GOETHE UNIVERSITÄT FRANKFURT AM MAIN

# WS2010/11:

# ‚Introduction to Nuclear and Particle Physics‘

**Lectures: Elena Bratkovskaya**

**Thursday, 14:00-16:15**
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**Script of lectures + tasks for homework:**
**http://th.physik.uni-frankfurt.de/~brat/index.html**

# Literature


1) Walter Greiner, Joachim A. Maruhn, „Nuclear models“; ISBN 3-540-59180-X Springer-Verlag
Berlin Heidelberg New York

2) „Particles and Nuclei. An Introduction to the Physical Concepts“
Bogdan Povh, Klaus Rith, Christoph Scholz, and Frank Zetsche - 2006
http://books.google.com/books?id=XyW97WGyVbkC&lpg=PR1&ots=zzIVa2OptV&dq=Bogdan%20Povh%2C%20Klaus%20Rith%2C%20Christoph%20Scholz%2C%20and%20Frank%20Zetsche&pg=PR1#v=onepage&q&f=false

3) "Introduction to nuclear and particle physics"
Ashok Das, Thomas Ferbel - 2003
http://books.google.de/books?id=E39OogsP0d4C&printsec=frontcover&dq=related:ISBN0521621968&lr=#v=onepage&q&f=false

4) "Elementary particles"
Ian Simpson Hughes - 1991
http://books.google.de/books?id=JN6qlZlGUG4C&printsec=frontcover&dq=related:ISBN0521621968&lr=#v=onepage&q&f=false

5) "Particles and nuclei: an introduction to the physical concepts"
Bogdan Povh, Klaus Rith - 2004
http://books.google.de/books?id=rJe4k8tkq7sC&printsec=frontcover&dq=related:ISBN0521621968&lr=#v=onepage&q&f=false

6) "Particle physics"
Brian Robert Martin, Graham Shaw - 2008
http://books.google.de/books?id=whIbrWJdEJQC&printsec=frontcover&dq=related:ISBN0521621968&lr=#v=onepage&q&f=false

7) "Nuclear and particle physics"
Brian Robert Martin - 2009
http://books.google.de/books?id=ws8QZ2M5OR8C&printsec=frontcover&dq=related:ISBN0521621968&lr=#v=onepage&q&f=false

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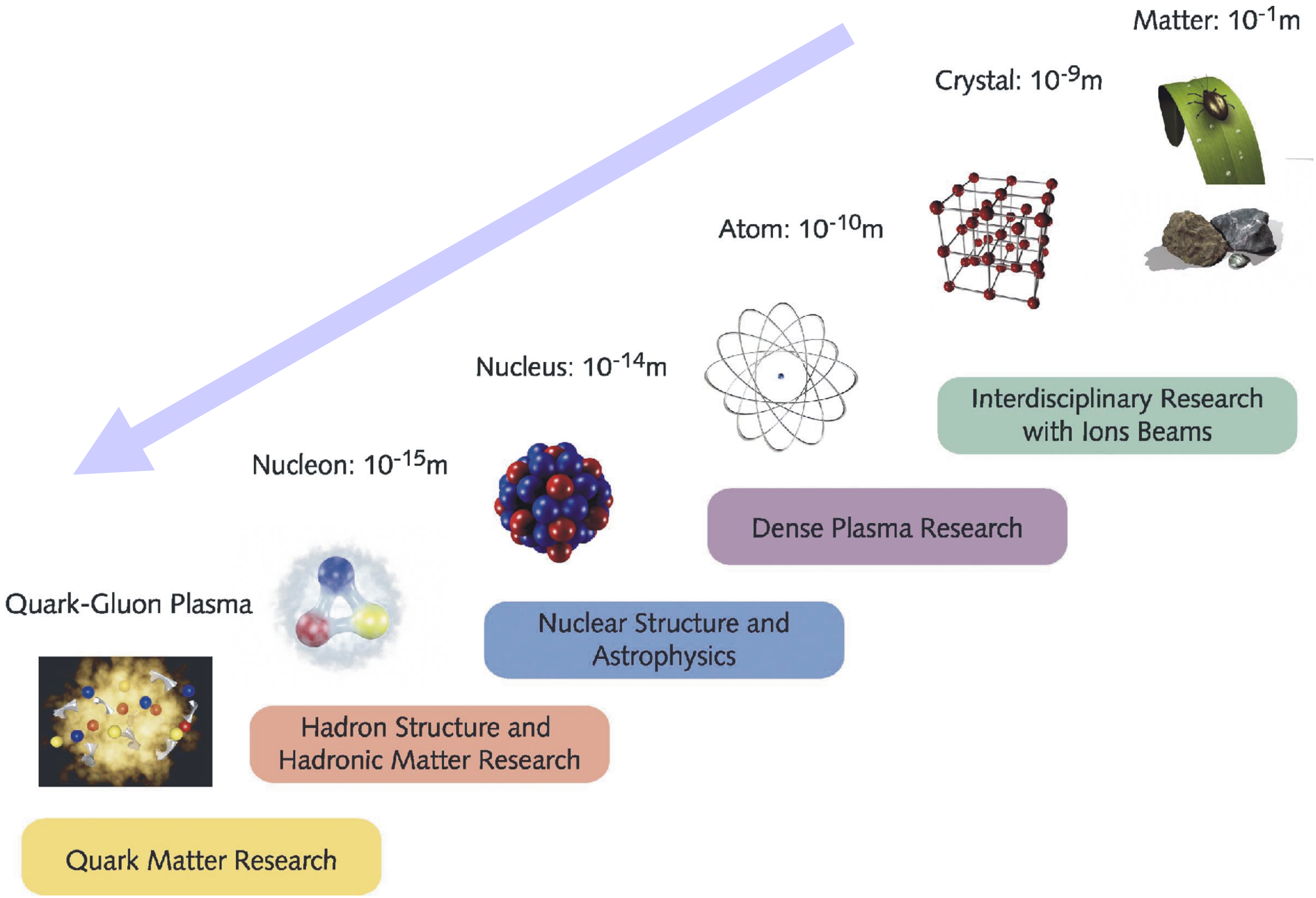
# Lecture 1
# Introduction: units, scales etc.
# Nuclear models

# Scales in the Universe



# Scales in nuclear physics

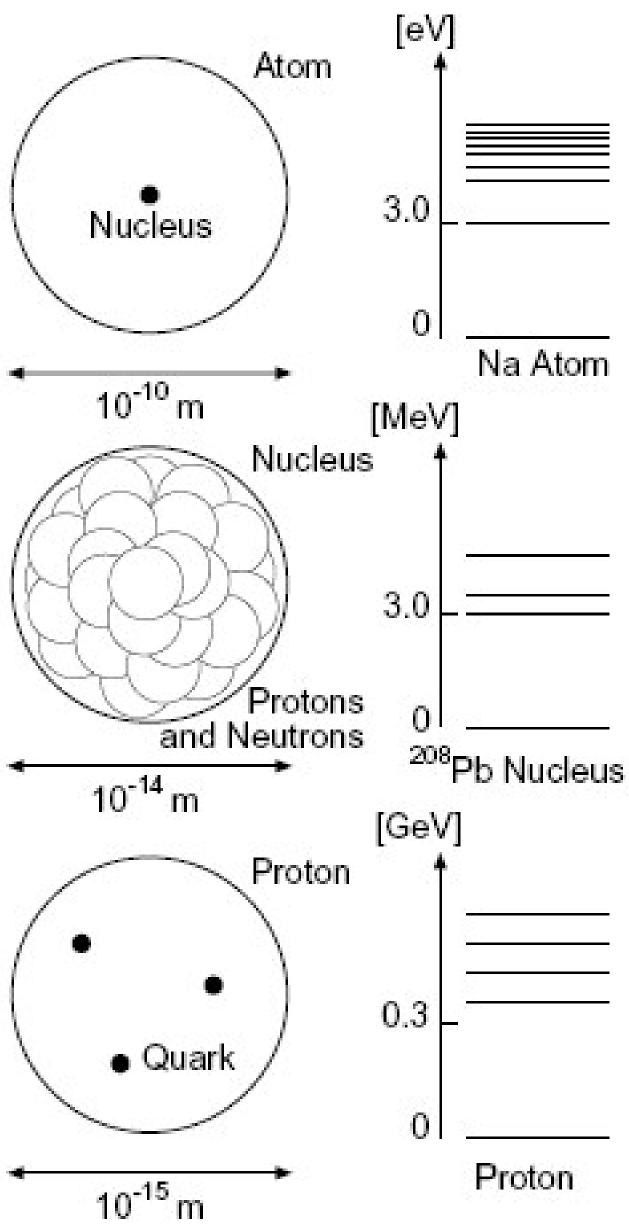



**Fig. 1.1.** Length scales and structural hierarchy in atomic structure. To the right, typical excitation energies and spectra are shown. Smaller bound systems possess larger excitation energies.

# Physical units

Common unit for length and energy:

- Length: fm (Fermi) – femtometer
  1 fm = $10^{-15}$m =$10^{-13}$ cm corresponds approximately to the size of the proton
- Energy: eV – electron volt
  1 eV = $1.602 \cdot 10^{-19}$ J is the energy gained by a particle with charge 1e by traversing a potential difference of 1V

Prefixes for the decimal multiples:

1keV = $10^3$ eV; 1MeV = $10^6$ eV; 1GeV = $10^9$ eV; 1TeV= $10^{12}$ eV

- Units for particle masses: MeV/$c^2$ or GeV/$c^2$ according to the mass-energy relation: $E=mc^2$, the total energy $E^2=mc^2+p^2c^2$

- speed of light in vacuum c=299 792 458 m/s

- Correspondence to the International System of Units (SI):
  1MeV/ $c^2$ = $1.783 \cdot 10^{-30}$ kg

# System of units in elementary particle physics

**Length and energy scales are connected by the uncertainty principle:**

$$\Delta x \, \Delta p \geq \frac{1}{2}\hbar$$

**The Planck constant *h* is a physical constant reflecting the sizes of quanta in quantum mechanics**

$$h = 6.626\ 068\ 96(33) \times 10^{-34}\ \mathrm{J\ s} = 4.135\ 667\ 33(10) \times 10^{-15}\ \mathrm{eV\ s}$$

