and the right hand side of this equation is a known number $(2(\alpha^{(i)}, \alpha^{(k)})/(\alpha^{(k)}, \alpha^{(k)}))$ is known from Dynkin diagram). In this way we obtain q, and if $q > 0$, $\alpha + \alpha^{(k)}$ is a root of the $(n+1)^{\text{th}}$ level. With this procedure, by (X), varying α and $\alpha^{(k)}$ we obtain all the $(n+1)^{\text{th}}$ level roots. Since we already know the roots of the 1^{th} level from Dynkin's diagram (i.e. the simple roots) and in addition $\alpha^{(i)} - \alpha^{(k)}$ is never a root when $\alpha^{(i)}$ and $\alpha^{(k)}$ are simple, this method can be used as a recurrence procedure to find all positive roots of the given algebra.
Let us try with G_2. The Dynkin diagram is:

and from (15) we obtain:

$$\frac{2(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)} = -1, \quad \frac{2(\alpha_1, \alpha_2)}{(\alpha_2, \alpha_2)} = -3.$$

1^{st} level: α_1, α_2

2^{nd} level: $\alpha_1 + \alpha_2$

3^{rd} level: $2\alpha_1 + \alpha_2$ is not a root, because:

$$\frac{2(\alpha_1 + \alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} = 2 - 1 = r - q;$$

but $r = 1$, so that $q = 0$.

$$\alpha_1 + 2\alpha_2 \quad \text{is a root.}$$

4^{th} level: $2\alpha_1 + 2\alpha_2 = 2(\alpha_1 + \alpha_2)$ is not a root (by V).

$\alpha_1 + 3\alpha_2$ is a root

5^{th} level: $2\alpha_1 + 3\alpha_2$ is a root: in fact $\dfrac{2(\alpha_1 + 3\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} = -1 = r - q$

and r $= 0$, so that $q = 1$

$\alpha_1 + 4\alpha_2$ is not a root: $\dfrac{2(\alpha_1 + 3\alpha_2, \alpha_2)}{(\alpha_2, \alpha_2)} = 3 = r - q$

but $r = 3$ so that $q = 0$.

6^{th} level: $3\alpha_1 + 3\alpha_2 = 3(\alpha_1 + \alpha_2)$ is not a root

$$2\alpha_1 + 4\alpha_2 = 2(\alpha_1 + 2\alpha_2) \text{ is not a root;}$$

so that we end with 5^{th} level, and the positive roots are

$$\alpha_1;\ \alpha_2;\ \alpha_1 + \alpha_2;\ \alpha_1 + 2\alpha_2;\ \alpha_1 + 3\alpha_2;\ 2\alpha_1 + 3\alpha_2.$$

The corresponding root diagram is reported in fig. 1.

8.8. Compact groups associated to classical simple Lie algebras:

Cartan's denomination of $\mathscr{L}$	Compact group G associated to L	Dimension of G = dimension of $\mathscr{L}$
A_l	SU_{l+1}: unitary unimodular complex matrices in $(l+1)$-dimensions	$l(l+2)$
B_l	O_{2l+1}: real orthogonal group in $(2l+1)$ dimensions	$l(2l+1)$
C_l	$Sp(2l)$: unitary $2l$-dimensional matrices leaving invariant a non singular antisymmetric matrix I: $U^T I U = I$ (sympletic group)	$l(2l+1)$
D_l	O_{2l}: real orthogonal group in $2l$ dimensions	$l(2l-1)$

9. Representations of Semisimple Lie Algebras

9.1. We recall here that by representation of a Lie algebra into a complex linear space L we mean a linear mapping $x \to T(x)$ where $x \in \mathscr{L}$, $T(x)$ is a linear operator in L, satisfying the condition:

$$T([x, y]) = T(x)\,T(y) - T(y)\,T(x).$$

We will treat here only finite dimensional representations, for which the following Weyl's theorem applies:
I. Any finite-dimensional representation of a semisimple Lie algebra is completely reducible. Hence we can limit ourselves to irreducible representations.
Chosen a basis $\{h_i, e_\alpha, e_{-\alpha}\}$ in $\mathscr{L}$, we will indicate with $\{H_i, E_\alpha, E_{-\alpha}\}$ the corresponding operators in any given representation.
II. It is possible to choose among equivalent representation, a particular one in a Hilbert space, in which: $H_i^+ = H_i$ and $E_\alpha^+ = E_{-\alpha}$[16]).

[16]) A representation of $\mathscr{L}$ gives us a representation of the associated compact real Lie algebra, which in turn generates a representation of the corresponding compact group. Call it $W(g)$. From what we said in sect. 3.4 we can always change $W(g)$ by an equivalence transformation $(W(g) \to W'(g) = A\,W(g)\,A^{-1})$ so to obtain an unitary representation.
Under the same equivalence transformation the operators F_i, F_α, G_α representing the compact basis (16) go into the operators:

$$F_i' = A\,F_i A^{-1} \quad \text{etc.}$$

which are antihermitian, so that $H_i, E_\alpha, E_{-\alpha}$ transform into operators satysfying:

$$(H_i')^+ = H_i',\ (E_\alpha')^+ = E_{-\alpha}'.$$

This result is not essential from a mathematical point of view, in that what really matters is the possibility of diagonalizing the operators H_i's which is assured by the fact that H_i's represent a Cartan subalgebra. (Continued on page 334.)

Then the H_i's, being commuting Hermitian operators, are simultaneously diagonalizable, and have real eigenvalues. If $M \equiv (M_1, \dots M_r)$ is a set of eigenvalues on a simultaneous eigenvector

$$H_i \, |M\rangle = M_i \, |M\rangle$$

M can be thought as an r-dimensional real vector (weight vector) by analogy with roots. Calling L_M the manifold spanned by eigenvectors belonging to the weight M, we have

$$L = \bigoplus L_M \text{ (the direct sum runs over all weights)}. \tag{1}$$

The L_M's, in general, are not one-dimensional, so that H_i's do not constitute a complete set of commuting operators. Hence some of the weights M can be degenerate.

III. No general prescriptions can be given to construct the operators commuting with the H_i's which remove this degeneracy.

However it can be shown [*20*] that their number is at most equal to

$$\eta = \frac{n - 3r}{2} \,^{17)} \qquad n = \text{dimension of } \mathscr{L} \qquad r = \text{rank of } \mathscr{L}$$

9.2. Properties of weights

Let $|M\rangle$ be a vector belonging to L_M, then by the commutation relations between H_i and E_α, we obtain:

$$H_i E_\alpha \, |M\rangle = (\alpha_i + M_i) \, E_\alpha \, |M\rangle. \tag{2}$$

Let us suppose $E_\alpha \, |M\rangle \neq 0$. Then (2) tells us that $M + \alpha \equiv (M_1 + \alpha_1 \dots M_r + \alpha_r)$ is a weight, and $E_\alpha \, |M\rangle$ belongs to $L_{M+\alpha}$. If $E_\alpha E_\alpha |M\rangle \neq 0$ we can repeat the reasoning concluding that $M + 2\alpha$ is a weight and that $E_\alpha E_\alpha \, |M\rangle$ belongs to $L_{M+2\alpha}$. By recurrence if $(E_\alpha)^k \, |M\rangle \neq 0$, then $M + (k\alpha)$ is a weight and $(E_\alpha)^k |M\rangle$ belongs to $L_{M+k\alpha}$. Being L finite dimensional this procedure must end, so that there exists an integer q such that $(E_\alpha)^q |M\rangle \neq 0$, i.e. $M + q\alpha$ is a weight, whereas $(E_\alpha)^{q+1} |M\rangle = 0$. By analogy we can work with $E_{-\alpha}$, obtaining an integer r such that

$$(E_{-\alpha})^r \, |M\rangle \neq 0$$

$$(E_{-\alpha})^{r+1} \, |M\rangle = 0.$$

From this follows that all the vectors

$$M - r\alpha, \dots M, \dots, M + q\alpha \tag{3}$$

are weights, but furthermore we have:

IV. these are the only weights of the form

$$M + k\alpha \quad (k = 0, \pm 1, \pm 2, \dots).$$

However because we will use in physical applications only unitary representations of the compact group associated to L, we have adopted this particular setting from the beginning.

17) In the case of A_2, $\eta = 1$ and we will give later the explicit expression of this operator which in the physical applications is identified with the square of the isotopic spin operator.

Hence weight vectors dispose into strings generated by roots and the E_α's behave as usual raising and lowering operators.

With the aid of the tensor g^{ij} we introduce a scalar products between weights and roots, as well as between weights:

$$(M, \alpha) = \sum_{ij} g^{ij} \alpha_i M_j \tag{4}$$

$$(M, M') = \sum_{ij} g^{ij} M_i M'_j. \tag{5}$$

V. If r and q are the integers defined through (3), we have

$$\frac{2(M, \alpha)}{(\alpha, \alpha)} = r - q \tag{6}$$

hence (see 8.3, VI):

$$M - \frac{2(M, \alpha)\,\alpha}{(\alpha, \alpha)} \text{ is a weight}. \tag{7}$$

We note that the close resemblance between the stated properties of weights, and the properties of roots listed in sect. 8.3 is not surprising in that roots are simply the weights of a particular representation of $\mathscr{L}$, i.e. the regular representation.

We introduce now an ordering between the weights of an arbitrary representation.

We recall that the r-simple roots constitute a basis in the space of the r-dimensional real vectors, so that for any weight M we can write:

$$M = \sum_i M_i \alpha^{(i)} \quad (\alpha^{(i)} = i^{\text{th}} \text{ simple root}). \tag{8}$$

We will say that $M > M'$ if the first non zero component of the vector $M - M'$ is greater than zero. Since there is only a finite number of distinct weights for any representation, among them there is a maximal weight, i.e. a weight which is greater than all the others. This definition has the consequence that if α is a positive root and $|\Lambda\rangle$ an eigenvector belonging to the maximal weight Λ, then $E_\alpha |\Lambda\rangle = 0$. (Otherwise $E_\alpha |\Lambda\rangle$ would be a vector belonging to the weight $\Lambda + \alpha$ which, since $\alpha > 0$, is greater than Λ).

Let R be a representation of $\mathscr{L}$ in the linear space L, and $|\Lambda, 1\rangle, |\Lambda, 2\rangle \dots |\Lambda, k\rangle$, be independent eigenvectors belonging to the maximal weight Λ. Consider the subspace l^1 spanned by vectors

$$E_{-\alpha} E_{-\beta} E_{-\gamma} \dots |\Lambda, 1\rangle \; (\alpha, \beta, \gamma, \dots \text{ positive roots}) \tag{9}$$

obtained applying to $|\Lambda, 1\rangle$ all finite products of $E_{-\alpha}$'s (including repetitions of the same operators). We claim that l^1 is invariant and irreducible.

In fact it is invariant under H_i's, and $E_{-\alpha}$'s $(\alpha > 0)$ whereas applying some $E_\alpha (\alpha > 0)$ to a vector of the form (9) we can move, using commutation relations, E_α to the right, until it reaches $|\Lambda, 1\rangle$ producing zero, and leaving a combination of vectors of the form (9). (It may happen that by commuting E_α with some $E_{-\beta}$ we obtain some $E_{\alpha-\beta}$ such that $\alpha - \beta > 0$. In this case we begin to move to the right $E_{\alpha-\beta}$ until it reaches $|\Lambda, 1\rangle$). Hence when R is irreducible, $l^1 \equiv R$. In l^1 there is only one independent vector with weight Λ. In fact any eigenvector of

H_i's is a linear combination of vectors (9) differing only by the order in which E_α's appear. The corresponding weight is

$$\Lambda - k_\alpha \alpha - k_\beta \beta - k_\gamma \gamma - \cdots = \Lambda - \sum_{\alpha>0} k_\alpha \alpha \ (k_\alpha \geq 0),$$

where k_α is the number of times $E_{-\alpha}$ appears in (9).
An eigenvector belonging to Λ is obtained when

$$\Lambda - \sum_{\alpha>0} k_\alpha \alpha = \Lambda \quad \text{i.e.} \quad \sum_{\alpha>0} k_\alpha \alpha = 0,$$

which implies $k_\alpha = 0$; hence it must be proportional to $|\Lambda, 1\rangle$.
From this it follows the irreducibility of l^1. In fact suppose l^1 to be reducible. Weyl's theorem (5.1) implies l^1 to be completely reducible. For example suppose:

$$l^1 = v_1 \oplus v_2 \ (v_1, v_2 \text{ invariant irreducible subspaces}).$$

Then:

$$|\Lambda, 1\rangle = |\Lambda, v_1\rangle + |\Lambda, v_2\rangle, \quad |\Lambda, v_r\rangle \in v_r$$

and in addition

$$H_i |\Lambda, 1\rangle = \Lambda_i |\Lambda, 1\rangle = H_i |\Lambda, v_1\rangle + H_i |\Lambda, v_2\rangle$$

i.e.

$$(H_i - \Lambda_i) |\Lambda, v_1\rangle + (H_i - \Lambda_i) |\Lambda, v_2\rangle = 0$$

$$(H_i - \Lambda_i) |\Lambda, v_1\rangle \in v_1; \ (H_i - \Lambda_i) |\Lambda, v_2\rangle \in v_2 \ \text{(by invariance of } v_1, v_2)$$

so that

$$(H_i - \Lambda_i) |\Lambda, v_1\rangle = 0$$

and the same for $|\Lambda, v_2\rangle$. Hence in l^1 there exist two independent vectors $|\Lambda, v_1\rangle$, $|\Lambda, v_2\rangle$ belonging to the weight Λ, which is impossible.
Hence $|\Lambda, v_1\rangle$ or $|\Lambda, v_2\rangle$ must vanish. Suppose $|\Lambda, v_1\rangle \neq 0$, then

$$|\Lambda, 1\rangle = |\Lambda, v_1\rangle$$

which implies $l^1 \subset v_1$ whereas by hypothesis $l^1 \supset v_1$. We conclude that $l^1 \equiv v_1$, $v_2 \equiv 0$, and l^1 is irreducible.
The great relevance of the concept of maximal weight suggested in part by previous considerations can be appreciated from the following theorems due to Cartan.
VI. Two irreducible representations having the same maximal weight are equivalent.
VII. An r-component vector Λ is the maximal weight for some irreducible representation of $\mathscr{L}$ if and only if

$$\Lambda_{\alpha_i} = \frac{2(\Lambda, \alpha^{(i)})}{(\alpha^{(i)}, \alpha^{(i)})}$$

is a non negative integer for any simple root $\alpha^{(i)}$ of $\mathscr{L}$.
Hence, once we have chosen a set of simple roots $\alpha^{(i)}$, any collection of non negative integers $(\Lambda_{\alpha_1} \Lambda_{\alpha_2}, \ldots \Lambda_{\alpha_r})$ uniquely defines an irreducible representation of $\mathscr{L}$ and all representations are obtained in this way.
VIII. If Λ is a maximal weight of a given irreducible representation of $\mathscr{L}$, then any other weight M has the form

$$M = \Lambda - \sum_i k_i \alpha^{(i)} \tag{10}$$

$$\alpha^{(i)} = i^{\text{th}} \text{ simple root}$$

$$k_i = \text{non negative integer}.$$

(The proof of this theorem easily follows from considerations preceeding result VI, and from property 8.4 X of simple roots).
(10) is quite analogous to X sect. 8.4 and a method similar to that devised in sect. 8.7 d can be based on it in order to construct all the weights of a given representation.
We dispose weights into levels according to the value of $\sum_i k_i$. Λ is assigned to the zero level.
A result analogous to sect. 8.4 XI holds, i.e.: if $M \neq \Lambda$ is a weight then there is at least one simple root such that $M + \alpha$ is a weight. This assures us that all the weights of the $(n+1)^{\text{th}}$ level are obtained subtracting some simple root to some weight of the n^{th} level and that when we reach an empty level, all the successive ones are unoccupied. Now suppose we know all the weights up to the n^{th} level and M be a weight belonging to such level. Then $M - \alpha^{(k)}$ is a weight if the integer r relative to the $\alpha^{(k)}$-string containing M is greater than zero. Moreover

$$r - q = \frac{2(M, \alpha^{(k)})}{(\alpha^{(k)}, \alpha^{(k)})} = \Lambda_{\alpha_k} - \sum_i k_i \frac{2(\alpha^{(i)}, \alpha^{(k)})}{(\alpha^{(k)}, \alpha^{(k)})}.$$

Now q is a known number (because we know all the weights up to the n^{th} level) and so is the right hand side of this equation, so that we can ascertain whether $M - \alpha^{(k)}$ is a weight or not. By varying M and $\alpha^{(k)}$ we obtain all the weights belonging to the $(n+1)^{\text{th}}$ level.
Hence starting from the zero level, i.e. from the maximal weight (there are no negative levels because $\Lambda + \alpha^{(k)}$ is not a weight for any $\alpha^{(k)}$) by this recurrence method we can construct all the weights of the representation. This method does not provide for each weight M the corresponding multiplicity, i.e. the dimensionality of the manifold L_M appearing in (1). Being the representation determined, up to an equivalence, by its maximal weight Λ, these multiplicities must be derivable from Λ, M and from the roots, but no simple formula can be given for them. Instead we will give later for SU_3 a simple rule which allows one to read directly these multiplicities from the weight diagram.
9.2.a) Till now we have analyzed properties which are the same for equivalent representations (in particular weight diagrams).
In practical calculations it is necessary to pick-up from each equivalence class a particular representative, i.e. standard matrix representation of the elements h_i, e_α, $e_{-\alpha}$. It is convenient to choose a representation in which a basis is constituted by normalized eigenvectors of H_i's as well as of the other commuting operators which are necessary to remove degeneracies (sect. 9.1). Of course with this choice the H_i's are represented by diagonal matrices, with coefficients determined by the weight diagram.
Using the commutation relations and the string property of weights, one can determine the matrix elements of E_α, $E_{-\alpha}$ up to certain phase factors. The procedure is quite analogous, although considerably more complicated, to that employed in usual angular momentum theory [*8*]. It is then necessary to make a definite phase convention. At the same time this convention fixes phases in the Clebsch-Gordan coefficients.

