 10^{-5}\,\eta_\tau^2, &(62)
\end{aligned}
$$

which replace the results of Eqs. (54)–(57). In the standard Hardy's proof of non–locality [55], the probabilities corresponding to our (60), (61) and (62) are perfectly vanishing. In our case they are very small but not zero. Nevertheless, this does not prevent us from deriving a contradiction between LR and QM. Indeed, as proved in Ref. [56], the well known criterion of physical reality of Einstein, Podolsky and Rosen [3] can be generalized to include predictions made with *almost* certainty, as it is required in the present case due to the nonvanishing values of probabilities (60)–(62).

According to this generalization, the following Eberhard inequality must be used to demonstrate the incompatibility between LR and QM:

$$H \equiv \frac{P(K^0,\bar{K}^0)}{P(K^0,K_L)+P(K_S,K_S)+P(K_L,\bar{K}^0)+P(K^0,U_{\rm Lif})+P(U_{\rm Lif},\bar{K}^0)} \leq 1. \tag{63}$$

Essentially, it is a different writing of the CH inequality:

$$Q \equiv \frac{P(K_S,\bar{K}^0)-P(K_S,K_S)+P(K^0,\bar{K}^0)+P(K^0,K_S)}{P(K^0,*)+P(*,\bar{K}^0)} \leq 1, \tag{64}$$

and the argument $U_{\rm Lif}$ refers to failures in lifetime detection. The QM expression for the probabilities containing lifetime undetection are:

$$P_{\rm QM}(K^0,U_{\rm Lif}) = \frac{1}{6}\eta\,(1-\eta_\tau)\,, \quad P_{\rm QM}(U_{\rm Lif},\bar{K}^0) = \frac{1}{6}\bar{\eta}\,(1-\eta_\tau)\,. \tag{65}$$

Note that the use of an inequality [57] allows for deviations, existing in real experiments, in the values of R_S and R_L required to prepare Hardy's state and, in addition, takes care of the difficulties associated to 'almost null' measurements, as is the case of probabilities (60)–(62). Both previous inequalities are actually derivable from LR for any value of R_S and R_L. However, Hardy's proof leads to inequality (63) only for Hardy's state ($R_S=0$ and $R_L=-1$). It is important to stress that the previous Eberhard and CH inequalities have been obtained *without invoking supplementary assumptions* on undetected events. They are both genuine Bell inequalities and provide the same restrictions on the efficiencies η, $\bar{\eta}$ and η_τ required for a detection loophole free experiment.

In order to discuss the feasibility of such an experiment, let us start considering a few ideal cases. Assume first that perfect discrimination between K_S and K_L were always possible ($\eta_\tau=1$ and $p_L=p_S=1$; see appendix of Ref. [41]); one could then make a conclusive test of LR for any nonvanishing values of η and $\bar{\eta}$: $H_{\rm QM}^{\eta_\tau=p_L=p_S=1} \to \infty$, $\forall\,\eta,\bar{\eta}\neq 0$. In a second ideal case with no undetected events, *i. e.* with $\eta=\bar{\eta}=\eta_\tau=1$, the inequalities are strongly

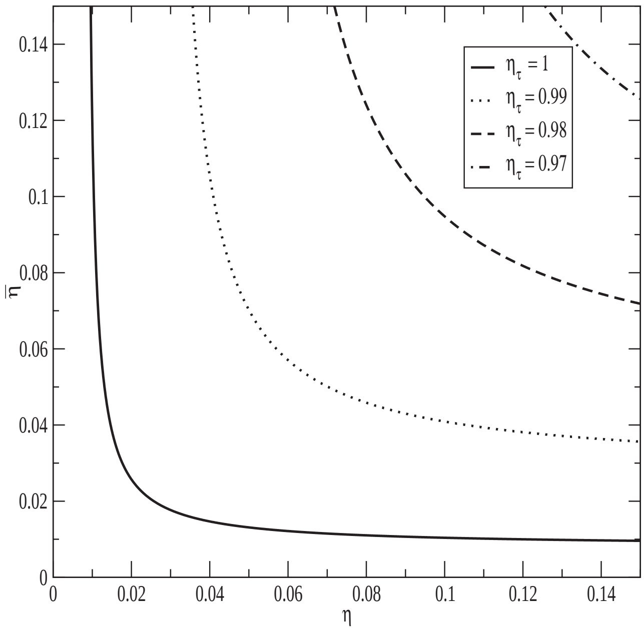

Figure 1: *The four curves (corresponding to* $\eta_\tau = 1$, $\eta_\tau = 0.99$, $\eta_\tau = 0.98$ *and* $\eta_\tau = 0.97$*) provide the values of* η *and* $\bar{\eta}$ *for which* $H_{\rm QM} = Q_{\rm QM} = 1$ *using Hardy's state. QM violates inequalities (63) and (64) for values of* η *and* $\bar{\eta}$ *situated above the corresponding curve.*

violated by QM, $H_{\rm QM}^{\eta=\bar{\eta}=\eta_\tau=1} \simeq 60.0$ and $Q_{\rm QM}^{\eta=\bar{\eta}=\eta_\tau=1} \simeq 1.25$, even if one allows for unavoidable K_S and K_L misidentifications. Finally, assuming that only the detection efficiency of kaon decay products is ideal ($\eta_\tau = 1$), for $\eta = \bar{\eta}$ ($\eta = \bar{\eta}/2$) Eberhard and CH inequalities are contradicted by QM whenever $\eta > 0.023$ ($\eta > 0.017$).

Let us now consider more realistic situations with small and possibly achievable values of η and $\bar{\eta}$. This implies that we have to consider large decay–product detection efficiencies such as $\eta_\tau = 0.97$, 0.98, 0.99 and, ideally, 1. For each η_τ, the values of η and $\bar{\eta}$ that permit a detection loophole free test ($H_{\rm QM}$, $Q_{\rm QM} > 1$) lie above the corresponding curve plotted in Fig. 1. As expected, when η_τ decreases, the region of η and $\bar{\eta}$ values which permits a conclusive test diminishes and larger values of η and $\bar{\eta}$ are required.

Note, however, that the strangeness detection efficiencies required for a conclusive test of LR vs QM with neutral kaons are considerably smaller than the limit ($\eta_0 = 0.67$) deduced by Eberhard [22] for non–maximally entangled photon states.

8 Conclusions

A series of proposals aiming to perform Bell inequality tests with entangled neutral kaon pairs has been reviewed. The relativistic velocities of these kaons and their strong interactions seem to offer the possibility of simultaneously closing the so–called locality and detection loopholes which affect analogous experiments performed with photons and ions. The real situation, however, is not a simple one.

All the proposal we discussed suffer from difficulties coming from the fact that the number of different complementary measurements on neutral kaons one can use for a Bell–test is reduced. Essentially, only strangeness and lifetime measurements are possible. The situation can be improved if the well known effects of kaon regeneration are taken into account. On the one hand, this amounts to an effective increase in the number of non–compatible measurements one can perform. On the other hand, by changing or removing the regenerators, the active presence of the experimenter is guaranteed. A final difficulty could still remain: the rather low efficiency of some of these neutral kaon measurements. A detailed analysis suggests that a Bell–test with neutral kaons free from the detection loophole would require a few % strangeness detection efficiencies and very high efficiencies for the detection of the kaon decay products.

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