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An Introduction to Group Theory and to Unitary Symmetry Models

G. DE FRANCESCHI — *Laboratori Nazionali del CNEN, Frascati (Roma)*

L. MAIANI — *Laboratori di Fisica, Istituto Superiore di Sanità, Roma*

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1. General Notions on Groups [*1*]

1.1. A group G is a set of elements together with a multiplication law which associates a third element to any pair of elements of G: $(a, b) \to (ab) = c$ in such a way that the following conditions hold:

i) associativity $(ab)c = a(bc) = abc$; $a, b, c \in G$;[1])

ii) there exists an unit element $e \in G$ such that for any $a \in G: ae = ea = a$;

iii) for any $a \in G$ there exists an element a^{-1}, called the inverse of a, such that:

$$aa^{-1} = a^{-1}a = e.$$

If $ab = ba$ for every pair of elements in G, the group is said to be commutative or Abelian.

1.2. A subset G' is called a subgroup if the set of its elements is by itself a group under the multiplication law of G. It is easy to see that a subset G' of G is a subgroup if and only if $ab^{-1} \in G'$ for any pair $a, b \in G'$.

[1]) The symbol $\in$ means "belongs to".

The subgroup G' of G is an invariant or normal subgroup if $hgh^{-1} \in G'$ for any $h \in G$ and any $g \in G'$.

1.3. A mapping φ of a group G_1 into another G_2 is called a homomorphism if:

$$a \to \varphi(a), \; b \to \varphi(b) \qquad \text{implies}$$

$$ab \to \varphi\,(ab) = \varphi(a)\varphi(b).$$

φ is called an *onto* homomorphism if for any $a' \in G_2$ there is an $a \in G_1$ such that $\varphi(a) = a'$.

Let e' be the unit element of G_2. The set K_φ of the elements of G_1 which are mapped into e', is called the kernel of the homomorphism φ. It is easily shown that K_φ is a normal subgroup of G_1.

A one-to-one homomorphism of G_1 onto G_2 is called an isomorphism. In this case $K_\varphi = \{e_1\}$[2]).

1.4. Let G' be a subgroup of G, and a any fixed element of G. The set of all products ah when h runs over the whole G', is called a right G' coset, indicated in what follows as aG'. If two cosets aG', bG' have one element in common they are in fact coincident. Therefore the whole group G decomposes into disjoint G' right cosets. In the same way one can define left G' cosets.

When G' is a normal subgroup of G, then for any $a \in G: aG' = G'a$. In fact the general element of aG' has the form:

$$ah \qquad h \in G'.$$

But $aha^{-1} = h', h' \in G'$; hence:

$$ah = h'a$$

that is any element of aG' is in $G'a$ and conversely. Let G' be a normal subgroup of G, and let us indicate with G/G' the set of all distinct G' cosets in G. The set G/G' equipped with the following multiplication law:

$$(aG')\,(bG') = (ab)G' \qquad a, b \in G$$

turns out to be a group, called the factor group of G with respect to G'. Due to the fact that G' is a normal subgroup, it is easy to see that the above introduced multiplication law satisfies 1.1 i-iii. We observe that the unit element of G/G' is eG'.

Given a homomorphism of a group G_1 onto a group G_2 we can form the factor group G_1/K_φ because K_φ is a normal subgroup of G_1. Observe that:

$$\varphi(a) = \varphi(b)$$

if and only if a and b belong to the same K_φ coset. In fact let e_2 be the unit element of G_2: the previous condition is equivalent to:

$$e_2 = \varphi(a)\,\varphi(b)^{-1} = \varphi(ab^{-1})$$

[2]) By means of $\{a, b, c, \ldots\}$ we denote the set of the elements $a, b, c, \ldots$

i.e. $ab^{-1} \in K_\varphi$. Hence there exists $h \in K_\varphi$ such that:

$$a = hb$$

i.e.

$$a \in K_\varphi b \equiv bK_\varphi.$$

Therefore we can define a mapping $\overline{\varphi}$ of G_1/K_φ onto G_2 as follows:

$$\overline{\varphi}(aK_\varphi) = \overline{\varphi}(b) \qquad b \in aK_\varphi$$

and this mapping is in fact an isomorphism of G_1/K_φ onto G_2.

1.5. Examples

a) Rotation group of the three dimensional euclidean space E_3.

We consider the set O_3 of transformationa of E_3 into itself, preserving distances and leaving unchanged a point O. Given $R_1, R_2 \in O_3$, R_1R_2 is defined as the transformation:

$$R_1R_2X = R_1(R_2X)$$

which obviously belongs to O_3. Furthermore the identity transformation $X \to X$ belongs to O_3. All the O_3 transformations are one–one so that for any of them it is possible to define an inverse which of course belongs to O_3. By definition this multiplication rule is also associative, so that it gives a group structure to O_3.

We choose a set of three orthogonal axes stemming from the fixed point O. Then to any transformation:

$$X \to X'$$

is associated a three by three real (non singular) matrix $\{R_{ik}\}$:

$$X'_i = \sum_k R_{ik}X_k$$

satisfying the orthogonality condition:

$$RR^T = 1 \tag{1}$$

(R^T is the transpose matrix, $R^T_{ik} = R_{ki}$).

For any $R \in O_3$ the correspondence $R \to \{R_{ik}\}$ is one-to-one; in terms of the matrices, the product in O_3 reduces to the usual matrix product so that O_3 is isomorphic to this matrix group. In what follows we will identify them.

It follows from (1) that $\det R = \pm 1$. The subset of the R's with determinant $+1$, is by itself a group, called R_3, and its elements are the proper rotations of E_3. R_3 is a normal subgroup of O_3, because for any element $R \in O_3$, and for any $R' \in R_3$:

$$\det\,(RR'R^{-1}) = +1.$$

The factor group O_3/R_3 has only two elements: the cosets $e_1 = R_3$ and $e_2 = -I \cdot R_3$, where I is the unit matrix. Product rules are:

$$e_1e_1 = e_2e_2 = e_1$$

$$e_1e_2 = e_2e_1 = e_2.$$

b) SU_2.

The set SU_2 of two dimensional unitary unimodular (determinant $= 1$) complex matrices forms a group with respect to the usual multiplication law for matrices. The general form of a SU_2 matrix is:

$$U \equiv \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}; \quad |\alpha|^2 + |\beta|^2 = 1 \tag{2}$$

(the bar denotes complex conjugation).
We can express U in terms of the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the unit matrix σ_0 as follows:

$$U \equiv \begin{pmatrix} \alpha_0 \ + i\alpha_3 & \alpha_2 + i\alpha_1 \\ -\alpha_2 + i\alpha_1 & \alpha_0 - i\alpha_3 \end{pmatrix} = \sigma_0\alpha_0 + i\ \boldsymbol{\alpha}\cdot\boldsymbol{\sigma} \tag{3}$$

where $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, \alpha_3)$ and the real numbers $\alpha_i (i = 0, \ldots, 3)$ satisfy:

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + {}_3^2 = 1\,. \tag{4}$$

Putting
$$\boldsymbol{\alpha} = -\lambda\boldsymbol{n}, \quad \boldsymbol{n}^2 = 1, \quad \lambda \geq 0, \quad \boldsymbol{n} = -\frac{\boldsymbol{\alpha}}{|\boldsymbol{\alpha}|}$$

by (4) we have:

$$\alpha_0^2 + \lambda^2 = 1$$

so that we can set:

$$\alpha_0 = \cos\frac{\theta}{2} \qquad 0 \leq \theta \leq 2\pi$$

$$\lambda = \sin\frac{\theta}{2}$$

so that:

$$U = \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\,\boldsymbol{n}\cdot\sigma\,. \tag{5}$$

We show now that SU_2 is homomorphic to the proper rotations group R_3.
For any three dimensional vector $\boldsymbol{X}$ we define the two by two hermitian matrix:

$$\tilde{X} = \boldsymbol{X}\cdot\boldsymbol{\sigma} \tag{6}$$

and observe that:

$$\det\tilde{X} = -|\boldsymbol{X}|^2$$

For any $U \in SU_2$, we define the transformation[3]):

$$\tilde{X}' = U\tilde{X}U^{+}. \tag{7}$$

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[3]) If $U \equiv \{U_{ik}\}$ is a square complex matrix, we define:

$$\bar{U}: (\bar{U})_{ik} = (\overline{U_{ik}})$$

$$U^{+}: (U^{+})_{ik} = (\overline{U_{ki}}).$$

The following properties are to be noted:

i) $\tilde{X}'$ is a hermitian matrix being U unitary:

ii) $\mathrm{Tr}\,\tilde{X}' = \mathrm{Tr}\,\tilde{X} = 0$

it follows then: $\tilde{X}' = \boldsymbol{X}'\cdot\boldsymbol{\sigma}$ because any hermitian two by two matrix is a linear combination of the Pauli matrices with real coefficients;

iii) $\det\tilde{X}' = \det\tilde{X} = -|\boldsymbol{X}'|^2 = -|\boldsymbol{X}|^2$.

It follows that the transformation $\boldsymbol{X} \to \boldsymbol{X}'$ is a mapping of E_3 into itself which preserves the distances and does not change the origin. Hence it is a transformation of O_3. Let us indicate with $R(U)$ the element of O_3 which corresponds to the matrix U. Then from (7) we have:

$$\widetilde{R(U_1U_2)\cdot\boldsymbol{X}} = U_1U_2\tilde{X}U_2^{+}U_1^{+} = \widetilde{U_1R(U_2)\cdot\boldsymbol{X}}U_1^{+} = \widetilde{R(U_1)R(U_2)\boldsymbol{X}}$$

i.e. $U \to R(U)$ is a homomerphism of SU_2 into O_3.
Observe that from (7) it follows that:

$$R(U) = R(-U).$$

Furthermore for any $U \in SU_2$ there exists a $V \in SU_2$ such that:

$$U = V^2. \tag{8}$$

If U is of the form (5), it is sufficient to choose:

$$V = \cos\frac{\theta}{4} - i\sin\frac{\theta}{4}\,\boldsymbol{n}\cdot\boldsymbol{\sigma}.$$

Then:

$$R(U) = R(V)\,R(V)$$

hence:

$$\det R(U) = (\det R(V))^2 = (\pm 1)^2 = 1\,.$$

We conclude that $U \to R(U)$ is a mapping of SU_2 into R_3 (the proper part of O_3).
Substituting expression (5) for U into (7) and carrying out the calculations, it is possible to derive an explicit expression of $\boldsymbol{X}'$ in terms of $\boldsymbol{X}$, $\boldsymbol{n}$, and θ:

$$\boldsymbol{X}' = (\boldsymbol{X}\cdot\boldsymbol{n})\cdot\boldsymbol{n} + \cos\theta\,[\boldsymbol{X} - (\boldsymbol{n}\cdot\boldsymbol{X})\,\boldsymbol{n}] + \sin\theta\,(\boldsymbol{n}\times\boldsymbol{X}). \tag{9}$$

So that $\boldsymbol{X}'$ is obtained from $\boldsymbol{X}$ by a counterclockwise rotation of θ around $\boldsymbol{n}$.
Now let R be the rotation uniquely defined by its rotation axis (with unit vector $\boldsymbol{n}$) and rotation angle $\theta (0 \leq \theta < 2\pi)$ in a specified sense, to be definite in the counterclockwise sense, around $\boldsymbol{n}$: the previous formula permits us immediately to find a SU_2 matrix U sich that $R(U)$ is the given rotation:

$$U = \cos\frac{\theta}{2} - i\,\sin\frac{\theta}{2}\,\boldsymbol{n}\cdot\boldsymbol{\sigma}.$$

This proves that the homomorphism $U \to R(U)$ is onto. It is easily shown that the only matrices U such that

$$R(U) = 1$$

are

$$U = \pm 1$$

which consequently constitute the kernel of the homomorphism.

c) SU_3.

The set of 3×3 complex, unitary unimodular matrices also forms a group with respect to the usual matrix multiplication law, and this group is called SU_3.

d) The set of proper Lorentz transformations forms a group indicated as $L_+^\uparrow$ which is isomorphic to a matrix group: the set of 4×4 real matrices Λ such that:

i) $\Lambda^T G \Lambda = G$

being G the matrix:

$$G \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

ii) $\det \Lambda = 1$, $\Lambda_{44} \geq 1$

d) Inhomogeneous proper Lorentz group.

Consider the set $P_+^\uparrow$ of the transformations of the Minkowski space defined as:

$$X \equiv (X_1, X_2, X_3, X_4) \rightarrow X' \equiv (X_1', X_2', X_3', X_4')$$

$$X_i' = \sum_k \Lambda_{ik} X_k + a_i$$

(where $\Lambda \in L_+^\uparrow$, and a is a four-vector) or simply:

$$X' = \Lambda X + a .$$

Applying a transformation determined by the pair (a, Λ), and then the transformation (a', Λ'), we obtain a new element of $P_+^\uparrow$

$$(a', \Lambda')(a, \Lambda) = (a' + \Lambda' a, \Lambda'\Lambda).$$

With this multiplication rule $P_+^\uparrow$ is a group.

It is easy to see that the unit element of $P_+^\uparrow$ is

$$(0, I) \qquad (I = \text{unit element of } L_+^\uparrow)$$

and that the inverse of (a, Λ) is

$$(a, \Lambda)^{-1} = (-\Lambda^{-1} a, \Lambda^{-1}).$$

The subset

$$\{(0, \Lambda)\}$$

is *a* subgroup of $P_+^\uparrow$ isomorphic to $L_+^\uparrow$, and the subset

$$\{(a, I)\}$$

is an abelian subgroup, isomorphic to the translations group in four-space. Moreover $\{(a, I)\}$ is an invariant subgroup.

1.6. Topological and Lie groups

Following our definition a group is an abstract set in which there is a multiplication rule satisfying i—iii. We have subsequently checked that certain sets of matrices are groups.

It is convenient to go somewhat further and introduce in these sets a notion of nearness of two elements in such a way that the group operations enjoy a "continuity" property in a sense to be specified later. (In mathematical language this procedure is referred to as the introduction of a topology in the group).

The reasons for doing so are that many results of the group representation theory, which are especially important for physics, are based on topological properties. We will not go over the general theory of topological groups (which is comprehensively treated in many textbooks; e.g. see reference (1)); instead we will concentrate on a particular class of groups; those for which it is possible to put a one-to-one correspondence between their elements and the points of a subset of a n-dimensional real euclidean space E_n.

Let G be the group under consideration and let $\varphi(G)$ be its image in E_n. If g is an element of G and $\varphi(g)$ its image, then for any spherical neighborhood S_ε of $\varphi(g)$

$$f \in S_\varepsilon, \quad |f - \varphi(g)| < \varepsilon$$

consider the intersection of S_ε with $\varphi(G)$ (indicated as $S_\varepsilon \cap \varphi(G)$). We define as neighborhood of g in G the set Σ_ε of the elements whose image points lie in $S_\varepsilon \cap \varphi(G)$. As ε runs over real positive numbers we obtain a family of neighborhoods for each element of G and with their help one can define the concept of limit and continuity of functions on the group in the same way as in the euclidean space E_n. One can also define open sets in G: a set $S \subset G$ is open if any point of S is included in a neighborhood entirely contained in S. If the group multiplication and the inversion are continuous with respect to this topology, we will call G a topological group.

A continuous correspondence (function) between real numbers x, $0 \leq x \leq 1$, and elements $g(x)$ of a topological group G, is called a continuous path on G. The group is said to be *connected* if for any pair of elements g and g' there exists a path having them as end points.

A path $g(x)$ on G is said to be closed if $g(0) = g(1)$. Two curves $f(x)$ and $g(x)$ are said to be reconciliable when there exists a function $\Gamma(x, y)$ $(0 \leq x, y \leq 1)$ continuous in both variables, with values in G, such that:

$$\Gamma(x, 0) = f(x)$$

$$\Gamma(x, 1) = g(x).$$

In particular a closed curve $g(x)$ will be reducible to the point f if it is reconciliable with the constant function:

$$f(x) = f, \quad f \in G, \quad 0 \leq x \leq 1.$$

A group G is said to be *simply connected* if any closed curve $g(x)$ is reducible to a point.

Let us consider again the image $\varphi(G)$ of G in E_n. If $\varphi(G)$ is a compact (i.e. closed and bounded) set, then G is said to be compact. In this case any continuous real function of the elements of G is bounded (Weierstrass theorem)[4].

[4]) It must be observed that the notion of compactness of a topological group is intrinsic, and can be given without referring to a particular parametrization of the group, just as the introduction of a topology in a group (also for this topic see [1]).

We specialize furthermore the concept of a topological group to that of a Lie group.
Let us suppose that there exists in a topological group a neighborhood N of the unit element e such that:
i) there is a one-to-one correspondence between elements of N and points of a subset of E_n. In addition we now require those parameters to be essential, i.e. it is not possible to express any of them in terms of the remaining $n-1$.
ii) if $a = a(x_1 \ldots x_n)$, $b = b(y_1 \ldots y_n)$ are elements of N such that ab and a^{-1} belong to N, and $(z_1 \ldots z_n)$, $(z'_1 \ldots z'_n)$ are respectively the parameters of ab and a^{-1}, then

$$z_i = z_i(x_1 \ldots x_n, y_1, \ldots y_n)$$

$$z'_k = z'_k(x_1 \ldots x_n)\,; \quad i, k = 1, \ldots n$$

are analytic functions of their arguments.
In this case G is said to be a n-dimensional Lie group. (In this connection a function $f(x_1, \ldots x_n)$ is said analytic at the point $(a_1, \ldots a_n)$ if there exists a neighbourhood of this point in which the function may be expressed as a converging power series of the differences $x_i - a_i$).
We will choose the parametrization of N in such a way that the set $(0, \ldots 0)$ corresponds to e.