9.3.a) Weight diagram of SU_2

The Lie algebra of SU_2 has rank one, so that its irreducible representations are characterized by a single non negative integer Λ_{α_1}. For any maximal weight Λ, we have:

$$\frac{2(\Lambda, \alpha^{(1)})}{(\alpha^{(1)}, \alpha^{(1)})} = \Lambda_{a_1} = \frac{2\,\frac{1}{g}\,\alpha_1\Lambda_1}{\frac{1}{g}\,\alpha_1\alpha_1} = \frac{2\Lambda_1}{\alpha_1}.$$

Since $\alpha_1 = 1$ (sect. 8.7a) we see that Λ_1 i.e. the covariant component of the maximal weight along $\alpha^{(1)}$ is an integer or half integer number which we call j. There is only one string i.e. the $\alpha^{(1)}$-string containing Λ, so that by applying $E_{-\alpha^{(1)}}$ to the eigenvector belonging to Λ, we generate all the vector space of the representation. There are $2j+1$ independent vectors at all, i.e.

$$|\Lambda\rangle,\ E_{-\alpha^{(1)}}|\Lambda\rangle,\ (E_{-\alpha^{(1)}})^2|\Lambda\rangle, \ldots, (E_{-\alpha^{(1)}})^{2j}|\Lambda\rangle;\ (E_{-\alpha^{(1)}})^{2j+1}|\Lambda\rangle = 0$$

so that an irreducible representation with $\Lambda_{a1} = 2j$ is $(2j+1)$-dimensional. The weight diagram is one dimensional and has the form:

$$-j \quad -j{+}1 \quad -j{+}2 \quad \cdots \quad j{-}2 \quad j{-}1 \quad j$$

so that the eigenvalues of $h_3 = iI_3$ range from j to $-j$.

9.3.b) Weight diagrams of SU_3

Being SU_3 of rank two, its irreducible representations can be labeled by two non negative integers (m, n) where, with the notations of fig. 2, we have

$$m = \frac{2(\Lambda, \alpha_2)}{(\alpha_2, \alpha_2)} \qquad n = \frac{2(\Lambda, \alpha_3)}{(\alpha_3, \alpha_3)} \tag{11}$$

and the corresponding weight diagrams are two-dimensional.
Now α_2, α_3 are linearly independent, so that we can write

$$\Lambda = n_2\alpha_2 + n_3\alpha_3$$

and, by taking the scalar product of both members with α_2 and with α_3, we obtain

$$m = 2n_2 - n_3$$

$$n = -n_2 + 2n_3$$

i.e.

$$n_2 = \frac{2m+n}{3} \qquad n_3 = \frac{m+2n}{3}.$$

Hence with respect to the basis chosen for roots in sect. 8.7c) we have

$$\Lambda \equiv \left(\frac{m+n}{2\sqrt{3}}\,;\ \frac{m-n}{6}\right). \tag{12}$$

These components are respectively the eigenvalues of H_1 and H_2 on the manifold L_Λ. We will construct now the weight diagram for an arbitrary irreducible representation (m, n) with a graphical method which lead to the resultmore quickly than the general one, outlined in sect. 9.2.

i) We draw with respect to two orthogonal axes the vector Λ of components $(m + n/2\sqrt{3},\ m - n/6)$ as well as the two simple roots α_2, α_3 of components

$$(1/2\sqrt{3},\ 1/2);\ (1/2\sqrt{3},\ -1/2).$$

(see fig. 3).

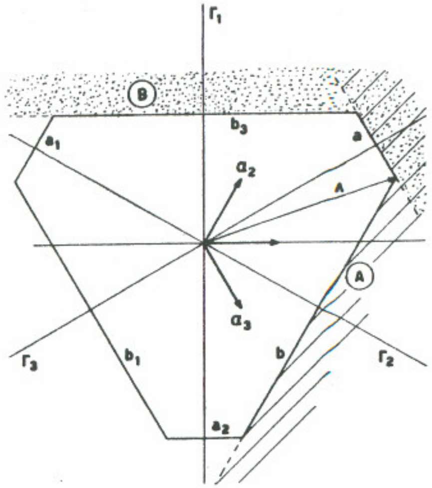


Fig. 3.

The α_2-string containing Λ consists of the $m+1$ weights

$$M_i = \Lambda - i\alpha_2;\ 0 \leqq i \leqq m.$$

The end point of M_i is obtained by reporting i times the vector $-\alpha_2$ starting from the end point of Λ. All these points lie on the segment b, spacing between two consecutive points being equal to $|\alpha_2|$, so that the lenght of b is equal to $m|\alpha_2|$. By considering the α_3-string containing Λ we obtain the segment a, of lenght $n|\alpha_3| = n|\alpha_3|$ in an analogous way.

ii) There are no weights ending in the dashed region A: in fact all the weights must be of the form

$$M = \Lambda - k\alpha_2 - h\alpha_3, \quad k, h \geqq 0.$$

iii) From (7) we see that if α is a root, by reflecting a weight through an axis orthogonal to α we obtain another weight, so that the weight diagram goes into itself by such operation.
Let us indicate with r_1, r_2, r_3 the axes through the origin orthogonal to $\alpha_1, \alpha_2, \alpha_3$. Reflection through r_3 must carry a into itself (so that r_3 intersects a in its midpoint) and send b into b_3. Hence in the region B there are no weights (by reflecting such weights through r_3 we would obtain weights ending in A). Furthermore b_3 contains the end points of new weights which are the reflected of those ending in b.
Segments a_2, a_1, b_1 are obtained by reflections through r_2 and r_1. The figure so obtained has the properties that no weight ends outside it and there are weights ending on its vertices and on its sides, distances between two consecutive end points being equal to $|\alpha_1| = |\alpha_2|$.
Let us see now how we can find all the weights of the diagram.
Consider an arbitrary weight $M = \Lambda - i\alpha_2 - h\alpha_3$ (i, h non negative integers). When $i \leqq m$, $\Lambda - i\alpha_2$ is a weight ending on b, so that all weights M with $i \leqq m$ can be obtained from strings starting from weights ending on b. When $i \geqq m$, we write

$$M = \Lambda - m\alpha_2 - (i-m)(\alpha_2 + \alpha_2) - [h - (i-m)]\alpha_3,$$

and now, being

$$0 \leq i - m \leq n$$

the vector

$$M_{i-m} = \Lambda - m\alpha_2 - (i - m)(\alpha_2 + \alpha_3)$$

is a weight ending on a_2, so that M belongs to the α_3 string containing M_{i-m}. Hence the α_3 strings of the weights ending on b and a_2 generate the whole diagram,

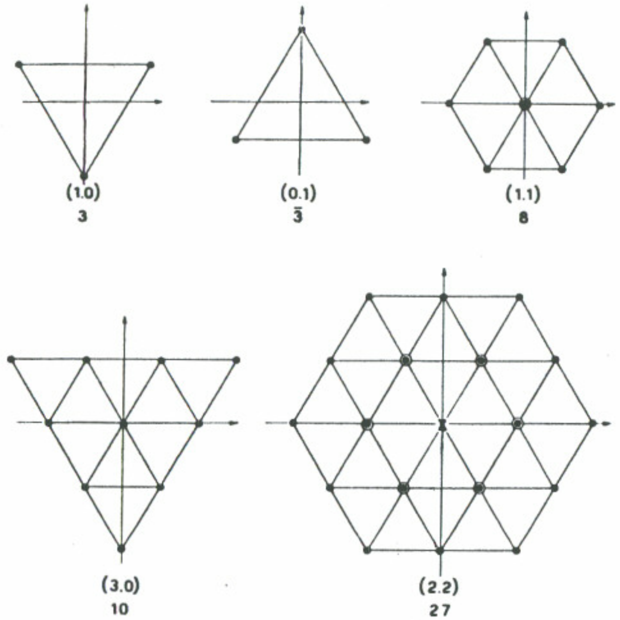


Fig 4. Weight diagrams of some SU_3 representations

and it is easy to see that the end points of the weights so obtained fill up the whole hexagon with a triangular pattern as shown in fig. 4. In addition this figure shows that the points are disposed in layers. Now the following rule applies:
Weights of the external layer are all simple, i.e. not degenerate.
Starting from the external layer, multiplicity increases by one at each layer, until a triangular path is reached: when this occurs multiplicity does not increase any more. When $n = 0$ or $m = 0$ the hexagon degenerates in a triangle and all the weights are simple.
When $n = m$ the diagram is a regular hexagon and multiplicity increases until one reach the last layer, which is constituted by a single point. In fig. 4 are given the weight diagrams of some between the most used representations of SU_3, with the respective multiplicities.
We said before that in any irreducible representation of SU_2, there is one operator, commuting with H_1 and H_2, which removes all the degeneracies.
It can be seen that the operator

$$T^2 = 3(H_1^2 + E_1 E_{-1} + E_{-1} E_1) \tag{13}$$

has just these properties. We note that

$$T^2 = H_1'^2 + E_1' E_{-1}' + E_{-1}' E_1' \qquad H_1' = \sqrt{3} H_1, \quad E_{\pm 1}' = \sqrt{3} E_{\pm 1}$$

where H_1', E_1', E_{-1}' satisfy the same commutation relations as the basis elements of a representation of A_1 (in fact they correspond to the elements h_1', e_1', e_{-1}' defined in sect. 8.7b). In any irreducible representation (m, n) of SU_3, the operators H_1', E_1', E_{-1}' generate by themselves a representation of SU_2, in general reducible. The irreducible subrepresentations of SU_2 inside (m, n) are characterized by the eigenvalue $T(T+1)$ of T^2(T = integer or half integer number). Considering beside T^2 the operator $2H_2$, one can show that in any (m, n) representation, for any pair of integers f, g such that

$$m + n \geq f \geq m \geq g \geq 0$$

there is exactly one SU_2 subrepresentation with

$$T = \frac{f - g}{2}, \qquad 2H_2 = f + g - \frac{2}{3}(m + 2n)$$

(Weyl's branching law).
For example in the (1, 1) representation there are:

1 submultiplet with	$T = 1$	$2H_2 = 0$
1 ,, ,,	$T = 1/2$	$2H_2 = 1$
1 ,, ,,	$T = 1/2$	$2H_2 = -1$
1 ,, ,,	$T = 0$	$2H_2 = 0$.

We shall see later that in the eightfold way model the operators H_1', T^2, H_2' are identified with T_3' T^2, Y, so that this rule gives us a decomposition of each SU_3 supermultiplet into isospin multiplets.

9.4. Tensor product of representations

Let ϱ_1 and ϱ_2 be two irreducible representations of $\mathcal{L}$ into the linear spaces L_1 and L_2:

$$x \in \mathcal{L} \quad x \to \varrho_1(x); \; \varrho_1(x) = \text{linear operator in } L_1$$

$$x \to \varrho_2(x); \; \varrho_2(x) = \text{linear operator in } L_2$$

and let $M^1, M^2, \ldots; N^1, N^2, \ldots;$ be the weights of the two representations, $|M^1\rangle, |M^2\rangle, \ldots |N^1\rangle, |N^2\rangle, \ldots$ the bases formed with the corresponding eigenvectors (for sake of simplicity we do not write explicitly the eigenvalues of the additional operators needed to remove all the degeneracies: they are however understood):

$$\varrho_1(hi)\,|M^k\rangle = H_i^{(1)}\,|M^k\rangle = M_i^k\,|M^k\rangle \tag{14}$$

$$\varrho_2(hi)\,|N^l\rangle = H_i^{(2)}\,|N^l\rangle = N_i^l\,N^l\rangle.$$

Then in the tensor product space $L = L_1 \otimes L_2$ which is spanned by the basis

$$|M^k\rangle\,|N^l\rangle$$

the tensor product representation ϱ of $\mathscr{L}$ is defined as

$$x \to (\varrho_1 \otimes \varrho_2)(x) = \varrho(x) \quad \varrho(x) = \text{linear operator in } L$$

$$(\varrho_1 \otimes \varrho_2)(x) \ |M^k\rangle |N^l\rangle = \varrho(x) |M^k\rangle |N^l\rangle = (\varrho_1(x) |M^k\rangle) |N^l\rangle + |M^k\rangle (\varrho_2(x) |N^l\rangle).$$

Recalling the definition of tensor product of two operators given in sect. 5.1 we see that

$$\varrho(x) = \varrho_1(x) \otimes 1^{(2)} + 1^{(1)} \otimes \varrho_2(x) \tag{15}$$

where $1^{(i)}$ is the identity operator in L_i. With this definition, the representation of the compact Lie group associated to $\mathscr{L}$ which is generated by ϱ, is just the tensor product of the representations of the same group generated by ϱ_1 and ϱ_2, as defined in sect. 5.1.
For the elements $\varrho(h_i)$ we have

$$\varrho(h_i) = H_i^{(1)} \otimes 1^{(1)} + 1^{(1)} \otimes H_i^{(2)}$$

so that

$$\varrho(h_i) |M^k\rangle |N^l\rangle = (M_i^k + N_i^l) |M^k\rangle |N^l\rangle,$$

i.e. the weights of ϱ are obtained by adding together the weights of ϱ_1 and ϱ_2 in all possible ways. In particular for the greatest weight we have

$$\Lambda = \Lambda^1 + \Lambda^2$$

$$\Lambda_{\alpha_i} = \Lambda^1_{\alpha_i} + \Lambda^2_{\alpha_i},$$

(for any simple root $\alpha^{(i)}$) where Λ^1 and Λ^2 are the maximal weights of ϱ_1 and ϱ_2. Moreover the eigenspace corresponding to Λ is always one-dimensional, and it is spanned by the vector

$$|\Lambda^1\rangle |\Lambda^2\rangle.$$

In general ϱ splits up in a direct sum of irreducible components

$$\varrho = \bigoplus_{\Lambda'} \varrho_{\Lambda'}. \tag{16}$$

Each of them, according to VI, is characterized by its maximal weight Λ', which we have chosen as a label in (16). In general in this formula will appear many irreducible equivalent components, i.e. terms with the same Λ'. Although general formulae can be given, characterizing which are the irreducible components and how many times they appear in (16) [*18*] yet these formulae are extremely complicated[18]) and are not used in practice[19]).

[18]) This is not the case of $A_1(SU_2)$ for which the decomposition (16) is explicity given by the well know Clebsch-Gordan formula:

$$\varrho = (\varrho_{j_1} \otimes \varrho_{j_2}) = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \varrho_j.$$

[19]) In practical calculations we need not only the decomposition (16), but also explicitly the matrix connecting the basis $|M^k\rangle\ |N^l\rangle$ to the basis spanning the irreducible components, i.e. we need all the Clebsch-Gordan coefficients involved. Such coefficients in general have not been calculated; however in the case of SU_3 they are tabulated [*29, 33*] for all tensor product of interest in physics.

Anyway it is a very simple task to isolate a particular term in (16), i.e. the component ϱ_Λ where Λ is the greatest weight of ϱ. In fact if we construct the manifold spanned by the vectors

$$E_{-\alpha} E_{-\beta} E_{-\gamma} \ldots |\Lambda\rangle$$

($\alpha, \beta, \gamma, \ldots$ = positive roots: $E_{-\alpha}, E_{-\beta}, E_{-\gamma}, \ldots$ lowering operators of ϱ) we obtain an irreducible invariant subspace (see sect. 9.2) and obviously if we consider the restriction of ϱ to this manifold, we obtain a representation having the maximal weight equal to Λ. Being Λ simple, ϱ_Λ occurs only once in (16).
Consider now the r-irreducible inequivalent representations ϱ_i with maximal weights $\Lambda^{(i)}$, such that

$$\frac{2(\Lambda^{(i)}, \alpha^{(k)})}{(\alpha^{(k)}, \alpha^{(k)})} = \delta_{ik}.$$

Then we have: any irreducible representation ϱ identified by the set of non negative integers $(\Lambda_{\alpha_1}, \Lambda_{\alpha_2}, \ldots \Lambda_{\alpha_r})$ can be obtained by making the tensor product

$$\underbrace{\varrho_1 \otimes \varrho_1 \cdots}_{\Lambda_{\alpha_1} \text{ terms}} \quad \underbrace{\otimes \varrho_2 \otimes \varrho_2 \cdots}_{\Lambda_{\alpha_2} \text{ terms}} \quad \underbrace{\otimes \varrho_2 \otimes \varrho_2 \cdots}_{\Lambda_{\alpha_r} \text{ terms}}, \tag{17}$$

and isolating the irreducible component of greatest weight. In fact this component belongs to the weight

$$\Lambda = \Lambda_{\alpha_1} \Lambda^{(1)} + \Lambda_{\alpha_2} \Lambda^{(2)} + \cdots + \Lambda_{\alpha_r} \Lambda^{(r)}$$

which is just the maximal weight of ϱ.

9.5. Contragradient representation

Given a representation ϱ of $\mathscr{L}$ in a linear space L, we can construct another representation $\bar{\varrho}$ which is called the contragradient (or adjoint) of ϱ.
We first fix in L a basis in which the operators $\varrho(x)$ are represented by certain matrices $(\varrho(x))^i_j$; then we consider a linear space L^* having the same dimensionality of L. The representation $\bar{\varrho}$ in L^* is constituted by the operators $\bar{\varrho}(x)$ which, with respect to a basis fixed in L^*, are represented by the matrices

$$(\bar{\varrho}(x))^i_j = -(\varrho(x))^j_i = -[\varrho(x)^T]^i_j. \tag{18}$$

Operators $\bar{\varrho}$ defined in (18) will be symbolically written as

$$\bar{\varrho} = -\varrho^T.$$

For the elements of $\mathscr{L}$: $h_i, e_\alpha, e_{-\alpha}$, we have

$$\left.\begin{aligned} h_i &\to \varrho(h_i) = H_i \\ e_\alpha &\to \varrho(e_\alpha) = E_\alpha \end{aligned}\right\} \text{ for the representation } \varrho$$

$$\left.\begin{aligned} h_i &\to \bar{\varrho}(h_i) = -H_i^T \\ e_\alpha &\to \bar{\varrho}(e_\alpha) = -E_\alpha^T \end{aligned}\right\} \text{ for the representation } \bar{\varrho},$$

so that the eigenvalue of $\bar{\varrho}(h_i)$ are just minus one times the eigenvalues of $\varrho(h_i)$, i.e. the weight diagram of $\bar{\varrho}$ is obtained by reflecting through the origin the diagram of ϱ. In particular $\bar{\varrho}$ is equivalent to ϱ if and only if its weight diagram is invariant under this reflection.

Consider a vector $x \in L$ with components (x^i) and a vector $y \in L^*$ with components (y_i): by definition applying transformations ϱ and $\bar{\varrho}$, we have

$$x'^i = \sum_k \varrho^i_k x^k$$

$$y'_i = \sum_k - \varrho^k_i y_k .$$

Having L and L^* the same dimension, there exists always a one-to-one correspondence between their elements. Let

$$x = A y \quad x \in L, y \in L^*$$

be such correspondence. If we transform x with ϱ and y with $\bar{\varrho}$, the resulting vectors will not be connected by A, unless ϱ is equivalent to $\bar{\varrho}$

$$x' = \varrho x, \; y' = \bar{\varrho} y, \quad \text{then}$$

$$x' = A y', \quad \text{implies}$$

$$\varrho = A \bar{\varrho} A^{-1}.$$

Hence only if $\varrho \sim \bar{\varrho}$ we can identify L and L^* in a way which is invariant under ϱ and $\bar{\varrho}$.