**the reduced Planck constant $\hbar$:** $\hbar = \frac{h}{2\pi}$

**Unit system used in elementary particle physics:** $\hbar = c = 1$

**➔ identical determination for masses, momentum, energy, inverse length and inverse time**

**[m]=[p]=[E]=[1/L]=[1/t]= keV, MeV,…**

**Typical masses:**

- **photon $m_\gamma$=0**
- **neutrino $m_\nu$ < 1 eV**
- **electron $m_e$=511 keV (= $9.10938215\times10^{-31}$ kg)**
- **nucleon (proton, neutron) $m_p$=938 MeV (=$1.672621637\times10^{-27}$ kg)**

# Angular momentum

**Spin – $\vec{S}$**

**The spin angular momentum $S$ of any physical system is quantized.**
**The allowed values of $S$ are:** $S = \hbar\sqrt{s(s+1)}$

**Spin quantum numbers $s$ is $n/2$, where $n$ can be any non-negative integer:**

$$s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$$

**Orbital angular momentum (or rotational momentum) – $\vec{L}$**
**Orbital angular momentum can only take on integer quantum numbers**

$$L = 0, 1, 2, \ldots$$

**Total angular momentum:** $\vec{J} = \vec{S} + \vec{L}$

**For each J exist 2J+1 projections of the angular momentum**

$$M = -J, -J+1, \ldots, J-1, J$$

# Statistics: fermions and bosons

**System of N particles: 1,2,…,N**
**Wavefunction:** $\psi(\vec{r}_1, ..., \vec{r}_N)$

**Symmetry: Replace two particles: 1←→2**

$$\psi(\vec{r}_2, \vec{r}_1, ..., \vec{r}_N) = C \cdot \psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N)$$

**Phase factor C ($C^2=1$):**
- **Bosons: C= +1**
- **Fermions: C= -1**

**Spin-statistics theorem:**
- **fermions have a half integer spin (1/2, 3/2, 5/2, …)**
- **bosons have an integer spin (0, 1, 2, …)**

**E.g.: Bosons: photons (γ) J=1, pions (π) J=0**
**Fermions: e,μ,ν,p,n J=1/2, Δ-resonance J=3/2**

# Electric charge and dipole moment

- The **electric charge** is quantized : **quanta – *e***

the **fine-structure constant** $\alpha$ as: $\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \approx \frac{1}{137}$

$\varepsilon_0$ is the electric constant. In particle physics $\varepsilon_0$=1, so $\alpha = e^2/4\pi$

- The **Bohr magneton and the nuclear magneton** are the physical constants and natural units which are used to describe the magnetic properties (**magnetic dipole moment**) of the electron and atomic nuclei respectively.

**Bohr magneton $\mu_B$** (in SI units) : $\mu_B = \frac{e\hbar}{2m_e}$

$\mu_B = 9.27400915\times10^{-24}$ J/T
$= 5.7883817555\times10^{-5}$ eV/T

$\mu_e = 1.001159652\,\mu_B$

**nuclear magneton $\mu_N$** : $\mu_N = \frac{e\hbar}{2m_p}$

$\mu_N < \mu_B$ by factor 1836

proton: $\mu_p = 2.79\,\mu_N$

neutron: $\mu_n = -1.91\,\mu_N \sim -2/3\,\mu_p$

# Fundamental interactions

| Interaction | Current Theory | Mediators | Relative Strength | Range (m) |
|---|---|---|---|---|
| Strong | Quantum chromodynamics(QCD) | gluons | $10^{38}$ | $10^{-15}$ |
| Electromagnetic | Quantum electrodynamics(QED) | photons | $10^{36}$ | $\infty$ |
| Weak | Electroweak Theory | W and Z bosons | $10^{25}$ | $10^{-18}$ |
| Gravitation | General Relativity(GR) | gravitons (not yet discovered) | 1 | $\infty$ |

| Particle | Electromagnetic interraction | Week interraction | Strong interraction | Statistic |
|---|---|---|---|---|
| Photon | + | (+) | | B |
| Lepton | + | + | | F |
| Baryon | + | + | + | F |
| Meson | + | + | + | B |
| Quarks | + | + | + | F |
| Gluons | | | + | B |

B=Bosons
F=Fermions

# Structure of atoms (history)

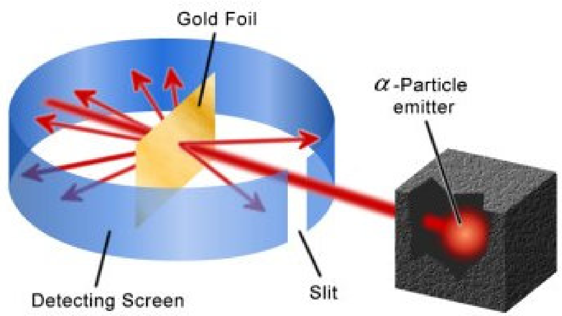
**The existance of atomic nucleus was discovered in 1911 by Ernest Rutherford, Hans Geiger and Ernest Marsden ➔**

**leads to the downfall of the plum pudding model (J.J. Thomson) of the atom, and the development of the Rutherford (or planetary) model.**

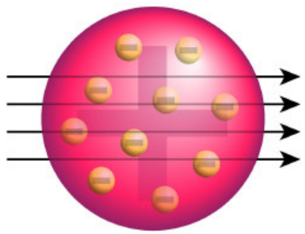
J.J. Thomson (1904) ‚**plum pudding model**' : the atom is composed of electrons surrounded by a soup of **uniformly distributed** positive charge (protons) to balance the electrons' negative charges

Ernest Rutherford
(1871-1937)

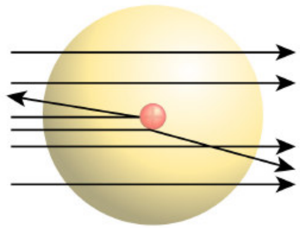
**Experiment 1909-1911: Rutherford bombarded gold foils with α-particles (ionized helium atom)**




**α-particles**



**Expected results from plum pudding model : alpha particles passing through the atom practically undisturbed.**

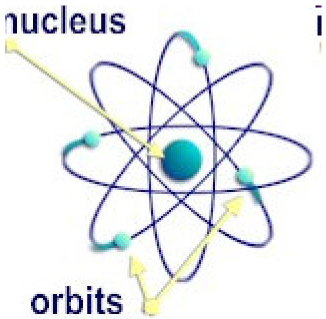


**Observed results: a small portion of the particles were deflected by large angles, indicating a small, concentrated positive charge ‚core'**

# Nuclear models

**Rutherford (or planetary) model:**

**the atom has very small positive 'core' – nucleus - containing protons with negatively charged electrons orbiting around it (as a solar sytem - planets around the sun).**



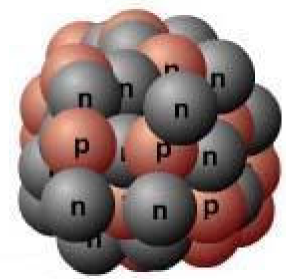

**⇒ the atom is 99.99% empty space ! The nucleus is approximately 100,000 times smaller than the atom. The diameter of the nucleus is in the range of 1.75 fm ($1.75 \times 10^{-15}$ m) for hydrogen (the diameter of a single proton) to about 15 fm for the heaviest atoms**

## Experimental discovery of the neutrons – James Chadwick in 1932

**(the existance of neutral particles (neutrons) has been predicted by Rutherford in 1921)**

**Experimentally found::**

**The nucleus consists of protons and neutrons**



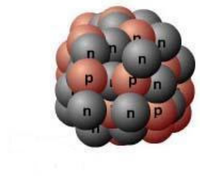

**Neutron: charge = 0, spin 1/2**

**$m_n$=939.56 MeV ($m_p$=938.27 MeV)**

**Mean life time $\tau_n$ = 885.7 s ~ 15 minutes** $n \rightarrow p + e^- + \overline{\nu}_e$

# Nuclear force

The atomic nucleus consists of protons and neutrons (two types of baryons) bound by the nuclear force (also known as the residual strong force).
The baryons are further composed of subatomic fundamental particles known as quarks bound by the strong interaction. The residual strong force is a minor residuum of the strong interaction which binds quarks together to form protons and neutrons.



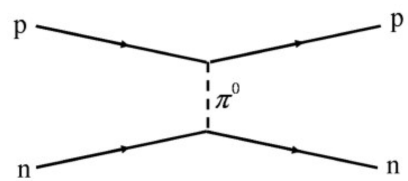
**Properties of nuclear forces :**

1. Nuclear forces are short range forces. For a distance of the order of 1 fm they are quite strong. It has to be strong to overcome the electric repulsion between the positively charged protons.
2. Magnitude of nuclear force is the same for *n-n, n-p* and *p-p* as it is charge independent.
3. These forces show the property of saturation. It means each nucleon interacts only with its immediate neighbours.
4. These forces are spin dependent forces.
5. Nuclear forces do not obey an inverse square law ($1/r^2$). They are non-central non-conservative forces (i.e. a noncentral or *tensor* component of the force does not conserve orbital angular momentum, which is a constant of motion under central forces).