1.7. Examples

We give here some examples to illustrate the relevant concepts introduced in the previous paragraph, as well as to establish some useful results concerning the groups which are of interest to us.
Let us begin with R_3. In sect 1.5 we have identified any rotation by a unit vector $\boldsymbol{n}$ and an angle $\theta\,(0 \leqq \theta \leqq 2\pi)$. The drawback of this is that: $R(\theta, \boldsymbol{n}) = R(2\pi - \theta, -\boldsymbol{n})$. We can instead obtain a one-to-one correspondence between rotations and three-dimensional vectors stemming from the origin of E_3, with lenght less than or equal to π: to any rotation of an angle θ $(0 \leqq \theta \leqq \pi)$ in the counter clockwise sense around a unit vector $\boldsymbol{n}$, we associate the vector $\alpha = \theta \cdot \boldsymbol{n}(|\alpha| \leqq \pi)$; conversely given α, θ and $\boldsymbol{n}$ can be obtained as:

$$\theta = |\boldsymbol{\alpha}|$$

$$\boldsymbol{n} = \frac{\boldsymbol{\alpha}}{|\boldsymbol{\alpha}|}\,(|\boldsymbol{\alpha}| \neq 0).$$

The end points of these vectors fill a sphere of radius π, and we note that the same rotation corresponds to points on the surface diametrically opposite. Hence it is necessary to identify those points in order to preserve the one-to-one correspondence property.
Since the set of parameters is a bounded closed connected subset of E_3, R_3 is a compact connected group. It is instead not simply connected, as can be seen if one considers a curve connecting two diametrically opposed points on the surface of the sphere. These two points correspond to the same element of R_3, so that the curve is effectively closed, but is cannot be reduced to a point.

Let us consider a rotation $R(\theta, \boldsymbol{n})$. From 1.5 (9) it follows that the matrix R which corresponds to the rotation R (1.5a) has the form

$$R_{ij} = (1 - \cos\theta)\, n_i n_j + \sum_k^{1,3} \sin\theta\, \epsilon_{ikj} n_k + \cos\theta\, \delta_{ij} \tag{10}$$

where n_k are the components of $\boldsymbol{n}$, ϵ_{ijk} is the Levi-Civita tensor, and δ_{ij} is the Kronecker tensor.
This expression is equivalent to

$$R = R(\boldsymbol{\alpha}) = e^{\theta \boldsymbol{n} \cdot \Sigma} = \sum_k^{0,\infty} \frac{1}{k!}\, \theta^k (\boldsymbol{n} \cdot \boldsymbol{\Sigma})^k = e^{\alpha \cdot \Sigma} \tag{11}$$

where the 3×3 matrices Σ_i are defined as follows:

$$\Sigma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad \Sigma_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}; \quad \Sigma_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This can be seen using in (11) the relations:

$$(\boldsymbol{n} \cdot \boldsymbol{\Sigma})^2 = -1 + |n)(n| \tag{12}$$

$$(\boldsymbol{n} \cdot \boldsymbol{\Sigma})^3 = -(\boldsymbol{n} \cdot \boldsymbol{\Sigma}) \tag{13}$$

$$(|n)(n|)_{ij} = n_i n_j\,; \quad \alpha = \theta \boldsymbol{n}.$$

It follows then that the coefficients R_{ij} of $R(\alpha)$ are nine analytic functions of $\boldsymbol{\alpha}$, and one can verify that the Jacobian matrix

$$\left(\frac{\partial R_{ij}}{\partial \alpha_k}\right) \alpha_1 = \alpha_2 = \alpha_3 = 0$$

has characteristic 3. Hence it is possible (see COHN, [2] appendix) in a suitable neighborhood N of the point $\boldsymbol{\alpha} = (0,0,0)$, to express α_i as analytic functions of three fixed coefficients R_{ij} (say R_{12}, R_{13}, R_{23}):

$$\alpha_i = \alpha_i(R_{ij}).$$

Now consider the product $R' \cdot R'' = R$ of two elements of R_3[5]), and call respectively $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}$ the parameters of R', R'', R. The coefficients R_{ij} of R are analytic functions of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Hence $\boldsymbol{\delta}$, being an analytic function of three of such coefficients, is an analytic function of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.
This demonstrates, together with the fact that the parameters of R^{-1} are obviously analytic functions of those of R, that R_3 is a 3-dimensional Lie group.
In an exactly analogous way it is possible, using (5) and the properties of Pauli matrices, to write any SU_2 matrix as:

$$U = \exp\left(-i\,\frac{\theta}{2}\,\boldsymbol{n} \cdot \boldsymbol{\sigma}\right) = \exp(-i\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}) \tag{14}$$

$$0 \leqq \theta \leqq 2\pi \quad \boldsymbol{\alpha} = \frac{\theta}{2} \cdot \boldsymbol{n}$$

[5]) such that $R', R'', R \in N$.

so that there is a correspondence between the elements of SU_2 and the points of the sphere of radius π centered at the origin of E_3, and this correspondence is one-to-one if one identifies *all* the points of the surface with the element -1 of SU_2. Due to this fact SU_2 is not only a compact and connected group, but it is also simply connected. The same arguments as before can be used to show that SU_2 is also a 3-dimensional Lie group.

SU_3. We may identify any matrix of SU_3 with the real and imaginary part of each coefficient U_{ij}, obtaining a one-to-one correspondence between the elements of SU_3 and the points of a set of E_{18}. These parameters however are not independent, because the matrix must be unitary and unimodular. We obtain 10 conditions at all[6]), so that we have only 8 independent parameters α_K. It is possible to write any element of SU_3 in a form like (11) and (14):

$$U = e^{i\sum_K^{1..8} \alpha_K F_K} \tag{15}$$

where $F_1, F_2 \ldots F_8$ are eight hermitian traceless independent matrices that are listed, with their commutation and anticommutation rules in [3].

Formula (15) shows that SU_3 is a connected, 8-dimensional group and in fact it may be shown that SU_3 is also *simply*-connected.

An example of a non compact Lie group is the proper Lorentz group (1.5d). If we choose as parameters of an element of $L_+^{\uparrow}$ its matrix coefficients, the subset of E_{16} so obtained is not bounded; in fact, in the coefficients of matrices belonging to the subgroup of *special* Lorentz transformations there appear expressions like:

$$\frac{1}{\sqrt{1-\beta^2}} \qquad 0 \leq \beta < 1$$

which of course are not bounded.

2. Linear Spaces

2.1. A set L of elements $x, y, z, \ldots$ is called a complex (real) linear or vector space if:

i) L is a commutative group with respect to a composition law (indicated with the symbol $+$) called sum:

$$x + y = y + x; \qquad 0 + x = x; \qquad x + (-x) = 0$$

ii) the product $\alpha x (x \in L, \alpha =$ complex (real) number, $\alpha x \in L)$ is defined so that the following conditions hold:

$$\alpha(x+y) = \alpha x + \alpha y$$

$$\alpha(\beta x) = (\alpha\beta)x = \alpha\beta x$$

$$(\alpha+\beta)x = \alpha x + \beta x$$

$$1 \cdot x = x.$$

[6]) Nine real conditions are the real and imaginary part of equations like

$$\Sigma_K u_{iK} \bar{u}_{jK} = \delta_{ij},$$

and the tenth is the condition: $\det(U_{ij}) = +1$.

2.2. n elements x_k of L are said to be linearly independent if

$$\sum_{K=1}^{n} \alpha_k x_k = 0$$

$\alpha_k =$ complex (real) number implies:

$$\alpha_k = 0 \qquad k = 1 \ldots n$$

otherwise they will be said linearly dependent.

We say that L is n-dimensional if there exist n linearly independent elements, whereas any $n+k$ vectors are always linearly dependent ($k \geqq 1$). Any set of n linearly independent vectors $e_i (i = 1, \ldots n)$ is called a basis for L, and we can express in a unique way any vector x as a linear combination of them:

$$x = \sum_{i=1}^{n} x_i e_i.$$

A transformation T

$$x \to x' = T(x) \qquad x \in L, \; x' \in L'$$

of the n-dimensional linear space L into the m-dimensional linear space L', is called a linear operator if

$$T(x+y) = T(x) + T(y) \qquad T(\alpha x) = \alpha T(x).$$

If $\{e_i\}$ is a basis in L and $\{e'_k\}$ is a basis in L', we have

$$x' = \sum_{k=1}^{m} x'_k e'_k = T(x) = \sum_{i=1}^{n} x_i T(e_i) = \sum_{i=1}^{n} x_i \sum_{k=1}^{m} T_{ki} e'_i$$

so that

$$x'_k = \sum_{i=1}^{n} T_{ki} x_i.$$

Hence the coordinates of the transformed vector x' are obtained from those of x, by means of a $(m \times n)$ matrix T_{ki}, that uniquely represents the given transformation in the bases $\{e_i\}$ and $\{e'_i\}$.

If in a linear space L, for any fixed n, there exist n linearly independent vectors, then L is said to be infinite dimensional.

2.3. A subset l of a linear space L, such that any linear combination of elements of l belongs to it, is called a subspace (or linear manifold) of L. L is said to be the direct sum of the subspaces $l_1, l_2, \ldots$ if it happens that any vector x of L can be expressed uniquely as a linear combination of vectors contained in $l_1, l_2 \ldots$ We will write:

$$L = l_1 \oplus l_2 \oplus l_3 \cdots$$

2.4. A correspondence of pairs of vectors of a complex linear space L into the complex numbers:

$$x, y \to (x, y)$$

satisfying the conditions:

$$(x,y) \text{ is linear in } x: \quad (\alpha z + \beta w, y) = \alpha(z,y) + \beta(w,y)$$

$$(x,y) = \overline{(y,x)}$$

$$(x,x) \geqq 0, \qquad (x,x) = 0 \quad \text{if and only if} \quad x = 0$$

is a scalar product in L.
We observe that in any finite-dimensional space it is always possible to define a scalar product (this is not true in general for infinite dimensional spaces: the existence of a scalar product must be assumed as an additional hypothesis): in fact if x_i are the coordinates of x in a fixed basis and y_i are those of y, we define

$$(x,y) = \sum_i^{1,n} x_i \overline{y}_i ,$$

and it is easy to see that all the previous conditions are satisfied.
Any linear space in which a scalar product can be defined is called a Hilbert space. In the infinite dimensional case the additional hypothesis of completeness is required (see [4]).
With the aid of the scalar product it is possible to introduce the concept of lenght of a vector x:

$$\|x\| = \sqrt{(x, x)} .$$

If now T is a linear operator which maps L into itself, then we will say that T is bounded if there exists a positive number C such that for any vector $x \neq 0$:

$$\frac{\|Tx\|}{\|x\|} \leqq C .$$

To any bounded operator T it is possible to associate another one, which is called the adjoint, defined as the operator satisfying the following condition:

$$(Tx, y) = (x, T^+ y)$$

$$\text{for any } x, y, \in L$$

when $T = T^+$, T is called self adjoint or hermitian; when $T^+T = TT^+ = 1$, T is said to be unitary.

3. Representation of Groups

3.1. Let G be a group and L a linear space. A representation of G in L is by definition a correspondence between elements of G and linear operators mapping L into itself, in such a way that:

$$\begin{aligned} T(g_1 g_2) &= T(g_1)\,T(g_2) \qquad g_1, g_2 \in G \\ T(e) &= I \end{aligned} \tag{1}$$

where I is the identity operator:

$$Ix = x \qquad x \in L .$$

It follows from (1) that the operator $T(g)$ $(g \in G)$ has an inverse that is $T(g^{-1})$; in fact:

$$T(g)\,T(g^{-1}) = T(gg^{-1}) = T(e) = I$$

$$T(g^{-1})\,T(g) = T(g^{-1}g) = T(e) = I .$$

The same group can be represented in finite dimensional spaces as well as in infinite dimensional spaces. In the first case we will speak of finite dimensional representation, the dimensionality of the representation being equal to that of the space.
When L is a Hilbert space, we can consider unitary representations of G, i.e. those for which $T(g)$ is a unitary operator.

3.2. Examples

Consider again R_3. We have seen that there is a one-to-one correspondence between rotation and 3×3 real orthogonal matrices with determinant equal to 1. Those matrices are operators in a 3-dimensional real linear space, and obviously this correspondence fullfills conditions (1), so that it is a representation of R_3. In addition this is a one-to-one representation, i. e. a faithful one.
Let now consider the set L^2 of the complex valued functions $\psi(\boldsymbol{x})$ defined in E_3, such that

$$\int |\psi(x)|^2 \, d^3x$$

exists. This is a vector space, and also a Hilbert space, with the scalar product defined as

$$(\psi, \varphi) = \int \psi(\boldsymbol{x})\, \overline{\varphi}(\boldsymbol{x})\, d^3x .$$

The Schrödinger equation for a particle of mass m in a given potential $V(\boldsymbol{x})$ is

$$\left[\frac{\hbar^2}{2m} \nabla^2 + (E - V(\boldsymbol{x}))\right] \psi(\boldsymbol{x}) = 0 \tag{2}$$

and for certain classes of potentials (for example Coulomb potential) there exist in L^2 solutions of (2) corresponding to bound states. In this case let us call L_E the subset ($\subset L^2$) of the solutions of (2) corresponding to the same eigenvalue E $\cdot$ L_E is obviously a linear space.
Define for any rotation $R \in R_3$ the operator $T(R)$ in L^2 as

$$(T(R)\psi)(\boldsymbol{x}) = \psi'(\boldsymbol{x}) = \psi(R^{-1}\boldsymbol{x}) .$$

If $V(x)$ is a central potential, i.e.

$$V(\boldsymbol{x}) = V(|\boldsymbol{x}|)$$

then $T(R)$ maps L_E into itself. In fact let $\psi(\boldsymbol{x}) \in L_E$ then

$$\left[\frac{\hbar^2}{2m} \nabla^2 + (E - V(\boldsymbol{x}))\right] \psi(R^{-1}\boldsymbol{x}) = \left[\frac{\hbar^2}{2m} \nabla'^2 + (E - V(\boldsymbol{x}'))\right] \psi(\boldsymbol{x}') = 0$$

where $\boldsymbol{x}' = R^{-1}\boldsymbol{x}$ and we have used the fact that

$$V(\boldsymbol{x}) = V(\boldsymbol{x}')$$

and

$$\nabla^2 = \sum_k \frac{\partial^2}{\partial x_k^2} = \sum_{i,k,j} R_{ij} R_{kj} \frac{\partial}{\partial x'_i} \frac{\partial}{\partial x'_k} = \sum_i \frac{\partial^2}{\partial x'^2_i} = \nabla'^2$$

due to orthogonality of R_{ij}.
In addition the correspondence $R \to T(R)$ satisfies (1)

$$(T(R_1 R_2)\psi)(\boldsymbol{x}) = \psi(R_2^{-1} R_1^{-1}\boldsymbol{x}) = (T(R_2)\psi)(R_1^{-1}\boldsymbol{x}) = (T(R_1)\,T(R_2)\psi)(\boldsymbol{x})$$

i.e.

$$T(R_1 R_2) = T(R_1)\,T(R_2)$$

and obviously the identity of R_3 is mapped into the unit operator. Hence we have a representation of R_3 in L_E, which is furthermore unitary.

3.3. a) Equivalence of representations

Let T_1 and T_2 be two representations of a given group in the spaces L_1 and L_2. They are equivalent if there exists a one-to-one linear mapping A of L_2 onto L_1 such that

$$T_1(g)\,A = A\,T_2(g)$$

for any $g \in G$. In this case we will write $T_1 \sim T_2$.
The set of all representations of G decomposes into classes of equivalent representations, and the fact that two equivalent representations are essentially the same thing, permits us to limit our study to inequivalent representations.
b) Reducible representations.
A subspace l of L is said to be invariant for a representation of G in L, if

$$T(g)\,x \in l \quad \text{when } x \in l$$

for any $g \in G$. (O and L are always invariant (trivial) subspaces). If a representation has no invariant subspaces other than O and L, it is said to be irreducible.
We observe that if $T(g)$ is a finite dimensional reducible representation of G in L, and l is an invariant subspace, we can choose a basis in L such that l is spanned by the first elements of the basis, while the remaining ones span a subspace l' ($L = l \oplus l'$).
The matrices corresponding to the operators $T(g)$ in this basis are of the block form

$$T(g) = \begin{pmatrix} T_l(g) & Q(g) \\ 0 & T_{l'}(g) \end{pmatrix} \tag{3}$$

where $T_l(g)$ maps l into itself, and $T_l(g)$ as well as $T_{l'}(g)$ define two representations of G. In fact:

$$T(g_1 g_2) = \begin{pmatrix} T_l(g_1)\,T_l(g_2) & T_l(g_1)\,Q(g_2) + Q(g_1)\,T_{l'}(g_2) \\ 0 & T_{l'}(g_1)\,T_{l'}(g_2) \end{pmatrix}.$$

When $Q(g) \equiv 0$, l and l' are both invariant. In this case we will say that $T(g)$ decomposes into the direct sum of $T_l(g)$ and $T_{l'}(g)$:

$$T = T_l \oplus T_{l'}.$$

In general we will say that a representation T of G in L is decomposable if it is possible to write L as the direct sum of invariant subspaces $l_1, l_2, \dots$ so that

$$L = l_1 \oplus l_2 \oplus \cdots = \bigoplus_i l_i$$

$$T = T_{l_1} \oplus T_{l_2} \oplus \cdots = \bigoplus_i T_{l_i}.$$

If in addition any component T_{l_i} of T is irreducible, then T is said to be completely reducible. We must observe that there are reducible representations of groups that are not decomposable. For example let us consider the set T of complex number which is a commutative group under the sum. The 2×2 matrices

$$T(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \qquad z \in C$$

constitute a representation in the 2-dimensional complex vector space of such a group. Obviously the subspace l of vectors like

$$\begin{pmatrix} u \\ 0 \end{pmatrix}$$

is an invariant one. However no other invariant subspace exists because

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \alpha \begin{pmatrix} u \\ v \end{pmatrix}$$

implies

$$\alpha = 1, \quad zv = 0$$

which are impossible to be satisfied for any z if $v \neq 0$. Hence $T(z)$ is not decomposable.
Furthermore a decomposable representation is not always completely reducible: for example the representation

$$z \in C \quad z \to T(z) = \begin{pmatrix} 1 & z & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is decomposable but obviously not completely reducible.
However for unitary finite dimensional representations it is always true that a reducible representation is completely reducible.
It sufficies to show that if l is an invariant subspace, the orthogonal complement $l^\perp$ of l is also invariant. In fact we have:

$$g \to T(g)$$

$$g^{-1} \to T(g^{-1}) = (T(g))^{-1} = (T(g))^+$$

so that if $x \in l, y \in l^\perp$

$$T(g^{-1})\,x \in l$$

hence:

$$0 = (T(g^{-1})x, y) = (x, T(g)y)$$

i.e.

$$T(g)y \in l^{\perp} \quad \text{when} \quad y \in l^{\perp}$$

for any $g \in G$. We may then write:

$$L = l \oplus l^{\perp}$$

$$T = T_l \oplus T_{l^{\perp}}.$$

If both T_l and $T_{l^{\perp}}$ are irreducible, the theorem is proved. Otherwise there will be invariant subspaces contained in l and (or) $l_{\perp}$. In this case we repeat the above arguments, decomposing T_l and (or) $T_{l^{\perp}}$. Being L finite dimensional the process must end after a finite number of steps leading to a full reduction of T. A weaker result holds in the infinite dimensional case, namely the unitarity of the representation guarantees only its decomposability.