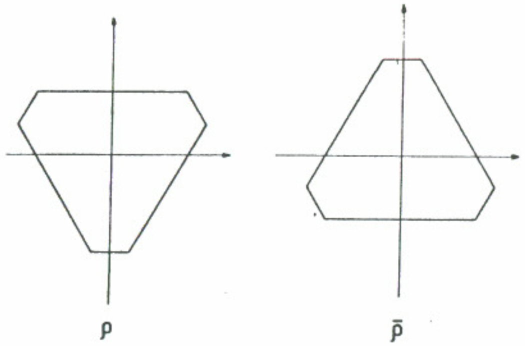


Fig. 5.

We note that the definition of adjoint representation has its analogous if we consider the group associated with $\mathscr{L}$, in that, if

$$g \to T(g)$$

is the representation of this group generated by ϱ, then

$$g \to \bar{T}(g) = (T(g)^{-1})^T$$

is that generated by $\bar{\varrho}$.

In the case of SU_3 we note that all weights diagrams are symmetrical for reflections around y-axis (this axis is in fact orthogonal to the root α_1) so that, to obtain the diagram of the adjoint representation, we have only to reverse it with respect to the x-axis.

A glance to figure 5 is sufficient to conclude that if ϱ is the representation (m, n), then $\bar{\varrho}$ is (n, m). In particular ϱ is equivalent to $\bar{\varrho}$ if and only if $m = n$ (in this case in fact they have the same maximal weight).

9.6. Explicit construction of SU_3 representations

The 3×3 antihermitian traceless matrices

$$\lambda_1 = -\frac{i}{2}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \quad \lambda_2 = -\frac{i}{2}\begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \quad \lambda_3 = -\frac{i}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\lambda_4 = -\frac{i}{2}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix} \quad \lambda_5 = -\frac{i}{2}\begin{pmatrix}0 & 0 & i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix} \quad \lambda_6 = -\frac{i}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \tag{19}$$

$$\lambda_7 = -\frac{i}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix} \quad \lambda_8 = -\frac{i}{2\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}$$

satisfy the commutation relations given in sect. 8.7c so that they are a 3-dimensional representation of the compact basis of A_2, and their real combinations span a representation (in fact irreducible) of the Lie algebra of SU_3. The operators representing h_1 and h_2, are (see sect. 8.7c)

$$H_1 = \frac{i\lambda_3}{\sqrt{3}} = \frac{1}{2\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$H_2 = \frac{i\lambda_8}{\sqrt{3}} = \frac{1}{6}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}.$$

From these we obtain three weights

$$\Lambda^1 = \left(\frac{1}{2\sqrt{3}}, \frac{1}{6}\right); \quad \Lambda^2 = \left(-\frac{1}{2\sqrt{3}}, \frac{1}{6}\right); \quad \Lambda^3 = \left(0, -\frac{1}{3}\right).$$

By considering the simple roots α_2 and α_3 we easily see that Λ^1 is the greatest weight and that

$$\frac{2(\Lambda^1, \alpha_2)}{(\alpha_2, \alpha_2)} = 1 \qquad \frac{2(\Lambda^1, \alpha_3)}{(\alpha_3, \alpha_3)} = 0,$$

so that this representation is the (1,0) one, whose diagram is reported in fig. 4.

The adjoint representation is obtained by taking the matrices $-\Lambda_i^T$, and is the (0,1) representation, inequivalent to the (1,0) one. These representations which are the fundamental ones, being the only 3-dimensional representations of A_2 will be called 3 and $\overline{3}$.
By taking the matrices

$$T(\alpha_1 \ldots \alpha_8) = e^{\sum_k^{1,8} \alpha_k \lambda_k} \qquad (\alpha_k \text{ real numbers}),$$

we obtain the set of 3×3 unitary unimodular matrices which constitute a group of operators in a 3-dimensional complex space. This group is by definition SU_3.
Representation $\overline{3}$ generates the operators

$$\overline{T}(\alpha_1 \ldots \alpha_1) = e^{-\sum_k \alpha_k \lambda_k^T} = \overline{\left(e^{\sum_k \alpha_k \lambda_k}\right)}.$$

This is the representation which to each matrix of SU_3 associates the complex conjugate matrix.
The linear spaces of representations 3 and $\overline{3}$ are

L_3: general vector x indicated as (x^1, x^2, x^3)

L_3^*: ,, ,, y ,, ,, (y_1, y_2, y_3).

If U_{ik} is a matrix of SU_3, then for the representation 3 we have

$$U \to T(U): x' = T(U)x$$
$$x'^i = \sum_k U_{ik} x^k,$$

whereas for $\overline{3}$

$$U \to \overline{T}(U), \quad y' = \overline{T}(U)y$$
$$y'_i = \sum_k \overline{U}_{ik} y_k.$$

The tensor product

$$L_n^m = \underbrace{L_3 \otimes L_3 \otimes \ldots L_3}_{m} \otimes \underbrace{L_3^* \otimes L_3^* \ldots \otimes L_3^*}_{n}$$

is just the vector space of the 3^{m+n} components objects (tensors) (sect. 5.1)

$$X_{j_1 \ldots j_n}^{i_1 \ldots i_m} \qquad (i_1, \ldots i_m; j_1, \ldots j_n = 1,2,3),$$

and the product representation of SU_3 in this space is constituted by the operators $T_{(n)}^{(m)}(U)$ defined as (sum over repeated indices is understood)

$$(T_{(n)}^{(m)}(U)X)_{j_1 \ldots j_n}^{i_1 \ldots i_m} = U_{i_1 i_1'} U_{i_2 i_2'} \ldots U_{i_m i_m'} \overline{U}_{j_1 j_1'} \ldots \overline{U}_{j_n j_n'} X_{j_1' \ldots j_n'}^{i_1' \ldots i_m'}. \tag{20}$$

From (15) sect. 9.4 we see that the compact basis of the Lie algebra of SU_3 is represented by the operators $\Lambda^{(k)}$ acting as

$$\begin{aligned}(\Lambda^{(k)}X)_{j_1 \ldots j_n}^{i_1 \ldots i_m} = (&\lambda_{i_1 i_1'}^{(k)} \delta_{i_2 i_2'} \ldots \delta_{i_m i_m'} \delta_{j_1 j_1'} \ldots \delta_{j_n j_n'} + \delta_{i_1 i_1'} \lambda_{i_2 i_2'}^{(k)} \times \\ &\times \delta_{i_3 i_3'} \ldots \delta_{j_n j_n'} + \ldots - \delta_{i_1 i_1'} \ldots \delta_{i_m i_m'} \lambda_{j_1' j_1}^{(k)} \times \\ &\times \delta_{j_2 j_2'} \ldots - \delta_{i_1 i_1'} \ldots \delta_{j_{n-1} j_{n-1}'} \lambda_{j_n' j_n}^{(k)}) X_{j_1' \ldots j_n'}^{i_1' \ldots i_m'},\end{aligned}$$

where $\lambda_{ij}^{(k)}$ are the matrices listed in (19).

9.7. Let us consider the tensor product representation (reducible when $k > 1$):

$$(3)^k = \underbrace{3 \otimes 3 \otimes 3 \otimes \ldots \otimes 3}_{k}.$$

Instead of considering the irreducible subspaces, we focus our attention on the operators which project on them. Suppose:

$$L_k = \bigotimes_\alpha L_\alpha \qquad \alpha = \text{labels irreducible subspaces.}$$

If Y_α projects over $L_\alpha (Y_\alpha^2 = Y_\alpha)$, then we have:

a) $Y_\alpha T^{(k)}(U) = T^{(k)}(U) Y_\alpha$ for any $U \in SU_3$;
b) there exists no $Y_{\alpha'}$, projecting on a subspace $L_{\alpha'} \subset L_\alpha$ commuting with $T^{(k)}(U)$'s, i.e. Y_α are minimal projections;
c) Y_α's are orthogonal ($Y_\alpha Y_\beta = 0$ when $\alpha \neq \beta$) and constitute a complete set $\left(\sum_\alpha Y_\alpha = 1\right)$.

To characterize Y_α, we have to consider the set of all operators commuting with $T^{(k)}(U)$'s.
Let us call p an arbitrary permutation of $1, 2 \ldots k$:

$$p: \quad 1 \to 1', 2 \to 2', \ldots k \to k',$$

where $1', 2' \ldots k'$ are again the numbers $1,2 \ldots k$ rearranged in some way specified by p. We associate to p a linear operator p acting on L_k, defined as:

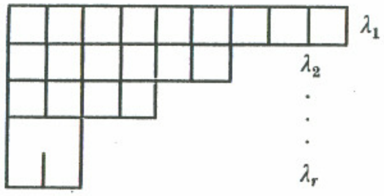
$$(pX)^{i_1 \ldots i_k} = X^{i_{1'} \ldots i_{k'}}.$$

Such operators constitute of course a representation of the group of all permutations of k-objects.
Moreover:

$$\begin{aligned}(pT^{(k)}(U)X)^{i_1 \ldots i_k} &= (T^{(k)}(U)X)^{i_{1'} \ldots i_{k'}} = U_{i_{1'} j_1} \ldots U_{i_{k'} j_k} X^{j_1 \ldots j_k} = \\ &= U_{i_{1'} j_{1'}} \ldots U_{i_{k'} j_{k'}} X^{j_{1'} \ldots j_{k'}} = U_{i_{1'} j_{1'}} \ldots U_{i_{k'} j_{k'}} (pX)^{j_1 \ldots j_k} = \\ &= U_{i_1 j_1} \ldots U_{i_k j_k} (pX)^{j_1 \ldots j_k} = (T^{(k)}(U) pX)^{i_1 \ldots i_k},\end{aligned}$$

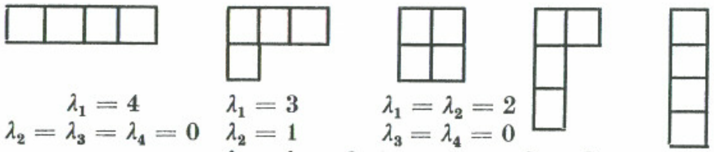
i.e. p commutes with $T^{(k)}(U)$. Let us call Σ_k the set of all the operators p and of all their linear combinations: clearly all the elements of Σ_k commute with

$T^{(k)}(U)$'s, but the importance of Σ_k lies in the fact that the converse is also true, i.e. [*21*]:
all the operators commuting with $T^{(k)}(U)$'s belong to Σ_k.
Hence Y_α's (which reduce the representation) are in Σ_k.
Their characterization can be achieved using the so called Young tableaux.
Consider an arbitrary partition of k objects into groups of $\lambda_1, \lambda_2 \ldots \lambda_r$ elements ($\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_r$). This partition is indicated by a tableau made of r rows containing $\lambda_1, \lambda_2, \ldots, \lambda_r$ boxes:



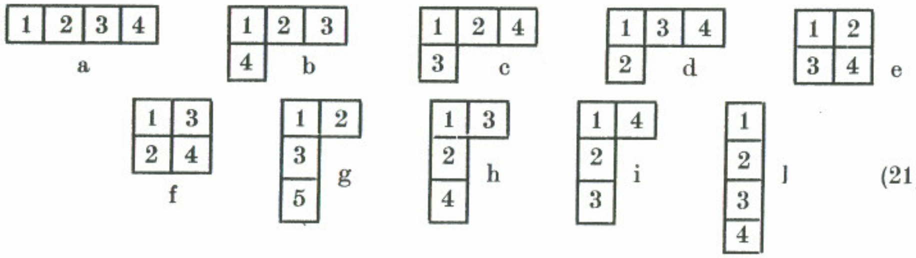
$$\lambda_1 + \lambda_2 + \cdots + \lambda_r = k.$$

Fill now this tableau with the numbers 1, 2, ... k in all possible ways consistent with the rule: numbers must increase, in each row from left and in each column from above.
Consider for example $k = 4$. Then we have five tableaux:



$\lambda_1 = 4$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$; $\lambda_1 = 3$, $\lambda_2 = 1$, $\lambda_3 = \lambda_4 = 0$; $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = \lambda_4 = 0$; $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 1$, $\lambda_4 = 0$; $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$

which, filled in all possible ways, give:



(21)

To each tableau determined by a partition $\lambda_1, \ldots \lambda_r$ ($\lambda_1 + \lambda_2 + \cdots = k$) and by a particular arrangement of the numbers $1, 2 \ldots k$ consistent with previous rule, we associate an operator Y (Young symmetrizer) defined as

$$Y = QP, \tag{22}$$

P = sum of all operators associated to permutations of $1, 2 \ldots k$ which leave unchanged the rows of the tableaux $= \sum_p p$, Q = sum over all permutations q which leave columns unchanged, each being multiplied by its signature $Q = \sum_p \delta_q q$.

Let us indicate permutations (as well as operators which represent them) in the cyclic notation: for example:

$$(1\ 2\ 4)\ (3) \quad \text{or simply } (1\ 2\ 4)$$

stands for:

$$1 \to 2$$
$$2 \to 4$$
$$4 \to 1$$
$$3 \to 3$$

and:

$$(1\ 4)\ (3\ 2)$$

stands for:

$$1 \to 4$$
$$4 \to 1$$
$$2 \to 3$$
$$3 \to 2.$$

Then e.g. tableau (21) f is associated to the operator:

$$Y_f = (e - (12) - (34) + (12)\,(34))\,(e + (13) + (24) + (13)\,(24))$$

(e is the identity permutation).
Applied to the gene"al tensor $X^{i_1 i_2 i_3 i_4}$, this operator gives the tensor

$$X_{\begin{array}{|cc|}\hline i_1 & i_2 \\ \hline i_3 & i_4 \\ \hline\end{array}} :$$

$$\begin{aligned} X_{\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline i_3 & i_4 \\ \hline\end{array}} = & X^{i_1 i_2 i_3 i_4} + X^{i_3 i_2 i_1 i_4} + X^{i_1 i_4 i_3 i_2} + X^{i_3 i_4 i_1 i_2} \\ & - X^{i_2 i_1 i_3 i_4} - X^{i_2 i_3 i_1 i_4} - X^{i_4 i_1 i_3 i_2} - X^{i_4 i_3 i_1 i_2} \\ & - X^{i_1 i_2 i_4 i_3} - X^{i_3 i_2 i_4 i_1} - X^{i_1 i_4 i_2 i_3} - X^{i_3 i_4 i_2 i_1} \\ & + X^{i_2 i_1 i_4 i_3} + X^{i_2 i_3 i_4 i_1} + X^{i_4 i_1 i_2 i_3} + X^{i_4 i_3 i_2 i_1}. \end{aligned}$$

It can be shown that:
i) all Y's are proportional to projections:

$$Y^2 = c\,Y \quad \text{so that } \frac{Y}{c} \text{ is a projection,}$$

ii) Y's are minimal projections, and

$$Y\,Y' = 0,$$

when Y' corresponds to a different tableau than Y (i.e. differing for the partition $\lambda_1, \lambda_2, \ldots \lambda_r$ or for a different arrangement of the numbers $1, 2, \ldots k$);
iii) for a given k the set of projections Y associated to all possible Young tableaux of order k completely reduces the representation $(3)^k$;

iv) Y's corresponding to the same partition $\lambda_1, \dots \lambda_r$, differing by the arrangement of $1, 2 \dots k$ project over equivalent representations.

It can be easily verified that any Young symmetrizer projects over tensors antisymmetrical in indices appearing in the same column of the corresponding tableau. Now our indices run over 1, 2, 3 so that each tableau with more than three rows gives an identically vanishing projection (this is the case of the tableau (21)1) and need not to be considered.

We want to see now how each irreducible (m, n) representation of SU_3 can be obtained isolating a particular irreducible component in a suitable product $3 \otimes 3 \dots \otimes 3$ [21].

Consider the tensor product $T^{(m+2n)} = (3)^{m+2n}$. We claim that the irreducible manifold L characterized by the Young tableau

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2\dots & n & & m+n \\ \hline m+n+1 & \dots & m+2n \\ \cline{1-3}\end{array} \tag{23}$$

transforms as the (m, n) representation.

Let us call $L^{(m)}_{(n)}$ the vector space of tensor with m upper and n lower indices; then the following mapping:

$$F^{i_{n+1}\dots i_{n+m}}_{j_1 \dots j_n} = \varepsilon_{j_1 i_1 i_{n+m+1}} \varepsilon_{j_2 i_2 i_{n+m+2}} X \begin{array}{|c|c|c|c|c|c|}\hline i_1 & i_2 & \dots & i_{n+1} & & i_{n+m} \\ \hline i_{n+m+1} & i_{n+m+2} & \dots \\ \cline{1-3}\end{array} \tag{24}$$

or, in short, $F^{(i)}_{(j)} = \varepsilon_{(j)(ke)} X \begin{array}{|c|c|}\hline k & i \\ \hline e \\ \cline{1-1}\end{array}$

(we contract with ε_{ijk} each pair of indices belonging to the same column) induces a one-to-one correspondence between L and a linear manifold (which we will specify later) contained in $L^{(m)}_{(n)}$. That (24) is a one-to-one mapping can be seen by showing directly that if

$$F^{(i)}_{(j)} = \varepsilon_{(i)(ke)} X \begin{array}{|c|c|}\hline k & i \\ \hline e \\ \cline{1-1}\end{array} = \varepsilon_{(i)(ke)} X' \begin{array}{|c|c|}\hline k & i \\ \hline e \\ \cline{1-1}\end{array},$$

then

$$X \begin{array}{|c|c|}\hline k & i \\ \hline e \\ \cline{1-1}\end{array} = X' \begin{array}{|c|c|}\hline k & i \\ \hline e \\ \cline{1-1}\end{array}.$$

To do this, write explicitly the expression

$$0 = \varepsilon_{j_1 i_1 i_{n+m+1}} \varepsilon_{j_2 i_2 i_{n+m+2}} \dots (X - X') \begin{array}{|c|c|c|c|c|c|}\hline i_1 & i_2 & \dots & i_{n+1} & \dots & i_{n+m} \\ \hline i_{n+m+1} & i_{n+m+2} & \dots \\ \cline{1-3}\end{array},$$

then contract indices $j_1, j_2, \dots j_n$ with $\varepsilon_{s_1 t_1 j_1}, \varepsilon_{s_2 t_2 j_2}, \dots$ Using the identity:

$$\varepsilon_{stj} \varepsilon_{jii'} = \delta_{si} \delta_{ti'} - \delta_{si'} \delta_{ti}$$

as well as antisymmetry of tensor X in the indices of the same column, the wanted result is obtained.