# Nuclear Yukawa potential

**Interactions between the particles must be carried by some quanta of interactions, e.g. a photon for the electromagnetic force.**

**Hideki Yukawa (1907–1981): Nuclear force between two nucleons can be considered as the result of exchanges of virtual mesons (pions) between them.**




**Yukawa potential (also called a *screened Coulomb potential*):**

$$V(r) = -g^2 \frac{e^{-mr}}{r}$$

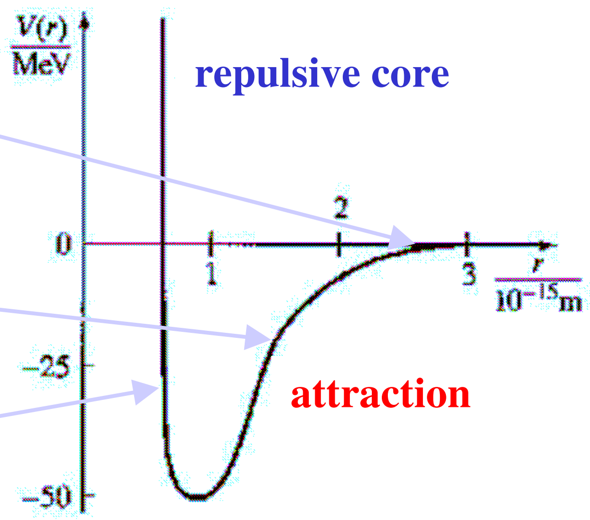
**where *g* is the coupling constant (strength of interraction).**

**Since the field quanta (pions) are massive (*m*) the nuclear force has a certain range, i.e. V→0 for large r.**

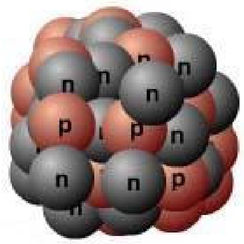
**At distances of a few fermi, the force between two nucleons is weakly attractive, indicated by a negative potential.**

**At distances below 1 fermi ($r_N$ ~1.12 fm): the force becomes strongly repulsive (repulsive core), preventing nucleons merging. The core relates to the quark structure of the nucleons.**

**Yukawa potential with a hard core:**



# Global Properties of Nuclei




*A* - mass number gives the number of nucleons in the nucleus

*Z* - number of protons in the nucleus (atomic number)

*N* – number of neutrons in the nucleus

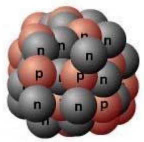
$$A = Z + N$$

In nuclear physics, nucleus is denoted as $^{A}_{Z}X$ , where X is the chemical symbol of the element, e.g. $^{1}_{1}H - hydrogen,$ $^{12}_{6}C - carbon,$ $^{197}_{79}Au - gold$

Different combinations of *Z* and *N* (or *Z* and *A*) are called nuclides

- nuclides with the same mass number *A* are called isobars $^{17}_{7}N,\ ^{17}_{8}O,\ ^{17}_{9}F$
- nuclides with the same atomic number *Z* are called isotopes $^{12}_{6}C,\ ^{13}_{6}C$
- nuclides with the same neutron number *N* are called isotones $^{13}_{6}C,\ ^{14}_{7}N$
- nuclides with neutron and proton number exchanged are called mirror nuclei $^{3}_{1}H,\ ^{3}_{2}He$
- nuclides with equal proton number and equal mass number, but different states of excitation (long-lived or stable) are calle nuclear isomers $^{180}_{73}Ta,\ ^{180m}_{73}Ta$

E.g.: The most long-lived non-ground state nuclear isomer is tantalum-180m, which has a half-life in excess of 1,000 trillion years



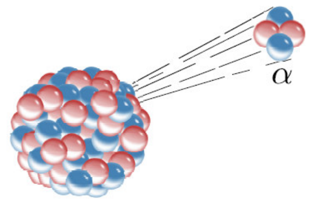
# Stability of nuclei

**Stability of nucleus**

**Stable nuclei** — **Unstable nuclei**

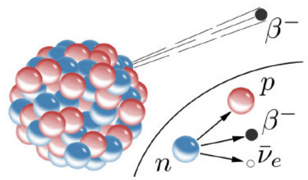
**Decay schemes :**

- **α-decay – emission of α-particle ($^4He$): $^{238}U \rightarrow {}^{234}Th + \alpha$**



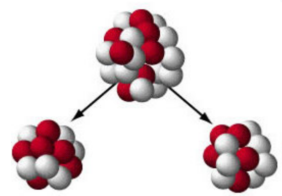

- **β-decay - emission of electron ($\beta^-$) or positron ($\beta^+$) by week interaction**

**$\beta^-$ decay: the weak interaction converts a neutron (n) into a proton (p) while emitting an electron ($e^-$) and an electron antineutrino ($\nu_e$):** $$n \rightarrow p + e^- + \bar{\nu}_e$$




**$\beta^+$ decay: the weak interaction converts a proton (p) into a neutron (n) while emitting a positron ($e^+$) and an electron neutrino ($\nu_e$):** $$p \rightarrow n + e^+ + \nu_e$$

- **fission - spontaneous decay into two or more lighter nuclei**

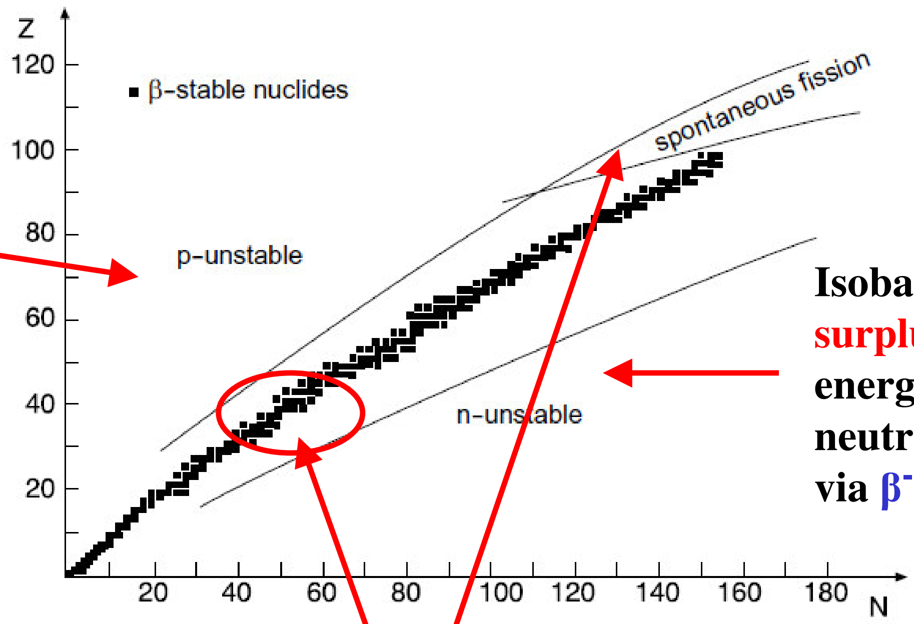


- **proton or neutron emission**

# Stability of nuclei

**Stable nuclei only occur in a very narrow band in the *Z* –*N* plane. All other nuclei are unstable and decay spontaneously in various ways.**

**In the case of a surplus of protons, the inverse reaction may occur: i.e., the conversion of a proton into a neutron via $\beta^+$-decays**




**Isobars with a large surplus of neutrons gain energy by converting a neutron into a proton via $\beta^-$-decays .**

**Fe- and Ni-isotopes possess the maximum binding energy per nucleon and they are therefore the most stable nuclides.**

**In heavier nuclei the binding energy is smaller because of the larger Coulomb repulsion. For still heavier masses, nuclei become unstable to fission and decay spontaneously into two or more lighter nuclei → the mass of the original atom should be larger than the sum of the masses of the daughter atoms.**

# Radionuclides

**Unstable nuclides are radioactive and are called radionuclides.**
**Their decay products ('daughter' products) are called radiogenic nuclides.**

**About 256 stable and about 83 unstable (radioactive) nuclides exist naturally on Earth.**

**The probability per unit time for a radioactive nucleus to decay is known as the decay constant *λ*. It is related to the lifetime *τ* and the half life $t_{1/2}$ by:**

$$\tau = 1/\lambda \quad \text{and} \quad t_{1/2} = \ln 2/\lambda$$

**The measurement of the decay constants of radioactive nuclei is based upon finding the activity (the number of decays per unit time):**

$$A = -\mathrm{d}N/\mathrm{d}t = \lambda N$$

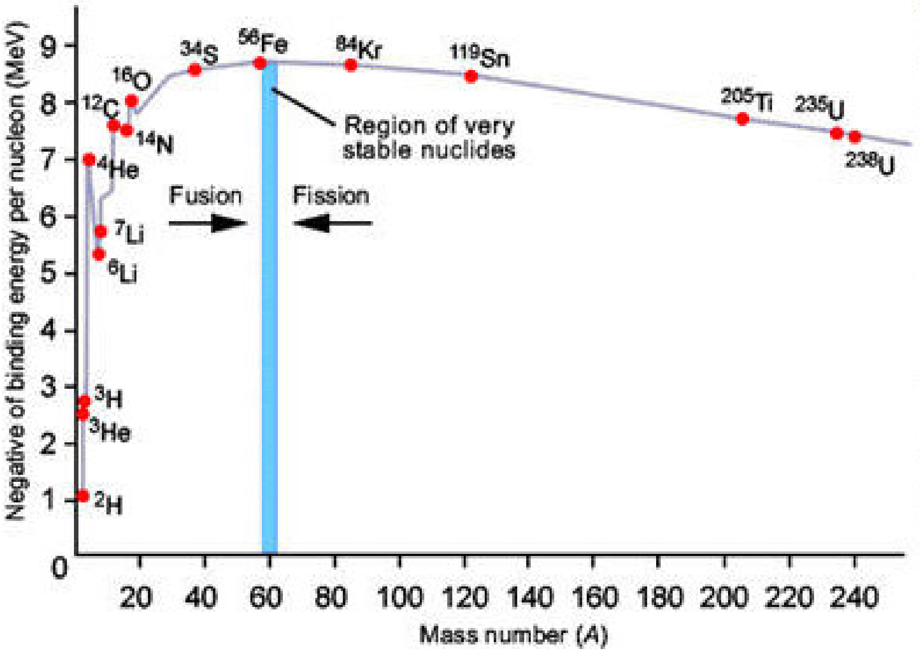
**where *N* is the number of radioactive nuclei in the sample.**