3.4. Characterization of the representations of compact or finite groups.

One important problem arising in the application of group theory in quantum physics is to know all the inequivalent representations of a given topological group G. This problem of the utmost importance from a purely mathematical point of view, has not been completely solved for an arbitrary topological group. However in the case of compact groups (and finite groups, i.e. those groups which contain a finite number of elements) the situation has been completely clarified by the works of Peter and Weyl, whose results we will summarize.
It is necessary in this connection to restrict our attention to those representations T satisfying the following requirements:
i) T is a continuous representation of G in a Hilbert space H; i.e. for any $g \in G$, from

$$g' \to g \quad g' \in G$$

it follows

$$\| T(g')x - T(g)\, x \| \to 0$$

for any vector $x \in H$;
ii) if H is infinite dimensional, $T(g)$ is a bounded operator (2.4.).
Then the following statements hold:
a) in any class of equivalent representations there is a unitary representation (U.R.);
b) any irreducible representation is finite-dimensional;
c) any U.R. is completely reducible.
It sufficies then for the groups under consideration to study the finite-dimensional irreducible U.R.

4. Representations of a Compact Lie Group

4.1. In this section we want to show that in the case of Lie groups, the problem of finding out the irreducible representations is essentially equivalent to that of finding finite sets of operators obeying certain commutation rules, or, more

technically expressed, to find the irreducible representations of the Lie algebra associated to the group.
We restrict our attention to the finite dimensional representations due to the fact that we are interested in irreducible ones.
Let then $g \to T(g)$ be a finite dimensional continuous representation of the compact n-dimensional Lie group G in a vector space L. If $(\alpha_1, \ldots, \alpha_n)$ is a parametrization of a neighbourhood N of the identity e of G, then we have

$$T(g) = T(\alpha_1, \ldots, \alpha_n) \quad g \in N.$$

It can be shown [5] that the operator functions $T = T(\alpha_1, \ldots, \alpha_n)$ are analytic [7]) so that there exist n operators I_k (they will be called infinitesimal generators), defined as

$$I_k = \left(\frac{\partial T}{\partial \alpha_k}\right) \quad \begin{matrix} k = 1, \ldots n \\ \alpha_1 = \alpha_2 = \ldots = 0. \end{matrix}$$

For example if we represent R_3 with 3×3 real orthogonal matrices R_{ij} as seen before, due to the fact that

$$R = R(\theta \cdot \mathbf{n}) = R(\boldsymbol{\alpha}) \quad R \in R_3$$

$$T(R) = e^{\alpha \cdot \Sigma}$$

with Σ_k defined as in sect. 1.7, we have

$$I_k = \Sigma_k.$$

These operators have the commutation rules

$$[I_i, I_j] = \sum_{k=1}^{3} \epsilon_{ijk}\, I_k. \tag{1}$$

In the same way, from eq. (14), for the 2-dimensional representation of SU_2 we find the generators

$$I_k = -i\,\frac{\sigma_k}{2}$$

which satisfy commutation rules identical with (1).
4.2. We will now deduce a differential equation satisfied by the operators $T(\alpha_1, \ldots, \alpha_n)$, connecting them to the I_k [6]. Let g and f belong to G; then for any $x \in L$, we can put

$$y(g^{-1}) = T(g^{-1})x. \tag{2}$$

From 3.1 (1) it follows that

$$T(fg)\, y(g^{-1}) = T(fg)\, T(g^{-1})x = T(f)\, x = y(f)$$

i.e.

$$y(f) = T(fg)\, y(g^{-1}). \tag{3}$$

---

[7]) For this we mean that any matrix element $T_{ij}(\alpha_1, \ldots, \alpha_n)$ is an analytic function of $(\alpha_1, \ldots, \alpha_n)$.

Let us fix now f in N in such a way that $f^{-1} \in N$. If g is in a suitable neighbourhood of f^{-1}, then

$$g \in N$$

$$gf \in N$$

and we can write eq. (3) as:

$$y(\alpha_h(f)) = T(\alpha_i(fg))\, y(\alpha_j(g^{-1})). \tag{4}$$

Taking derivatives of (4) with respect to the parameters $\alpha_l(f)$, we obtain:

$$\frac{\partial y(\alpha_1(f), \ldots \alpha_n(f))}{\partial \alpha_l(f)} = \sum_{k=1}^{n} \frac{T(fg)}{\partial \alpha_k(fg)} \frac{\partial \alpha_k(fg)}{\partial \alpha_l(f)}\, y(g^{-1}). \tag{5}$$

It is important to note that the real functions $S_{kl} = (\partial \alpha_k(fg)/\partial \alpha_l(f))$ depend on f, g, on the group multiplication rule, on the parametrization given to N, but not on the representation $T(g)$. Letting $g \to f^{-1}$ in (5), we obtain the equation

$$\frac{\partial y(\alpha_1, \ldots \alpha_n)}{\partial \alpha_l} = \sum_{k=1}^{n} I_k S_{kl}(\alpha_1, \ldots, \alpha_n)\, y(\alpha_1, \ldots, \alpha_n) \tag{6}$$

together with the boundary condition

$$y(0, \ldots 0) = x$$

or, in an equivalent way

$$\frac{\partial T(\alpha_1, \ldots \alpha_n)}{\partial \alpha_l} = \sum_{k=1}^{n} I_k S_{kl}(\alpha_1, \ldots \alpha_n)\, T(\alpha_1, \ldots \alpha_n) \tag{7}$$

$$T(0, \ldots 0) = 1.$$

We can now demonstrate the following theorem.
If $T_1(g)$ and $T_2(g)$ are representations of a connected Lie group G in the same linear space L, and they have the same infinitesimal generators I_k, then for any $g \in G$, $T_1(g) = T_2(g)$.
In fact $T_1(g)$ and $T_2(g)$ are solutions, in a certain neighbourhood N of the unit element, of the same differential equation (7), with the same boundary condition, so that, for any element $g \in N$ we have $T_1(g) = T_2(g)$. Now in the theory of topological groups it is shown that any element g of a connected group can be expressed as a product of a finite number of elements $g_1, \ldots g_k$ belonging to an arbitrary neighbourhood of the identity: let now N be such a neighbourhood. For any $g \in G$ we have

$$g = g_1 g_2 \ldots g_k \qquad g_i \in N$$

$$T_1(g) = T_1(g_1)\, T_1(g_2) \ldots T_1(g_k) = T_2(g_1) \ldots T_2(g_k) = T_2(g)$$

which proves the theorem.

4.3. Going back to (7), we must have, for any solution $T(g)$

$$\frac{\partial^2 T(g)}{\partial \alpha_k\, \partial \alpha_l} = \frac{\partial^2 T(g)}{\partial \alpha_l\, \partial \alpha_k} \qquad g \in N \tag{8}$$

(integrability conditions). For $g = e$, from (8) it follows [6]

$$[T_k, I_l] = \sum_h C^h_{kl} I_h \tag{9}$$

where the real numbers C^h_{kl} (structure constants) depend upon the derivatives of S_{kl} evaluated in $\alpha_1 = \alpha_2 = \ldots = 0$, i.e. they are independent from the particular representation chosen. By virtue of (9), C^h_{kl} satisfy

$$C^h_{kl} = -C^h_{lk}$$

$$\sum_h (C^h_{sl} C^k_{hi} + C^h_{is} C^k_{hl} + C^h_{li} C^k_{hs}) = 0. \tag{10}$$

Consider now a real vector space S of a dimension n equal to the dimension of the Lie group G, and let $\{\lambda_k\}$ $(k = 1 \ldots n)$ be a basis in S. With the aid of C^h_{kl} a composition rule in S can be defined as follows:

$$(\lambda_i, \lambda_k) \to [\lambda_i, \lambda_k] = (\text{by definition}) = \sum_k C^h_{ik} \lambda_h. \tag{11}$$

If

$$x = \sum_k x_k \lambda_k$$

and

$$y = \sum_i y_i \lambda_i \qquad x, y \in S$$

we define:

$$(x, y) \to [x, y] = \sum_{i,k} x_k y_i [\lambda_k, \lambda_i] = $$
$$= \sum_s \Big(\sum_{i,k} x_k y_i C^s_{ki}\Big) \lambda_s.$$

Due to (10), this multiplication rule has the properties

$$[x, y] = -[y, x] \qquad y, x \in S$$

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \text{ (Jacobi identity)}$$

which are analogous to the usual properties of the commutator of two operators. The vector space S equipped with the composition law just defined is called the Lie algebra $\Lambda(G)$ associated to the group G. It seems that this definition depends (through C^l_{ki}) upon the particular parametrization of N. However if we make an analytic change of variables in N (i.e. if $g \in N$ and $g \equiv (\alpha'_1, \ldots \alpha'_n)$, then

$$\alpha'_i = \alpha'_i(\alpha_1, \ldots \alpha_n)$$

are invertible analytic functions in all the arguments), we obtain a set of new structure constants C'^h_{ik}, which are related to C^s_{ql} through a non singular matrix a_{ij} in the following way:

$$C'^h_{ki} = \sum_{q,l,S} a_{kq} a_{il} C^s_{ql} (a^{-1})_{sh}$$

so that it is possible to find in $\Lambda(G)$ a set of n linearly independent vectors $\lambda'_k = \sum_h a_{kh} \lambda_h$ such that

$$[\lambda'_k, \lambda'_i] = \sum_h C'^h_{ki} \lambda'_h .$$

We see then that to a change of parametrization in N, there corresponds in $\Lambda(G)$ only a change of basis, so that in fact $\Lambda(G)$, is uniquely determined by G. We can at this point introduce independently from the group G the concept of representation of the Lie algebra $\Lambda(G)$. By this we mean a mapping of $\Lambda(G)$ into a set of linear operators defined in a vector space L

$$x \to A(x)$$

such that

i) $A(\alpha x + \beta y) = \alpha A(x) + \beta A(y)$ (linearity)

ii) $A([x, y]) = [A(x), A(y)]$

where now $[A, B]$ means the commutator of A and B.
We then see that starting from a representation of G

$$g \to T(g)$$

its infinitesimal generators can be thought as a representation of a basis in $\Lambda(G)$, which extends by linearity to a representation of $\Lambda(G)$. The usefulness of introducing $\Lambda(G)$ is that the converse is also essentially true, in a way to be explained below.
Let then $\{\lambda_k\}$ be a basis in $\Lambda(G)$ and let

$$\lambda_k \to A_k$$

$$[A_i, A_k] = \sum_h C^h_{ik} A_h \tag{12}$$

in a finite dimensional representation of $\Lambda(G)$.
C^h_{ik} are the structure constants associated to G through a given parametrization of N. Let us consider the differential equation

$$\frac{\partial T(\alpha_1, \ldots, \alpha_n)}{\partial \alpha_k} = \sum_h S_{hk}(\alpha_1, \ldots, \alpha_n) A_h T(\alpha_1, \ldots, \alpha_n) \tag{13}$$

$$T(0, \ldots 0) = 1 .$$

The integrability conditions of (13), can be expressed in terms of the $S_{hk}(\alpha_1, \ldots \alpha_n)$ and it can be shown ([1] cap. IX) that they are satisfied in a neighbourhood of the point $(0, \ldots 0)$ due to the definition of S_{hk} (see eq. (6)) and to eq. (12), which is the form that the integrability condition assumes in the point $(0, \ldots 0)$. Hence (13) is solvable in a suitable neighbourhood N' of the point $(0, \ldots 0)$, giving us a correspondence between the elements $g \in G$, contained in a neighbourhood of the unit element, and linear operators $T(g)$ in L.
It may be verified, in a rather cumbersome way, that if $g, g', gg' \in N'$, then

$$T(gg') = T(g)\, T(g'),$$

and for this we refer the reader to [6].

Let now g be an arbitrary element of G; by a result quoted previously, if G is connected, we can express g as a product of elements of N':

$$g = g_1 \ldots g_k \quad g_1, g_2, \ldots, g_k \in N' .$$

It would be tempting at this point to define

$$T(g) = T(g_1) \ldots T(g_k),$$

obtaining in this way a representation of the full group in L (it is obvious that this definition satisfies 3.1 (1)). However, if

$$g = g_1 \ldots g_k = g'_1 \ldots g'_{k'}, \quad g_i, g'_i \in N'$$

we are not sure that

$$T(g_1)\, T(g_2) \ldots T(g_k) = T(g'_1) \ldots T(g'_{k'}), \tag{14}$$

and in fact there are many cases in which they differ.
As an example of this we may consider the group G of the rotations of E_3 around the z axis. We parametrize this group with the values assumed by the rotation angle $\theta(-\pi \leq \theta \leq \pi)$.
To the points $\theta = \pm\pi$ it corresponds a unique element so that they must be identified. If $g(\theta_1), g(\theta_2)$ are elements of a neighbourhood N of the identity, and $g(\theta_1) \cdot g(\theta_2) \in N$, then

$$g(\theta_1)\, g(\theta_2) = g(\theta_1 + \theta_2) .$$

This gives to G a structure of Lie group, and in addition G is compact and connected. This group is one dimensional and so is $\Lambda(G)$; hence any operator A on a linear space L, determines a representation of $\Lambda(G)$.
The simplest example at hand is the representation of $\Lambda(G)$ over the one dimensional complex linear space. Consider the linear operator ik (i.e. the operator that multiplies by ik, k real number). We take ik as the representative of the generator of $\Lambda(G)$, and equation (13) reads

$$\frac{\partial T(\theta)}{\partial \theta} = ikT(\theta)$$

$$T(\theta) = 1$$

$$T(\theta) = e^{ik\theta}$$

where θ belongs to a suitable neighbourhood N of $\theta = 0$. Let now $g(\theta)$ be an element of G. There exists an integer n such that θ/n belongs to this neighbourhood, and we can define:

$$T(g(\theta)) = T(g(\theta/n))^n = e^{ik\theta} .$$

However the element $\bar{g}$ which corresponds to $\theta = \pm\pi$ can be written (with a suitable n) as

$$g\left(\frac{\pi}{n}\right)^n \text{ and } g\left(-\frac{\pi}{n}\right)^n$$

with $g(\pi/n)$ and $g(-\pi/n)$ belonging to N; but now we have

$$T\left(g\left(\frac{\pi}{n}\right)\right)^n = e^{ik\pi} \neq T\left(g\left(-\frac{\pi}{n}\right)\right)^n = e^{-ik\pi}.$$

The failure of this method is due to the fact that G is not simply connected. For simply connected groups instead eq. (14) is fulfilled (see [7]).
In any case given a connected Lie groups G, with a Lie algebra $\Lambda(G)$, it is possible in an essentially unique way, to construct a connected Lie group $\tilde{G}$, having the same Lie algebra $\Lambda(G)$, which is in addition simply connected [1].
From this it follows the existence in $\tilde{G}$ and G of two neighbourhoods $\tilde{N}$ and N of the unit elements which are in a one-to-one bicontinuous correspondence, in such a way that if

$$\tilde{g}, \tilde{f}, \tilde{g}\cdot\tilde{f} \in N$$

$$g, f \in N$$

then

$$\tilde{g} \leftrightarrow g, \tilde{f} \leftrightarrow f$$

$$gf \in N \text{ and } \tilde{g}\tilde{f} \leftrightarrow gf.$$

This "local" isomorphism, since $\tilde{G}$ is simply connected, can be extended to a homomorphism φ of $\tilde{G}$ onto G ([1] cap. VIII). $\tilde{G}$ is called the universal covering group of G.
When G is simply connected this homomorphism reduces to an isomorphism, i.e. G and $\tilde{G}$ are essentially the same group.
At this point it should be clear that from a representation of the Lie algebra $\Lambda(G)$ of a connected Lie group G, we can construct a representation of the universal covering group of G. Let us see how to sort out from the representations of $\tilde{G}$, representations of G.
If $\tilde{g} \to T(\tilde{g})$ is a representation of $\tilde{G}$ we consider the set of those operators which correspond to the kernel K_φ of the homomorphism (see 1.3) $\tilde{G} \to G$. If this set reduces to the identity operator, then $T(\tilde{g}) = T(\tilde{f})$ when $\tilde{g}$ and $\tilde{f}$ belong to the same K_φ coset, i.e. the function $T(\tilde{g})$ is constant over each K_φ coset. Hence to any element $\tilde{g}K_\varphi$ of $\tilde{G}/K_\varphi$ it corresponds a linear operator

$$T(\tilde{g}K_\varphi) = T(\tilde{g})$$

in a way that preserves the associativity of the multiplication law in $\tilde{G}/K_\varphi$. This correspondence is then a representation of $\tilde{G}/K_\varphi$ in L and also, being $\tilde{G}/K_\varphi$ isomorphic to G, a representation of G in L.
Summarizing we can say that:
i) if G is simply connected, a representation of $\Lambda(G)$ in L determines uniquely a representation of G;
ii) if G is not simply connected, a representation of $\Lambda(G)$ in L determines a representation of $\tilde{G}$ which reduces to a representation of G if and only if the kernel K_φ of the homomorphism $\tilde{G} \to G$ is mapped into the identity operator. To exemplify let us consider again the case of R_3 and SU_2. We have seen that the infinitesimal generators of SU_2 verify the same commutation rules as the generators of R_3, i.e. they have the same Lie algebra.