Moreover a simple but rather tedious calculation leads to the relation

$$T^{(m)}_{(n)}(U)\, F = T^{(m)}_{(n)}\, \varepsilon X = \varepsilon T^{m+2n}(U) X,$$

where $T^{(m)}_{(n)}(U)$ is any element of the representation $(3)^m \times (\bar{3})^n = \underbrace{3 \otimes 3 \dots 3}_{m} \otimes \underbrace{\bar{3} \dots \otimes \bar{3}}_{n}$ and with ε we have indicated the mapping (24). This shows that the image L^ε of L into $L^{(m)}_{(n)}$ is an irreducible subspace for the representation $(3)^m \times (\bar{3})^n$, whose restriction to L^ε is equivalent to the restriction of $T^{(m+2n)}$ to L.

Proof will be complete if we show that the vector belonging to the maximal weight of $(3)^m \times (\bar{3})^n$ is contained in L^ε (see sect. 5.4).

Now, according to the matrices of sect. 9.6, the vector belonging to the maximal weight of the representation 3 is

$$(\bar{x})^i = \delta^{i1} = (1, 0, 0),$$

whereas for the representation $\bar{3}$ it is

$$(\bar{y})_i = \delta_{i2} = (0, 1, 0).$$

Hence the tensor of $L^{(m)}_{(n)}$ belonging to the maximal weight of $(3)^m \otimes (3)^n$ is

$$(\bar{F})^{i_{n+1}\dots i_{n+m}}_{j_1 \dots j_n} = \delta^{i_{n+1} 1} \delta^{i_{n+2} 1} \dots \delta^{i_{n+m} 1} \cdot \delta_{j_1 2} \dots \delta_{j_n 2}$$

and we have to show that $\bar{F}$ belongs to L^ε.

Consider the tensor $\bar{X}$ (belonging to L) defined as

$$\bar{X} \begin{array}{|c|c|c|c|}\hline 1 & \dots & 1 & \dots 1 \\ \hline 3 & \dots & 3 \\ \cline{1-3}\end{array} = \left(\frac{-1}{2}\right)^n$$

$$\bar{X} \begin{array}{|c|c|c|}\hline i_1 & \dots & \dots i_{n+m} \\ \hline i_{n+m+1} & \dots \\ \cline{1-2}\end{array} = 0,$$

when indices appearing in columns are not permutations of 1 and 3, and some of the last m indices in first row is different from 1. Then obviously we have

$$(\bar{F})^{(i)}_{(j)} = \varepsilon_{(j)(ke)} \bar{X} \begin{array}{|c|c|}\hline k & i \\ \hline e \\ \cline{1-1}\end{array}.$$

i.e. F belongs to L^ε, Q.E.D.

For example previous theorem allows one to conclude that linear manifolds on which tableaux (21 b, c, d) project, transform as the (2,1) representation, whereas those corresponding to (21 e, f) transform as (0,2).
In the composition of the tensor product $(3)^k$, also tableaux with three rows appear. To which irreducible representation do they correspond? We will see this referring to an example.
Consider the manifold of tensors

$$X_{\begin{array}{|c|c|}\hline i_1 & i_4 \\ \hline i_2 \\ \cline{1-1} i_3 \\ \cline{1-1}\end{array}}$$

belonging to the product $(3)^4$ (tableau (21 i)).
It transforms under SU_3 as

$$(T^{(4)}(U)\,X)_{\begin{array}{|c|c|}\hline i_1 & i_4 \\ \hline i_2 \\ \cline{1-1} i_3 \\ \cline{1-1}\end{array}} = U_{i_1 j_1} U_{i_2 j_2} U_{i_3 j_3} U_{i_4 j_4} X_{\begin{array}{|c|c|}\hline j_1 & j_4 \\ \hline j_2 \\ \cline{1-1} j_3 \\ \cline{1-1}\end{array}}$$

Being X antisymmetrical in j_1, j_2, j_3, we can write

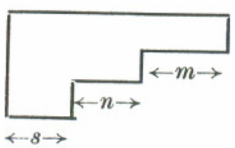
$$X_{\begin{array}{|c|c|}\hline j_1 & j_4 \\ \hline j_2 \\ \cline{1-1} j_3 \\ \cline{1-1}\end{array}} = \varepsilon_{j_1 j_2 j_3} X_{\begin{array}{|c|c|}\hline 1 & j_4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}},$$

i.e.

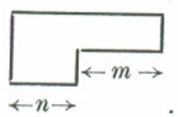
$$T^{(4)}(U)\,X_{\begin{array}{|c|c|}\hline i_1 & i_4 \\ \hline i_2 \\ \cline{1-1} i_3 \\ \cline{1-1}\end{array}} = (U_{i_1 j_1} U_{i_2 j_2} U_{i_3 j_3} \varepsilon_{j_1 j_2 j_3})\, U_{i_4 j_4} X_{\begin{array}{|c|c|}\hline 1 & j_4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}}$$

$$= \det U\, \varepsilon_{i_1 i_2 i_3} U_{i_4 j_4} X_{\begin{array}{|c|c|}\hline 1 & j_4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}} = U_{i_4 j_4} X_{\begin{array}{|c|c|}\hline i_1 & j_4 \\ \hline i_2 \\ \cline{1-1} i_3 \\ \cline{1-1}\end{array}}.$$

so that the irreducible component (21 i) is equivalent to the representation 3, which corresponds to the tableau □. This reasoning can be repeated for tableaux with an arbitrary number s of columns with three boxes, leading to the conclusion that:



is equivalent to



Using these arguments we find, for example, the decompositions:

$$(3)^4 = \underset{a}{(4,0)} + \underbrace{(2,1) + (2,1) + (2,1)}_{bcd} + \underbrace{(0,2) + (0,2)}_{ef} + \underbrace{(1,0) + (1,0) + (1,0)}_{ghi}$$

$$(3)^2 = (0,1) + (2,0)$$

$$(3)^3 = (3,0) + (1,1) + (1,1) + (0,0).$$

A formula can be given for the dimension of the (m, n) representation [*21*] by counting independent tensors of a given tableau

$$d_{mn} = (1+m)(1+n)\left(1 + \frac{m+n}{2}\right). \tag{25}$$

Note that, when this produces no ambiguity, we will use d_{mn} to indicate the representation (m, n) $(m \geq n)$ and $\bar{d}_{mn}$ to indicate its contragradient (n, m) (for example 10 for (3,0) and $\overline{10}$ for (0,3).
We want now to characterize the manifold L^s into which L is mapped by (24).
More definitely we show that L^s is the linear manifold of all tensors $F^{i_1 \ldots i_m}_{j_1 \ldots j_n}$ symmetrical in upper and lower indices and traceless, i.e. such that

$$F^{i\, i_2 \ldots i_m}_{i\, j_2 \ldots j_n} = 0. \tag{26}$$

We show this in two steps:
i) any tensor in L^s is a linear combination of tensors with the stated properties. In fact we proved before that in L^s there is the tensor

$$(\bar{F})^{i_1 \ldots i_m}_{j_1 \ldots j_n} = \delta^{i_1 1} \ldots \delta^{i_m 1} \delta_{j_1 2} \ldots \delta_{j_n 2},$$

which of course is symmetrical and traceless. Consider tensors

$$F(U) = T^{(m)}_{(n)}(U)\, \bar{F},$$

where U runs over the whole SU_3. We have

$$F(U)^{i_1 \ldots i_m}_{j_1 \ldots j_n} = U_{i_1 i_1'} \ldots U_{i_m i_m'} \bar{U}_{j_1 j_1'} \ldots \bar{U}_{j_n j_n'} \delta^{i_1' 1} \ldots \delta^{i_m' 1} \ldots \delta_{j_n' 2} =$$

$$= U_{i_1 1} \ldots U_{i_m 1} \bar{U}_{j_1 2} \ldots \bar{U}_{j_n 2}.$$

$F(U)$'s are of course symmetrical in upper and lower indices and, due to unitarity of the U's, they are traceless. The linear manifold spanned by $F(U)$'s is contained in L^s and is obvously invariant for the representation $(3)^m \times (\bar{3})^n$. Since L^s is irreducible (as we saw before) and this manifold is surely not zero, we conclude that it is identical with L^s; i.e. L^s contains tensors symmetrical and traceless.
ii) Call L^m_n the manifold of all tensors symmetrical in upper and lower indices and traceless. From i) $L^m_n \supset L^s$. We show now that the dimension of L^m_n equals that of L^s, so that $L^m_n = L^s$.

A tensor $F^{i_1 \dots i_m}_{j_1 \dots j_n}$ symmetrical in upper and lower indices, has

$$\binom{m+3-1}{m}\binom{n+3-1}{n} = \frac{(m+2)!}{m!\,2!}\,\frac{(n+2)!}{n!\,2!} = $$
$$= \frac{(m+2)(m+1)(n+2)(n+1)}{4} \tag{27}$$

independent components (remember that indices i, j go over 1, 2, 3), and the number of independent conditions (26) equals the number of independent components of a tensor symmetrical in $(m-1)$ upper an $(n-1)$ lower indices, i.e. it is equal to:

$$\binom{(m-1)+3-1}{m-1}\binom{(n-1)+3-1}{n-1} = \frac{m(m+1)\,n\,(n+1)}{4}. \tag{28}$$

Substracting (28) from (27) we obtain:

$$\dim\,(L^m_n) = (m+1)\,(n+1)\left(1+\frac{m+n}{2}\right) = \dim\,L.$$

Because $\dim L^\varepsilon = \dim L$ being (24) a one-to-one mapping, we conclude:

$$\dim L^\varepsilon = \dim L^m_n$$

i.e.

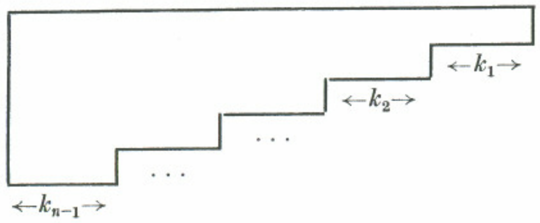
$$L^\varepsilon = L^m_n.$$

Hence we have found another possible realization of the $(m, n)SU_3$ representation: the space of tensors $F^{i_1 \dots i_m}_{j_1 \dots j_m}$ symmetrical in upper and lower indices and traceless, transforming as:

$$F^{i_1 \dots i_m}_{j_1 \dots j_n} \to (F')^{i_1 \dots i_m}_{j_1 \dots j_n} = U_{i_1 i_1'} \dots U_{i_m i_m'}\,\bar{U}_{j_1 j_1'} \dots \bar{U}_{j_n j_n'}\,F^{i_1' \dots i_m'}_{j_1' \dots j_n'}.$$

A large part of the above results can be generalized to representations of an arbitrary group SU_n. According to the general results of sect. 9.2 the irreducible SU_n representations can be labeled by $(n-1)$ positive integers $(k_1, k_2, .., k_{n-1})$. In particular the representation $(1, 0, 0,..,0)$ is always made up with SU_n matrices themselves, and is n dimensional. Results i) ii) applied to the reduction of the tensor product $(n)^k = \underset{1}{n} \otimes \underset{2}{n} \otimes \dots \otimes \underset{k}{n}$ hold unchanged: in this case of course indices $i_1 \dots i_k$ run over $1,2,\dots n$, and we have to consider tableaux containing up to n rows.

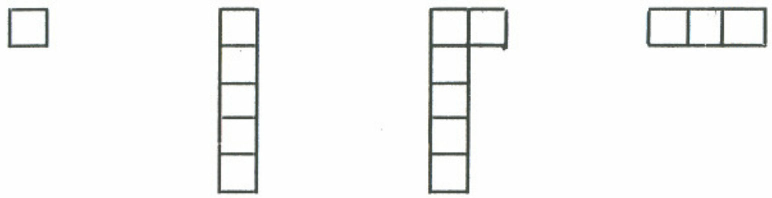
A tableau of the form



is associated to the representation $(k_1,..k_{n-1})$, and tableaux differing by an arbitrary number of columns with n rows characterize equivalent representations. The dimensionality of the representation $(k_1,..k_{n-1})$ is given by the formula:

$$d(k_1,..k_{n-1}) = \prod_{l}^{0,n-2}\ \prod_{s}^{1,n-l-1}\left(1+\frac{k_s+k_{s+1}+\dots+k_{s+l}}{l+1}\right).$$

We list here for example some SU_6 representations, together with their dimensionality (the contragradient of $(k_1 \dots k_{n-1})$ is $(k_{n-1}, k_{n-2}, .., k_1)$)



	(1, 0, 0, 0, 0)	(0, 0, 0, 0, 1)	(1, 0, 0, 0, 1)	(3, 0, 0, 0, 0)
Dim:	6	$\bar{6}$	35	56

Reduction of tensor product of arbitrary SU_3 representations.

In finding the irreducible components of the product $(m, n) \otimes (m', n')$ again the technique of Young tableaux can be used. We give here without proof a simple rule for making this reduction [22]. We illustrate this rule referring to the product $(2,2) \otimes (1,1)$.

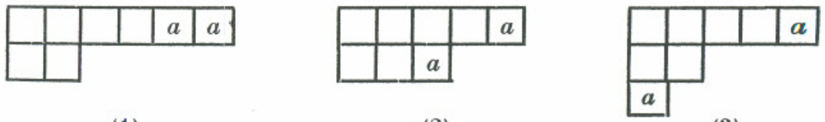
Write down the corresponding tableaux, having filled with symbols a and b first and second row of one of the two, arbitrarily selected:

Then add to the empty tableau the boxes of the first row of the second one in all possible ways consistent with the rules:

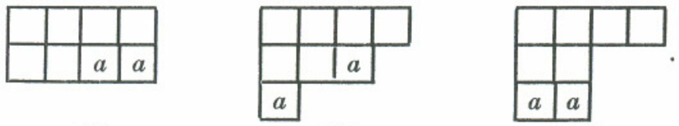
i) there must never appear two a's in the same column;

ii) for each resulting tableau, containing $\lambda_1, \lambda_2, \lambda_3$ boxes in first, second, third row, it must be $\lambda_1 \geqq \lambda_2 \geqq \lambda_3$.

In our case we have six possible tableaux:

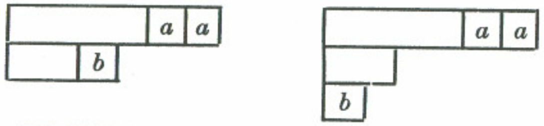


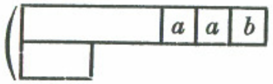
(1) (2) (3)



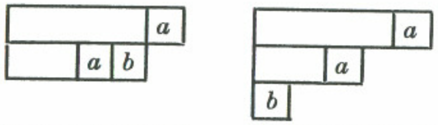
(4) (5) (6)

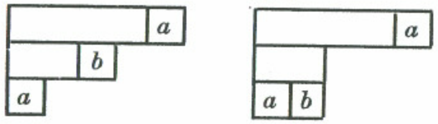
To all these tableaux we add now, in all possible ways, all boxes containing b, again consistently with ii) and with the additional rules;
iii) there must never appear two b's in the same column;
iv) for each tableau so obtained write the sequence of a's and b's obtained reading first row from right to left, then the second row from right to left and so on: only tableaux are allowed such that at any stage of this sequence the number of a's already read is greater than or equal to the corresponding number of b's.
From tableau (1) we obtain the following allowed tableaux

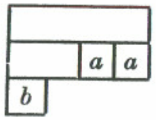
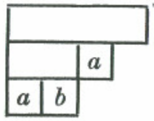
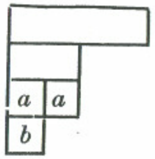


( is not allowed by iv))

from (2), (3), (4), (5) we obtain respectively:

(2) → 

(3) → 

(4) →  (5) →  (6) → 

Representations corresponding to the so-obtained tableaux are then the irreducible components of the tensor product considered (as previously observed we have not to consider tableaux with more than three rows):

$$(2, 2) \otimes (1, 1) = 27 \otimes 8 = (3, 3) + (4, 1) + (4, 4) + (2, 2) + (2, 2) + (3, 0) + \\ + (0, 3) + (1,1).$$

The dimension of (2, 2) ⊗ (1, 1) is equal to $27 \cdot 8 = 216$, which using (25) can be checked with the dimension of the second member.
Reader can easily obtain the following decompositions:

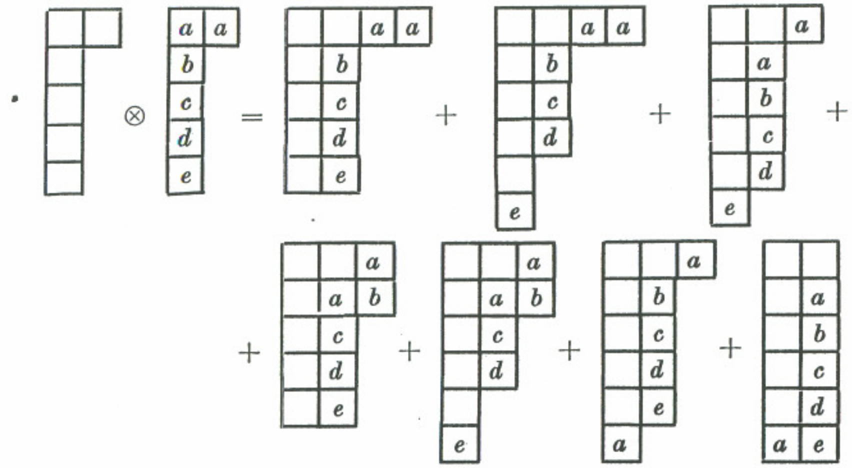
$$8 \otimes 8 = 1 + 8 + 10 + \overline{10} + 27$$

$$8 \otimes 10 = 35 + 27 + 10 + 8$$

$$8 \otimes 10 = 8 + \overline{10} + 27 + 35.$$

The same procedure can be applied to the reduction of the tensor product of arbitrary SU_n representations, labeling with $c, d, \ldots$ boxes in third, fourth, .. rows. At the end of course we will drop tableaux containing more than n rows.
Consider the case of SU_6 and the product:

$$35 \otimes 35 = (1, 0, 0, 0, 1) \otimes (1, 0, 0, 0, 1)$$



i.e.

$$35 \otimes 35 = (2, 0, 0, 0, 2) + (2, 0, 0, 1, 0) + (1, 0, 0, 0, 1) + (0, 1, 0, 0, 2) \\ + (0, 1, 0, 1, 0) + (1, 0, 0, 0, 1) + (0, 0, 0, 0, 0).$$

10. Eightfold Way

10.1. We are now in position to build up a concrete theory for strongly interacting particles. The first thing to do, after what we have said in sect. 7.5, is to identify particles with the same spin and the same parity with linearly independent vectors inside certain irreducible representations of

$$G \times U_1(N),$$

where G is a Lie group of rank two and $U_1(N)$ is the baryonic number gauge group. This identification fixes the connection between infinitesimal generators of the group and isospin and hypercharge operators.
Of course we must preserve the isospin structure of the particles, in that particles belonging to the same isomultiplet must go into the same irreducible representation.
The procedure is not straightforward since there are three non isomorphic rank two groups, and in addition particles do not exhibit any impressive regularity, apart from the isospin multiplet structure.
Hence at a pure classificatory level one has no clear indication on which the underlying symmetry group, actually is. Furthermore, once a group has been chosen, it is not clear which are the correct assignments of the particles to its irreducible representations.
In fact many different models have been proposed and in principle the right choice should emerge from a comparison of theoretical predictions with experiments.