**The unit of activity is defined 1 Bq [Becquerel] = 1 decay/s.**

# Binding energy of nuclei

**The mass of the nucleus:** $M(Z,N) = m_N \cdot N + m_P \cdot Z - E_B$

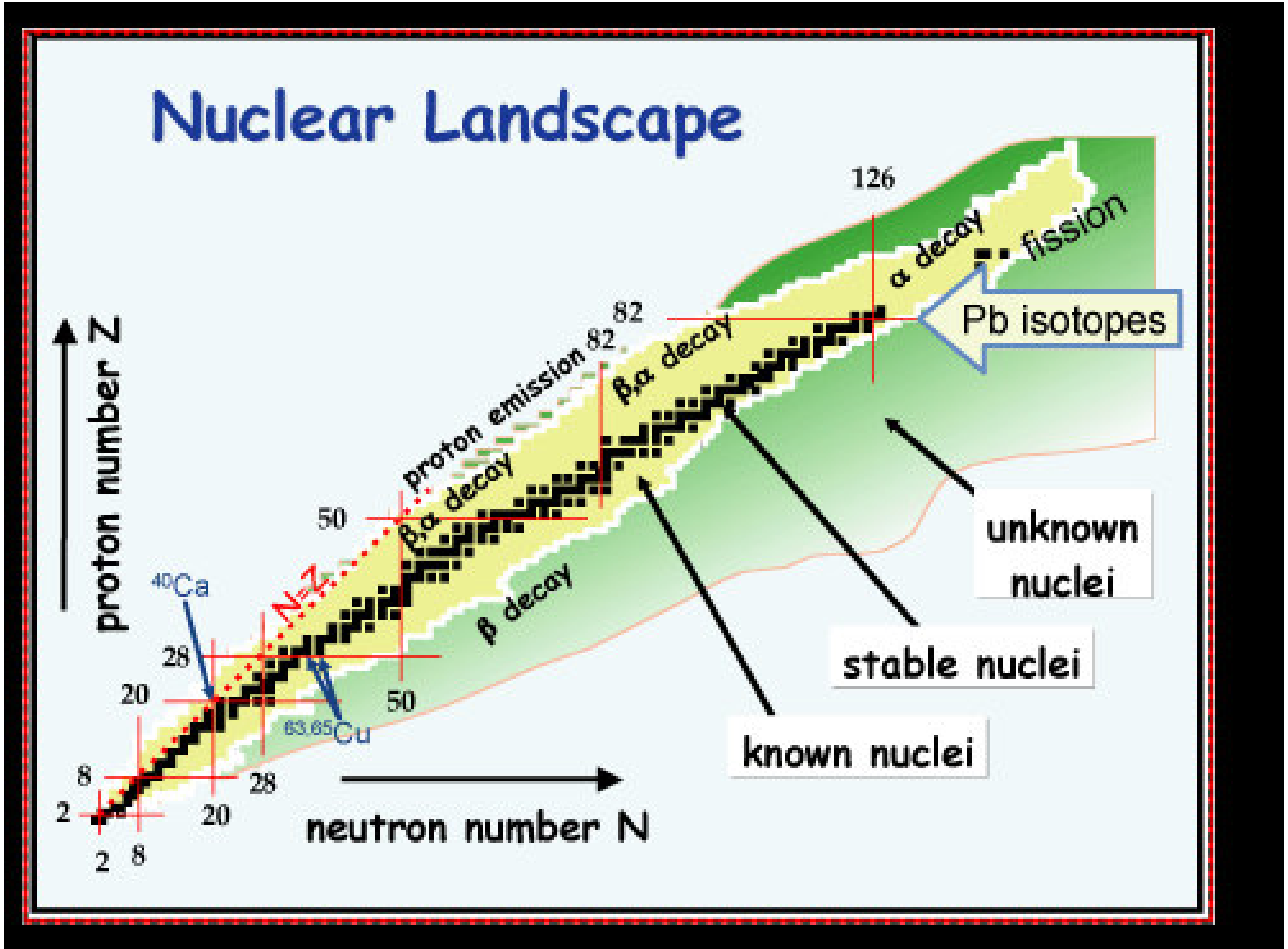
**$E_B$ is the binding energy per nucleon or mass defect (the strength of the nucleon binding ). The mass defect reflects the fact that the total mass of the nucleus is less than the sum of the masses of the individual neutrons and protons that formed it. The difference in mass is equivalent to the energy released in forming the nucleus.**




**The general decrease in binding energy beyond iron ($^{58}$Fe) is due to the fact that, as nuclei get bigger, the ability of the strong force to counteract the electrostatic repulsion between protons becomes weaker.**
**The most tightly bound isotopes are $^{62}$Ni, $^{58}$Fe, and $^{56}$Fe, which have binding energies of 8.8 MeV per nucleon.**
**Elements heavier than these isotopes can yield energy by nuclear fission; lighter isotopes can yield energy by fusion.**

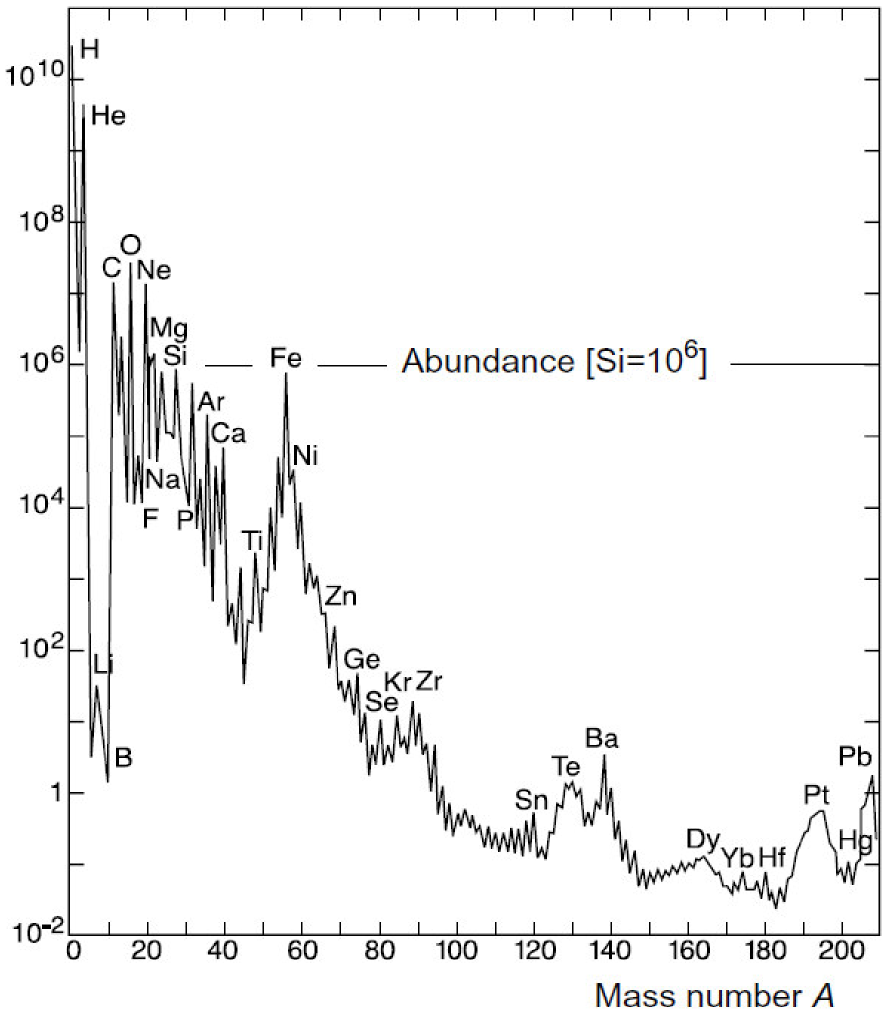
**Fusion -** **two atomic nuclei fuse together to form a heavier nucleus**
**Fission -** **the breaking of a heavy nucleus into two (or more rarely three) lighter nuclei**

# Nuclear Landscape



# Nuclear abundance

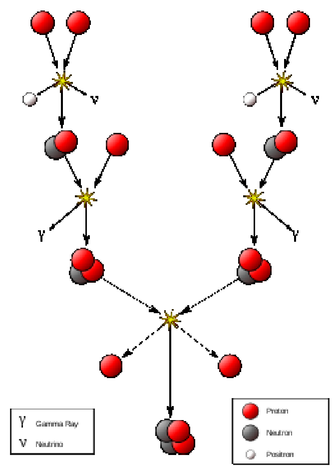
**Abundance of the elements in the solar system as a function of their mass number *A*, normalized to the abundance of silicon Si (= $10^6$):**




**Light nuclei:** **the synthesis of the presently existing deuterium $^2$H and helium $^4$He from hydrogen $^1$H fusion mainly took place at the beginning of the universe (minutes after the big bang).**

**Nuclei up to $^{56}$Fe, the most stable nucleus, were produced by nuclear fusion in stars.**

**Nuclei heavier than this last were created in the explosion of very heavy stars (supernovae)**

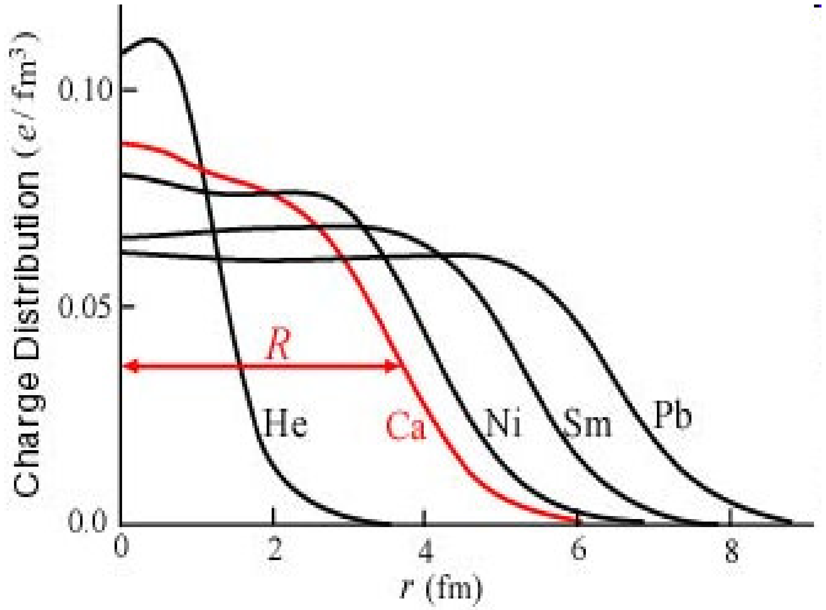


# Size of nuclei

**The diameter of the nucleus is in the range of 1.75 fm ($1.75\times10^{-15}$ m) for hydrogen (the diameter of a single proton) to about 15 fm for the heaviest atoms, such as uranium.**

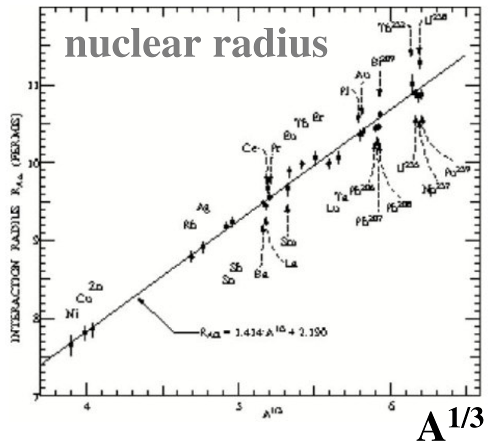
**The charge distribution function of a nucleus**
**= Woods-Saxon distribution:**

$$\rho(r) = \frac{\rho_0}{1 + e^{\frac{r-R}{a}}}$$




**where r is the distance from the center of nucleus; the parameters are adjusted to the experimental data:**
**a=0.5 fm**
**$\rho_0$ =0.17 fm – normal nuclear density**

**$R = R_0 \cdot A^{1/3}$ fm - nuclear radius**
**where the radius of nucleon is $R_0$=1.2 fm**




**Experimental data show that $R \sim A^{1/3}$ ➔**
**Stable nuclei have approximately a constant density in the interior**

# Nuclear models

## Nuclear models

### Models with strong interaction between the nucleons

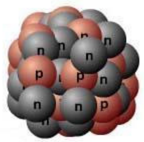
- Liquid drop model
- α-particle model
- Shell model
- …

Nucleons interact with the nearest neighbors and practically don‘t move: mean free path $\lambda << R_A$ nuclear radius

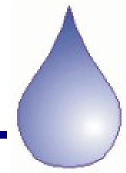
### Models of non-interacting nucleons

- Fermi gas model
- Optical model
- …

Nucleons move freely inside the nucleus: mean free path $\lambda \sim R_A$ nuclear radius



# The liquid drop model



The **liquid drop model** is a model in nuclear physics which treats the nucleus as a **drop of incompressible nuclear fluid**, first proposed by George Gamow and developed by Niels Bohr and John Archibald Wheeler. The fluid is made of nucleons (protons and neutrons), which are held together by the strong nuclear force. This is a crude model that does not explain all the properties of the nucleus, but does explain the **spherical shape** of most nuclei. It also helps to predict the binding energy of the nucleus.