In addition SU_2 is simply connected, so that SU_2 is the covering group of R_3. In the section 1.5b, the homomorphism of SU_2 onto R_3 has been proved, together with the fact that the kernel K_φ is the set of the two matrices ± 1. We will see later that any irreducible representation of $\Lambda(SU)_2$ is uniquely determined by an integer or half integer non negative number j, in such a way that its dimension is $2j+1$. When j is an integer the elements ± 1 are mapped into unity, so that these representations are in fact representations of R_3.
4.4. Quite independently from Lie groups, we can define a real, n-dimensional Lie algebra as a set $\mathscr{L}$ of elements such that:
i) $\mathscr{L}$ is a real n-dimensional linear space;
ii) there exists in $\mathscr{L}$ a composition law indicated as $x, y \to [x, y]$ linear in x and y, antisymmetric, satisfying Jacobi's identity (see sect. 4.3.).
As before we define a representation of $\mathscr{L}$ in a linear space L to be a mapping $x \to A(x)$ of the elements of $\mathscr{L}$ into linear operators of L satisfying conditions:

a) $A(\alpha x + \beta y) = \alpha A(x) + \beta A(y)$;

b) $A([x, y])$ = commutator of $A(x)$ and $A(y) = [A(x), A(y)]$.

Of particular importance is the so-called regular representation of $\mathscr{L}$. In this representation $\mathscr{L}$ plays a double role in that it is the linear space in which the representation is contructed and at the same time it supplies the element to be represented

$$x \in \mathscr{L}$$

$$x \to \text{ad}(x)$$

where $\text{ad}(x)$ is the linear mapping of $\mathscr{L}$ into itself (qua vector space) defined as

$$\text{ad}(x)y = [x, y].$$

Furthermore

$$\text{ad}(\alpha x + \beta z) = \alpha\, \text{ad}(x) + \beta\, \text{ad}(z)$$

$$[\text{ad}(x), \text{ad}(z)] = \text{ad}([x, z])$$

by virtue of Jacobi identity. Hence the correspondence

$$x \to \text{ad}(x)$$

is in fact a representation of $\mathscr{L}$.
An invariant subspace for the regular representation is called an ideal $\mathscr{I}$:

if

$$y \in \mathscr{I}$$

$$\text{ad}(x)y = [x, y] \in \mathscr{I} \text{ for any } x \in \mathscr{L}.$$

In particular $\mathscr{I}$ is abelian when $x \in \mathscr{I}$ $y \in \mathscr{I}$ implies $[x, y] = 0$.
Presence or absence of ideals has extremely important consequences for the structure of the Lie algebra itself and Lie algebras are divided in three classes accordingly:

i) simple Lie algebra: no ideals other than $\mathcal{L}$ and zero;
ii) semisimple Lie algebras: no abelian ideals other than zero;
iii) all the other Lie algebras.

Whereas for the first two classes there is a complete mathematical theory, the third one is not as well assessed till now.
A subalgebra of $\mathcal{L}$ is a linear subspace l such that

$$x, y \in l$$

implies

$$[x, y] \in l;$$

if in addition for any $x, y \in l$

$$[x, y] = 0,$$

we say l to be an abelian subalgebra.
Finally we mention that for any Lie algebra $\mathcal{L}$ there exists always a connected, simply connected Lie group $\tilde{G}$, such that

$$\Lambda(\tilde{G}) = \mathcal{L}.$$

Hence it is equivalent to speak of Lie algebras or of connected simply connected Lie groups. Groups related to simple (semisimple) Lie algebras, are called in turn simple (semisimple). In addition the correspondence between Lie groups and Lie algebras is such that:

there is in G		there is in $\mathcal{L}$
subgroup	$\leftrightarrow$	subalgebra
abelian subgroup	$\leftrightarrow$	abelian subalgebra
invariant subgroup	$\leftrightarrow$	ideal
abelian invariant subgroup	$\leftrightarrow$	abelian ideal

We introduce another useful concept: a subalgebra $\mathcal{C}$ is called a Cartan subalgebra, if it has the properties:
a) $\mathcal{C}$ is a maximal abelian subalgebra, i.e. there exists no other abelian subalgebra containing $\mathcal{C}$;
b) if $h \in \mathcal{C}$, then in any representation of $\mathcal{C}$ over a complex linear space, $A(h)$ is a diagonalizable operator.
For semisimple Lie algebras associated to compact Lie groups, i.e. those algebras we will use later in physical applications, one can show that any $\mathcal{L} = 0$ admits non zero Cartan subalgebras and that all Cartan subalgebras of $\mathcal{L}$ have the same dimensionality. The common dimensionality is called the rank of $\mathcal{L}$.
Finally we make a remark concerning the characterization of the connected Lie groups admitting the same Lie algebra $\mathcal{L}$. The simply connected group $\tilde{G}$ uniquely identified by $\mathcal{L}$ must be of course the covering group of all them. Hence for each of them there is a particular homomorphism $\tilde{G} \underset{\varphi}{\rightarrow} G$, so that G is isomorphic to the group $\tilde{G}/K_\varphi$. The essential feature of K_φ are:
a) K_φ is a central subgroup, i.e. its elements commute with all elements of $\tilde{G}$;
b) K_φ is a discrete subgroup, i.e. K_φ is made up with isolated elements (in $\tilde{G}$). In particular if $\tilde{G}$ is compact, K_φ has a finite number of elements.

Hence given $\tilde{G}$, one has to identify all its central discrete subgroups, which is a relatively simple task. Then by making the corresponding quotient groups, one find all the required groups.
For example see the case of R_3 and SU_2. ± 1 is the only discrete central subgroup of SU_2, so that R_3 is the only non simply connected group having the same Lie algebra as SU_2.

5. Kronecker Product of Representations

5.1. Let L and L' be two linear spaces, respectively of dimensions n and n'. Consider the set $L \otimes L'$ of all formal sums[8])

$$|x\rangle = \sum_{mm'} c_{mm'} |m\rangle\, |m'\rangle \qquad \begin{matrix} m = 1 \ldots n \\ m' = 1 \ldots n' \end{matrix}$$

where $|m\rangle$ and $|m'\rangle$ are bases in L and L' and $c_{mm'}$, are arbitrary complex numbers. Defining linear combinations of elements $x, y \in L \otimes L'$ as

$$\alpha |x\rangle + \beta |y\rangle = \sum_{m,m'} (\alpha c_{mm'} + \beta b_{mm'})\, |m\rangle\, |m'\rangle$$

($b_{mm'}$ are the coefficients pertaining to y).
$L \otimes L'$ acquires a structure of linear space (of $n \cdot n'$ dimensions) and will be called the tensor or Kronecker product of L times L'.
In $L \otimes L'$ a scalar product can be defined as

$$(x, y) = \sum_{m,m'} c_{mm'}\, \bar{b}_{mm'}.$$

If now $g \to T(g)$ and $g \to T'(g)$ are two representations of the same group G in L and L', a new representation of G in $L \otimes L'$ (indicated as $T \otimes T'$) can be defined as follows:

$$g \to T(g) \otimes T'(g)$$

where

$$(T(g) \otimes T'(g))\, |m\rangle\, |m'\rangle = \sum_{ss'} T_{sm}(g)\, T'_{s'm'}(g)\, |s\rangle\, |s'\rangle$$

and for an arbitrary vector $|x\rangle$:

$$(T(g) \otimes T'(g))\, |x\rangle = \sum_{mm'} c_{mm'}\, (T(g) \otimes T'(g))\, |m\rangle\, |m'\rangle =$$

$$= \sum_{ss'} \Big(\sum_{mm'} c_{mm'}\, T_{ms}(g)\, T'_{m's'}(g) \Big)\, |s\rangle\, |s'\rangle.$$

Hence we see that each element x of $L \otimes L'$ is uniquely determined by a set of $n \cdot n'$ complex numbers $c_{mm'}$, and under the representation $T \otimes T'$ these components transform as

$$c'_{mm'} = \sum_{ee'} T_{em}\, T'_{e'm'}\, c_{ee'}.$$

---

[8]) Where it is convenient we use for vectors the Dirac notation: $|m\rangle$ for e_m (sect. 2.2).

It is easy to see that $T \otimes T'$ is unitary when T and T' are; but even in the case that T and T' are irreducible, $T \otimes T'$ is not so. However, according to the general statements in section (3.4b—c) being $T \otimes T'$ unitary, it will be fully reducible: the space $L \otimes L'$ can be written as a direct sum of invariant subspaces $l_{i,\alpha}$ which transform according to irreducible representations $T_{i,\alpha}(g)$[9]:

$$L \otimes L' = \bigoplus_{i,\alpha} l_{i,\alpha}$$

$$T \otimes T' = \bigoplus_{i,\alpha} T_{i,\alpha} \qquad i = 1 \dots K.$$

Let us choose in each $l_{i,\alpha}$ an orthonormal basis of vectors indicated as

$$|r; i, \alpha\rangle$$

where the index r labels the basis vectors of $l_{i,\alpha}$. Collecting all these vectors we obtain an orthonormal basis in $L \otimes L$; hence there exists a unitary matrix

$$C(m, m'; r, i, \alpha)$$

connecting this basis to that previously introduced

$$|m\rangle\,|m'\rangle = \sum_{r,i,\alpha} C(m, m'; r, i, \alpha)\,|r; i. \alpha\rangle$$

$$|r; i, \alpha\rangle = \sum_{mm'} \bar{C}(m, m'; r, i, \alpha)\,|m\rangle\,|m'\rangle.$$

The quantities $C(m, m'; r, i, \alpha)$ are called Clebsch-Gordan coefficients of the group G.
We observe that due the fact that $l_{i\alpha}$ and $l_{i\beta}$ transform according to equivalent representations (sect. 3.3a), their elements can be put in a linear one-to-one correspondence, so that we can choose vectors $|r, i\alpha\rangle$, $|s, i\,\beta\rangle$ in such a way that indices r and s run over the same range. In addition these vectors can be choosen so to satisfy:

$$\langle i\alpha r'|T(g)|i\alpha r\rangle = \langle i\alpha r'|T_{i\alpha}(g)|i\alpha r\rangle = \langle i\beta r'|T_{i\beta}(g)|i\beta r\rangle = \langle i\beta r'|T(g)|i\beta r\rangle.$$

We will always refer to a basis selected in this way whenever we will have to deal with reducible representations of a group G, calling it the standard basis.

6. Schur's Lemma and Wigner-Eckart Theorem

6.1. Schur's lemma

i) Let $T_1(g)$ and $T_2(g)$ be two irreducible, inequivalent, finite dimensional representations of a group G in the linear spaces L_1 and L_2.

[9]) In general there will be several irreducible subspaces transforming according the same irreducible representation. These subspaces are distinguished by the additional label α, whereas i distinguishes between group of subspaces transforming according inequivalent representations.

Any linear operator A mapping L_1 into L_2, such that

$$T_2(g)\,A = A\,T_1(g) \quad \text{for any} \quad y \in G$$

is the null operator.
Proof: Let N_A be the set of vectors in L_1 such that

$$A x = 0 \quad x \in N_A.$$

N_A is an invariant subspace for $T_1(g)$. In fact if $x \in N_A$, then

$$A\,T_1(g) x = T_2(g) A x = 0$$

i.e.

$$T_1(g) x \in N_A \quad \text{for any} \quad g \in G.$$

Then $N_A = L_1$ or $N_A = 0$. In the first case the theorem is proved. In the second case let us call R_A the image of L_1 into L_2. R_A is invariant for $T_2(g)$:

$$y \in R_A$$

i.e.

$$y = A x$$

$$T_2(g) y = T_2(g) A x = A\,T_1(g) x$$

i.e.

$$T_2(g) y \in R_A \quad \text{when} \quad y \in R_A.$$

Hence $R_A = 0$ or $R_A = L_2$. The second case is excluded being T_1 and T_2 inequivalent, which proves the theorem.
This theorem can be extended to infinite dimensional representations, provided A is a bounded operator.
ii) If T_1 and $T_2(g)$ are irreducible, equivalent, finite dimensional representations of G in the complex vector spaces L_1 and L_2, i.e. there exists a one-to-one mapping U of L_1 into L_2 such that $T_2 U = U T_1$, then any linear operator A mapping L_1 into L_2 and satisfying

$$T_1 A = A T_2$$

is a multiple of U: $A = \lambda U$.
Proof: From

$$T_2(g) A = A\,T_1(g) \quad \text{and}$$

$$U^{-1} T_2(g)\,U = T_1(g) \quad \text{it follows}$$

$$T_2(g)\,A = A\,U^{-1} T_2(g)\,U$$

i.e.

$$T_2(g) A\,U^{-1} = A\,U^{-1} T_2(g)$$

so that the theorem is proved if we show that any operator A' which commutes with all the operators of an irreducible representation of G is a multiple of the unit element (in fact if this is true we have $A\,U^{-1} = \lambda I$ i.e. $A = \lambda U$).
Any operator A' has at least an eigenvector $x \neq 0$ belonging to some eigenvalue λ:

$$A' x = \lambda x.$$

Let V be the linear manifold spanned by the vectors belonging to this eigenvalue ($V \neq 0$). V is invariant under $T_2(g)$ in fact:

$$x \in V$$

implies

$$A' T_2(g)x = T_2(g)A'x = \lambda T_2(g)x$$

i.e.

$$T_2(g)x \in V .$$

But $V \neq 0$, and T_2 is irreducible. Hence it follows $V = L_2$, i.e.

$$A' = \lambda I .$$

As before, the theorem is true for any bounded operator A' commuting with the operators $T(g)$ of an unitary irreducible representation of G in any Hilbert space.
6.2. We use the results of sect. 6.1 to determine the structure of an operator T, which is invariant under an arbitrary unitary representation of a group G.
In particular we will determine the form of its matrix elements, with respect to a fixed basis.
To be definite let $g \to U(g)$ be an unitary representation of G in a Hilbert space $\mathscr{H}$. We require it to be completely reducible, which is always the case when G is compact or finite. Let T be an operator mapping $\mathscr{H}$ into itself such that:

$$T\,U(g) = U(g)T \quad \text{for any} \quad g \in G . \tag{1}$$

$\mathscr{H}$ can be decomposed (sect. 5.1) into a direct sum of invariant irreducible subspaces:

$$\mathscr{H} = \bigoplus_\alpha (\bigoplus_i l_{i\alpha}) = \bigoplus_{i\alpha} l_{i\alpha} . \tag{2}$$

For any vector $\Phi_{i\alpha} \in l_{i\alpha}$, we have

$$U(g)\,\Phi_{i\alpha} \in l_{i\alpha}$$

so that we may define an operator $U^{i\alpha}(g)$ mapping $l_{i\alpha}$ into itself as

$$U^{(i\alpha)}(g)\,\Phi_{i\alpha} = U(g)\,\Phi_{i\alpha} .$$

$U^{(i\alpha)}(g)$, called the restriction of $U(g)$ to $l_{i\alpha}$, by hypothesis constitute an irreducible representation of G specified by the labels $i\alpha$

$$(U^{(i\alpha)}(g) \sim U^{(i\beta)}(g)) .$$

Consider now the vector $T\Phi_{j\alpha}(\Phi_{i\alpha} \in l_{i\alpha})$. By (2) we can uniquely write

$$T\,\Phi_{i\alpha} = \sum_{j\beta} \psi_{j\beta}\, (\psi_{j\beta} \in l_{j\beta}) \tag{3}$$

and define the operators $T^{(i\alpha)}_{(j\beta)}$, mapping $l_{i\alpha}$ into $l_{j\beta}$ as

$$T^{(i\alpha)}_{(j\beta)}\,\Phi_{i\alpha} = \psi_{j\beta}; \quad T\,\Phi_{i\alpha} = \sum_{j\beta} T^{(i\alpha)}_{(j\beta)}\,\Phi_{i\alpha} . \tag{4}$$

We use now (1), which gives:

$$U\,T\,\Phi_{i\alpha} = \sum_{j\beta} \psi\, U_{j\beta} = \sum_{i\beta} U^{(j\beta)}\,\psi_{j\beta} = \sum_{j\beta} U^{(j\beta)}\, T^{(i\alpha)}_{(j\beta)}\,\Phi_{i\alpha} = \sum_{j\beta} \psi'_{j\beta} =$$

$$= T\,U\Phi_{i\alpha} = T\,U^{(i\alpha)}\,\Phi_{i\alpha} = \sum_{j\beta} T^{(i\alpha)}_{(j\beta)}\, U^{(i\alpha)}\,\Phi_{i\alpha} = \sum_{j\beta} \psi''_{j\beta} .$$

By definition of direct sum, we then obtain:

$$\psi'_{j\beta} = \psi''_{j\beta}$$

i.e.

$$U^{(j\beta)}\, T^{(i\alpha)}_{(j\beta)} = T^{(i\alpha)}_{(j\beta)}\, U^{(i\alpha)} . \tag{5}$$

$U^{(i\alpha)}$, $U^{(j\beta)}$ are irreducible representations of G on $l_{i\alpha}$ and $l_{j\beta}$, so that we can apply Schur's lemma to the operator $T^{(i\alpha)}_{(j\beta)}$ concluding that:

a) $T^{(i\alpha)}_{(j\beta)} \equiv 0$ when $i \neq j$
b) $T^{(i\alpha)}_{(i\beta)} = \lambda(i\alpha\beta)\, V^{i\alpha}_{i\beta}$ where $\lambda(i\alpha\beta)$ depends upon i, α, β and $V^{i\alpha}_{i\beta}$ is a fixed operator mapping $l_{i\alpha}$ into $l_{i\beta}$, such that

$$U^{i\beta}\, V^{i\alpha}_{i\beta} = V^{i\alpha}_{i\beta}\, U^{i\alpha} . \tag{6}$$

In particular, choosing inside each $l_{i\alpha}$ a standard basis $\{\Phi^{i\alpha}_r\}$ as in sect. 5.1, the operator defined as

$$V^{i\alpha}_{i\beta}\,\Phi^{i\alpha}_r = \Phi^{i\beta}_r$$

satisfies (6).
In conclusion, using a), b), (3) we see that:

$$(\Phi^{j\beta}_s, T\Phi^{i\alpha}_r) = \lambda(i, \alpha\beta)\, \delta_{ij}\delta_{rs} . \tag{7}$$

6.3 Wigner-Eckart theorem

This theorem, which is valid for any compact group, can be seen as a generalization of the preceeding statements on matrix elements of invariant operators. Let $g \to U(g)$ be an unitary representation of G into the Hilbert space $\mathscr{H}$. Suppose we have a finite number of operators T_k such that

$$U(g)\, T_k\, U(g^{-1}) = (D^j(g))_{k'k} T_{k'}$$

where $D^j(g)$ is a matrix of the irreducible representation of G labeled by j. Operators of this kind are called irreducible tensor operators transforming as the representation j.
Consider now a decomposition of $\mathscr{H}$ into irreducible subspaces $l_{i\alpha}$, and a standard basis $\{\Phi^{i\alpha}_r\}$.
Wigner-Eckart theorem states:
Given a set of irreducible operators T^l_k transforming as the representation of G specified by l, then:

i) the matrix element:

$$(T^l_k\, \Phi^{i\alpha}_r, \Phi^{j\beta}_s) \tag{8}$$

vanishes whenever the representation j is not contained in the Kronecker product of the representations l and i.

ii) when the representation j is contained in this tensor product, then:

$$(T^l_k\, \Phi^{i\alpha}_r, \Phi^{j\beta}_s) = \sum_\gamma C(i\, r, l k; j s \gamma)\, \langle i\alpha || T^l || j\beta\rangle_\gamma \tag{9}$$

where $C(ir, lk; js\gamma)$ are Clebsch-Gordan coefficients which project the vector $\Phi^i_r \Phi^l_k$ of the tensor product of the representations i and l into the vector $\Phi^{j\gamma}_s$ transforming as the γ^{th} irrducible component equivalent to the representation j. $\langle i\alpha || T^l || j\beta\rangle_\gamma$ is a symbolic way of writing a number which depends no more upon the "magnetic" quantum numbers k, r, s, and is called a reduced matrix element of T^l_k. In this way the dependence of (8) upon the magnetic quantum numbers is lumped into Clebsch-Gordan coefficients, i.e. is the same for any set of irreducible operators transforming in a fixed way, and here is the main importance of the theorem.