However the great difficulty is that even large discrepancies could be due not to a substantial failure of the model, but to the effects of the symmetry breaking interactions.

As stressed by A. SALAM [23], it is hoped that a deeper understanding of the latter will finally lead to a non-ambiguous interpretation of many possible tests.

The main informations a symmetry model provides us, are of the following type:

i) whatever the model is, one finds that the known particles alone do not arrange themselves in complete supermultiplets, and one is then led to assume the existence of new particles with definite values of T_3 and Y. One can even guess their masses and the most likely production and decay modes.

However there is no a-priori reason to require completely filled supermultiplets. in fact one can imagine situations in which the symmetry breaking interactions, which are certainly present (see sect. 7.5), make some members of a supermultiplet so unstable, that they are not practically observable. On the contrary the discovery of a predicted particle provides strong evidence in favour of the implied model;

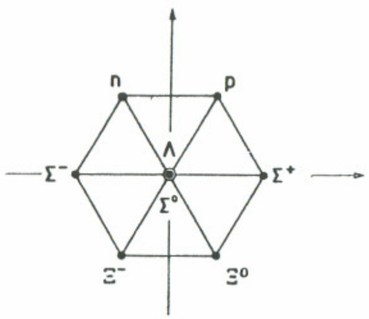


Fig. 6.

ii) from the symmetry scheme one can deduce relations between amplitudes of different processes; (see the example of iso-spin, sect. 7.4).

iii) by assuming certain transformation properties of weak and electromagnetic interaction Lagrangians of hadrons under the symmetry group, with the aid of Wigner-Eckart theorem (sect. 6.3), one can derive relations between the amplitudes of weak and electromagnetic processes involving such particles.

In what follows, we will focus our attention on the "eightfold way" model proposed by NE'EMAN [24] and GELL-MANN [3], which has proved to be the most successful one[20]).

10.2. a) In the eightfold way model one chooses SU_3 as the underlying symmetry group, and associates the eight "stable" baryons $N, \Sigma, \Lambda^\circ, \Xi$ to the basis vectors of its (1, 1) eight dimensional representation, as indicated in fig. 6.

We can easily deduce the relations between T_3, Y, and the diagonal elements H_1, H_2, by remembering (sect. 9.3 b) that in the representation (m, n) of SU_3 the eigenvalues of H_1 and H_2 corresponding to the maximal weight Λ are

$$\Lambda \equiv \left(\frac{m+n}{2\sqrt{3}}, \frac{m-n}{6}\right).$$

Hence, with $m = n = 1$ we have

$$\Lambda \equiv \left(\frac{1}{\sqrt{3}}, 0\right).$$

If we want to associate to $|\Lambda\rangle$ the Σ^+ particle, we have to put

$$T_3 = \sqrt{3}\ H_1 \qquad (1)$$

[20]) For a concise discussion concerning the other symmetry models see [25, 26].

and in addition, by requiring p to correspond to $|\Lambda - \alpha_3|\rangle$,

$$Y = 2H_2.$$

The operator T^2 defined in sect 9.3b, which removes all degeneracies inside a supermultiplet, is just the square of the isospin. We will choose normalized simultaneous eigenvectors of T^2, T_3, Y as a basis in any irreducible representation.

The pseudoscalar mesons also fit very nicely into an octet according to the scheme reported in fig. 7.

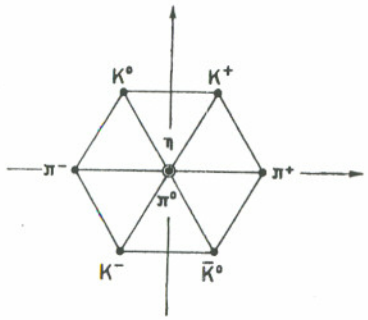


Fig. 7.

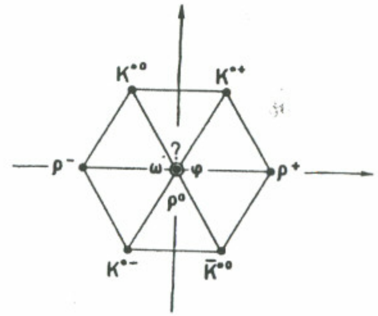


Fig. 8.

It should be noted that the η meson, which quite naturally completes the octet, has been discovered after the introduction of the eighfold way.

When we try to arrange the vector mesons into the scheme, we get in trouble. In fact we would like to assign the nine mesons $\rho(T = 1, Y = 0)$, $K^*(T = 1/2, Y = 1)$, $\bar{K}^*(T = 1/2, Y = -1)$, $\omega = (T = 0, Y = 0)$, $\varphi(T = 0, Y = 0)$ to the same irreducible representation. However, since T_3, T^2, Y are a complete set of commuting operators inside each irreducible representation, it is not possible to fit in the same supermultiplet two distinct isosinglets (ω, φ) with the same hypercharge. The usual assignment is to put eight mesons into an octet (fig. 8) and the remaining one into a singlet (i. e., in the (0, 0) representation), and the question arises whether the ω or the φ particle is to be put in the singlet.

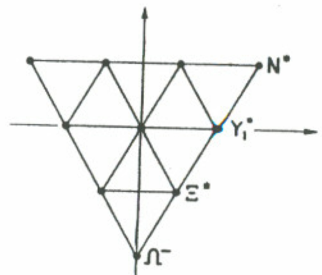


Fig. 9.

We shall see in the following that this question can be consistently resolved.

In considering the baryon-meson resonances, we have to find a representation containing an isospin $^3/_2$ multiplet with hypercharge equal to one, corresponding to the well known $N^*_{3/2}$, $\pi - N$ resonance. The lowest representations containing this isomultiplet are 10 and 27. The first one accomodates wery well the $N = 1$, $J^P = {}^3/_2{}^+$ resonances $N^*_{3/2}$, Y^*_1, Ξ^*, Ω^- (fig. 9).

Whereas $N^*_{3/2}$, Y^*_1, Ξ^* were at hand when it was proposed to assign them to the decuplet, the Ω^- was not yet known: the model predicted its quantum numbers $(N = 1, T = 0, Y = -2)$ as well as its mass $M_\Omega = 1676$ MeV.

On this indications extensive searches for the Ω^- have been carried out, until a particle with right mass and hypercharge has been found studying K^-p reactions at 5.0 GeV/c [27].
The positive result of the search has been considered as one of the most brilliant successes of the eightfold way model.
There is another particle, namely the Y_0^* (mass = 1405 MeV, $T = 0$, $Y = 0$, $J^P = 1/2^-$) which usually is assigned to a (0,0) representation.

Name of the particle	N	J^P	Mass (MeV)	Y	T
N	1	$1/2^+$	(p) 938.2 (n) 939.6	1	$1/2$
Λ^0	1	$1/2^+$	1115.4	0	0
Σ	1	$1/2^+$	+ 1189.4 − 1197.1 1192.4	0	1
Ξ	1	$1/2^+$	− 1321 1314	−1	$1/2$
Y_0^*	1	$1/2^-$?	1405	0	0
N*	1	$3/2^+$	1236 ± 2	1	$3/2$
Y_1^*	1	$3/2^+$	1382.1 ± 0.9	0	1
Ξ^*	1	$3/2^+$	1529.1 ± 1.0	−1	$1/2$
Ω^-	1	$3/2^+$?	1675 ± 3	−2	0
π	0	0^-	± 139.6 135.0	0	1
η	0	0^-	548.7 ± 0.5	0	0
K	0	0^-	+ 493.8 498.0	1	$1/2$
ρ	0	1^-	763 ± 4	0	1
ω	0	1^-	782.8 ± 0.5	0	0
φ	0	1^-	1019.5 ± 0.3	0	0
K*	0	1^-	891 ± 1	1	$1/2$

We report a table deduced from [28]: in it we report the quantum numbers (including masses) of the particles we have arranged in the scheme. In the same reference many other particles are listed which we have not considered here mainly for two reasons: first of all the quantum numbers of many of them are at this time not well established (sometimes even their existence is doubtful). Secondly it appears that the situation is so incomplete to make not very useful an even tentative grouping of such particles into supermultiplets.
10.2b) It is convenient for further applications, to identify the eigenstates of T^2, T_3, Y, (to which particles are assigned) inside the tensor representations we have described in sect. 9.6, 9.7.
The particular symmetrical tensors $T^{i_1 \dots i_m}_{j_1 \dots j_n}$ defined as

$$T^{i_1 \dots i_m}_{j_1 \dots j_n} = \begin{cases} 1 \text{ when } i_1 \dots i_m \text{ is a permutation of } \underbrace{11\dots1}_{k_1}\ \underbrace{22\dots2}_{k_2}\ \underbrace{33\dots3}_{k_3} \text{ and } j_1 \dots j_n \text{ is} \\ \text{a permutation of } \underbrace{11\dots1}_{h_1}\ \underbrace{22\dots2}_{h_2}\ \underbrace{33\dots3}_{h_3} \\ 0 \text{ otherwise} \end{cases}$$

are, according to sect. 9.6, eigenstates of H_1, H_2 (inside the representation $(3)^m \otimes (\bar{3})^n$) with weights

$$(k_1 - h_1)\Lambda^1 + (k_2 - h_2)\Lambda^2 + (k_3 - h_3)\Lambda^3,$$

where $\Lambda^{1,2,3}$ are the weights of representation 3. Hence they are eigenstates of T_3, Y with eigenvalues

$$T_3 = \frac{1}{2}(k_1 - h_1) - \frac{1}{2}(k_2 - h_2)$$

$$Y = \frac{1}{3}(k_1 - h_1) + \frac{1}{3}(k_2 - h_2) - \frac{2}{3}(k_3 - h_3).$$

They are a basis in the manifold of the tensors which are symmetrical in upper and lower indices, so that to obtain a basis in the (m, n) representation, which diagonalizes T_3 and Y, it is necessary to take those linear combinations of (1) which are traceless. Among them we will select those linear combinations, which correspond to eigenstates of T^2.
In what follows we will be concerned only with the eight and ten dimensional representations.
In the case of the (8) representation we have to consider tensors with two indices. We have not to impose any particular symmetry property, but only the trace condition:

$$T^i_i = 0.$$

There are nine independent tensors T^i_j:

	T_3	Y
$(T_{(1)})^i_j = \delta^{i1}\delta_{j1}$	0	0
$(T_{(2)})^i_j = \delta^{i1}\delta_{j2}$	1	0
$(T_{(3)})^i_j = \delta^{i1}\delta_{j3}$	$1/2$	1

	T_3	Y
$(T_{(4)})^i_j = \delta^{i2}\delta_{j1}$	-1	0
$(T_{(5)})^i_j = \delta^{i2}\delta_{j2}$	0	0
$(T_{(6)})^i_j = \delta^{i2}\delta_{j3}$	$-1/2$	1
$(T_{(7)})^i_j = \delta^{i3}\delta_{j2}$	$-1/2$	-1
$(T_{(8)})^i_j = \delta^{i3}\delta_{j2}$	$1/2$	-1
$(T_{(9)})^i_j = \delta^{i3}\delta_{j3}$	0	$0.$

$T_{(1)}, T_{(5)}, T_{(9)}$ are not traceless. There are only two independet traceless linea combinations

$$T_{(1)} - T_{(5)}$$

$$T_{(1)} + T_{(5)} - 2\,T_{(9)} \qquad (2)$$

which span, together with $T_{(2)}, T_{(3)}, T_{(4)}, T_{(6)}, T_{(7)}, T_{(8)}$ the (1.1) representation. It is worth noticing that the general tensor of (1,1) can be thought as a three by three traceless matrix, and the transformation law 9.6 (20) can be written as a matrix product

$$T \to T' = U\,T\,U^{-1} \quad U \in SU_3.$$

Then the operators representing the SU_3 Lie algebra act as

$$(\Lambda T) = [\lambda, T].$$

where λ is any linear combination of the matrices given in sect 9.6 (19). It is then very easy to see that the tensors (2) are eigenstates of T^2 with eigenvalues, respectively, 2 and 0.

From this point of view, recalling what we said in the previous subsection, we see that:

i) in the case of stable baryons, we have the following assignments:

$$\Sigma^+ \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \Lambda^0 \to \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix} \quad \text{etc.}$$

Which are symbolically summarized in the matrix:

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & \Sigma^+ & \mathrm{p} \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & \mathrm{n} \\ \Xi^- & \Xi^0 & -\frac{2\Lambda^0}{\sqrt{6}} \end{pmatrix} \qquad (3)$$

Factors $1/\sqrt{2}$, $1/\sqrt{6}$ have been introduced so that our basis tensors are normalized with respect to the scalar product

$$(T, T') = \sum_{ij} T^i_j\,(T'^j_i) = \text{Trace}\,[T\,(T')^+]\,.$$

i) In the same way for pseudoscalar mesons we have the matrix:

$$M = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & \mathrm{K}^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \mathrm{K}^0 \\ \mathrm{K}^- & \mathrm{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \qquad (4)$$

For what concerns the 10-dimensional (3,0) representation, we notice that it is spanned by the symmetrical tensors:

$$(T_{\alpha\beta\gamma})^{ijk} = \begin{cases} 1 \text{ when } ijk \text{ is a permutation of } (\alpha,\beta,\gamma = 1,2,3) \\ 0 \text{ otherwise} \end{cases}$$

$\alpha\beta\gamma$	T_3	Y	T^2
111	$3/2$	1	$15/4$
112	$1/2$	1	$15/4$
122	$-1/2$	1	$15/4$
222	$-3/2$	1	$15/4$
113	1	0	2
123	0	0	2
223	-1	0	2
133	$1/2$	-1	$3/4$
233	$-1/2$	-1	$3/4$
333	0	-2	0

10.3. Antiparticle multiplets

So far we have not explicity considered the assignments of antiparticles.

If $|a\rangle$ is a particle state, then the corresponding antiparticle state $|\bar{a}\rangle$ has the same space-time properties, but the values of all charges (Q, N, Y, etc.) are reversed. $|a\rangle$ and $|\bar{a}\rangle$ are connected by the charge conjugation operator C

$$|\bar{a}\rangle = C\,|a\rangle,$$

which can be assumed to be a unitary operator satisfying:

$$C^2 = 1\,.$$

It is a well known fact that strong interactions are symmetrical under C, so that if particles exhibit, the SU_3 symmetry, relative to strong interactions the same behaviour must be displayed by the corresponding antiparticles.

In particular for each supermultiplet of baryonic number N, the corresponding antiparticle states must arrange according, to a supermultiplet with baryonic number $-N$. By definition C satisfies

$$\begin{aligned} CQ + QC = \{C,Q\} &= 0 \\ \{C,N\} &= 0 \\ \{C,Y\} &= 0, \end{aligned} \tag{5}$$

so that, using the Gell-Mann-Nishijima formula ($Q = T_3 + {}^1/_2 Y$), we get

$$\{C,T_3\} = 0. \tag{6}$$

Let us indicate with

$$L = \{|(m, nN), T^2, t_3, y\rangle\}$$

the linear manifold spanned by the basis vectors of the (m, n) supermultiplet of particles with baryonic number N particles and with

$$L_C = \{C\,|(m, n\,N)\,T^2, t_3, y\rangle\}$$

the corresponding antiparticle states. Then by (5), (6), we have

$$\begin{aligned} T_3\, C\,|(m, n\,N)\,T^2, t_3, y\rangle &= -t_3\, C\,|(m, n\,N)\,T^2, t_3, y\rangle \\ Y\ \ C\,|(m, n\,N)\,T^2, t_3, y\rangle &= -y\, C\,|(m, n\,N)\,T^2\, t_3, y\rangle \\ N\ \ C\,|(m, n\,N)\,T^2, t_3, y\rangle &= -NC\,|(m, n\,N)\,T^2, t_3, y\rangle, \end{aligned} \tag{7}$$

i.e. all the weights of $\{C\,|(m, n\,N)\,T^2, t_3, y\rangle\}$ are opposite to those of $\{|(m, n\,N)\,T^2, t, y\rangle\}$. From what we said in sect. 9.5 we conclude that antiparticles transform under $SU_3 \otimes U_1(N)$ as members of the (n, m) representation (with baryonic number-N).

In the tensor formalism we have employed before, it is possible to define the C-operation simply as the interchange of lower and upper indices: for example $T^{ij} \xrightarrow{c} T_{ij}$.

We note that in general this is a mapping between two different tensor spaces, unless the number of upper and lower indices are equal, i.e. the representation is self-conjugate.

In the case of antibaryons we have the following assignments:

$$B \equiv \begin{pmatrix} \dfrac{\bar\Sigma^0}{\sqrt2} + \dfrac{\bar\Lambda^0}{\sqrt6} & \bar\Sigma^- & \bar\Xi^- \\ \bar\Sigma^+ & -\dfrac{\bar\Sigma^0}{\sqrt2} + \dfrac{\bar\Lambda^0}{\sqrt6} & \bar\Xi^0 \\ \bar p & \mathrm{n} & -\dfrac{2\bar\Lambda^0}{\sqrt6} \end{pmatrix} \tag{8}$$

(the bar indicates that these particles have baryonic number -1).

The pseudoscalar meson multiplet is mapped into itself by C since mesons have baryonic number equal to zero:

$$M \to (\bar M)^T = \begin{pmatrix} \dfrac{\bar\pi^0}{\sqrt2} + \dfrac{\bar\eta}{\sqrt6} & \bar\pi^- & \dots \\ \bar\pi^+ & \dots\dots & \dots \\ \dots & \dots\dots & \dots \end{pmatrix}.$$

In the case of vector mesons a minus sign appears due to the fact that neutral vector mesons have negative charge conjugation.

10.4. Mass formulae

Till now we have grouped particles into supermultiplets, without worrying about the very large mass difference involved. The mass of a particle is the mean value of its Hamiltonian H in its rest frame: if SU_3 were a symmetry group in the strict sense, H would be an invariant operator so that, by Schur's lemma, inside each irreducible representation, H would be a multiple of the unit operator, i.e. particles inside the same multiplet would have the same mass. This is not the case and, as stressed before, we are forced to assume that a component of H violates unitary symmetry. We write

$$H = H_0 + H_1,$$

where H_0 is invariant under SU_3. For what concerns H_1, it must be charge independent and hypercharge conserving

$$[H_1, T_i] = [H_1, Y] = 0, \quad i = 1, 2, 3$$

so that H_1 is constant inside each isomultiplet (we do not consider at this point electromagnetic and weak contributions to the mass).