The parametrisation of nuclear masses as a function of $A$ and $Z$, which is known as the **Weizsäcker formula** or the **semi-empirical mass formula**, was first introduced in 1935

$$M(A,Z) = Zm_p + Nm_n - E_B$$

$E_B$ is the **binding energy of the nucleus :**

$$E_B = \underbrace{a_V \cdot A}_{\text{Volum term}} - \underbrace{a_S \cdot A^{\frac{2}{3}}}_{\text{Surface term}} - \underbrace{a_C \cdot \frac{Z^2}{A^{\frac{1}{3}}}}_{\text{Coulomb term}} - \underbrace{a_{Sym} \cdot \frac{(N-Z)^2}{A}}_{\text{Assymetry term}} - \underbrace{\frac{\delta}{A^{1/2}}}_{\text{Pairing term}}$$

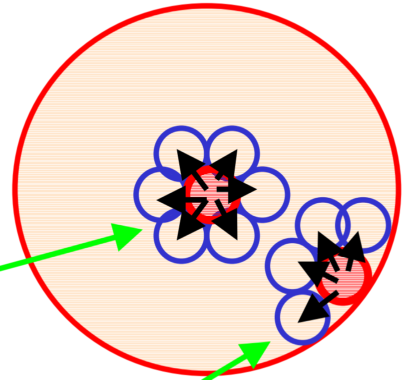
**Empirical parameters:**

$$a_V \approx 16MeV$$
$$a_S \approx 20MeV$$
$$a_C \approx 0,75MeV$$
$$a_{Sym} \approx 21MeV$$

$$\delta = \begin{cases} -11.2 \text{ MeV}/c^2 \text{ for even } Z \text{ and } N \text{ (even-even nuclei)} \\ 0 \text{ MeV}/c^2 \text{ for odd } A \text{ (odd-even nuclei)} \\ +11.2 \text{ MeV}/c^2 \text{ for odd } Z \text{ and } N \text{ (odd-odd nuclei).} \end{cases}$$

# Binding energy of the nucleus

**Volume energy (dominant term):** $E_V = a_V A$



**The basis for this term is the strong nuclear force. The strong force affects both protons and neutrons ➔ this term is independent of *Z*.**

**The strong force has a very limited range, and a given nucleon may only interact strongly with its nearest neighbors and next nearest neighbors. Therefore, the number of pairs of particles that actually interact is roughly proportional to *A*.**

**Surface energy:** $E_S = -a_S A^{2/3}$

**This term, also based on the strong force, is a correction to the volume term. A nucleon at the surface of a nucleus interacts with less number of nucleons than one in the interior of the nucleus, so its binding energy is less. The surface energy term is therefore negative and is proportional to the surface area :**
**If the volume of the nucleus is proportional to *A* ($V=4/3\pi R^3$), then the radius should be proportional to $A^{1/3}$ ($R \sim A^{1/3}$) and the surface area to $A^{2/3}$ ($S=\pi R^2 = \pi A^{2/3}$).**

**Coulomb (or electric) energy:** $E_C = -a_C \frac{Z^2}{A^{1/3}}$

**The electric repulsion between each pair of protons in a nucleus contributes toward decreasing its binding energy:**
**from QED - interaction energy for the charges $q_1,q_2$ inside the ball** $E_{int} \sim \frac{q_1 q_2}{R}$

# Binding energy of the nucleus

**Asymmetry energy (also called Pauli Energy):** $E_{asym} = -a_{Sym}\dfrac{(N-Z)^2}{A}$

**An energy associated with the Pauli exclusion principle: two fermions can not occupy exactly the same quantum state . At a given energy level, there is only a finite number of quantum states available for particles.**

**As long as mass numbers are small, nuclei tend to have the same number of protons and neutrons. Heavier nuclei accumulate more and more neutrons, to partly compensate for the increasing Coulomb repulsion by increasing the nuclear force. This creates an asymmetry in the number of neutrons and protons.**

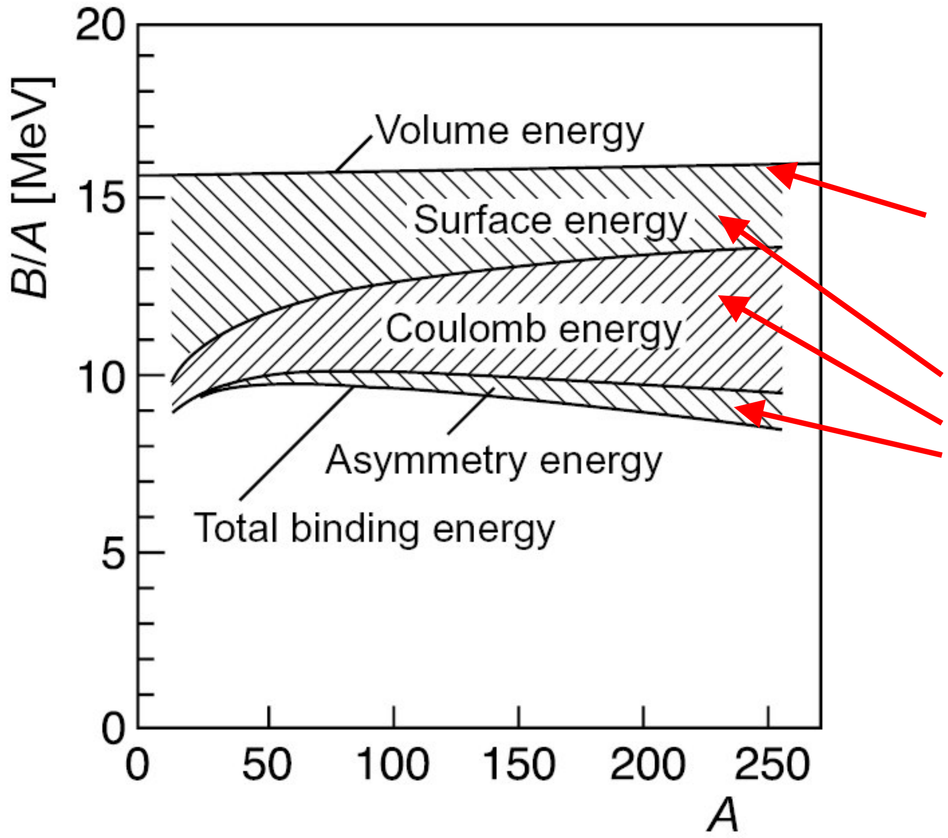
**For, e.g., $^{208}Pb$ it amounts to N –Z = 44. The dependence of the nuclear force on the surplus of neutrons is described by the asymmetry term. This shows that the symmetry decreases as the nuclear mass increases.**

**If it wasn't for the Coulomb energy, the most stable form of nuclear matter would have $N=Z$, since unequal values of $N$ and $Z$ imply filling higher energy levels for one type of particle, while leaving lower energy levels vacant for the other type**

**Pairing energy:** $E_{pair} = -\dfrac{\delta}{A^{1/3}}$

**An energy which is a correction term that arises from the effect of spin-coupling. Due to the Pauli exclusion principle the nucleus would have a lower energy if the number of protons with spin up will be equal to the number of protons with spin down. This is also true for neutrons. Only if both $Z$ and $N$ are even, both protons and neutrons can have equal numbers of spin up and spin down particles An even number of particles is more stable ($\delta<0$ for even-even nuclei) than an odd number ($\delta>0$).**

# The liquid drop model




**The different contributions to the binding energy per nucleon versus mass number *A*:**

**The horizontal line at ≈ 16 MeV represents the contribution of the volume energy.**

**This is reduced by the surface energy, the asymmetry energy and the Coulomb energy to the effective binding energy of ≈ 8 MeV(*lower line*).**

**The contributions of the asymmetry and Coulomb terms increase rapidly with *A*, while the contribution of the surface term decreases.**

**The Weizsäcker formula is often mentioned in connection with the liquid drop model. In fact, the formula is based on some properties known from liquid drops: constant density, short-range forces, saturation, deformability and surface tension. An essential difference, however, is found in the mean free path of the particles: for molecules in liquid drops, this is far smaller than the size of the drop; but for nucleons in the nucleus, it is large.**

# Lecture 2

# Nuclear models:
# Fermi-Gas Model
# Shell Model

# The basic concept of the Fermi-gas model

The theoretical concept of a Fermi-gas may be applied for systems of weakly interacting fermions, i.e. particles obeying Fermi-Dirac statistics leading to the Pauli exclusion principle ➔

• Simple picture of the nucleus:

— Protons and neutrons are considered as moving freely within the nuclear volume. The binding potential is generated by all nucleons

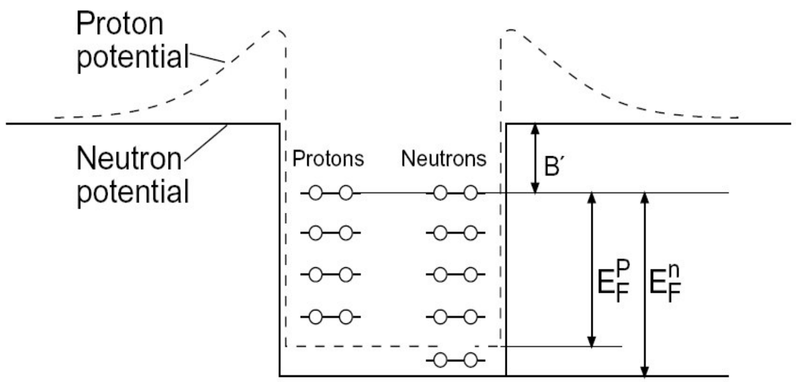
— In a first approximation, these nuclear potential wells are considered as rectangular: it is constant inside the nucleus and stops sharply at its edge

— Neutrons and protons are distinguishable fermions and are therefore situated in two separate potential wells

— Each energy state can be ocupied by two nucleons with different spin projections

— All available energy states are filled by the pairs of nucleons ➔ no free states , no transitions between the states

— The energy of the highest occupied state is the Fermi energy $E_F$



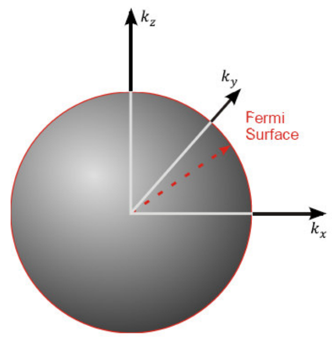

— The difference *B'* between the top of the well and the Fermi level is constant for most nuclei and is just the average binding energy per nucleon *B'/A* = 7–8 MeV.