The number of terms appearing in (9) is simply the number of times the representation j appears in the tensor product of representations i and l.

6.4. We want to visualize the important results obtained in last two sections with a simple example.

Let us consider a representation of SU_2 which is the direct sum of two irreducible representations of $j = 1, 1/2$.

In this case:

$$H = l_1 \oplus l_{1/2} \qquad \text{(no need for any index like } \alpha)$$

$$U(g) = U^1(g) \oplus U^{1/2}(g).$$

A standard basis is one in which the third generator of SU_2 is diagonal: vectors of this basis will be indicated as

$$|j, m_j\rangle \qquad j = 1, {}^1/_2$$

$$m_j = \begin{cases} +1,\ 0,\ -1 & j = 1 \\ {}^1/_2,\quad -{}^1/_2 & j = {}^1/_2. \end{cases}$$

Consider a set of two operators $T^{1/2}_i (i = \pm{}^1/_2)$ transforming as the $j = {}^1/_2$ representation, i.e.

$$U(g)\, T^{1/2}_i\, U(g)^{-1} = \sum_k (U^{1/2}(g))_{ki}\, T^{1/2}_k, \quad g \in SU_2.$$

Let us find the structure of matrix elements of $T^{1/2}_i$, using Wigner-Eckart theorem. From (35) we have

$$\langle j', m'_{j'} \mid T^{1/2}_i \mid j, m_j\rangle = C^{j\,{}^1/_2\,j'}_{m_j\,i\,m'_{j'}}\, \lambda(j, j').$$

The decomposition of tensor products involved here is as follows:

$$U^{1/2} \otimes U^{1/2} = U^1 \oplus U^0$$

$$U^1 \otimes U^{1/2} = U^{3/2} \oplus U^{1/2}.$$

Hence when $j = 1$ the matrix elements vanish unless $j' = {}^1/_2$ and viceversa. Non vanishing matrix elements are:

$$\langle 1\, m' \,| T^{1/2}_i |\, {}^1/_2 m\rangle = C^{{}^1/_2\,{}^1/_2\,1}_{m\,i\,m'}\, \lambda(1, {}^1/_2)$$

$$\langle {}^1/_2\, m' \,| T^{1/2}_i |\, 1\, m\rangle = C^{1\,{}^1/_2\,{}^1/_2}_{m\,j\,m'}\, \lambda({}^1/_2, 1).$$

Using Clebsch-Gordan coefficients of CONDON and SHORTLEY we find the following structure:

$$T^{1/2}_{1/2}: \quad \begin{pmatrix} 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \frac{\lambda}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\mu}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{2}{3}}\mu & 0 & 0 \end{pmatrix} \begin{matrix} m & j \\ 1 & \\ 0 & 1 \\ -1 & \\ {}^1/_2 & \\ -{}^1/_2 & {}^1/_2 \end{matrix}$$

$$m: \ \underbrace{1 \quad 0 \quad -1}_{j:\ 1} \quad \underbrace{{}^1/_2 \quad -{}^1/_2}_{{}^1/_2}$$

$$T^{1/2}_{-1/2}: \quad \begin{matrix} 1\ \begin{cases} 1 \\ 0 \\ -1 \end{cases} \\ {}^1/_2\ \begin{cases} {}^1/_2 \\ -{}^1/_2 \end{cases} \end{matrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & \lambda \\ \sqrt{\frac{2}{3}}\mu & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu}{\sqrt{3}} & 0 & 0 & 0 \end{pmatrix}$$

$$\underbrace{1 \quad 0 \quad -1}_{1} \quad \underbrace{{}^1/_2 \quad -{}^1/_2}_{{}^1/_2}$$

(we set $\lambda(1\ {}^1/_2) = \lambda$, $\lambda({}^1/_2, 1) = \mu$).

Finally we consider the case of an invariant operator, i.e. an operator T such that:

$$U(g)\, T\, U(g^{-1}) = T.$$

In this case, using formulas of sect. 6.2, we find the following structure:

$$\begin{matrix} 1\ \begin{cases} 1 \\ 0 \\ -1 \end{cases} \\ {}^1/_2\ \begin{cases} {}^1/_2 \\ -{}^1/_2 \end{cases} \end{matrix} \left(\begin{array}{ccc|cc} \lambda^1 & 0 & 0 & & \\ 0 & \lambda^1 & 0 & & 0 \\ 0 & 0 & \lambda^1 & & \\ \hline & & & \lambda^{1/2} & 0 \\ & 0 & & 0 & \lambda^{1/2} \end{array}\right)$$

$$\underbrace{1 \quad 0 \quad -1}_{1} \quad \underbrace{{}^1/_2 \quad -{}^1/_2}_{{}^1/_2}$$

7. Symmetry Principles in Elementary Particles Physics

7.1. In the development of Quantum Physics increasing attention has been given, at various stages, to invariance principles. Particularly, in the physics of elementary particles the systematic inquiry and exploitation of the so called internal symmetries led, in the last few years, to remarkable successes in contrast to the unsatisfactory status of dynamical calculations. It appears that, due to peculiar features of the quantum mechanical description, the theory of group representations provides the natural device to handle such invariance principles.

Physical laws express correlations among observable events. The latter of course take place in space and time so that to specify any of them we need certain coordinates with respect to a fixed space-time frame of reference. In addition we need some other "internal parameters" (such as charge, baryonic number etc.) which uniquely determine the nature of the objects under consideration.

Physical laws are then functional relations among sets of such "coordinates". Loosely speaking a symmetry is a transformation on the coordinates which leaves invariant these relations.

There are invariance principles which we believe to be valid for any kind of phenomena, and these are: invariance under translations in space and time and spatial rotations. As stressed by WIGNER [8], the mere possibility of comparing results of experiments made in different places and at different times (i.e. the reproducibility of phenomena) is based on this assumption.

At the same level of universality we accept the assumption that the physical laws are the same in all frames of reference differing for an uniform rectilinear motion. All this is summarized in the statement that Physics is invariant under inhomogeneous proper Lorentz transformations.

There are many other symmetries in elementary particle physics which are shared only by certain kinds of processes (as for example is the case for the SU_2 or isotopic spin symmetry which is valid only for strong interactions). We postpone the study of these topics to a brief sketch of Wigner's analysis of relativistic invariance in quantum mechanics.

7.2. In the formalism of quantum mechanics, there is a normalized vector ψ in a Hilbert space $\mathscr{H}$, corresponding to any physical situation we can set up in laboratory. The normalization condition determines ψ only up to a phase factor, so that what is really relevant is a set Ψ of vectors different from one another by phase factors. Ψ is called a unit in $\mathscr{H}$. There is in addition a self-adjoint operator A corresponding to any observable quantity a and the connection between theory and experiments is contained in the statement that the average value of a in the situation Ψ is

$$m_{\Psi}(a) = (A\psi, \psi)$$

where ψ is an arbitrary vector of the ray Ψ. As it is well known, any physical quantity can be written in terms of expressions like:

$$|(\psi, \varphi)|^2$$

and these depend only upon the rays Ψ, Φ to which ψ, φ belong.

It is an experimental fact that physically realizable states always correspond to definite values of charge, baryonic number (N) and leptonic number (l). This has the consequence that a vector in $\mathscr{H}$ which is a superposition of two states with different eigenvalues of Q or N or l cannot correspond to a physically realizable situation. Hence $\mathscr{H}$ breaks up into subspaces called coherent sectors such that the superposition principle holds only within each coherent sector.

Such phenomenon is the manifestation of the so-called superselection rules [9, 10]. Consider now a system prepared in a certain state Ψ. If g is an element of the proper inhomogeneous Lorentz group $P_+^{\nearrow}$ we can apply this transformation to the instruments with which we have prepared Ψ, obtaining a new physical situation of the same system, described by a ray Ψ^g. As stated by Wigner, the theory is relativistically invariant if the following requirements are satisfied.

i) The mapping $\Psi \to \Psi^g$ is one-to-one, and maps each coherent sector into a coherent sector;

ii) if $\Psi \to \Psi^g$, $\Phi \to \Phi^g$, then:

$$|(\psi, \varphi)| = |(\psi^g, \varphi^g)|$$

where ψ, φ, ψ^g, φ^g are vectors belonging to Ψ, Φ, Ψ^g, Φ^g;

iii) $(\Psi^g)^{g'} = \Psi^{(gg')}$ and

$$\text{when } g' \to g \quad \Psi^{g'} \to \Psi^g.$$

The existence of Ψ^g for any Ψ and g is equivalent to what is called the homogeneity of Minkowski space-time, and combined with i) makes us sure that all physical operations possible in a given frame, are possible in any other frame connected to it by a Lorentz transformation. Condition ii) tells us that the connections between any two states depend only upon their relative motion or position. Finally iii) merely expressed the fact that $P_+^{\nearrow}$ is a group and that two slightly different transformations must produce nearly the same effect. From i) and iii) it follows that Ψ and Ψ^g belong to the same coherent sector.

From i)—ii) it can be shown [11, 12] that there is a unitary operator $U(g)$ corresponding to each $g \in P_+^{\nearrow}$, such that

$$U(g)\psi \in \Psi^g \quad \text{when} \quad \psi \in \Psi.$$

$U(g)$ is defined up to a factor of modulus one, in that the substitution $U(g) \to U'(g) = \omega(g)\, U(g)$ ($|\omega(g)| = 1$) gives us another set of admissible operators. However there exist a neighborhood N of the unit element in $P_+^{\nearrow}$ and a particular choice of these phase factors such that

a) $g \in N$, $g \to U(g)$ is a continuous mapping;

b) $g, g', g \cdot g' \in N$ implies $g \cdot g' \to U(gg') = U(g)\, U(g')$,

so that we have a local representation of $P_+^{\nearrow}$ in $\mathscr{H}$. Being $P_+^{\nearrow}$ not simply connected this local representation extends to a representation of the covering group $\widetilde{P_+^{\nearrow}}$. Its infinitesimal generators, multiplied by $-i$, are ten self adjoint operators P_μ and $M_{\mu\nu} = -M_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$), P_μ being identified with the total momentum (so that P_0 is the total hamiltonian), whereas

$$J^i = \sum_{h,k} \varepsilon^{ihk} M_{hk}, \quad h, k = 1, 2, 3$$

are the total angular momentum operators of the system. P_μ and $M_{\mu\nu}$ are a representation of the Lie algebra of $P_+^{\nearrow}$. Their commutation relations are listed in [9]. In particular

$$[P_\mu, P_0] = [J^i, P_0] = 0$$

so that momentum and angular momentum conservation follows from Lorentz invariance.
From previous considerations we have seen that the only possible relativistically invariant descriptions of a quantum system are given in terms of unitary representation of $\widetilde{P}_{+}^{\nearrow}$ in the Hilbert space of state vectors. When these representations are irreducible we speak of elementary system. In this case any state can be reached from a fixed one by means of a Lorentz transformation, and there is no way to divide the Hilbert space into subsets transforming independently under $\widetilde{P}_{+}^{\nearrow}$.
The study of the irreducible unitary representations of $\widetilde{P}_{+}^{\nearrow}$ has been carried out by WIGNER [12] and the results are as follows: each irreducible representation is characterized by two numbers m and s. m^2 is the eigenvalue of $\sum_{\mu} P^{\mu} P_{\mu}$ in the representation[10]) and is, according to our previous identification of P_{μ} with the total momentum, the mass squared of the particle. Hence the only cases of interest for physics are the irreducible representations with $m^2 > 0$ or $m^2 = 0$. In the first case s is an integer or half integer non negative number, and is equal to the spin of the particle. In the second case $s = 0, \pm {}^1/_2, \pm 1, \dots$ and is the component of the particle spin along the direction of motion (helicity).
A very detailed analysis of Lorentz group representations as well as their application to scattering processes can be found in [13].
We conclude emphasizing that from the previous considerations we have extracted a very precise definition of elementary particle in a quantum theory, at least as far as its space-time behaviour is concerned: it is a system whose states transform like an irreducible representation of $\widetilde{P}_{+}^{\nearrow}$, and it is thus uniquely determined by its mass and spin.
7.3. Apart from space-time symmetries, some kind of interactions between particles exhibit peculiar invariance properties: in particular we will focus our attention on symmetries of strong interactions.
Should we know the actual dynamical structure of strong interactions then it would be possible to check directly what are the transformation on internal labels which leave such dynamics invariant (in the sense of sect. 7.1).
Let us see in a particular model (field theory) what type of conclusions can be drawn from the existence of a Lie group of such transformations.
This will be a guide for us in the actual situation where we do not know the dynamics involved, in order to be able to guess, from certain experimental observations, the existence of strong interactions symmetries.
Suppose that strongly interacting particles are described by certain fields $\psi_{\alpha}(x)$ and by a Lagrangian $\mathcal{L}(\psi_{\alpha}, \partial_{\mu}\psi_{\alpha})$.
Moreover suppose that there exists a certain n-dimensional Lie group G of transformations on the fields

$$\psi_{\alpha}(x) \to \psi'_{\alpha}(x) = U\psi_{\alpha}(x)U^{-1} = \sum_{\beta} t_{\alpha\beta}\psi_{\beta}(x) \tag{1}$$

which leave $\mathcal{L}$ invariant (here the labels α, β refer only to internal degrees of freedom such as charge, baryonic number, hypercharge etc. whereas space-time labels are neglected).

[10]) Due to the fact that $\sum_{\mu} P_{\mu} P^{\mu}$ commutes with all the infinitesimal generators, in any irreducible representation is must be a multiple of the unit operator.

Such transformations are induced on the fields by unitary operators $U(g)$ $(g \in G)$ which are a representation of G in the Hilbert space of the strongly interacting particles. The infinitesimal generators multiplied by $-i$ are n selfadjoint operators Q_k. Their expression in terms of the fields can be found by requiring that the infinitesimal transformations leave unchanged the Lagrangian. In fact one can construct in terms of the fields n divergence-free currents

$$J_{\mu}^{k}(x) \qquad k = 1, \dots n$$
$$\partial^{\mu} J_{\mu}^{k}(x) = 0 \tag{2}$$

such that the operators

$$\int J_4^k(x)\, d^3x$$

satisfy the commutation relations characterizing the Lie algebra of G, and are just the infinitesimal generators of the transformation (1). From (2) it follows that such operators are constant of motion (this result is just the quantum counterpart of the classical Noether's theorem [14]).
Consider now one particle states. They are obtained by applying to the vacuum state a creation operator a_{α}^{+} (again space-time labels are omitted) which obviously satisfies

$$U a_{\alpha}^{+} U^{-1} = \sum_{\beta} t_{\alpha\beta} a_{\beta}^{+}$$

so that

$$U\,(a_{\alpha}^{+} \mid 0\rangle) = \sum_{\beta} t_{\alpha\beta} (a_{\beta}^{+} \mid 0\rangle)$$

i.e. one particle states transform like a representation of G. Moreover, since the Q_k's are constant operators, the Hamiltonian which is the time displacements generator, commutes with them, being therefore an invariant operator under G; the same applies to the mass operator. If G is compact, then the representation of G over one particle states is completely reducible and the irreducible components correspond to states of particles with the same mass.
We can simultaneously diagonalize a number of Q_k equal to the rank of G, then we find multiplets of particles with equal masses, distinguished (apart from possible degeneracies[11]) by the eigenvalues of the diagonal Q_k's, i.e. by certain "internal" quantum numbers.
Passing to multiparticle states we observe that they transform under G as follows

$$U(a_{1\alpha}^{+} a_{2\beta}^{+} \dots \mid 0\rangle) = U a_{1\alpha}^{+} U^{-1} U a_{2\beta}^{+} U^{-1} \dots \mid 0\rangle =$$
$$= \sum_{\alpha',\beta'} t_{\alpha\alpha'}, t_{\beta\beta'} \dots (a_{1\alpha'}^{+}, a_{2\beta'}^{+} \dots \mid 0\rangle)$$

(1, 2 ... take into account space time as well as other degrees of freedom which are uneffected by G) i.e. as tensor product of one particle representations.
Suppose

$$\mid q_1^k, 1\rangle; \ \mid q_2^k, 2\rangle; \dots$$

to be one particle states which are eigenstates of Q_k with eigenvalues $q_1^k, q_2^k, \dots$ Then, under the unitary transformation $U = (1 + i\varepsilon Q_k)$, the multiparticle

[11]) This happens e.g. in the case of SU_3.

state

$$| q_1^k, 1\rangle \, | q_2^k, 2\rangle \ldots \tag{4}$$

transforms as

$$(1 + i\varepsilon Q_k) \, | q_1^k, 1\rangle \, | q_2^k, 2\rangle \cdots = (1 + i\varepsilon Q_k) \, | q_1^k, 1\rangle \, (1 + i\varepsilon Q_k) \, | q_2^k, 2\rangle \cdots =$$

$$= | q_1^k, 1\rangle \, | q_2^k, 2\rangle \cdots + i\varepsilon (q_1^k + q_2^k + \cdots) \, | q_1^k, 1\rangle \, | q_2^k, 2\rangle \cdots + O(\varepsilon^2)$$

and we conclude that the state (4) is an eigenstate of Q_k with eigenvalue $q_1^k + q_2^k + \cdots$, i.e. the Q_k's are *additive conserved quantities.*

In conclusion: starting from invariance under an n-dimensional Lie group G, we have found:

a) n additive conserved quantities (in general not all simultaneously diagonalizable);

b) a multiplet structure for the one particle states;

c) consider furthermore the scattering of two particles into an arbitrary multiparticle state. The corresponding amplitudes are given by the matrix elements of the S-operator which, being a function of the Lagrangian $\mathcal{L}$, turns out to be an invariant operator under G, as $\mathcal{L}$ is. We already saw in sect. 6.2 the general structure of matrix elements of such an operator between states belonging to arbitrary representations of the group G. That analysis tells us that symmetry under G severely restricts the form of the S-matrix, leading to relations between amplitudes of a-priori uncorrelated processes.