Let us label each strongly interacting particle in its rest frame with the quantum numbers $N, J^P, \lambda, T_3, T^2, Y$, where λ is the label of the SU_3 representation to which it is assigned. Physical masses are then the eigenvalues of the matrix:

$$\langle N', J'^{P'}, \lambda', T^2, T'^2, Y'\,|H|\,N, J^P, \lambda, T_3, T^2, Y\rangle.$$

We have written N and J^P for completeness but obviously H commutes with them, so that the relevant matrix elements are:

$$\begin{aligned} \langle NJ^P, \lambda', T_3', T'^2, Y'\,|H|\,N, J^P, \lambda, T_3, T^2, Y\rangle &= \langle \lambda' T_3' T'^2 Y'\,|H|\,\lambda T_3 T^2 Y\rangle = \\ &= \langle \lambda' T_3' T'^2 Y'\,|H_0|\,\lambda T_3 T^2 Y\rangle + \langle \lambda' T_3' T'^2 Y'\,|H_1|\,\lambda T_3 T^2 Y\rangle. \end{aligned} \tag{9}$$

To simplify notations we assume that particles with the particular values of N and J^P considered, group together in only two supermultiplets λ and λ'. This case easily extends to the general one, and in addition this is the most complicated situation that has been found till now.

Using the assumptions made on H_0 and H_1, the matrix element (9) writes as:

$$m_0(\lambda)\, \delta_{\lambda\lambda'}\, \delta_{T^2T'^2}\, \delta_{YY'}\, \delta_{T_3T_3'} + m_1(T^2, Y)_{\lambda\lambda'}\, \delta_{T^2T'^2}\, \delta_{YY'}\, \delta_{T_3T'_3}. \tag{10}$$

Hence H_0 is diagonal as it has to be. Matrix elements of H_1 do not depend on T_3 so that we will not menti on it further on. Moreover H_1 can connect only states belonging to isomultiplets with the same T^2 and Y. Recalling what we said in sect. 9.3 b, we see that in λ as well as in λ' isomultiplets with given values of T^2 and Y can occur at most once, so that the matrix representing H_1, has the form:

$$\begin{array}{c|cc|cc|} & & T^2, Y & & T^2, Y \\ \hline & & & & \\ \lambda \; \{ T^2, Y & & m_1(T^2Y)_{\lambda\lambda} & & m_1(T^2Y)_{\lambda\lambda'} \\ & & & & \\ & & m_1(T^2Y)_{\lambda'\lambda} & & m_1(T^2Y)_{\lambda'\lambda'} \\ \lambda' \; \{ T^2, Y & & & & \\ \hline & & \lambda & & \lambda' \end{array} \tag{11}$$

To diagonalize this matrix is equivalent to diagonalize each submatrix

$$\begin{pmatrix} m_1(T^2, Y)_{\lambda\lambda} & m_1(T^2, Y)_{\lambda\lambda'} \\ m_1(T^2, Y)_{\lambda'\lambda} & m_1(T^2, Y)_{\lambda'\lambda'} \end{pmatrix} \tag{12}$$

for each value of T^2 and Y occurring both in λ and λ'.
In the case of stable baryons, pseudoscalar mesons, and decuplet resonances, for each value of N and J^P, only one representation is present, so that H_1 has the form

$$H_1 \equiv \begin{pmatrix} m_1(T^2, Y) & & 0 \\ & m_1(T^2, Y') & \\ 0 & & \ddots \end{pmatrix} \tag{13}$$

and each isomultiplet has the mass

$$m = m_0 + m_1(T^2, Y).$$

We have found nothing else but that particles with same T^2, Y have the same mass. Significant results are obtained by assuming H_1 to be the $Y = 0$, $T_3 = 0$, $T^2 = 0$ member of a set of tensor irreducible operators transforming as the regular SU_3 representation.
In a field theoretical treatment one would describe the symmetry breaking interactions by adding to the symmetrical Lagrangian $\mathscr{L}_0$ a term $\mathscr{L}_{\text{MS}}$ which is required to be hypercharge and isospin conserving:

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\text{MS}}.$$

Using perturbation theory in $\mathscr{L}_{\text{MS}}$ one would write mass corrections as the expectation value of a power serie in $\mathscr{L}_{\text{MS}}$. If we assume $\mathscr{L}_{\text{MS}}$ to transform like the $T^2 = T_3 = Y = 0$ member of an octet, we are led in first order approximation to our assumption on H_1.

With this proviso we can obtain the general form of matrix elements (13), using Wigner-Eckart theorem (sect. 6.3.).
We write H_1 as $T^{(8)}_{0,0,0}$ and introduce the notation

$$C(\lambda, T^2, T_3, Y;\; 8, 0, 0, 0;\; \mu_\gamma T^2 T_3 Y)$$

for the Clebsch-Gordan coefficient connecting the vectors

$$|\lambda, T^2 T_3 Y\rangle \;\; |8, 0, 0, 0\rangle; \quad |\mu_\gamma T^2 T_3 Y\rangle,$$

where the latter vectors span the standard basis which decomposes the tensor product $\lambda \otimes 8$. The suffix γ distinguishes between equivalent representations appearing in the decomposition.
Then, according to Wigner-Eckart theorem, we have:

$$m = m_0 + m_1(T^2, y) = m_0 + \sum_\gamma C(\lambda, T^2, T_3, Y; 8, 0, 0, 0; \mu_\gamma T^2 T_3 Y) \cdot \langle \lambda \| T^{(8)} \| \mu_\gamma \rangle. \tag{14}$$

Consider stable baryons. In this case $\lambda = 8$, and $8 \otimes 8 = 1 + 8 + 8 + 10 + \overline{10} + 27$, so that the regular representation appears twice in this reduction, and we have two reduced matrix elements.
Using the Clebsch-Gordan coefficients reported in [*29*] one finds:

$$\begin{aligned} m_{\text{N}} &= m_0 - \frac{\sqrt{5}}{10} \langle 8 \| T^{(8)} \| 8_1 \rangle + \frac{1}{2} \langle 8 \| T^{(8)} \| 8_2 \rangle \\ m_{\Xi} &= m_0 - \frac{\sqrt{5}}{10} \langle 8 \| T^{(8)} \| 8_1 \rangle - \frac{1}{2} \langle 8 \| T^{(8)} \| 8_2 \rangle \\ m_{\Lambda} &= m_0 - \frac{\sqrt{5}}{5} \langle 8 \| T^{(8)} \| 8_1 \rangle \\ m_{\Sigma} &= m_0 + \frac{\sqrt{5}}{5} \langle 8 \| T^{(8)} \| 8_1 \rangle. \end{aligned} \tag{15}$$

We have four masses and three unknown parameters. Their elimination leads to the relation (firstly given by GELL-MANN):

$$m_{\text{N}} + m_{\Xi} = \frac{3}{2} m_{\Lambda} + \frac{1}{2} m_{\Sigma} \qquad \text{(2256 MeV)} \quad \text{(2268 MeV)}. \tag{16}$$

The values of the masses have been taken from the table listed before.
We apply the same procedure to the decuplet resonances. Since in the reduction $8 \otimes 10 = 8 + 10 + 27 + 35$ the 10 representation appears only once, in the formula (14) we have only one reduced matrix element.
Using again Clebsch-Gordan coefficients, one obtains:

$$M_{\text{N}^*} = M_0 - \frac{1}{\sqrt{8}} \langle 10 \| T^8 \| 10 \rangle; \; M_{\text{Y}_1^*} = M_0; \; M_{\Xi^*} = M^0 + \frac{1}{\sqrt{8}} \langle 10 \| \text{T}^{(8)} \| 10 \rangle;$$

$$M_{\Omega^-} = M_0 + \frac{2}{\sqrt{8}} \langle 10 \| T^{(8)} \| 10 \rangle.$$

Since there are two unknown parameters and four masses, one can obtain:

$$\begin{gathered} M_{N^*} - M_{Y^*} = M_{\Xi^*} - M_{\Omega^-} \\ (146 \pm 3\ \text{MeV}) \quad (146 \pm 4\ \text{MeV}) \\ 2M_{\Xi^*}\, M = {}_{Y_1^*} + M_{\Omega^-} \\ (2058 \pm 2\ \text{MeV}) \quad (2057 \pm 3\ \text{MeV}). \end{gathered} \tag{17}$$

These formulae simply state that masses in the decuplet are equally spaced.
The mass of Ω^- can be obtained from the known masses of N^* and Y_1^*. This predicted value, as we said, is amazingly close to the experimental one.
For the octet of the pseudoscalar mesons the argument is the same as that made for the stable baryons. Thus we get a formula analogous to (16):

$$\begin{gathered} m_K + m_{\bar{K}} = \frac{3}{2} m_\eta + \frac{1}{2} m_\pi \\ (992\ \text{MeV}) = (891\ \text{MeV}). \end{gathered} \tag{18}$$

Two remarks are in order:
1) $m_K = m_{\bar{K}}$ by charge-conjugation invariance of strong interactions.
2) (18) is in very poor agreement with experiment.
Substantial improvement can be achieved if we put in (18) the squared masses instead of the masses:

$$\begin{gathered} 2m_K^2 = \frac{3}{2} m_\eta^2 + \frac{1}{2} m_\pi^2 \\ (49.2 \cdot 10^4\ (\text{MeV})^2) \quad (46.2 \cdot 10^4\ (\text{MeV})^2). \end{gathered} \tag{19}$$

An argument given by Feynman to support this substitution is that in any field theoretical model, corrections to bare boson masses affect directly m^2.
The same remarks apply to the vector mesons which will be treated in next section.

10.5. $\omega - \varphi$ mixing

We have previously assigned eight vector mesons to an octet and the ninth to an SU_3 singlet; but we had no way to decide whether the ω or the φ had to be placed in the singlet. With the aid of the mass formula for the octet, which has of course the same form as (19), we find:

$$2m_{K^*}^2 = \frac{3}{2} m_8^2 + \frac{1}{2} m_\rho^2 \tag{20}$$

(m_8 = mass of the isosinglet i.e. of the $T^2 = Y = 0$ member of the octet)
Inserting the known values for m_{K^*}, m_ρ, one obtains $m_8 = 930 \pm 3$ MeV. This value is intermediate between $m_\varphi = 1019,5 \pm 3$ MeV and $m_\omega = 782. \pm 0.5$ MeV; however both differences $m_\varphi^2 - m_8^2 = 18 \cdot 10^4\ \text{MeV}^2$, $m_8^2 - m_\omega^2 = 25 \cdot 10^4\ \text{MeV}^2$ are very large.
If we insist that the mass formula must hold even in this case to an accuracy comparable to that obtained for the three other cases considered, we must conclude that neither ω nor φ can be identified sic et simpliciter with the isosinglet of the octet.

However physical particles are eigenstates of:

$$H = H_0 + H_1.$$

And H in the representation $8 \oplus 1$ to which the vector mesons are assigned is represented by the matrix:

$$H \equiv \begin{pmatrix} M_0^2 + (K^*|H_1|K^*) & 0 & 0 & 0 & 0 \\ 0 & M_0^2 + (\rho|H_1|\rho) & 0 & 0 & 0 \\ 0 & 0 & M_0^2 + (\overline{K^*}|H_1|\overline{K^*}) & 0 & 0 \\ 0 & 0 & 0 & M_0^2 + (8|H_1|8) & (8|H_1|1) \\ 0 & 0 & 0 & (1|H_1|8) & M_1^2 + (1|H_1|1) \end{pmatrix}.$$

If now M_0 is nearly equal to M_1, we can expect that the off-diagonal elements of H are important, so that the $T = 0$, $Y = 0$ eigenstates of H which we want to identify with ω and φ, are considerably different from the corresponding eigenstates of H_0.
We have then to diagonalize the submatrix

$$\begin{pmatrix} M_0^2 + \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & M_1^2 + \varepsilon_{22} \end{pmatrix} \tag{21}$$

where

$$\varepsilon_{11} = \langle 8|H_1|8\rangle,\ \varepsilon_{12} = \langle 8|H_1|1\rangle \text{ etc.}$$

Changing, if necessary, phases of the states $|8\rangle, |1\rangle$ we can take ε_{12} real so that from the hermiticity of H_1, $\varepsilon_{12} = \varepsilon_{21}$.
We write:

$$\begin{aligned} |\varphi\rangle &= \cos\theta\,|8\rangle + \sin\theta\,|1\rangle. \\ |\omega\rangle &= -\sin\theta\,|8\rangle + \cos\theta\,|1\rangle. \end{aligned} \tag{22}$$

The angle θ must be determined requiring $|\varphi\rangle$ and $|\omega\rangle$ to be eigenstates of (21) with eigenvalues equal to their physical masses squared [*30*].
From this one finds:

$$(\tan\theta)^2 = \frac{m_\varphi^2 - m_8^2}{m_8^2 - m_\omega^2} = 0{,}69;\ \theta = 39°50' \tag{23}$$

where $m_8^2 = M_0^2 + \varepsilon_{11}$ is the value given by the mass formula for the octect.
According to these results φ and ω appear to be superpositions of pure SU_3 states, in contrast to all the other particles we have considered [*31*].
It is important to note that, in contrast to the other cases, the introduction of the mixing angle θ rises to four the number of parameters needed to describe the vector mesons mass spectrum $(\theta, M_0, M_1, \langle 8|H_1|8\rangle)$. Having four masses at our disposal $m_{K^*}, m_\rho, m_\omega, m_\varphi$ it is not possible in this context to have any test of the theory. The possibility of determining θ in an independent way will be discussed later.

Our derivation of the sum rules (16), (17), (19), 20), was based on the knowledge of the Clebsch-Gordan coefficients involved.
Alternatively one can find the general structure inside each irreducible representation of an operator, such as H_1, commuting with T_1, T_2, T_3, Y and transforming according to the regular representation of SU_3.
Such an operator must have the form:

$$H_1 = f(T^2, Y) = a\,1 + bY + cT^2 + dY^2 + \cdots .$$

The stated transformation property further restricts H_1 to the form

$$H_1 = bY + c\left(T^2 - \frac{1}{4}Y^2\right) + a' \cdot 1,$$

so that in each irreducible representation the Hamiltonian is

$$H = H_0 + H_1 = a \cdot 1 + bY + c\left(T^2 - \frac{1}{4}Y^2\right). \tag{24}$$

For fermions (24) gives the mass formula:

$$M = a \cdot 1 + bY + c\left(T(T+1) - \frac{1}{4}Y^2\right). \tag{25}$$

For bosons, taking into account conditions $CHC^{-1} = H$, $CYC^{-1} = -Y$ (where C is the charge-conjugation operator), b must vanish:

$$m^2 = a + c\left(T(T+1) - \frac{1}{4}Y^2\right). \tag{26}$$

For the decuplet resonance one has the relation $T = 1/2\,Y + 1$, which reduces (26) to

$$m = a' + bY, \tag{27}$$

giving the equal spacing rule.
The general formula (24) has been given by S. OKUBO [32]

10.6. Baryon-meson Yukawa couplings

It is interesting to find out all the Yukawa-type couplings between baryons and pseudoscalar mesons which are invariant under SU_3. In fact we will find that these couplings involve only two constants so that writing them in the usual isotopic spin form, one can derive relations between coupling constants such as $g_{NN\pi}$, $g_{\Sigma\Lambda\pi}$, $g_{N\Sigma K}$ etc.
In a field-theoretical model one assumes baryons and mesons fields to behave under SU_3 as tensor operators (see sect. 6.3) belonging to the eight dimensional representation.
The Yukawa-type interaction Lagrangian has the form

$$\mathscr{L} = \sum_{\alpha\beta\gamma} g_{\alpha\beta\gamma}\, \bar{B}_\alpha \gamma_5\, B_\beta\, M_\gamma$$

where B_β and $\bar{B}_\alpha$ are, respectively, baryon and antibaryon fields specified by the labels β and α (i.e. p, n, ... $\bar{\text{p}}$, $\bar{\text{n}}$, ...) and M_γ is a meson field specified by γ (i.e. π^+, π^-, ...).
Invariance under SU_3 demands $\mathscr{L}$ to transform as a scalar operator. Now the operators

$$\bar{B}_\alpha \gamma_5 B_\beta M_\gamma$$

transform like members of the $8 \otimes 8 \otimes 8$ representation, and obviously the same holds for $\mathscr{L}$. Coefficients $g_{\alpha\beta\gamma}$ have to be determined by requiring $\mathscr{L}$ to be one of those vectors which in the decomposition of $8 \otimes 8 \otimes 8$ belong to the irreducible (0, 0) components. Using the method described in sect. 9.7 one can see that there are two (0, 0) components, i.e. only two possible invariant Yukawa-couplings.
According to sect. 10.3, antibaryon fields transform as

$$(\bar{B}) \to U\bar{B}U^{-1} \qquad U \in SU_3$$

i.e. just like baryons, due to the fact that the 8-representation is selfconjugate. It is then easy to verify that the tensor operator

$$\text{Trace}\,(\bar{B}\gamma_5 BM) \tag{28}$$

(trace involves summation only over SU_3 indices, so that γ_5 must be treated as a number) is invariant. In the same way one can see that also the operator

$$\text{Trace}\,(\bar{B}\gamma_5 MB) \tag{29}$$

is invariant.
Instead of (28) and (29) we will use the so called F and D combinations defined as

$$(F)\ \text{Trace}\,(\bar{B}\gamma_5[B, M]); \quad (D)\ \text{Trace}\,(\bar{B}\gamma_5\{B, M\}),$$

which are obviously linearly independent invariant operators. Recalling that in the product $8 \otimes 8 \otimes 8$ there are exactly two operators of such kind, we conclude that the most general invariant trilinear operator in mesons, baryons and antibaryons fields is a linear combination of F and D. In particular:

$$\mathscr{L} = \sum_{\alpha\beta\gamma} g_{\alpha\beta\gamma}\, \bar{B}_{\alpha\gamma_5}\, B_\beta M_\gamma = g_F\, \text{Tr}\,(\bar{B}\gamma_5\,[B, M]) + g_D\, \text{Tr}\,(\bar{B}\gamma_5\,\{B, M\}). \tag{30}$$

To deduce couplings in terms of isomultiplets one has merely to substitute (3), (4), (8) into (30), to carry out the trace and to rearrange terms in order to have a combination of isospin couplings (for example of the form $g_{NN\pi}\, \overline{N\tau}\, N\pi$, $\overline{\Sigma}\Sigma\eta$ etc.).
If we define:

$$g_{NN\pi} = g = \frac{g_D - g_F}{\sqrt{2}}$$

$$\alpha = \frac{-g_F}{\sqrt{2}\,g},$$

then all coupling constants can be expressed in terms of α and g. We refer the reader to [33] for a complete list of these relations.