# Number of nucleon states

**Heisenberg Uncertainty Principle:** $\Delta x \, \Delta p \geq \frac{1}{2}\hbar$

**The volume of one particle in phase space:** $2\pi\hbar$

**The number of nucleon states in a volume V:**




$$n = \frac{\int\int d^3r \, d^3p}{(2\pi\hbar)^3} = \frac{V \cdot 4\pi \int_0^{p_{max}} p^2 dp}{(2\pi\hbar)^3} \qquad (1)$$

**At temperature T = 0, i.e. for the nucleus in its ground state, the lowest states will be filled up to a maximum momentum, called the Fermi momentum $p_F$. The number of these states follows from integrating eq.(1) from 0 to $p_{max}=p_F$:**

$$n = \frac{V \cdot 4\pi \int_0^{p_F} p^2 dp}{(2\pi\hbar)^3} = \frac{V \cdot 4\pi \, p_F^3}{(2\pi\hbar)^3 \cdot 3} \Rightarrow n = \frac{V \cdot p_F^3}{6\pi^2\hbar^3} \qquad (2)$$

**Since an energy state can contain two fermions of the same species, we can have**

**Neutrons:** $N = \frac{V \cdot (p_F^n)^3}{3\pi^2\hbar^3}$ **Protons:** $Z = \frac{V \cdot (p_F^p)^3}{3\pi^2\hbar^3}$

**$p_F^n$ is the fermi momentum for neutrons, $p_F^p$ – for protons**

# Fermi momentum

Use R = $R_0 \cdot A^{1/3}$ fm, $$V = \frac{4\pi}{3}R^3 = \frac{4\pi}{3}R_0{}^3 A$$

The density of nucleons in a nucleus = number of nucleons in a volume V:

$$n = 2\cdot\frac{V\cdot p_F^3}{6\pi^2\hbar^3} = 2\cdot\frac{4\pi}{3}R_0{}^3 A\cdot\frac{p_F^3}{6\pi^2\hbar^3} = \frac{4A}{9\pi}\frac{R_0{}^3 p_F^3}{\hbar^3} \qquad (3)$$

two spin states

**Fermi momentum $p_F$:**

$$p_F = \left(\frac{6\pi^2\hbar^3 n}{2V}\right)^{1/3} = \left(\frac{9\pi\hbar^3}{4A}\frac{n}{R_0{}^3}\right)^{1/3} = \left(\frac{9\pi\cdot n}{4A}\right)^{1/3}\cdot\frac{\hbar}{R_0} \qquad (4)$$

After assuming that the proton and neutron potential wells have the same radius, we find for a nucleus with *n=Z=N =A/2* the Fermi momentum $p_F$:

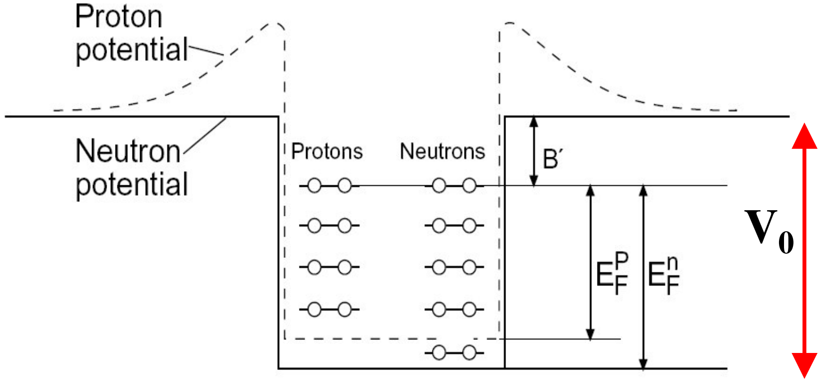
$$p_F = p_F^n = p_F^p = \left(\frac{9\pi}{8}\right)^{1/3}\cdot\frac{\hbar}{R_0} \approx 250\, MeV/c$$

The nucleons move freely inside the nucleus with large momenta.

**Fermi energy:** $$E_F = \frac{p_F^2}{2M} \approx 33\ MeV$$

*M* =938 MeV- the mass of nucleon

# Nucleon potential




The difference B‘ between the top of the well and the Fermi level is constant for most nuclei and is just the average binding energy per nucleon B/A = 7–8 MeV.

➔ The depth of the potential $V_0$ and the Fermi energy are independent of the mass number A:

$$V_0 = E_F + B' \approx 40 \ \ MeV$$

Heavy nuclei have a surplus of neutrons. Since the Fermi level of the protons and neutrons in a stable nucleus have to be equal (otherwise the nucleus would enter a more energetically favourable state through $\beta$-decay) this implies that the depth of the potential well as it is experienced by the neutron gas has to be larger than of the proton gas (cf Fig.).

Protons are therefore on average less strongly bound in nuclei than neutrons. This may be understood as a consequence of the Coulomb repulsion of the charged protons and leads to an extra term in the potential:

$$V_C = (Z-1)\frac{\alpha \cdot \hbar c}{R}$$

# Kinetic energy

The dependence of the binding energy on the surplus of neutrons may be calculated within the Fermi gas model.

First we find the average kinetic energy per nucleon:

$$\langle E \rangle = \frac{\int_0^{E_F} E \cdot \frac{dn}{dE} dE}{\int_0^{E_F} \frac{dn}{dE} dE} = \frac{\int_0^{p_F} E \cdot \frac{dn}{dp} dp}{\int_0^{p_F} \frac{dn}{dp} dp}$$

where

$$\frac{dn}{dp} = Const \cdot p^2 \qquad n = \frac{V \cdot p_F^3}{6\pi\hbar^3}$$

distribution function of the nucleons

$$\langle E_{\text{kin}} \rangle = \frac{\int_0^{p_F} E_{\text{kin}}\, p^2 \mathrm{d}p}{\int_0^{p_F} p^2 \mathrm{d}p} = \frac{3}{5} \cdot \frac{p_F^2}{2M} \approx 20 \text{ MeV}$$

The total kinetic energy of the nucleus is therefore

$$E_{\text{kin}}(N,Z) = N\langle E_n \rangle + Z\langle E_p \rangle = \frac{3}{10M}\left(N \cdot (p_F^n)^2 + Z \cdot (p_F^p)^2\right)$$

$$E_{\text{kin}}(N,Z) = \frac{3}{10M} \frac{\hbar^2}{R_0^2} \left(\frac{9\pi}{4}\right)^{2/3} \frac{N^{5/3} + Z^{5/3}}{A^{2/3}} \qquad (5)$$

where the radii of the proton and the neutron potential well have again been taken the same.

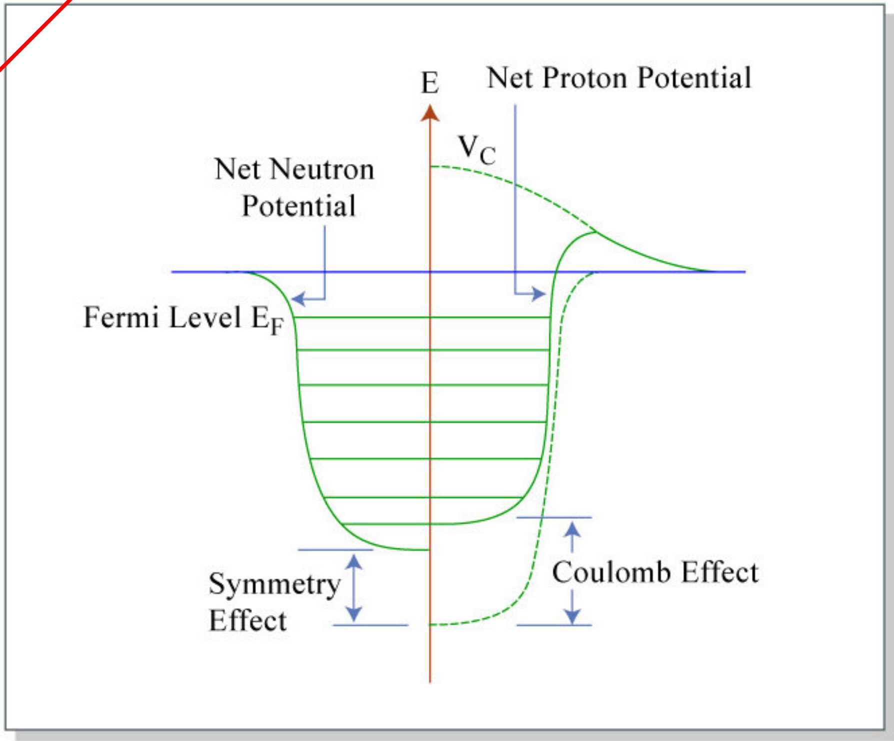
# Binding energy

This **average kinetic energy** has a **minimum at $N = Z$ for fixed mass number $A$** (but varying $N$ or, equivalently, $Z$). Hence the **binding energy gets maximal for $N = Z$**.