At this point one remark is in order. It may well be (and this is the case for isotopic spin or SU_2 symmetry) that not all the Q_k's commute with charge, or with some among the other observables which define a superselection rule. When this is the case these Q_k's do not have a complete set of observable states and their conservation cannot be directly observed. If the symmetry has to be useful, at least a number equal to the rank of G among the Q_k's has to sommute with each observable defining a super selction rule (as well as among themselves), in order to use their common eigenvalues as labels for physical states. Point a) is then reduced to the existence of at least r additive mutually commuting conserved quantities (r = rank of G).

A simple example of an invariance principle which can be treated in this way is the so called first kind gauge invariance.

Suppose the following transformations to leave unchanged the Lagrangian: for any Hermitian field φ: $\varphi \to \varphi$

for any non Hermitian field ψ corresponding to $+1$ charged particles: $\psi \to e^{i\alpha}\psi$

for the adjoint field $\overline{\psi}$: $\overline{\psi} \to e^{-i\alpha}\overline{\psi}$ (α real).

They constitute a one-dimensional Abelian Lie group[12]).

[12]) This group called U_1 has as elements the complex number $e^{i\alpha}$ (α mod 2π), with the multiplication law:

$$e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)}.$$

As for any Abelian group, its irreducible representations are one dimensional: in fact let $e^{i\alpha} \to U(\alpha)$ be an irreducible representation over a linear space L; then for any β

$$U(\alpha)\, U(\beta) = U(\beta)\, U(\alpha)$$

The corresponding conserved current is the charge-density current, and point a) expresses charge conservation. However being this group Abelian, its irreducible representations are 1-dimensional, so that each resulting multiplet contains only one particle. In this case there are no consequences other than mere charge conservation.

Conservation of any additive charge-like quantity (e.g. baryonic number or hypercharge) can be obtained in this way.

7.4. On the basis of previous considerations it should be clear how things actually go. Strong interactions display additive conservation laws, and moreover the great variety of strongly interacting particles (hadrons) seems to divide naturally into sets of particles with very analoguos properties (for example π^+, π^-, π^0; K^+, K^0; p, n and so on). One takes these experimental facts as an indication that the underlying dynamics possesses a non abelian symmetry group. Then one tries with some Lie group and compares with experiments the relations which can be found in the way outlined before.

Imagine a world in which only proton, neutron, π^+, π^-, π^0, are present, as was the situation at the time when the isotopic spin was introduced.

The mass spectrum of these particles provides a very clear evidence for the existence of a non Abelian symmetry group, whose representation on one particle states splits up into two components, the pion and the nucleon. Inside each multiplet, particles are distinguished by only one quantum number (charge), so that we are led to a non Abelian Lie group of rank one. As we will see later there is only one compact simple group of this kind, i.e. SU_2. Its Lie algebra is spanned by three elements T_1, T_2, T_3, satisfying the product rules of angular momentum, and its irreducible representations are labeled by a number $T \geqq 0$ such that:

i) $2T$ is an integer:

ii) the dimension of the representation is $2T + 1$;

iii) the spectrum of T_3 consists of numbers $T, T-1, \cdots, -T$.

We have then to assign the nucleon to the $T = {}^1/_2$, and the pion to the $T = 1$ representations. T is called isotopic spin (I-spin) [*15*].

If we assume, as a convention, that particles correspond to eigenstates of T_3, then T_3 has to be connected to the charge operator. In fact the following relation holds for pions and nucleons:

$$Q - \frac{1}{2} N = T_3 \qquad (N = \text{baryonic number}). \tag{5}$$

In this context we have two additive conservation laws: charge and baryonic number. The latter is derived as invariance under a gauge group, whereas charge conservation is included in isotopic spin conservation. Note that T_1 and T_2 do not commute neither with T_3 nor with charge, so that they do not correspond to observable quantities.

so that (Schur's lemma) $U(\alpha) = \lambda(\alpha) \cdot 1$. Hence L has to be one-dimensional. Irreducible representations are of the form

$$e^{i\alpha} \to \lambda_k(\alpha) = e^{ik\alpha}, \quad \text{where } k \text{ is an integer.}$$

Consider now a $\pi - \mathrm{N}$ system. Under SU_2 it transforms as an element of the tensor product of the I-spin 1 and I-spin $^1/_2$ representations. It is well known that this product splits up into an I-spin $^3/_2$ and an I-spin $^1/_2$ representation. If we want to study a scattering process of the type

$$\pi + N \to \pi' + N' \tag{6}$$

we have to evaluate matrix elements of the type:

$$\langle \pi N \mid S \mid \pi' N' \rangle \tag{7}$$

where S is invariant under SU_2. Writing

$$\mid \pi N \rangle = C^3_{\pi N} \mid {}^3/_2\, T_3 \rangle + C^1_{\pi N} \mid {}^1/_2\, T_3 \rangle$$

with the aid of Clebsch-Gordan coefficients, (7) writes as:

$$C^3_{\pi N}\, C^3_{\pi' N'} \langle {}^3/_2\, T_3 \mid S \mid {}^3/_2\, T_3' \rangle + C^3_{\pi N}\, C^1_{\pi' N'} \langle {}^3/_2\, T_3 \mid S \mid {}^1/_2\, T_3' \rangle + + C^1_{\pi N}\, C^1_{\pi' N'} \langle {}^1/_2\, T_3 \mid S \mid {}^1/_2\, T_3' \rangle + C^1_{N\pi}\, C^3_{\pi' N'} \langle {}^1/_2\, T_3 \mid S \mid {}^1/_2\, T_3' \rangle . \tag{8}$$

Using the analysis of sect. 6.2, we see that

$$\langle {}^3/_2\, T_3 \mid S \mid {}^3/_2\, T_3' \rangle = \delta_{T_3 T_3'} A^3$$

$$\langle {}^1/_2\, T_3 \mid S \mid {}^1/_2\, T_3' \rangle = \delta_{T_3 T_3'} A^1 ,$$

where A^3 and A^1 depend only upon space-time labels. All the other matrix elements vanish. In conclusion we can express the amplitudes of all processes like (6) in terms of only two amplitudes which are function of space-time variables, but do not depend anymore on the charge variables.

It is a well known fact that experimentally at an energy near 190 MeV for the incident pion the amplitude A^3 greatly dominates: neglecting A^1 we find at that energy a well determined ratio for the following processes:

a) $\pi^+ + \mathrm{p} \to \pi^+ + \mathrm{p}$
b) $\pi^- + \mathrm{p} \to \pi^- + \mathrm{p}$
c) $\pi^- + \mathrm{p} \to \pi^0 + \mathrm{n}$.

Rate a: Rate b: Rate $c = 9:1:2$ which is well verified experimentally.

Since isotopic spin has been introduced many other hadrons have been found, together with another additive conservation law: hypercharge (Y) conservation. However all hadrons still fit well into isomultiplets when relation (5) is modified as

$$Q - \frac{1}{2}\, Y = T_3$$

and all the experimental findings are consistent with the assumption that strong interactions are invariant under SU_2 [*16*]. In this context, as for baryonic number, hypercharge conservation is considered to derive from invariance under a gauge group, quite independently from isotopic spin conservation.

This situation can be summarized stating that the symmetry group for strong interactions is the direct product[13]) of two gauge groups (N and Y-conservation) times SU_2 (I-spin conservation). As a consequence N, (T, T_3) and Y quantum numbers appear in a completely uncorrelated way.

7.5. Unitary symmetry models

In unitary symmetry models one tries to derive T_3 and Y conservation from invariance under a group which does not break into the direct product $SU_2 \times U_1(Y)$, introducing in this way relations between particles belonging to isomultiplets with different hypercharge. It is a fact that as yet nobody has succeeded in extending such procedure to include N-conservation which, as before, is derived from a separated gauge group [*17*].

The feeling for such a higher symmetry is not strongly substantiated at first sight by experimental evidence. In fact according to results derived in sect. 7.3) particles would be organized into supermultiplets, (i.e. irreducible representations) behaving as elementary objects under strong interactions; but now inside same supermultiplets there would appear particles differing by Y as well as by T_3. This is in conflict with the experimental evidence in that $\Delta m/m$ between particles differing by Y are quite large and not imputable to non strong interactions (for example $m_\Lambda - m_N \cong 175$ MeV, $m_\Xi - m_N \cong 380$ MeV). Hence we must conclude that the idea of a higher symmetry in the sense above specified, cannot be literally applied.

Nevertheless the following interpretation has been proposed: there is a symmetrical component in strong interactions which is responsible of the gross structure of particles world; in addition there is a weaker component to be treated as a perturbation responsible of the departures from the exact symmetry. It is understood that both components are charge independent as well as strangeness conserving.

[13]) Given two groups G_1 and G_2, their direct product $G_1 \times G_2$ is defined as the set of ordered pairs $(g_1, g_2)\,(g_i \in G_i)$ with the multiplication law

$$(g_1, g_2)(f_1, f_2) = (g_1 f_1, g_2 f_2).$$

This definition satisfies all the required axioms. Given a representation $g_i \to U(g_i)$ of G_i on the linear spaces L_i, we can find a representation of $G_1 \times G_2$ in the direct product $L_1 \times L_2$ as follows

$$(g_1, g_2) \to U(g_1) \otimes U(g_2) = U(g_1, g_2)$$

and it can be shown that all the representations of $G = G_1 \times G_2$ can be put into this form. $U(g_1, g_2)$ is irreducible if and only if $U(g_1)$ and $U(g_2)$ are. In our particular case we have the group

$$SU_2 \times U_1(N) \times U_1(Y)$$

specified by the triplets (α, β, g) (α, β real, $g \in SU_2$).
In a space spanned by $2\,T + 1$ vectors

$$|N, Y; T, T_3), \; -T \leqq T_3 \leqq T, N, Y \text{ fixed}$$

an irreducible representation of $SU_2 \times U_1(N) \times U_1(Y)$ has the form:

$$(\alpha, \beta, g) \to e^{i\beta N}\, e^{i\alpha Y}\, D^{(T)}(g)$$

where $D^{(T)}(g)$ are matrices defining an irreducible representation of SU_2.

It should be noted that the weaker component has not been till now satisfactory identified. However the idea of a simmetry breaking interaction treated as first order perturbation, has provided us with corrections to the predictions derived from a pure symmetrical model, which are consistent with experimental findings.

7.6. Previous reasoning obviously do not indicate us what the symmetry group for strong interactions actually is. Following general requirements however seem to be quite reasonable, and are usually imposed on possible candidates:

i) this group must be a Lie group. In fact we want to identify additive conserved quantities such as T_3 and Y with its infinitesimal generators;

ii) is must be compact: this assures that its irreducible unitary representations are finite-dimensional, so that wen can fill up resulting supermultiplets with a finite number of particles (see sect. 2.2);

iii) it must be semi-simple (see sect. 4.4). This restriction is mainly due to practical reasons: for semisimple Lie groups in fact there is a complete mathematical theory, which is not the case non for semisimple groups;

iv) the rank of the group, i.e. the rank of its Lie algebra, must be two, because we require two conserved commuting quantities i.e. T_3 and Y;

v) is must contain a subgroup isomorphic to SU_2 in order to recover the isotopic spin symmetry. Actually this does not bear any restriction in that any semisimple Lie group has this property (see later sect. 9.7 b).

To construct a concrete theory we need at this point a characterization of Lie groups as well as a classification of their irreducible representations.

In next sections we will study these topics with some detail.

8. Structure of Semisimple Lie Algebras

In sect. 4 we studied the relations between Lie groups and Lie algebras, and the conclusion achieved was that there is a one-to-one correspondence between Lie algebras and Lie groups, so that instead of studying Lie groups one can study the corresponding Lie algebras and their representations, which is more convenient by a mathematical point of view.

8.1. We have given in sect. 4.4 the definition of n-dimensional Lie algebra of rank r. At the same time we noted that the mapping

$$x \to \operatorname{ad} x \qquad x \in \mathscr{L}$$

$$\operatorname{ad} x(y) = [x, y]$$

is a representation of $\mathscr{L}$ called its regular representation.

In terms of it we can define in $\mathscr{L}$ a bilinear form as follows

$$(x, y) = \operatorname{Tr}(\operatorname{ad}x, \operatorname{ad}y). \tag{1}$$

Obvious properties are (α = real number)

$$(\alpha x, y) = \alpha(x, y); \qquad (x + y, z) = (x, z) = (y, z) \tag{1'}$$

$$(x, y) = (y, x)$$

$$([z, x], y) = -(x, [z, y]).$$

The following very important theorem has been proved by Cartan:

$\mathscr{L}$ is semisimple if and only if (x, y) is not degenerate, i.e. if $(x, y) = 0$ for any $y \in \mathscr{L}$, implies $x = 0$.

This criterion is essential in the classical theory of semisimple Lie algebras. Furthermore the most important results of this theory heavily rest on the possibility of diagonalizing operators $\operatorname{ad} x$ where x runs over the elements of a Cartan subalgebra $\mathscr{C}$ of $\mathscr{L}$. Now in $\mathscr{L}$ there are surely eigenvectors of $\operatorname{ad}(x)$ (in fact for each element $y \in \mathscr{C}$ we have:

$$\operatorname{ad} x(y) = [x, y] = 0 \qquad x, y \in \mathscr{C}$$

so that $\mathscr{C}$ is an eigenspace of $\operatorname{ad}(x)$ belonging to the eigenvalue zero) but in general in $\mathscr{L}$ (which is a real vector space) a complete system of such eigenvectors does not exists (see later the example reported in sect. 8.7 a). The way out of this difficulty is to englarge $\mathscr{L}$ to a complex Lie algebra in which structure theory can be easily carried out. From it, as we shall see later, corresponding results for the real semisimple Lie algebras associated with compact Lie groups can be deduced.

8.2. Complexification

If $\mathscr{L}$ is a real semisimple Lie algebra, we can construct its complex extension $\mathscr{L}_c$ by choosing a basis x_k in $\mathscr{L}$ and considering the set of all linear combinations of this basis, with the product between two elements defined as follows. If

$$x = \sum_k^{1,n} \lambda_k x_k \qquad (\lambda_k, \mu_h = \text{complex numbers})$$

$$y = \sum_k^{1,n} \mu_h x_h$$

then

$$[x, y] = \sum_{h,k}^{1,n} \lambda_k \mu_h [x_k, x_h] = \sum_{hks}^{1,n} \lambda_h \mu_h C_{kh}^s x_s$$

where C_{kh}^s are the structure constants relative to the basis choosen in $\mathscr{L}$. This product satisfies condition 4.4 ii), and obviously this definition of $\mathscr{L}_c$ does not depend upon the particular basis choosen. In $\mathscr{L}_c$ the form (x, y) is simply

$$(x, y) = \sum_{h,k} \lambda_h \mu_k (x_h, x_k).$$

This form is not degenerate if and only if $\det(x_h, x_k) \neq 0$; this condition is the same whether we consider $\mathscr{L}$ or $\mathscr{L}_c$, so that if $\mathscr{L}$ is semisimple so is $\mathscr{L}_c$. In addition the rank of $\mathscr{L}_c$ is the same of that of $\mathscr{L}$.

8.3. Structure theory

We will be content here to state without proof all results relevant for applications. A complete derivation can be found in the excellent book by Jacobson [*18*][14]).

[14]) An easier treatment can be found in [*19*]. See in addition the very readable paper by Racah [*20*].

In the following by $\mathscr{L}$ we mean a complex semisimple n-dimensional Lie algebra of rank r and by $\mathscr{C}$ one of its Cartan subalgebras.

I. Any operator $\mathrm{ad}\, h$, $h \in \mathscr{C}$, is diagonalizable in $\mathscr{L}$. If $h_1, \ldots h_r$ is a basis in $\mathscr{L}$, and $e_{\alpha_1 \ldots \alpha_r}$ is a common eigenvector of ad h_i's(ad $h_i(e_{\alpha_1\ldots\alpha_r}) = \alpha_i e_{\alpha_1\ldots\alpha_r}$), we have for any element $h \in \mathscr{C}$:

$$h = \sum_i^{i,r} \lambda_i h_i$$

$$\mathrm{ad}\, h e_{\alpha_1\ldots\alpha_r} = \Big(\sum_i \lambda_i \alpha_i\Big) e_{\alpha_1\ldots\alpha_r}$$

so that it sufficies to sonsider only the diagonalization of operators $\mathrm{ad}\, h_i$.

II. h_i's can be choosen in such a way that the α_i are all real. Upon introducing the notation $\alpha = (\alpha_1 \ldots, \alpha_r)$ we write

$$\mathrm{ad}\, h_i e_\alpha = \alpha_i e_\alpha .$$

The real r-components object α is called a root vector, or simply a root. $\mathscr{L}$ splits up into a direct sum of common eigenspaces of $\mathrm{ad}\, h_i (i = 1, 2 \ldots r)$

$$\mathscr{L} = \mathscr{L}_0 \bigoplus_\alpha \mathscr{L}_\alpha \tag{2}$$

where the direct sum runs over all non vanishing roots and $\mathscr{L}_0$ is the eigenspace belonging to the root $(0, \ldots, 0)$. Obviously $\mathscr{L}_0$ contains $\mathscr{C}$, but a stronger result holds, namely

III. $$\mathscr{L}_0 = \mathscr{C}, \tag{3}$$

and furthermore each $\mathscr{L}_\alpha (\alpha \neq 0)$ is one dimensional.

It follows the there are $n - r$ non zero roots.

We can extend the basis $\{h_i\}$ to a basis in $\mathscr{L}$, adding to these elements all the vectors e_α where α is a non zero root and e_α is a vector spanning $\mathscr{L}_\alpha$.

IV. Consider the restriction of the trace form (1) to $\mathscr{C}$. It is determined by the matrix:

$$g_{ij} = (h_i, h_j) = \mathrm{Tr}(\mathrm{ad}\, h_i\, \mathrm{ad}\, h_j), \qquad i, j = 1, 2, \ldots r. \tag{4}$$

In the basis $\{h_i, e_\alpha\}$, the operators

$$\mathrm{ad}\, h_i\, \mathrm{ad}\, h_j$$

are diagonal, and have eigenvalues equal to $\alpha_i \alpha_j$. Hence

$$g_{ij} = \sum_\alpha \alpha_i \alpha_j \quad \text{(the sum runs over all roots)} \tag{5}$$

which implies g_{ij} to be a real symmetric matrix.

As a consequence g_{ij} can be diagonalized with a real orthogonal substitution:

$$g'_{ij} = \sum_{kh} A_{ik} A_{jh} g_{kh} = \lambda_j \delta_{ij}.$$

Upon introducing

$$h'_i = \sum_k A_{ik} h_k$$

which corresponds to a change of basis in $\mathscr{C}$, we see that:

$$\mathrm{ad}\, h'_i e_\alpha = \Big(\sum_k A_{ik} \alpha_k\Big) e_\alpha = \alpha'_i e_\alpha$$

and α'_i also are real numbers. Then:

$$\mathrm{Tr}(\mathrm{ad}\, h'_i\, \mathrm{ad}\, h'_j) = \sum_\alpha \alpha_i \alpha_j = \sum_{h,k} A_{ik} A_{jh} g_{hk} = g'_{ij} = \lambda_i \delta_{ij}.$$

Hence

$$\lambda_i = g_{ii} = \sum_\alpha (\alpha'_i)^2 \geqq 0.$$

Suppose that for some value of i, $\lambda_i = 0$. Then $\alpha'_i = 0$ for any α, so that

$$[h'_i, e_\alpha] = 0.$$

Moreover

$$[h'_i, h'_j] = 0,$$

so that $\mathrm{ad}\, h'_i$ is represented in the basis $\{h'_i, e_\alpha\}$ by the null matrix.

This has the consequence

$$(h'_i, x) = 0 \quad \text{for any} \quad x \in \mathscr{L}$$

i.e. (by Cartan's criterion) $h'_i = 0$ which is excluded.

In conclusion we see that g_{ij} is a non singular, positive definite, real matrix. We will indicate with g^{ij} its inverse

$$\sum_k g_{ik} g^{kj} = \delta_{ij}.$$

With the aid of this metric tensor we define a scalar product between roots:

$$(\alpha, \beta) = \sum_i \alpha^i \beta_i = \sum_i \alpha_i \beta^i = \sum_{ij} \alpha_i \beta_j g^{ij}. \tag{6}$$

The difference between $\alpha^i = \Sigma g^{ij} \alpha_j$ and α_i can be removed by performing a real linear transformation on h_i's, which reduces g_{ij} into the form δ_{ij}. In this basis the scalar product between roots is written as

$$(\alpha, \beta) = \sum_i \alpha_i \beta_i .$$

In the basis $\{h_i, e_\alpha\}$ part of the multiplication rules are defined in terms of roots

$$[h_i, h_j] = 0$$

$$[h_i, e_\alpha] = \alpha_i e_\alpha .$$

We will see that the same applies to all multiplication rules between h_i and e_α, in the sense that roots determine all the structure constant relative to the basis $\{h_i, e_\alpha\}$. This depends on peculiar properties of roots, which are the very heart of structure theory.

Properties of roots.

In the following $k\alpha$ and $\alpha+\beta$ are defined as

$$k\alpha = (k\alpha_1, \ldots, k\alpha_r)$$

$$\alpha+\beta = (\alpha_1+\beta_1, \ldots, \alpha_r+\beta_r).$$

V. If α is a non zero root, then $k\alpha$ is a root (k = real number) if and only if $k = \pm 1,0$.
Hence the $n-r$ non zero roots are distributed in pairs α, $-\alpha$. (From this we see that $n-r$ is an even integer for any semisimple Lie algebra).
VI. Any two non zero roots $\alpha, \beta, (\alpha+\beta \neq 0)$ uniquely determine two non negative integers r, q such that

$$\beta - r\alpha, \beta-(r-1)\alpha, \ldots, \beta, \beta+\alpha, \ldots \beta+q\alpha$$

are the only non zero roots of the form $\beta+k\alpha$. This serie of roots is called the α-string containing β. Interchanging α and β we obtain two other integers q', r' characterizing the β-string containing α.
Numbers r, q satisfy the condition:

$$r-q = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \qquad (r'-q' = 2\frac{(\alpha,\beta)}{(\beta,\beta)}. \tag{7}$$

Being in addition

$$-r \leqq q-r \leqq q \quad (-r' \leqq q'-r' \leqq q')$$

we obtain that if α and β are non zero roots, then

$$\beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha, \qquad \alpha - \frac{2(\alpha,\beta)}{(\beta,\beta)}\beta \tag{8}$$

are non zero roots. The first (second) os obtained by reflecting $\beta(\alpha)$ with respect to the plane orthogonal to $\alpha(\beta)$.
The general feature of the multiplication table can be understood in terms of the following relation:

$$\mathrm{ad}\, h_i[e_\alpha, e_\beta] = [h_i,[e_\alpha, e_\beta]] = (\alpha_i+\beta_i)[e_\alpha, e_\beta] \tag{9}$$

which is a simple consequence of Jacobi identity.
We distinguish three cases:
a) $\alpha+\beta \neq 0$, $\alpha+\beta$ is not a root.
In this case eq. (9) implies, $[e_\alpha, e_\beta] = 0$. Otherwise $[e_\alpha, e_\beta]$ would be an eigenvector of h_i and $\alpha+\beta$ would be a root.
b) $\alpha+\beta = 0$.
Hence ad $h_i[e_\alpha, e_{-\alpha}] = 0$ so that $[e_\alpha, e_\beta] \in \mathscr{C}$ and we can write:

$$[e_\alpha, e_{-\alpha}] = \sum_i \lambda^i h_i.$$

VII. It is possible to normalize $e_\alpha, e_{-\alpha}$ so that $(e_\alpha, e_{-\alpha}) = \mathrm{Tr}(\mathrm{ad}\, e_\alpha\, \mathrm{ad}\, e_{-\alpha}) = 1$. With this choice it results:

$$\lambda^j = \alpha^i = \sum_j g^{ij}\alpha_j$$

$$[e_\alpha, e_{-\alpha}] = \sum_j g^{ij}\alpha_j h_i = \sum_i \alpha^i h_i. \tag{10}$$

We note that this normalization determines e_α and $e_{-\alpha}$ up to factors $d_\alpha, d_{-\alpha}$ such that $d_\alpha d_{-\alpha} = 1$.
c) $\alpha+\beta \neq 0$, $\alpha+\beta$ is a root. By eq. (9) and by unidimensionality of $\mathscr{L}_{\alpha+\beta}$ it follows:

$$[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}.$$

One can show that

$$N_{\alpha,\beta} = N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha} = -N_{\beta\alpha} \tag{11}$$

and in addition, by disposing of factors $d_\alpha, d_{-\alpha}$ in $e_\alpha, e_{-\alpha}$, one can assume

$$N_{\alpha\beta} = -N_{-\alpha,-\beta}. \tag{12}$$

In this case

VIII.
$$N^2_{\alpha\beta} = \frac{q(r+1)}{2}(\beta,\beta) \tag{13}$$

where q, and r are the integers determining the α-string containing β. We see that when $\alpha+\beta$ is a root, $N_{\alpha\beta} \neq 0$.
This relation determines $N_{\alpha\beta}$ up to a sign which must be chosen so to satisfy (11), (12).
Collecting all these results, we can write the complete multiplication table relative to the basis $\{h_i, e_\alpha, e_{-\alpha}\}$:

$$\begin{aligned}
&[h_i, h_j] = 0\\
&[h_i, e_{\pm\alpha}] = \pm\alpha_i e_{\pm\alpha}\\
&[e_\alpha, e_{-\alpha}] = \sum_j \alpha^i h_i = \sum_{ij} \alpha_i h_i g^{ij}\\
&[e_\alpha, e_\beta] = \begin{cases} (\alpha \neq -\beta, \alpha+\beta & \text{is not a root}) = 0\\ (\alpha \neq -\beta, \alpha+\beta & \text{is a root}) = N_{\alpha\beta}\, e_{\alpha+\beta}.\end{cases}
\end{aligned} \tag{14}$$

All structure constants relative to this basis are determined by roots and are all real. In addition a linear non singular transformation on the h_i's does not change the form of these products. In fact if we have

$$h_i' = \sum_j A_{ij} h_j, \quad h_j = \sum_i = (A^{-1})_{ji} h_i',$$

then

$$[h_i', e_{\pm\alpha}] = \sum_j A_{ij}[h_j, e_{\pm\alpha}] = \pm\Big(\sum_i A_{ij}\alpha_j\Big) e_{\pm\alpha} = \alpha_i' e_{\pm\alpha}$$

$$[e_\alpha, e_{-\alpha}] = \sum_i \alpha^i h_i = \sum_{i,l} \alpha^i (A^{-1})_{il} h_l' = \sum_l \alpha'^l h_l',$$

where

$$\alpha'_i = \Sigma_j A_{ij} \alpha_j ; \quad \alpha'^l = \sum_i \alpha^i (A^{-1})_{il} ,$$

so that α_i and α^j transform respectively as the covariant and contravariants components of a vector. g_{ij} transforms as a covariant tensor:

$$g'_{ij} = (h'_i, h'_j) = \sum_{l,k} A_{il} A_{ik} g_{lk} , \tag{15}$$

so that the scalar product between roots is dependent upon the particular basis choosen in $\mathscr{C}$. From this it follows that although there are in $\mathscr{C}$ bases in which $\alpha'_i s$ are complex numbers, in any case (α, α) is a positive number.
By (14) we see that the roots of a Lie algebra, determine uniquely its structure. Hence a classification of semisimple Lie algebras of rank r is equivalente to find all sets of r-dimensional real vectors which satisfy V, VI, (7). This is the argument of next section.
8.4. To begin with we introduce now an ordering between roots in the following way.
Given two roots α and β, α is said to be greater than β if the first non vanishing component of $\alpha - \beta \equiv (\alpha_i - \beta_i)$ is a positive number. In particular a root is positive if it is greater than zero.
Of course this ordering depends on the basis choosen in $\mathscr{C}$ and it is the same that the ordering of words in a dictionary.

We introduce another useful concept. A root α is simple if:

a) α is a positive root,

b) α cannot be written as sum of two positive roots.

Two important properties of simple roots are nearly immediate.
If α and β are simple roots, then:
i) $\alpha - \beta$ is not a root. If $\alpha - \beta$ were a positive root, then α would be equal to $(\alpha - \beta) + \beta$, i.e. would not be simple. Conversely if $\alpha - \beta$ were a negative root, then $\beta - \alpha$ would be positive, and β would be equal to $(\beta - \alpha) + \alpha$, i.e. β would not be simple;
ii) $(\alpha, \beta) \leqq 0$. By i) if we consider the α-string containing β, we see that $r = 0$, so that

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} = r - q = -q \leqq 0 .$$

The usefulness of considering simple roots lies in the following theorem:
IX. There are exactly r linearly independent simple roots which we will indicate with $\alpha^{(1)}, \ldots \alpha^{(r)}$. Furthermore any positive root can be written as a linear combination of simple roots with non negative integers as coefficients.
X. if $\alpha > 0$ is a non simple root, there exists a simple root $\alpha^{(k)}$ such that $\alpha - \alpha^{(k)}$ is a positive root.
We will see later that properties IX), X) enable one to construct all roots starting form simple roots. This limits further analysis only to simple roots.
From (7) we can derive very severe restriction on the angle between two roots as well as on the ratio of their lenghts. In fact it is:

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} = m , \quad \frac{2(\alpha, \beta)}{(\beta, \beta)} = n$$

i.e.

$$\frac{(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} = \frac{m \cdot n}{4} = \cos^2 \varphi_{\alpha\beta} \leqq 1 ;$$

if $m, n \neq 0$

$$\frac{(\alpha, \alpha)}{(\beta, \beta)} = \frac{n}{m} .$$

When α and β are simple roots, m and n are non positive by 8.4 ii) and furthermore $\alpha \neq \beta$ so that the only possibilities we are left with are:

m	n	$\varphi_{\alpha\beta}$	$(\alpha, \alpha)/(\beta, \beta)$
-1	-1	120°	1
-1	-2	135°	2
-1	-3	150°	3
-2	-1	135°	$1/2$
-3	-1	150°	$1/3$
0	0	90°	arbitrary

In addition in the latter case ($\varphi_{\alpha\beta} = 90°$) neither $\alpha - \beta$, nor $\alpha + \beta$ are roots (see properties 8.4i, ii).
8.5. On the basis of stated properties of simple roots, we illustrate the classification of (complex) semisimple Lie algebras.
Let us first examine the case in which simple roots split up into groups of simple roots, such that any root of each group is orthogonal to all roots belonging to different groups, whereas inside each group there is no root orthogonal to all the other ones:

$$\alpha^{(1)}, \ldots \alpha^{(r_1)}; \quad \beta^{(1)}, \ldots \beta^{(r_2)}; \quad \gamma^{(1)}, \ldots \gamma^{(r_m)}$$

$$r_1 + r_2 + \cdots + r_m = r .$$

The whole root diagram splits up into mutually orthogonal parts $\alpha, \ldots, \beta \ldots ; \ldots ;$ $\gamma, \ldots$ and in addition $\alpha \pm \beta \ldots ; \ldots \alpha \pm \gamma, \ldots ; \beta \pm \gamma \ldots$ are not roots, so that for the corresponding $e_\alpha, \ldots ; e_\beta, \ldots ; e_\gamma \ldots$ we have

$$[e_\alpha, e_\beta] = 0, \quad [e_\alpha, e_\gamma] = 0 \quad \text{and so on.}$$

Furthermore one can choose in $\mathscr{C}$ a basis of h'_is which also decomposes into groups of vectors $\{h_1^{(1)}, \ldots h_{r_1}^{(1)}; h_2^{(2)}, \ldots h_{r_2}^{(2)}; \ldots$ such that for example

$$[h_i^{(1)}, e_\beta] = 0 \ldots$$

$$[h_i^{(1)}, e_\gamma] = 0 \ldots$$

Summarizing we see that the basis $\{h_i e_\alpha, e_{-\alpha}\}$ can be decomposed into the direct sum of bases $\{h_i^{(1)}, e_\alpha, e_{-\alpha}\}$, $\{h_j^{(2)}, e_\beta, e_{-\beta}\}, \ldots \{h_l^{m}, e_\gamma, e_{-\gamma}\}$ such that all products between elements of different groups vanish.
Each linear manifold spanned by such bases is evidently a subalgebra in $\mathscr{L}$ and it is even an ideal, so that the existence of roots orthogonal to all the others implies $\mathscr{L}$ to be not simple. The converse is also true, i.e.

XI. Necessary as well as sufficient condition for a semisimple Lie algebra to be simple is that $\mathcal{L}$ has no simple root orthogonal to all the others.
From XI and from the previous considerations it follows that any semisimple Lie algebra is the direct sum of simple Lie algebras (Weyl's theorem) so that the classi fication of semisimple Lie algebras is reduced to that of simple one.

Classification of simple Lie algebras

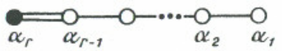
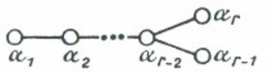
As it should be clear from last section, the problem of determining all simple Lie algebras of a fixed rank r, is equivalent to finding all sets of r simple roots satisfying V, VI, (7), and the condition, that no one of them is orthogonal to all the others.
The essential results of the structure theory can be formulated as following.
XII. Lenght of simple roots can assume at most two values.
Keeping this in mind, the set of simple roots of a simple Lie algebra can be conveniently described in a graphical way introduced by E. B. DYNKIN:

(1) to any simple root we associate a circle:

(2) two circles are connected by one, two, or three lines when the angle between corresponding roots is respectively 120°, 135°, or 150°.
If the roots are orthogonal, circles are not connected.

(3) Circles corresponding to shorter roots are blackened.

The only simple algebras are then defined by following diagrams (for any fixed r):

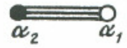
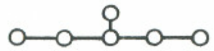
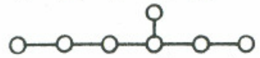
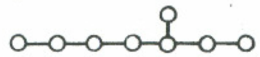
Nome of the algebra	Dynkin diagram	Dimensionality	Remarks
A_r		$r(r+2)$	$r \geqq 1$
B_r		$r(2r+1)$	$r \geqq 2$
C_r		$r(2r+1)$	$r \geqq 2$
D_r		$r(2r-1)$	$r \geqq 3$

These algebras are all distinct when $r \geqq 4$, wherease we note that:

i) when $r = 1$ there is only one simple Lie algebra, i.e. A_1;

ii) when $r = 2$, Dynkin diagrams of B_2 and C_2 are identical, i.e. B_2 and C_2, having same dimensionality and same structure constants are identical;

iii) when $r = 3$, A_3 and D_3 have the same Dynkin diagram so that again $A_3 = D_3$.
Apart from these four classes, there are five exceptional Lie algebras, named G_2, F_4, E_6, E_7, E_8, defined by the following diagrams:

Name	Diagram	Dimensionality
G_2		14
F_4		52
E_6		78
E_7		133
E_8		248

In particular there are only three distinct Lie algebras of rank two, i.e. A_2, $C_2 = B_2$, G_2.

8.6. Classification of simple, compact Lie groups

As said in sect. 7.5 these groups are of the main concern in unitary symmetry models. To any of them we can uniquely associate a real Lie algebra $\mathcal{L}_r$, and it is remarkable that, as showed by H. WEYL, the compactness of the group reflects in $\mathcal{L}_r$ in that its trace form (1) is negative definite. In view of this circumstance, $\mathcal{L}_r$ itself is called a compact (real) Lie algebra.
To carry out structure theory it has been convenient to consider complex Lie algebras. Now, whereas any real algebra $\mathcal{L}_r$ uniquely define its complex extension $\mathcal{L}_c$, the converse is not true, in that the same $\mathcal{L}_c$ can be obtained starting from different real Lie algebras i.e. from different Lie groups (this is for example the case of R_3 and of the 3-dimensional Lorentz group, which have the same complex Lie algebra A_1).
However, as again has been showed by H. WEYL, for any semisimple complex Lie algebra $\mathcal{L}_c$, there is essentially one real semisimple compact Lie algebra whose complex extension is $\mathcal{L}_c$. This has the meaning that in $\mathcal{L}_c$ there exists a basis such that:
i) all its structure constants are real;
ii) the real Lie algebra spanned by this basis is compact.
In particular starting from the canonical basis $\{h_i, e_\alpha, e_{-\alpha}\}$ it can be easily shown that the basis

$$f_i = -ih_i; \quad f_\alpha = -i(e_\alpha + e_{-\alpha}); \quad g_\alpha = -(e_\alpha - e_{-\alpha}) \tag{16}$$

(α runs over positive roots)

is compact[15]).

8.7. Examples

a) SU_2

We have seen in sect. 4.1 that the real Lie algebra associated to $SU_2(R_3)$ is spanned by three elements I_1, I_2, I_3 with the product rules:

$$[I_1, I_2] = I_3, \quad [I_2, I_3] = I_1, \quad [I_3, I_1] = I_2.$$

[15]) To verify that $(x, x) < 0\,(x \neq 0)$ whenever x is a real linear combination of the elements (16), one has simply to use the orthogonality relations $(h_i, e_\alpha) = 0$, $(e_\alpha, e_\beta) = \delta_{-\alpha,\beta}$ which follow from (1′) and from the normalization condition $(e_\alpha, e_{-\alpha}) = 1$.

This is a 3-dimensional simple Lie algebra of rank 1. For the general element $x = \sum_i c_i I_i$ (c_i real) it is easy to see that $\mathrm{Tr}(\mathrm{ad}\,x\,\mathrm{ad}\,x) < 0$ as it must be, being SU_2 compact. Let us choose I_3 as the element spanning $\mathscr{C}$. Finding roots is equivalent to solve the following eigenvalue equation (with $\alpha \neq 0$):

$$\mathrm{ad}\,I_3(x_\alpha) = [I_3, x_\alpha] = \alpha x_\alpha, \qquad x_\alpha = c_1 I_1 + c_2 I_2 + c_3 I_3 .$$

which is equivalent to:

$$\begin{cases} c_1 = \alpha c_2, \\ c_2 = -\alpha c_1 \end{cases} \quad \text{i.e.} \quad \alpha = \pm i, \quad c_2 = \mp i c_1 .$$

If we introduce the elements $I_\pm = I_1 \pm i I_2$ (which are in the complexification of our algebra!), we have

$$[I_3, I_\pm] = \mp i I_\pm, \qquad [I_+, I_-] = -2 i I_3$$

so that, by posing

$$h_3 = i I_3; \quad e'_\pm = \frac{i}{2} I_\pm$$

we obtain the product rule

$$[h_3, e'_\pm] = \pm e'_\pm .$$

We observe that $\mathrm{Tr}(\mathrm{ad}\,h_3\,\mathrm{ad}\,h_3) = 2$ so that the metric tensor (which reduces to a number g) is $g = 2$. The contravariant component of the single positive root is

$$\alpha^1 = \frac{1}{2}\alpha_1 = \frac{1}{2}$$

so that we have

$$[e'_+, e'_-] = \frac{1}{2}[e'_+, e'_-] \qquad h_3 = \frac{1}{2} h_3$$

being $(e'_+, e'_-) = 1$ as one can easily verify.
Root space is one-dimensional, and we have two non zero roots: ± 1, and one simple root. The corresponding Dynkin diagram is made of a single circle, so that this Lie algebra is just A_1.
Usually as basis are taken the elements

$$h_3, e_\pm = \sqrt{2}\, e'_\pm \qquad ((e_+, e_-) = 2)$$

whose product rules are

$$[h_3, e_\pm] = \pm e_\pm$$
$$[e_+, e_-] = h_3$$

and this we will do in the following.
b) Recalling the product rules (14) which are the canonical rules for any semi-simple Lie algebra $\mathscr{L}$, we see that the elements of $\mathscr{L}$, defined as:

$$h'_\alpha = \frac{\Sigma_i \alpha^i h_i}{(\alpha, \alpha)} \qquad e'_{\pm\alpha} = \frac{e_{\pm\alpha}}{(\alpha, \alpha)^{1/2}}$$

satisfy the relations:

$$[h'_\alpha, e'_{\pm\alpha}] = \pm e'_{\pm\alpha}$$
$$(e'_\alpha, e'_{-\alpha}) = h'_\alpha .$$

Hence h'_α, $e'_{\pm\alpha}$ span a subalgebra in $\mathscr{L}$ which is identical with A_1, so that by the considerations made in sect. 4.4 we see that the compact group associated to any $\mathscr{L}$ contains subgroups isomorphic to SU_2.

c) A_2

We start from the Dynkin diagram: O—O
From this we see that there are two simple roots $\alpha^{(1)}$ and $\alpha^{(2)}$ of equal lenghts, making, an angle of 120 degrees:

$$\frac{2(\alpha^{(1)}, \alpha^{(2)})}{(\alpha^{(1)}, \alpha^{(1)})} = \frac{2(\alpha^{(1)}, \alpha^{(2)})}{(\alpha^{(2)}, \alpha^{(2)})} = -1 .$$

Hence the $\alpha^{(1)}$ string containing $\alpha^{(2)}$ consists of the two elements $\alpha^{(2)}$, $\alpha^{(1)} + \alpha^{(2)}$ and the reversed string contains $\alpha^{(1)}$ and $\alpha^{(1)} + \alpha^{(2)}$. A_2 is 8-dimensional and has rank 2, so that we expect six non zero roots at all: in fact they are

$$\pm\alpha^{(1)}, \quad \pm\alpha^{(2)}, \quad \pm(\alpha^{(1)} + \alpha^{(2)}) .$$

The root diagram is a regular hexagon (see fig. 1).

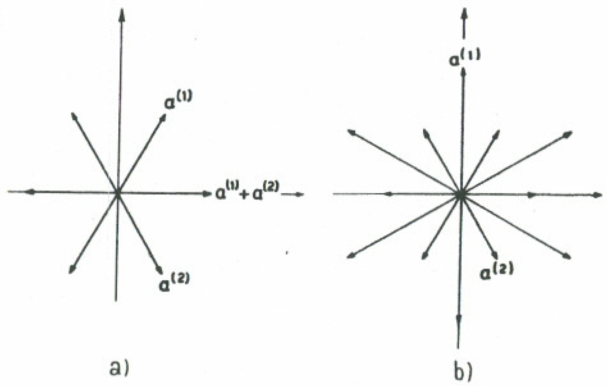


Fig. 1. a) Root diagramm of SU_3 b) Root diagram of G_2

In order to construct explicitly our Lie algebra, we evaluate now covariant components of α's (α_i) in a fixed frame of reference in $\mathscr{C}$. Each choice of the frame will lead us to a well defined set of structure constants in a certain basis $\{h_i, e_\alpha, e_{-\alpha}\}$. The most convenient choice is to refer α's to orthogonal axes, i.e. to axes such that:

$$(h_i, h_j) = g_{ij} = \sum_\alpha \alpha_i \alpha_j = \delta_{ij} .$$

In this case $\alpha^i = \alpha_i$.

In the notation of the fig. 2 (were we have relabeled the roots and for simplicity we use lower indices):

$$\alpha_1 = \left(\frac{1}{\sqrt{3}}, 0\right) \qquad g_{ij} = \sum_\alpha \alpha_i \alpha_j = \delta_{ij}$$

$$\alpha_2 = \left(\frac{1}{2\sqrt{3}}, \frac{1}{2}\right)$$

$$(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = (\alpha_3, \alpha_3) = \frac{1}{3}$$

$$\alpha_3 = \left(\frac{1}{2\sqrt{3}}, -\frac{1}{2}\right)$$

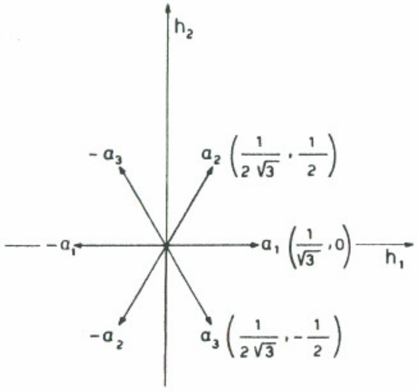


Fig. 2.

In terms of these components, we can write the following product rules:

$$[h_1, e_{\pm 1}] = \pm \frac{1}{\sqrt{3}} e_{\pm 1}, \quad [h_1, e_{\pm 2}] = \pm \frac{1}{2\sqrt{3}} e_{\pm 2}, \quad [h_1, e_{\pm 3}] = \pm \frac{1}{2\sqrt{3}} e_{\pm 3}$$

$$[h_2, e_{\pm 1}] = 0, \quad [h_2, e_{\pm 2}] = \pm \frac{1}{2\sqrt{3}} e_{\pm 2}, \; [h_1, e_{\pm 3}] = \mp \frac{1}{2} e_{\pm 3}$$

$$[e_1, e_{-1}] = \frac{1}{\sqrt{3}} h_1, \quad [e_2, e_{-2}] = \frac{1}{2\sqrt{3}} h_1 + \frac{1}{2} h_2, \quad [e_3, e_{-3}] = \frac{1}{2\sqrt{3}} h_1 - \frac{1}{2} h_2.$$

To complete the multiplication table we need the quantitaties $N_{\alpha\beta}$. By relations (11), (12) we see that we can arbitrarily fix only one sign in $N_{\alpha\beta}$; for example we can fix the sign of N_{23}. Using (13), we have

$$N_{23}^2 = \frac{q(r+1)}{2} (\alpha_3, \alpha_3) = \frac{1}{6}$$

being $q = 1$, $r = 0$. We choose

$$N_{23} = + \frac{1}{\sqrt{6}}$$

so that the remaining multiplication rules are

$$[e_2, e_3] = \frac{1}{\sqrt{6}} e_1 \qquad [e_2, e_{-1}] = -\frac{1}{\sqrt{6}} e_{-3}$$

$$[e_1, e_{-3}] = \frac{1}{\sqrt{6}} e_2 \qquad [e_1, e_{-2}] = -\frac{1}{\sqrt{6}} e_3$$

$$[e_3, e_{-1}] = \frac{1}{\sqrt{6}} e_{-2} \qquad [e_{-3}, e_{-2}] = \frac{1}{\sqrt{6}} e_{-1}.$$

The compact basis is:

$$\lambda_3 = -i\sqrt{3}\, h_1 \qquad \lambda_1 = -i\sqrt{\frac{3}{2}}\,(e_1 + e_{-1}) \qquad \lambda_2 = -\sqrt{\frac{3}{2}}\,(e_1 - e_{-1})$$

$$\lambda_8 = -i\sqrt{3}\, h_2 \qquad \lambda_4 = -i\sqrt{\frac{3}{2}}\,(e_2 + e_{-2}) \qquad \lambda_5 = -\sqrt{\frac{3}{2}}\,(e_2 - e_{-3})$$

$$\lambda_6 = -i\sqrt{\frac{3}{2}}\,(e_3 + e_{-3}) \qquad \lambda_7 = -\sqrt{\frac{3}{2}}\,(e_3 - e_{-3}).$$

We note that this basis differs from that given in (16) only by real factors which do not affect its compactness and have been introduced in order to have product rules of the form

$$[\lambda_l, \lambda_k] = \sum_m^{1,8} f_{lkm} \lambda_m \qquad l, k = 1, \dots 8$$

where f_{lkm} is the completely antisymmetric tensor given in [3]. f_{lkm} defines the structure constants of SU_3, (see sect. 1.7), which is then the compact group associated to A_2.

d) Calculation of roots

We outline here a method of calculating the roots of a simple Lie algebra based on properties IX, X of simple roots.

If α is a positive root we will say that α lies in the n^{th} level when:

$$\alpha = \sum_i k_i \alpha^{(i)}, \quad n = \sum_i k_i.$$

Property X) makes us sure that any root of the n^{th} level is obtained by adding a simple root to some positive root belonging to the $(n-1)^{\text{th}}$ level. In particular if the n^{th} level is empty, all the successive levels are also empty.

Suppose we know all roots up to the n^{th} level, and let $\alpha = \sum_j k_j \alpha^{(j)}$ belong to such level. Then we can ascertain whether $\alpha - l\alpha^{(k)}$, for any non negative integer l, ($\alpha^{(k)}$ is a simple root) is a root or not, so that we know the number r, relative to the $\alpha^{(k)}$-string containing α. Furthermore:

$$r - q = \frac{2(\alpha, \alpha^{(k)})}{(\alpha^{(k)}, \alpha^{(k)})} = \sum_i k_i \frac{2(\alpha^{(i)}, \alpha^{(k)})}{(\alpha^{(k)}, \alpha^{(k)})}$$

and the right hand side of this equation is a known number $(2(\alpha^{(i)}, \alpha^{(k)})/(\alpha^{(k)}, \alpha^{(k)}))$ is known from Dynkin diagram). In this way we obtain q, and if $q > 0$, $\alpha + \alpha^{(k)}$ is a root of the $(n+1)^{th}$ level. With this procedure, by (X), varying α and $\alpha^{(k)}$ we obtain all the $(n+1)^{th}$ level roots. Since we already know the roots of the 1^{th} level from Dynkin's diagram (i.e. the simple roots) and in addition $\alpha^{(i)} - \alpha^{(k)}$ is never a root when $\alpha^{(i)}$ and $\alpha^{(k)}$ are simple, this method can be used as a recurrence procedure to find all positive roots of the given algebra.
Let us try with G_2. The Dynkin diagram is:

and from (15) we obtain:

$$\frac{2(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)} = -1, \quad \frac{2(\alpha_1, \alpha_2)}{(\alpha_2, \alpha_2)} = -3.$$

1^{st} level: α_1, α_2

2^{nd} level: $\alpha_1 + \alpha_2$

3^{rd} level: $2\alpha_1 + \alpha_2$ is not a root, because:

$$\frac{2(\alpha_1 + \alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} = 2 - 1 = r - q;$$

but $r = 1$, so that $q = 0$.

$\alpha_1 + 2\alpha_2$ is a root.

4^{th} level: $2\alpha_1 + 2\alpha_2 = 2(\alpha_1 + \alpha_2)$ is not a root (by V).

$\alpha_1 + 3\alpha_2$ is a root

5^{th} level: $2\alpha_1 + 3\alpha_2$ is a root: in fact $\dfrac{2(\alpha_1 + 3\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} = -1 = r - q$

and $r = 0$, so that $q = 1$

$\alpha_1 + 4\alpha_2$ is not a root: $\dfrac{2(\alpha_1 + 3\alpha_2, \alpha_2)}{(\alpha_2, \alpha_2)} = 3 = r - q$

but $r = 3$ so that $q = 0$.

6^{th} level: $3\alpha_1 + 3\alpha_2 = 3(\alpha_1 + \alpha_2)$ is not a root

$2\alpha_1 + 4\alpha_2 = 2(\alpha_1 + 2\alpha_2)$ is not a root;

so that we end with 5^{th} level, and the positive roots are

$$\alpha_1;\ \alpha_2;\ \alpha_1 + \alpha_2;\ \alpha_1 + 2\alpha_2;\ \alpha_1 + 3\alpha_2;\ 2\alpha_1 + 3\alpha_2.$$

The corresponding root diagram is reported in fig. 1.

8.8. Compact groups associated to classical simple Lie algebras:

Cartan's denomination of $\mathscr{L}$	Compact group G associated to L	Dimension of G = dimension of $\mathscr{L}$
A_l	SU_{l+1}: unitary unimodular complex matrices in $(l+1)$-dimensions	$l(l+2)$
B_l	O_{2l+1}: real orthogonal group in $(2l+1)$ dimensions	$l(2l+1)$
C_l	$Sp(2l)$: unitary $2l$-dimensional matrices leaving invariant a non singular antisymmetric matrix I: $U^T I U = I$ (sympletic group)	$l(2l+1)$
D_l	O_{2l}: real orthogonal group in $2l$ dimensions	$l(2l-1)$

9. Representations of Semisimple Lie Algebras

9.1. We recall here that by representation of a Lie algebra into a complex linear space L we mean a linear mapping $x \to T(x)$ where $x \in \mathscr{L}$, $T(x)$ is a linear operator in L, satisfying the condition:

$$T([x, y]) = T(x)\,T(y) - T(y)\,T(x).$$

We will treat here only finite dimensional representations, for which the following Weyl's theorem applies:
I. Any finite-dimensional representation of a semisimple Lie algebra is completely reducible. Hence we can limit ourselves to irreducible representations.
Chosen a basis $\{h_i, e_\alpha, e_{-\alpha}\}$ in $\mathscr{L}$, we will indicate with $\{H_i, E_\alpha, E_{-\alpha}\}$ the corresponding operators in any given representation.
II. It is possible to choose among equivalent representation, a particular one in a Hilbert space, in which: $H_i^+ = H_i$ and $E_\alpha^+ = E_{-\alpha}$ [16]).

[16]) A representation of $\mathscr{L}$ gives us a representation of the associated compact real Lie algebra, which in turn generates a representation of the corresponding compact group. Call it $W(g)$. From what we said in sect. 3.4 we can always change $W(g)$ by an equivalence transformation $(W(g) \to W'(g) = A\,W(g)\,A^{-1})$ so to obtain an unitary representation.
Under the same equivalence transformation the operators F_i, F_α, G_α representing the compact basis (16) go into the operators:

$$F_i' = A\,F_i A^{-1} \quad \text{etc.}$$

which are antihermitian, so that $H_i, E_\alpha, E_{-\alpha}$ transform into operators satysfying:

$$(H_i')^+ = H_i', \ (E_\alpha')^+ = E_{-\alpha}'.$$

This result is not essential from a mathematical point of view, in that what really matters is the possibility of diagonalizing the operators H_i's which is assured by the fact that H_i's represent a Cartan subalgebra. (Continued on page 334.)