The experimental situation does not allow a precise determination of α. The value $\alpha = 1$ (i.e. pure F-type coupling) seems to be excluded by hyperfragments binding, which requires $g_{\Lambda\Sigma\pi} = 2/\sqrt{3}\, g\,(1-\alpha) \neq 0$. The dynamical calculations of MARTIN and WALI [34] indicate $0.15 \leq \alpha \leq 0.56$ i.e. a prevalence of the D-type coupling. With this set of values $g_{N\Lambda K}$ turns out to be of the same order of magnitude than $g_{NN\pi}$. This seems to be in contrast with K-photoproduction data, which suggest $g_{N\Lambda K}$ an order of magnitude smaller than $g_{NN\pi}$.
These discrepancies can be in principle accounted for by symmetry breaking interactions. One other possibility has been pointed out in [35].
Same considerations can be applied to the baryon-vector meson couplings, leading to two possible Lagrangians: one F-type, and the other D-type.
If one writes the F-type coupling in terms of isospin multiplets, [36] the ρ appears to be coupled to the isospin current (i.e. to terms like $\bar{N}\boldsymbol{\tau}N$, $\bar{\boldsymbol{\Sigma}}\times\boldsymbol{\Sigma}$, etc.) and the ω_8 (i.e. the $T_3 = Y = T = 0$ member of the vector meson-octet) to the hypercharge current (i.e. to the term $\bar{N}N + \bar{\Xi}\Xi$) whereas in the D-type these peculiar couplings do not appear. Now we know (for example from the isovector part of the electromagnetic form factors of the nucleon) that ρ is actually coupled to the isospin current, so that in this case we have to assume only F-type coupling. The interaction Lagrangian then is

$$\mathcal{L} = g_{BBV}\,\mathrm{Tr}\,(B\gamma_\mu\,[B, V_\mu]),$$

where V_μ is a matrix analogous to (4), with the substitutions:

$$\pi \to \rho_\mu$$
$$K \to K^*_\mu$$
$$\eta \to (\omega_8)_\mu$$

We have now only one parameter.
The vector meson singlet (i.e. the φ_0) is coupled with the baryonic number current:

$$\mathcal{L} = g_{BBV}\,(\varphi_0)_\mu\,\mathrm{Tr}\,(\bar{B}\gamma_\mu B)$$

Consider now the pseudoscalar-vector meson couplings: also in this case we may have two invariant combinations of terms like

$$\left(\frac{\partial}{\partial x_\mu} M_\alpha\right) M_\beta\,(V_\mu)_\gamma$$

(α, β, γ, are SU_3 indices which label the various mesons.), namely

$$\mathrm{Trace}\;(V_\mu M\,(\partial_\mu M) - V_\mu\,(\partial_\mu M)\,M) = Tr\,(V_\mu[M, \partial_\mu M])$$
$$\mathrm{Trace}\;(V_\mu M\,\partial_\mu M + V_\mu \partial_\mu M M) = Tr\,(V_\mu\{M, \partial_\mu M\}).$$

However the second term, by virtue of the condition:

$$\partial_\mu V_\mu = 0,$$

is equivalent to a divergence, i.e. it adds a divergence to the Lagrangian, so that it can be assumed to vanish. In fact:

$$V_\mu M(\partial_\mu M) + V_\mu(\partial_\mu M)M = V_\mu\partial_\mu(MM) = \partial_\mu(V_\mu MM) + (\partial_\mu V_\mu)MM =$$
$$= \partial_\mu(V_\mu MM).$$

Hence also this coupling is pure F. In contrast with the previous case the vector meson singlet cannot be coupled to pseudoscalar mesons: in fact the only possible coupling would be

$$(\varphi_0)_\mu\,\mathrm{Tr}\,((\partial_\mu M)M),$$

but under charge conjugation $M \to M^T$, $\partial_\mu M \to \partial_\mu M^T$, $\varphi_{0\mu} \to -\varphi_{0\mu}$ (charge conjugation of ρ_0, ω, and φ is -1) so that this coupling is not invariant under C. This has the consequence that the decay

$$\varphi_0 \to K + \bar{K}$$

is forbidden, whereas

$$\omega_8 \to K + \bar{K}$$

is allowed, i.e. only the componend of the φ particle on the octet can decay in in $K\bar{K}$. This fact can be used in principle to determine the $\omega - \varphi$ mixing independently from mass formulae [37].

10.7 Decuplet decays

The decays of the baryon decuplet resonances allowed by energy, T_3, T^2 and Y conservation are

$$N^* \to N + \pi; \quad Y_1^* \to \begin{cases}\Lambda^0\pi \\ \Sigma\,\pi\end{cases}; \quad \Xi^* \to \Xi\pi$$

(Ω^- is stable against electromagnetic and strong decays because of its mass, which is less than the threshold of the $\Xi\bar{K}$ channel which is the only open for these interactions). These decays, in the limit of exact SU_3, are described by a single amplitude. In fact in this case the matrix elements involved are of the type

$$M\,(N^* \to N\pi) = (N^*\,|S|\,N\pi), \tag{31}$$

where S is scalar under SU_3. By reducing the product $8 \otimes 8$, with Clebsch-Gordan coefficients, we can write (31) as the matrix element of a scalar operator between vectors of irreducible representations.
Such matrix elements are zero for vectors belonging to irreducible components of $8 \otimes 8$ different from 10, whereas they are equal to

$$a\,\delta_{T^2T'^2}\,\delta_{T_3T_3'}\,\delta_{YY'}$$

for vectors of the 10 component. Hence we have

$$M(N^{*++} \to p + \pi^+) = -\frac{1}{\sqrt{2}}\,a; \qquad M(Y_1^{*+} \to \Lambda\pi) = -\frac{1}{2}\,a$$

$$M(Y_1^{*+} \to \Sigma_0\pi^+) = \frac{1}{\sqrt{12}}\,a; \qquad M(Y_1^{*+} \to \Sigma^+\pi^0) = -\frac{1}{\sqrt{12}}\,a \tag{32}$$

$$M(\Xi^{*0} \to \Xi^- + \pi^+) = -\frac{1}{\sqrt{6}}\,a; \qquad M(\Xi^{*0} \to \Xi^0\pi^0) = \frac{1}{2\sqrt{3}}\,a,$$

(all the other amplitudes can be obtained from these by isotopic spin symmetry) where factors multiplying a are the proper Clebsch-Gordan coefficients.
If we want to compare (32) with experiments we have to take into account mass differences. The most simple thing to do is to introduce mass differences into the phase space factors which multiply $|M|^2$ in the expression for rates, leaving untouched relations (32).
Predictions so obtained are in an unpleasant disagreement with experiments. For example one has:

$$\frac{\text{Rate }(Y_1^{*+} \to \Sigma\pi)}{\text{Rate }(Y_1^{*+} \to \Lambda\pi)} = (\text{phase space ratio}) \times \frac{4}{6} = \frac{P_\Sigma^3}{P_\Lambda^3} \cdot \frac{2}{3} \simeq 12\%$$

$(p_{\Sigma,\Lambda}$ = momentum of Σ or Λ particle),

whereas experimentelly the ratio is consistent with zero $(\sim 2 \pm 2\%)$.
This discrepancy can be thought as due to large non symmetrical interactions, which must be properly accounted for.
In fact it has been shown by V. GUPTA and V. SINGH [38] and by C. BECCHI, E. EBERLE, G. MORPURGO that inserting a symmetry breaking interaction of the type used for mass formulae, one can derive relations between decuplet decay amplitudes which agree well with the experimental data.
10.8. The main test of the unitary symmetry model in strong interactions would be to check experimentally the relations which one can derive between amplitudes of different scattering processes. However relations obtained assuming full SU_3 symmetry widely disagree with experimental data [23] and again one has to take into account the role of symmetry breaking interactions. This role has not been till now satisfactorily understood so that, partly for this reason, partly for lack of experimental data, we do not know at present how to make meaningful tests of the eightfold way with scattering processes.

11. Electromagnetic Interactions

11.1. We know that the electromagnetic field interacts with hadrons in such a way to conserve T_3 and Y; moreover, due to the smallness of the coupling constant, we can describe such interaction with a pertubative method, starting from an interaction Lagrangian of the form:

$$\mathcal{L}_{\text{int}} = e j_\mu(x) A^\mu(x). \tag{1}$$

Here $j_\mu(x)$ is the electromagnetic current of hadrons. The structure of this operator depends upon the dynamics of strong interactions themselves, which we at present do not know in detail.
Hence the Lagrangian (1) has to be considered as a phenomenological device which allows us to explicitly consider the dependence on electromagnetic field of these interactions, whereas unknown effects of strong interactions are lumped into the local operator $j_\mu(x)$. We note that the matrix elements of $j_\mu(x)$ between physical states are connected to measurable quantities (form factors).
By definition we have

$$\int j_4(x)\, d^3x = Q \tag{2}$$

and by charge, Y, T_3 conservation:

$$\partial^\mu j_\mu(x) = 0$$

$$[j_\mu(x), T_3] = [j_\mu(x), Y] = 0. \tag{3}$$

In the context of the eightfold way model, relations between matrix elements of $j_\mu(x)$ can be obtained by the knowledge of the commutation relations between $j_\mu(x)$ and the generators of SU_3.
From the Gell-Mann Nishijima relation we have:

$$\int j_4(x)\, d^3x = Q = T_3 + \frac{1}{2}\, Y = 3H_1 + H_2.$$

This suggests $j_\mu(x)$ to be composed of two parts: the current of the third component of total isospin plus one half the hypercharge current

$$j_\mu(x) = j_\mu^{(T_3)}(x) + \frac{1}{2}\, j_\mu^{(Y)}(k), \tag{4}$$

and now $j_\mu^{(T_3)}(x)$ and $j_\mu^{(Y)}(x)$ have the same transformation properties (under SU_3) as T_3 and Y. From this it is easy to see that $j_\mu(x)$ commutes with $E_{\pm 3}$ and the same for $\mathcal{L}_{\text{int}}$. If we put (see sect. 8.7.c)

$$U_3 = \frac{3}{4}\, Y - \frac{1}{2}\, T_3 = \frac{3}{2}\, H_2 - \frac{\sqrt{3}}{2}\, H_1$$

$$U_\pm = \sqrt{3}\, E_{\mp 3},$$

we see that U_3, $U_\pm$ have the same commutation relations of the isotopic spin generators, and are called the U-spin generators. Hence electromagnetic interactions at all perturbative orders conserve charge and U-spin, which in this case play the same role as hypercharge and I-spin for medium strong interactions.
From the previous analysis we see that everything we said for medium strong interactions can be applied to electromagnetic interactions, substituting the partition into I-spin and Y-multiplets of irreducible SU_3 representations with a partition into U-spin and Q-multiplets.
We give for reference the decomposition of the pseudoscalar mesons octet into U-spin multiplets (Fig. 10).

One sees that π^+, K^+ and K^-, π^- constitute two U-spin doublets whereas K_0 and $\bar{K}_0$ are members of an U-spin triplet. π^0 and η are eigenstates of U_3, with eigenvalue zero, but they are not eigenstates of U^2. Instead the combinations

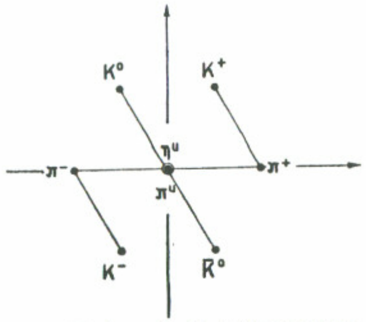


Fig. 10. U-spin multiplets in the pseudoscalar mesons octet

$$|\pi^u\rangle = \frac{1}{2}\,|\pi^0\rangle - \frac{\sqrt{3}}{2}\,|\eta\rangle$$

$$|\eta^u\rangle = \frac{\sqrt{3}}{2}\,|\pi^0\rangle + \frac{1}{2}\,|\eta\rangle$$

are eigenstates of U^2 with eigenvalues 2 and 0, so that π^u, K^0, $\bar{K}^0$ constitute an U-spin triplet and η^u is an U-spin singlet.
As a general rule in the weight diagram U-spin multiplets are directed along the α_3 root.
From what we said, we can draw very easily a certain number of consequences:

a) *Electromagnetic mass splittings* — In calculating mass splittings due to symmetry breaking interactions we neglected the electromagnetic effects, which we want now to take into account.
We start from a situation in which, apart from SU_3-invariant strong interactions, only electromagnetic interactions are present. In this case the masses of the particles inside an SU_3 multiplet split up according to U-spin multiplets, being (1) invariant under U-spin. Hence masses obey a law of the form

$$m = m_0 + m(Q, U) \qquad (m_0 = \text{common mass of the } SU_3 \text{ multiplet}).$$

In the case of stable baryons we get in particular the relations:

$$\begin{aligned} m_p &= m_{\Sigma^+} \\ m_n &= m_{\Xi^0} \\ m_{\Xi^-} &= m_{\Sigma^-}. \end{aligned} \tag{5}$$

Relations (5) are of course not satisfied by the actual masses; this is natural because we have neclected the important contribution of medium strong interactions. We can however deduce from (5) the relation:

$$m_n - m_p = (m_{\Xi^0} - m_{\Xi^-}) - (m_{\Sigma^+} + m_{\Sigma^-}) \tag{6}$$

which has been given firstly by S. COLEMAN and S. L. GLASHOW [41].
If we now turn on the medium strong symmetry breaking interactions, masses of particles lying inside the same I-spin multiplet are shifted of the same amount, so that we way expect relation (6) to be left unchanged, since it compares mass differences for particles with same T^2. In fact inserting the experimental values (6) reads:

$$(m_{\Sigma^-} - m_{\Sigma^+}) - (m_n - m_p) = 6.38 \pm 0.3 \text{ MeV}$$

$$m_{\Xi^-} - m_{\Xi^0} = 6.5 \pm 1.2 \text{ MeV (data from [28]).}$$

The agreement is excellent. One should remark that the previous derivation of (6) rests on the possibility of treating in an independent way the medium strong and electromagnetic effects. This seems not to be in general a legitimate procedure. In a field theoretical treatment we would write a Lagrangian made up of three terms:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{MS} + \mathcal{L}_{e.m.}$$

$\mathcal{L}_0$ is the symmetrical part, whereas $\mathcal{L}_{MS}$ and $\mathcal{L}_{e.m.}$ are responsible of medium strong and electromagnetic interactions. Using perturbative methods, the correction to the symmetrical mass m_0 is expressed as the expectation value of a power series in $\mathcal{L}_{MS} + \mathcal{L}_{e.m.}$. Hence we see that COLEMAN and GLASHOW relation follows if we neglect all terms which contain powers of the product $\mathcal{L}_{MS}\mathcal{L}_{e.m.}$, retaining all orders in $\mathcal{L}_{MS}$ and $\mathcal{L}_{e.m.}$.
From this point of view the validity of (6) is quite unexpected (see however [23]).
b) *Magnetic moments* — Just in the same way, neglecting medium strong interactions and assuming only U-spin and charge conservations, we obtain the following relations between magnetic moments of stable baryons [39, 40] (the same relations apply to the electric and magnetic form factors)

$$\begin{aligned} \mu(\Sigma^+) &= \mu(p) \\ \mu(\Sigma^-) &= \mu(\Xi^-) \\ \mu(\Xi^0) &= \mu(n) \\ -\frac{1}{\sqrt{3}}\,\mu(\Sigma^0\Lambda) &= \mu(n) - \mu(\Lambda) \\ -\sqrt{3}\,\mu(\Lambda\Sigma^0) &= \mu(n) - \mu(\Sigma^0), \end{aligned} \tag{7}$$

where $\mu(\Sigma^0\Lambda)$ is called the transition magnetic moment between Σ^0 and Λ, and appears for example in the amplitude of the decay $\Sigma^0 \to \Lambda + \gamma$.
The experimental information available up to now does not allow to test any-one of the relations (7).
For pseudoscalar mesons we obtain easily the results [40] that the form factors of K^+ and π^+ are equal, whereas the form factors of K^0 and $\bar{K}^0$ are zero.
In fact:

form factor (K^0) = form factor ($\bar{K}^0$) by U-spin
form factor (K^0) = — form factor ($\bar{K}^0$) by Charge-conjugation

c) *η and π^0 — two photons decay* — We said previously that the $U_3 = Q = 0$ eigenstates of U^2 are (in the pseudoscalar meson octet)

$$|\pi^u\rangle = \frac{1}{2}\,|\pi^0\rangle - \frac{\sqrt{3}}{2}\,|\eta\rangle \quad (U = 1)$$

$$|\eta^u\rangle = \frac{\sqrt{3}}{2}\,|\pi^0\rangle + \frac{1}{2}\,|\eta\rangle \quad (U = 0),$$

conversely one has:

$$|\pi^0\rangle = \frac{1}{2}\,|\pi^u\rangle + \frac{\sqrt{3}}{2}\,|\eta^u\rangle$$

$$|\eta\rangle = -\frac{\sqrt{3}}{2}\,|\pi^u\rangle + \frac{1}{2}\,|\eta^u\rangle.$$

Observe now that $|\pi^u\rangle$ cannot decay, by U-conservation, into two photons (which have $U = 0$) so that the amplitudes for $(\pi^0, \eta) \to 2\gamma$ are equal to

$$A(\pi^0 \to 2\gamma) = \frac{\sqrt{3}}{2} A(\eta^u \to 2\gamma)$$

$$A(\eta \to 2\gamma) = \frac{1}{2} A(\eta^u \to 2\gamma),$$

i.e.

$$A(\pi^0 \to 2\gamma) = \sqrt{3}\, A(\eta \to 2\gamma).$$

After phase space corrections, assuming the lifetime of π^0 to be equal to $1.5 \cdot 10^{-16}$s. we obtain a width for $\eta \to 2\gamma$ of 140 eV [*41*], which is not inconsistent with present data.

d) *First order relations* — Up to this point we have only used U-spin invariance. More detailed informations can be obtained if we retain in the perturbative expansions considered only first order terms (in the electromagnetic coupling) i.e. terms containing the elctromagnetic current only once.

In this case we can exploit the assumption contained in formula (4), i.e. the fact that $j_\mu(x)$ transforms as a member of the eight-dimensional representation of SU_3. By using Wigner-Eckart theorem, for example, we can express all magnetic moments of stable baryons in terms of only two magnetic moments; (the same applies to electric form factors). For example we find

$$\mu_\Lambda = \frac{1}{2}\mu_n = -0.95 \text{ nuclear magnetons},$$

whereas experimentally

$$\mu_\Lambda = -0.66 \pm 0.35 \text{ nuclear magnetons}.$$

All the other explicit relations are contained in [*40*].
Finally, let us consider the electromagnetic decays:

$$\omega \to e^+ + e^- (\mu^+ + \mu^-)$$
$$\varphi \to e^+ + e^- (\mu^+ + \mu^-).$$

Both decays can be thought to go through the one-photon channel [*37*]

(8)

In the amplitude for the process (8) the matrix element of the electromagnetic current between the ω (or φ) state and the vacuum is involved. Now if we write (sect. 10.5 (22))

$$|\varphi\rangle = \cos\theta\,|8\rangle + \sin\theta\,|1\rangle$$
$$|\omega\rangle = \sin\theta\,|8\rangle + \cos\theta\,|1\rangle,$$

we see that we have to evaluate the matrix elements:

$$\langle 1|\, j_\mu(x)\,|0\rangle$$
$$\langle 8|\, j_\mu(x)\,|0\rangle.$$

$j_\mu(x)$ transforms as the 8 representation and, whereas the product $8 \otimes 8$ contains the singlet representation (to which the vacuum is assigned) this representation is not contained in the product $8 \otimes 1$. Hence, by Wigner-Eckart theorem, the first matrix element vanishes. We conclude that an SU_3 singlet cannot decay through the one photon channel, so that only the components of ω and φ over the octet can go (at first order) into $e^+ + e^- (\mu^+ + \mu^-)$.

The ratio

$$\frac{\Gamma(\varphi \to e^+e^-)}{\Gamma(\omega \to e^+e^-)} = (\text{phase space corrections}) \tan^2\theta$$

provides in principle a measure of $\tan^2\theta$ independent from mass formulae.
Up to now however only the $\omega \to e^+ + e^-$ decay has been observed, so that we cannot make any comparison of the theory with experiment.

12. Leptonic Decays of Hadrons

12.1. Very exciting results have been obtained by the application of SU_3 symmetry to the field of weak interactions of hadrons. We will not give here an extensive discussion of all the topics involved, limiting ourselves to sketch the theory for leptonic decays[21]). These processes have the general form

$$A \to l + \nu_l + B + B' + \cdots$$
$$A \to l + \nu_l,$$

where $A, B, B' \ldots$ are strongly interacting particles, l is a lepton (e, μ), ν_l the corresponding neutrino. A few significant examples are:

		ΔT	ΔS	ΔQ
$\Delta S = 0$	$\pi^+ \to \pi^0 + e^+ + \nu_e$	-1	0	-1
	$\pi^+ \to \mu^+ + \nu_\mu$	-1	0	-1
	$n \to p + e^- + \bar\nu_e$	1	0	1
	$\Sigma^+ \to \Lambda + e^+ + \nu_e$	-1	0	-1
$\Delta S = 0$	$K^+ \to \pi^0 + e^+ + \nu_e$	$-1/2$	-1	-1
	$K^+ \to \mu^+ + \nu_\mu$	$-1/2$	-1	-1
	$\Lambda \to p + e^- + \bar\nu_e$	$1/2$	1	1
	$\Sigma^- \to n + e^- + \bar\nu_e$	$1/2$	1	1

(1)

All these processes can be described starting from an interaction Lagrangian of the form:

$$\mathcal{L}_I = \frac{G}{\sqrt{2}} [J_\mu (j_\mu)^+ + \text{H. c.}] \tag{2}$$

J_μ and j_μ are the weak currents associated to hadrons and leptons, and G is the weak coupling constant determined from μ-decay.

21) For a more detailed treatment of weak interactions see [*42, 43, 44*].

For what concerns j_μ, expermental findings agree with the form (in terms of lepton fields)

$$j_\mu = \bar{e}\gamma_\mu(1+\gamma_5)\nu_e + \bar{\mu}\gamma_\mu(1+\gamma_5)\nu_\mu , \tag{3}$$

whereas the structure of J_μ, which is thought to be determined by strong interactions, is not known in detail. The amplitudes of the processes we are considering are expressed as matrix elements of (2)

$$\frac{G}{\sqrt{2}}[\langle B + B' + .. |J_\mu| A\rangle \langle l\,\nu_l|j_\mu^+|0\rangle + \langle BB' .. |J_\mu^+| A\rangle \langle l\,\nu_l|j_\mu|0\rangle];$$

the matrix elements of j_μ can be calculated, so that the actual difficultyis constituted by the other terms.

For what concerns space-time properties, experiments indicate that J_μ can be splitted up in two terms: one transforming as a vector and the other as a pseudovector: we will refer to them as to the vector and axial-vector currents:

$$J_\mu = J_\mu^V + J_\mu^A .$$

The first one is responsible e.g. for the β-decay of π^+:

$$\pi^+ \to \pi^0 + e^+ + \nu_e ,$$

and the other for the usual π decay:

$$\pi^+ \to \mu^+ + \nu_\mu .$$

Let us define ΔS, ΔT_3, ΔQ respectively as the changes of strangeness, isospin, charge suffered by hadrons[22]) (see (1)). Then we can divide leptonic decays into two classes: $\Delta S = 0$, and $\Delta S \neq 0$ decays. Experiments indicate [43] that the following selection rules are satisfied within errors (which are however rather large):

i) for $\Delta S = 0$ decays, $|\Delta T_3| = 1$ (hence $\Delta Q = \pm 1$)

ii) for $\Delta S \neq 0$ decays $\Delta S = \Delta Q$, $|\Delta S| = 1$ (hence $|\Delta T_3| = 1/2$).

This selection rule forbids for example the process:

$$\Sigma^+ \to n + e^+ + \nu_e$$

which actually has not been seen [45].

We are then led to write J_μ as:

$$J_\mu = J_\mu^{V(0)} + J_\mu^{V(1)} + J_\mu^{A(0)} + J_\mu^{A(1)} , \tag{4}$$

where $J_\mu^{V(0)}$ is the strangeness conserving and $J_\mu^{V(1)}$ is the $\Delta S = 1$ part of the vector current, and the same for axial currents.

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22) These changes are not independent. From the Gell-Mann Nishijima formula:

$$\Delta Q = \Delta T_3 + \frac{1}{2}\Delta S .$$

In the context of isotopic spin symmetry one could attempt to explain selection rule i), assuming the $\Delta S = 0$ current to tranform under SU_2 as a tensor operator, belonging to the isospin one representation.

The CVC (conserved vector current) theory of FEYNMANN and GELL-MANN [46] embodies this assumption in a much stronger statement; they identify the $\Delta S = 0$ vector currents $J_\mu^{V(0)}$, $(J_\mu^{V(0)})^+$ with the $T_3 = \pm 1$ components of the isospin current, i.e. of the current which arises from SU_2 invariance (see sect. 7.3.).

The $T_3 = 0$ component is the isovector part of the electromagnetic current. As a consequence, insofar we neglect electromagnetic effects, $J_\mu^{V(0)}$ and $(J_\mu^{V(0)})^+$ are conserved:

$$\partial_\mu J_\mu^{V(0)} = 0,\ \partial_\mu (J_\mu^{V(0)})^+ = 0 .$$

We quote three significant examples of predictions made by CVC theory [42]:

a) the agreement between the Fermi constant in β-decay of nuclei and the Fermi constant in μ-decay;

b) the rate of the pion β-decay can be calculated from the neutron β-decay obtaing a result in agreement with experiments:

c) using Wigner-Eckart theorem one can express the matrix elements of $J_\mu^{V(0)}$, $(J_\mu^{V(0)})^+$, for example between nucleon states, in terms of the electromagnetic current matrix elements, i.e. of nucleon e. m. form factors which are known from $e - N$ scattering experiments. Preliminary data on neutrino experiments seem to support this prediction.

The $\Delta S = 0$ axial current is also assumed to transform as an isovector, but in contrast to $J_\mu^{V(0)}$ it is not conserved.

The simplest generalization to include strange particles decays is to assume J_μ^V and J_μ^A to possess well-defined transformation properties under SU_3. In particular if we assume J_μ^V and J_μ^A to belong to octets of tensor operators, it is obvious that selection rules i), ii) are fulfilled (of course the converse is not true: i) ii) do not imply octet currents).

Now SU_3 symmetry provides us an octet of vector currents (see sect. 7.3) which, in absence of symmetry breaking interactions, are conserved. We could then identify various pieces of J_μ^V with such currents. This however would imply each current to be coupled to leptons with the same strenght, i.e. with the same coupling constant as the $\Delta S = 0$ part. On the contrary experiments give coupling constants for strange particle decays which are smaller of an order of magnitude than the $\Delta S = 0$ couplings.

N. CABIBBO has assumed [47—48] that the vector current coupled to leptons has the form

$$J_\mu^V = \cos\theta\, J_\mu^{V(0)} + \sin\theta\, J_\mu^{V(1)} , \tag{5}$$

where $J_\mu^{V(0)}$, $(J_\mu^{V(0)})^+$, $J_\mu^{V(1)}$, $(J_\mu^{V(1)})^+$, are the $T_3 = \pm 1$, $Y = 0$, $T_3 = \pm 1/2$, $Y = \pm 1$ members of the octet of currents deriving from SU_3 invariance (to which electromagnetic current belongs, sect. 11.1). θ is an angle which characterizes weak interactions of all hadrons. Moreover the axial current is assumed to have the form

$$J_\mu^A = \cos\theta\, J_\mu^{A(0)} + \sin\theta\, J_\mu^{A(1)} \tag{6}$$

with the same θ as (5). $J_\mu^{A(0)}$, $(J_\mu^{A(0)})^+$, $J_\mu^{A(1)}$, $(J_\mu^{A(1)})^+$ are tensor operators transforming as the $T_3 = \pm 1$, $Y = 0$, $T_3 = \pm 1/2$, $Y = \pm 1$ members of an octet.

We give here a brief account of predictions which can be obtained from this theory in the case of baryons decays. For a detailed discussion see [43, 49].
Matrix elements of J_μ^V between baryon states can be expressed in terms of θ and of two reduced matrices ($8 \otimes 8$ contains the eight representation twice). By analogy with sect. 10.6 let us call them F_V, D_V. In the limit of zero momentum transfer, they can be written as:

$$(F_V)_\mu = \bar{u}(p_f)\gamma_\mu F_V(k^2)\, u(p_i)$$
$$(D_V)_\mu = \bar{u}(p_f)\gamma_\mu D_V(k^2)\, u(p_i) \qquad (7)$$

p_i, p_f: baryons initial and final momentum
$k = p_f - p_i$, $k^2 \to 0$.
And in the limit in which such currents are conserved

$$F_V(0) = 1, \quad D_V(0) = 0,$$

so that the matrix elements of J_μ^V are determined by θ.
For what concerns axial currents, again we have two reduced matrices, F, D which in the same limit as (7) can be written as

$$F_\mu = \bar{u}(p_f) F(k^2)\gamma_\mu\gamma_5 u(p_i) \; (k^2 \to 0)$$
$$D_\mu = \bar{u}(p_f)\, D(k^2)\gamma_\mu\gamma_5\, u(p_i), \qquad (8)$$

but now the lack of conservation of J_μ^A does not allow to obtain additional conditions on $F(0)$, $D(0)$.
Concluding we see that the baryons decays (in the limit $k^2 \to 0$, absence of symmetry breaking interactions[23]) are described in terms of three numbers: $F(0)$, $D(0)$, θ.
H. COURANT et al. [45] find two sets of parameters consistent with data, both with near the same value of θ, but differing for the ratio F/D.
It is remarkable that near the same value of θ ($\theta \simeq 0.25$) has been given by CABIBBO in the paper previously quoted, comparing the decays:

$$K^+ \to \mu^+ + \nu_\mu$$
$$\pi^+ \to \mu^+ + \nu_\mu.$$

In these decays only axial currents contribute ($(J_\mu^{A(1)})^+$ and $(J_\mu^{A(0)})^+$) respectively) so that the branching ratio:

$$\frac{R(K^+ \to \mu^+ + \nu_\mu)}{R(\pi^+ \to \mu^+ + \nu_\mu)}$$

is proportional to

$$\left|\frac{\langle K^+ | (J^{A(1)})^+ | 0\rangle}{\langle \pi^+ | (J^{A(0)})^+ | 0\rangle}\right|^2,$$

i.e. to $\tan^2\theta$.

[23]) When the symmetry breaking interaction is taken into account at the first order, it can be shown that our conclusions on the vector current remain correct [50].

13. Concluding Remarks

13.1. The idea of a higher symmetry, in the concrete formulation of the eightfold way model, undoubtely greatly improves the phenomenological description of the behaviour of strongly interacting particles. However the very fact that it works rises the need of understanding at a deeper, dynamical level, how the symmetry is brought about (as well as its partial violation).
For an up-to-date discussion of the various attempts made in this direction, using bootstrap technique as well as field theoretical methods, see [23]. To the latter class belongs SCHWINGER's W_3 model [51] as well as the popular "quarks" or "aces" model (proposed by ZWEIG and GELL-MANN].
Another interesting problem is that of the connection between internal and space-time symmetries.
In this context very promising is the SU_6 model proposed by GÜRSEY, RADICATI and PAIS [52]; see also [53]) who treat on the same footing spin, isospin and hypercharge.

References

[1] L. PONTRJAGIN, Topological groups, Princeton Univ. Press.
[2] P. COHN, Lie groups, Cambridge Univ. Press (1961); C. CHEVALLEY, Theory of Lie Groups, Vol. 1, Princeton Univ. Press.
[3] M. GELL-MANN, Phys. Rev. **125**, 1067 (1962).
[4] J. VON NEUMANN, Mathematical Foundations of Quantum Mechanics, Princeton Univ. Press (1955); M. A. NAIMARK, Normed Rings, P. Noordhoff N. V.-Groningen.
[5] M. A. NAIMARK, Les representations lineaire du groupe de Lorentz — Dunod, chapter 2.
[6] G. YA. LYUBARSKII, The Applications of Group Theory in Physics, Pergamon Press, cap. IX.
[7] V. BARGMANN, Ann. Math. **1**, 1 (1954).
[8] E. P. WIGNER, Rendiconti della Scuola Int. di Fisica „Enrico Fermi", XXIX Course (1964).
[9] S. SCHWEBER, An Introduction to Relativistic Quantum Field Theory, J. Wiley and Sons., N. Y. (1961).
[10] G. WICK, A. S. WIGHTMAN, E. P. WIGNER, Phys. Rev. **88**, 101 (1952).
[11] A. S. WIGHTMAN, Dispersion Relations and Elementary Particles, J. Wiley, N. Y. (1960).
[12] E. P. WIGNER, Ann. Math. **40**, 149 (1939).
[13] H. JOOS, Fortschr. Phys. **10**, 65 (1962).
[14] N. N. BOGOLIOUBOV, D. V. SHIRKOV, Introduction to the Theory of Quantized Fields, Interscience, N. Y. (1960).
[15] J. M. BLATT, V. F. WEISSKOPF, Theoretical Nuclear Physics. — J. Wiley and Sons, N. Y. (1958).
[16] K. NISHIJIMA, Fundamental Particles, Benjamin (1963).
[17] L. MICHEL, Invariance in Quantum Mechanics and Group Extension, Instambul (1962); preprint version Ecole Polytechnique, Paris.
[18] N. JACOBSON, Lie algebras, Interscience (1962).
[19] E. B. DYNKIN, Structure of semisimple Lie algebras, Math. Soc. Transl. series 1, vol. 9.
[20] G. RACAH, Group Theory and Spectroscopy. Institute for Advanced Study, Princeton (1951), Reprint CERN 61 - 8 (1961).
[21] H. BÖRNER, Darstellungen von Gruppen, Springer Verlag — Berlin (1955).
[22] G. DE ROBINSON, Representation Theory of the Symmetric Group, Toronto Univ. Press.

[23] A. SALAM, Symmetry of Strong Interactions, Intern. Conf. on High Ener. Phys., Dubna (1964).
[24] Y. NE'EMAN, Nuclear Phys. **26**, 222 (1961).
[25] R. E. BEHRENDS, J. DREITLEIN, C. FRONSDAL, W. LEE, Rev. mod. Phys. **34**, 1 (1962).
[26] R. GATTO, Theoretical Physics — Seminar IAEA (1963).
[27] V. E. BARNER et al., Phys. Rev. Letters **12**, 134 (1964).
[28] A. H. ROSENFELD et al., Rev. mod. Phys. **36**, 977 (1964).
[29] P. MCNAMEE, S. J. and FRANK CHILTON, Rev. mod. Phys. **36**, 1005 (1964).
[30] S. OKUBO, Phys. Letters **5**, 165 (1963).
[31] J. J. SAKURAI, Phys. Rev. Letters **8**, 472 (1962).
[32] S. OKUBO, Progr. theor. Phys. (Kyoto), **27**, 949 (1962).
[33] J. DE SWART, Rev. mod. Phys. **35**, 916 (1963).
[34] A. W. MARTIN, K. C. WALI, Phys. Rev. **130**, 2455 (1963).
[35] S. HATSUKADE et al., Photoproduction of Strange Particles and Unitarity Symmetry, Rochester. U-R-875-52 (1964).
[36] J. J. SAKURAI, Proc. of the Intern. School of Phys. „Enrico Fermi" Course XXVI.
[37] R. H. DALITZ, Proc. of the Sienna Conf. on Elem. Part. (1963), Plenary Sessions, vol. II.
[38] V. GUPTA, V. SINGH, Phys. Rev. **135**, B 1442 (1964).
[39] S. COLEMAN, S. L. GLASHOW, Phys. Rev. Letters **6**, 423 (1961).
[40] N. CABIBBO, R. GATTO, Nuovo Cimento **21**, 872 (1961).
[41] R. GATTO, Proc. of the Intern. Sienna Conf. on Elem. Part. (1963), Plenary Sessions, vol. II.
[42] J. D. JACKSON, Weak Interactions, Brandeis Summer Institute 1962, Benjamin, New York (1963).
[43] L. B. OKUN, Weak Interactions and Unitary Symmetry, Institute of Theor. and Exper. Phys., Moscow (1964) (in russian).
[44] Weak Interactions and Higher Symmetries, Acta Physica Austriaca. Suppl. I Springer Verlag Wien — New York (1964).
[45] H. COURANT et al., Phys. Rev. **136**, B 1791 (1964).
[46] R. P. FEYNMAN, M. GELL-MANN, Phys. Rev. **109**, 193 (1958).
[47] N. CABIBBO, Phys. Rev. Letters **10**, 531 (1963).
[48] Proc. of the Intern. School of Phys. „Ettore Majorana", Benjamin, New York (1964).
[49] L. WOLFENSTEIN, Phys. Rev. **135**, B 1436 (1964).
[50] M. ADEMOLLO, R. GATTO, Phys. Rev. Letters **13**, 264 (1964).
[51] J. SCHWINGER, Phys. Rev. Letters **12**, 237 (1964).
[52] F. GÜRSEY, L. A. RADICATI, Phys. Rev. Letters **13**, 173 (1964).
A. PAIS, Phys. Rev. Letters **13**, 175 (1964);
F. GÜRSEY, A. PAIS, L. A. RADICATI, Phys. Rev. Letters **13**, 299 (1964).
[53] B. SAKITA, Phys. Rev. **135**, B 1756 (1964).

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