If we expand (5) in the difference $N - Z$ we obtain

$$E_{\text{kin}}(N,Z) = \frac{3}{10M}\frac{\hbar^2}{R_0^2}\left(\frac{9\pi}{8}\right)^{2/3}\left(A + \frac{5}{9}\frac{(N-Z)^2}{A} + \cdots\right)$$

The first term corresponds to the **volume energy** in the **Weizsäcker mass formula,** the second one to the **asymmetry energy**. The asymmetry energy grows with the neutron (or proton) surplus, thereby reducing the binding energy




**Note**: this consideration neglected the change of the nuclear potential connected to a change of N on cost of Z. This additional correction turns out to be as important as the change in kinetic energy.

# Shell model

**Magic numbers: Nuclides with certain proton and/or neutron numbers are found to be exceptionally stable. These so-called magic numbers are**

**2, 8, 20, 28, 50, 82, 126**

**— The doubly magic nuclei:** $^{4}_{2}He_{2}$, $^{16}_{8}O_{8}$, $^{40}_{20}Ca_{20}$, $^{48}_{20}Ca_{28}$, $^{208}_{82}Pb_{126}$

**— Nuclei with magic proton or neutron number have an unusually large number of stable or long lived nuclides.**

**— A nucleus with a magic neutron (proton) number requires a lot of energy to separate a neutron (proton) from it.**

**— A nucleus with one more neutron (proton) than a magic number is very easy to separate.**

**— The first exitation level is very high: a lot of energy is needed to excite such nuclei**

**— The doubly magic nuclei have a spherical form**

**→ nucleons are arranged into complete shells within the atomic nucleus**

# Excitation energy for magic nuclei

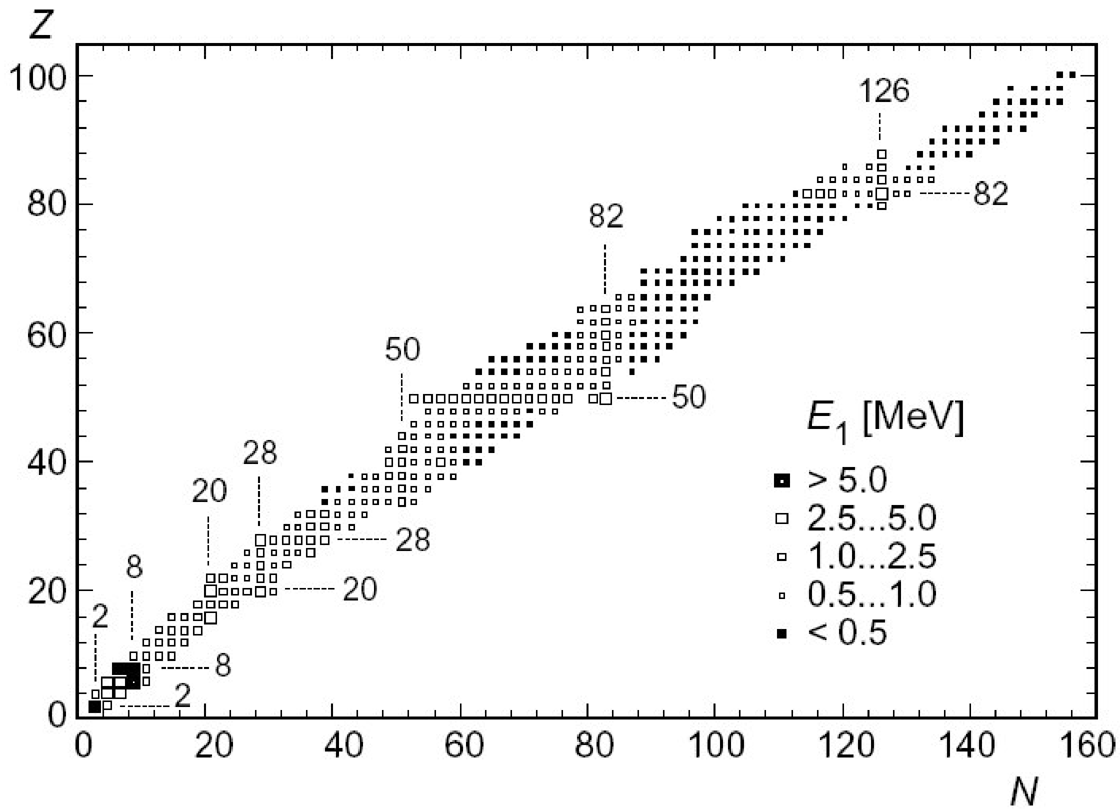



**Fig. 17.5.** The energy $E_1$ of the first excited state of even-even nuclei. Note that it is particularly big for nuclei with "magic" proton or neutron number. The excited states generally have the quantum numbers $J^P = 2^+$. The following nuclei are exceptions to this rule: ${}^4_2\mathrm{He}_2$, ${}^{16}_8\mathrm{O}_8$, ${}^{40}_{20}\mathrm{Ca}_{20}$, ${}^{72}_{32}\mathrm{Ge}_{40}$, ${}^{90}_{40}\mathrm{Zr}_{50}$ $(0^+)$, ${}^{132}_{50}\mathrm{Sn}_{82}$, ${}^{208}_{82}\mathrm{Pb}_{126}$ $(3^-)$ and ${}^{14}_6\mathrm{C}_8$, ${}^{14}_8\mathrm{O}_6$ $(1^-)$. $E_1$ is small further away from the "magic" numbers – and is generally smaller for heavier nuclei (data from [Le78]).

# Nuclear potential

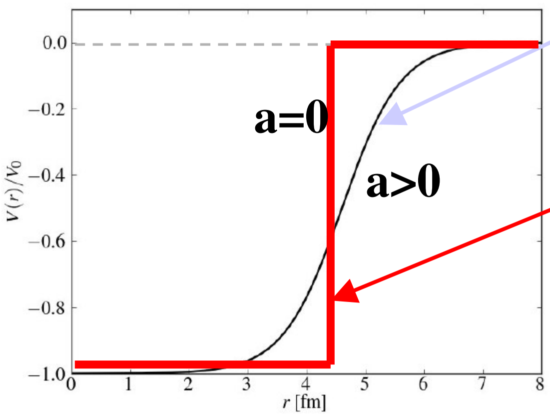
**The energy spectrum is defined by the nuclear potential**
**➔ solution of Schrödinger equation for a realistic potential**

**The nuclear force is very short-ranged => the form of the potential follows the density distribution of the nucleons within the nucleus:**

- **for very light nuclei (A < 7), the nucleon distribution has Gaussian form (corresponding to a harmonic oscillator potential)**
- **for heavier nuclei it can be parameterised by a Fermi distribution. The latter corresponds to the Woods-Saxon potential**

**1) Woods-Saxon potential:**

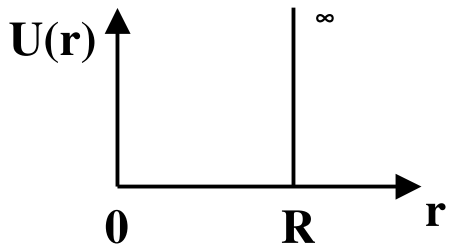
$$U(r) = -\frac{U_0}{1+e^{\frac{r-R}{a}}}$$




**2) a→0: approximation by the rectangular potential well :**

$$U(r) = \begin{cases} -U_0, & r < R \\ 0, & r \geq R \end{cases}$$

**e.g. 3) approximation by the rectangular potential well with infinite barrier energy :**




$$U(r) = \begin{cases} 0, & r < R \\ \infty, & r \geq R \end{cases}$$

# Schrödinger equation

Schrödinger equation: $$\hat{H}\Psi = E\Psi \qquad (1)$$

Single-particle Hamiltonian operator: $$\hat{H} = -\frac{\hbar\nabla^2}{2M} + U(r) \qquad (2)$$

Eigenstates: Ψ(r) - wave function
Eigenvalues: E - energy
U(r) is a nuclear potential – spherically symmetric ➔

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

Angular part: $$\hat{\lambda} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$-\hbar^2\hat{\lambda} = \hat{L}^2$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\frac{\hat{L}^2}{\hbar^2}$$

L – operator for the orbital angular momentum

$$\hat{L}^2 Y_{lm}(\theta,\varphi) = \hbar^2 l(l+1) Y_{lm}(\theta,\varphi) \qquad (3)$$

Eigenstates: $Y_{lm}$ – spherical harmonics

# Radial part

The wave function of the particles in the nuclear potential can be decomposed into two parts: a radial one $\Psi_1(r)$, which only depends on the radius r, and an angular part $Y_{lm}(\theta,\varphi)$ which only depends on the orientation (this decomposition is possible for all spherically symmetric potentials):

$$\Psi(r,\theta,\varphi) = \Psi_1(r)\cdot \Upsilon_{lm}(\theta,\varphi) \qquad (4)$$

From (4) and (1) =>

$$\left[-\frac{\hbar^2}{2M}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\hat{L}^2}{2M}\right]\Psi_1(r)\cancel{\Upsilon_{lm}(\theta,\varphi)} = E\,\Psi_1(r)\cancel{\Upsilon_{lm}(\theta,\varphi)}$$

=> eq. for the radial part:

$$\left[-\frac{\hbar^2}{2M}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{\hbar^2}{r^2}\frac{l(l+1)}{2M}\right]\Psi_1(r) = E\,\Psi_1(r) \qquad (5)$$

Substitute in (5):

$$\Psi_1(r) = \frac{R(r)}{r}$$

$$-\frac{\hbar^2}{2M}\frac{d^2R(r)}{dr^2}+\frac{\hbar^2}{r^2}\frac{l(l+1)}{2M}R(r) = E\,R(r) \qquad (6)$$

# Constraints on E

Eq. for the radial part: $$\frac{\hbar^2}{2M}\frac{d^2R(r)}{dr^2}+\left[E-\frac{\hbar^2}{r^2}\frac{l(l+1)}{2M}\right]R(r)=0 \qquad (7)$$

From (7) ➔

1) Energy eigenvalues for orbital angular momentum *l*:

*E:*

*l=0* *s*

*l=1* *p* states

*l=2* *d*

*l=3* *f*

…

2) For each *l*: -*l*< *m* <*l* => (2*l*+1) projections *m* of angular momentum.

The energy is independent of the *m* quantum number, which can be any integer value between ± *l*. Since nucleons also have two possible spin directions, this means that the *l* levels are 2·(2*l*+1) times degenerate if a spin-orbit interaction is neglected.

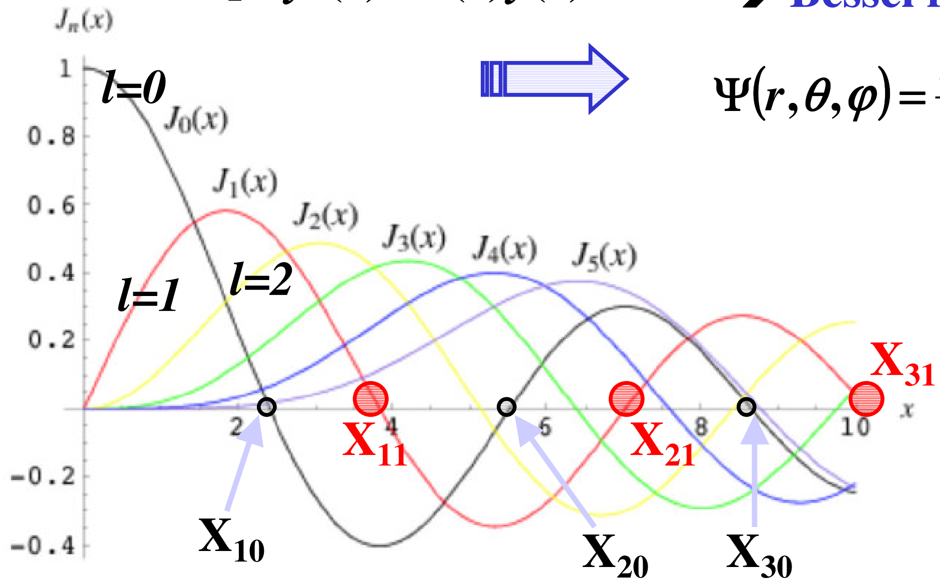
3) The parity of the wave function is fixed by the spherical wave function $Y_{ml}$ and reads $(-1)^l$: $$\hat{P}\,\Psi(r,\theta,\varphi)=P\,\Psi_1(r)\cdot \Upsilon_{lm}(\theta,\varphi)=(-1)^l\;\Psi_1(r)\cdot \Upsilon_{lm}(\theta,\varphi)$$

$\Rightarrow$ *s,d,..* - even states; *p,f,…* - odd states

# Main quantum number *n*

**Eq. for the radial part:** $$\frac{\hbar^2}{2M}\frac{d^2R(r)}{dr^2}+\left[E-\frac{\hbar^2}{r^2}\frac{l(l+1)}{2M}\right]R(r)=0 \qquad (7)$$

**Solution of differential eq:** $y''(r)+\lambda(r)y(r)=0$ ➔ **Bessel functions $j_l(kr)$**




$$\Psi(r,\theta,\varphi)=\frac{A}{r}j_l(kr)\cdot Y_{lm}(\theta,\varphi)$$

$$k^2=\frac{2ME}{\hbar^2}$$

**Boundary condition for the surface, i.e. at r=R: Ψ(R,θ,φ) =0**
**➔ restrictions on k in Bessel functions:** $j_l(kr)=0$
**⇒ main quantum number *n* – corresponds to nodes of the Bessel function : $X_{nl}$**

$$k\cdot R(r)=X_{nl} \qquad k^2R^2=X_{nl} \qquad \frac{2ME}{\hbar^2}R^2=X_{nm}^2 \qquad (8)$$

# Shell model

Thus, according to Eq. (8) : $$E_{nl} = \frac{X_{nl}^2 \hbar^2}{2MR^2} \iff E_{nl} = Const \cdot X_{nl}^2 \qquad (9)$$

Nodes of Bessel function

Energy states are quantized ➔ structure of energy states $E_{nl}$

$l = 0 \quad s-states \quad j_0$

$n = 1 \quad X_{10} = 3.14$

$n = 2 \quad X_{20} = 6.28$

$n = 3 \quad X_{30} = 9.42$

$l = 1 \quad p-states \quad j_1$

$n = 1 \quad X_{11} = 4.49$

$n = 2 \quad X_{21} = 7.72$

$l = 2 \quad d-states \quad j_2$

$n = 1 \quad X_{12} = 5.76$

$l = 3 \quad f-states \quad j_3$

$n = 1 \quad X_{13} = 6.99$

$l = 4 \quad g-states \quad j_4$

$n = 1 \quad X_{14} = 8.1$

| *state* | $E_{nl} = C \cdot X_{nl}^2$ | *degeneracy* | *states with* $E \le E_{nl}$ |
|---|---|---|---|
| 1s | $E_{1s} = C \cdot 9.86$ | 2 $2\cdot(2l+1)$ | 2 |
| 1p | $E_{1p} = C \cdot 20.2$ | 6 | 8 |
| 1d | $E_{1d} = C \cdot 33.2$ | 10 | 18 |
| 2s | $E_{2s} = C \cdot 39.5$ | 2 | 20 |
| 1f | $E_{1f} = C \cdot 48.8$ | 14 | 34 |
| 2p | $E_{2p} = C \cdot 59.7$ | 6 | 40 |
| 1g | $E_{1g} = C \cdot 64$ | 18 | 58 |

First 3 magic numbers are reproduced, higher – not!

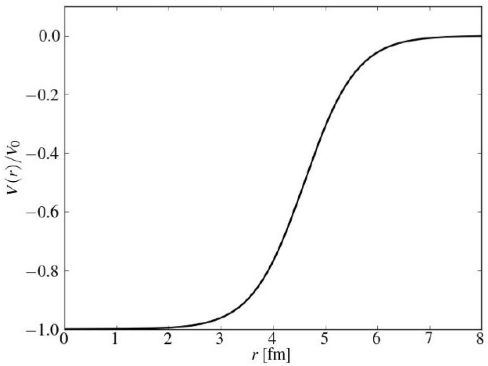
2, 8, 20, 28, 50, 82, 126

Note: here for U(r) = rectangular potential well with infinite barrier energy

# Shell model with Woods-Saxon potential

**Woods-Saxon potential:**

$$U(r) = -\frac{U_0}{1+e^{\frac{r-R}{a}}}$$




| $N$ | 0 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 4 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $n\ell$ | 1s | 1p | 1d | 2s | 1f | 2p | 1g | 2d | 3s | $\cdots$ |
| Degeneracy | 2 | 6 | 10 | 2 | 14 | 6 | 18 | 10 | 2 | $\cdots$ |
| States with $E \leq E_{n\ell}$ | 2 | 8 | 18 | 20 | 34 | 40 | 58 | 68 | 70 | $\cdots$ |

**The first three magic numbers (2, 8 and 20) can then be understood as nucleon numbers for full shells. This simple model does not work for the higher magic numbers. For them it is necessary to include spin-orbit coupling effects which further split the *nl* shells.**

# Spin-orbit interaction

**Introduce the spin-orbit interaction $V_{ls}$ – a coupling of the spin and the orbital angular momentum:**

$$\hat{H} = -\frac{\hbar\nabla^2}{2M} + U(r) + \hat{V}_{ls}$$

$$\left[-\frac{\hbar\nabla^2}{2M} + U(r) + \hat{V}_{ls}\right]\Psi(r,\theta,\varphi) = E\ \Psi(r,\theta,\varphi)$$

$$\left[-\frac{\hbar\nabla^2}{2M} + U(r)\right]\Psi(r,\theta,\varphi) = \left(E - V_{ls}\right)\ \Psi(r,\theta,\varphi)$$

**where** $\hat{V}_{ls}\Psi(r,\theta,\varphi) = V_{ls}\Psi(r,\theta,\varphi)$

**Eigenstates** **eigenvalues**

**spin-orbit interaction:** $\hat{V}_{ls} = C_{ls}(\vec{l}, \vec{s})$

**total angular momentum:** $\vec{j} = \vec{l} + \vec{s}$

$$\vec{j} \cdot \vec{j} = (\vec{l} + \vec{s})(\vec{l} + \vec{s}) = \vec{l}^2 + \vec{s}^2 + 2\vec{l}\vec{s}$$

$$\vec{l} \cdot \vec{s} = \frac{1}{2}(\vec{j}^2 - \vec{l}^2 - \vec{s}^2)$$

# Spin-orbit interaction

$$C_{ls}\cdot\frac{1}{2}(\vec{j}^{\,2}-\vec{l}^{\,2}-\vec{s}^{\,2})\Psi(r,\theta,\varphi)=V_{ls}\Psi(r,\theta,\varphi)$$

$$\boxed{V_{ls}=C_{ls}\frac{\hbar^2}{2}\left[j(j+1)-l(l+1)-s(s+1)\right]}$$

**Consider:**

$$j=l+\frac{1}{2}:\qquad V_{ls}=C_{ls}\frac{\hbar^2}{2}\left[(l+\frac{1}{2})(l+\frac{1}{2})-l^2-l-\frac{1}{2}\cdot\frac{3}{2}\right]=\underline{C_{ls}\frac{\hbar^2}{2}l}$$

$$j=l-\frac{1}{2}:\qquad V_{ls}=C_{ls}\frac{\hbar^2}{2}\left[(l-\frac{1}{2})(l+\frac{1}{2})-l^2-l-\frac{1}{2}\cdot\frac{3}{2}\right]=\underline{-C_{ls}\frac{\hbar^2}{2}(l+1)}$$

**This leads to an energy splitting $\Delta E_{ls}$ which linearly increases with the angular momentum as**

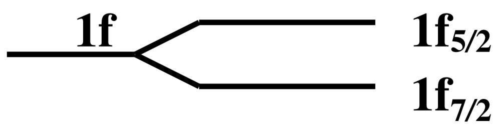
$$\Delta E_{ls}=\frac{2l+1}{2}\langle V_{ls}\rangle$$

**It is found experimentally that $V_{ls}$ is negative, which means that the state with $j=l+1/2$ is always energetically below the $j=l-1/2$ level.**

# Spin-orbit interaction

The total angular momentum quantum number $j = l \pm 1/2$ of the nucleon is denoted by an extra index j: $nl_j$

e.g. the 1f state splits into a $1f_{7/2}$ and a $1f_{5/2}$ state




The *nlj* level is (2*j* + 1) times degenerate

➔ Spin-orbit interaction leads to a sizeable splitting of the energy states which are indeed comparable with the gaps between the *nl* shells themselves.

Magic numbers appear when the gaps between successive energy shells are particularly large.

2, 8, 20, 28, 50, 82, 126

Single particle energy levels: