Some notes (under construction) on the first part of the course (prerequisites necessary to understand the second part)

ONDE NON LINEARI 2

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1 Dynamical systems and vector fields

Here we summarise some known facts on dynamical systems (systems of ODEs).

The **dynamical system**

$$\begin{array}{l} \frac{dx^i}{dt} = u^i(\vec{x},t), \quad i=1,\dots,N \qquad \left(\frac{d\vec{x}}{dt} = \vec{u}(\vec{x},t)\right). \\ \vec{u}(\vec{x},t) = (u^1(\vec{x},t),\dots,u^N(\vec{x},t)) \in \mathbb{R}^N, \\ \vec{x} = (x^1,\dots,x^N) \in \mathbb{R}^N, \quad \nabla_{\vec{x}} = (\partial_{x^1},\dots,\partial_{x^N}), \end{array} \tag{1}$$

together with the initial condition $\vec{x}(t_0) = \vec{x}_0 \in \mathbb{R}^N$, define a flow (a trajectory in the phase space $\mathbb{R}^N$), tangent to the **vector field** $\vec{u}(\vec{x},t)$ (see Fig. 1).

The general solution of (1), depending on N arbitrary constants $\vec{c}$, is characterized by the system of nondifferential equations:

$$\varphi_j(\vec{x},t) = c_j, \quad j=1,\dots,N, \tag{2}$$

where the c_j's are N independent constants. Solving the system wrt $\vec{x}$, if $\frac{\partial(\varphi_1,\dots,\varphi_N)}{\partial(x_1,\dots,x_N)}$, one obtains the general solution of (1):

$$\vec{x} = \vec{X}(t,\vec{c}). \tag{3}$$

Definition 1. $I(\vec{x},t)$ is an integral of motion of (1) iff I satisfies the linear PDE:

$$I_t + \vec{u}\cdot\nabla_{\vec{x}} I = 0 \tag{4}$$

(I is constant on the characteristic curves (integral curves) of (1)).

Definition 2. Equation (4) can be written as

$$\begin{array}{l} \hat{u} I = 0, \\ \hat{u} := \partial_t + \vec{u}\cdot\nabla_{\vec{x}} = \sum\limits_{k=0}^{N} u^k \partial_{x^k}, \quad u^0 = 1,\ x^0 = t. \end{array} \tag{5}$$

where **the first order linear operator $\hat{u}$ is also called vector field** associated with the ODE (1).

The application of $\hat{u}$ to a scalar differentiable function $f(\vec{x},t)$:

$$\hat{u}f(\vec{x},t) \tag{6}$$

is the "directional derivative of f, at the point $(\vec{x},t) \in \mathbb{R}^{N+1}$ of the extended phase space, in the direction of the vector $(1,\vec{u}(\vec{x},t))$ (whose components are the coefficients of the vector field $\hat{u}$)". We are therefore identifying the extended vector field $(1,\vec{u}(\vec{x},t)) = (1,u^1,\dots,u^N)$ with the operator $\hat{u}$, that takes a directional derivative in the direction of $(1,\vec{u})$! One of the advantages of such identification is that $\hat{u}$ does not depend on coordinates.

Definition 3. A dynamical system (1) is Hamiltonian, and/or the associated vector field $\hat{u}$ is Hamiltonian, iff N is even and there exists a function $H(\underline{q},\underline{p},t)$ such that the ODE (1) and $\hat{u}$ can be written in the form:

$$\begin{array}{l} \frac{dq_i}{dt} = \frac{\partial H(\underline{q},\underline{p},t)}{\partial p_i}, \quad i = 1,\dots,n = \frac{N}{2}, \\ \frac{dp_i}{dt} = -\frac{\partial H(\underline{q},\underline{p},t)}{\partial q_i}, \quad i = 1,\dots,n, \end{array} \tag{7}$$

$$\hat{u} = \partial_t + \{H,\cdot\}_{\underline{p},\underline{q}}, \tag{8}$$

where $\vec{x} = (q_1,\dots,q_n,p_1,\dots,p_n)^T$, and the expression $\{f,g\}_{\underline{p},\underline{q}}$ is the well-known Poisson bracket of f and g wrt $\underline{p},\underline{q}$

$$\{f,g\}_{\underline{p},\underline{q}} = \sum_{n=1}^{N/2}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right) = J\begin{pmatrix} \nabla_{\underline{q}} f \\ \nabla_{\underline{p}} f \end{pmatrix} \cdot \begin{pmatrix} \nabla_{\underline{q}} g \\ \nabla_{\underline{p}} g \end{pmatrix}. \tag{9}$$

More precisely, the vector $\vec{u}(\vec{x},t)$ can be written in the form:

$$\vec{u}(\vec{x},t) = \begin{pmatrix} \nabla_{\underline{p}} H \\ -\nabla_{\underline{q}} H \end{pmatrix} = J\begin{pmatrix} \nabla_{\underline{q}} H \\ \nabla_{\underline{p}} H \end{pmatrix}, \tag{10}$$

where

$$J := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}, \tag{11}$$

$$\vec{x} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix}, \quad \nabla_{\vec{x}} = \begin{pmatrix} \nabla_{\underline{q}} \\ \nabla_{\underline{p}} \end{pmatrix} = \begin{pmatrix} \partial_{q_1} \\ \vdots \\ \partial_{q_n} \\ \partial_{p_1} \\ \vdots \\ \partial_{p_n} \end{pmatrix}, \tag{12}$$

so that

$$\vec{u}(\vec{x},t)\cdot\nabla_{\vec{x}} = (\nabla_{\underline{p}}H)\cdot\nabla_{\underline{q}} - (\nabla_{\underline{q}}H)\cdot\nabla_{\underline{p}} = J\begin{pmatrix}\nabla_{\underline{q}}H\\ \nabla_{\underline{p}}H\end{pmatrix}\cdot\begin{pmatrix}\nabla_{\underline{q}}\\ \nabla_{\underline{p}}\end{pmatrix}, \tag{13}$$

and the directional derivative of f in the direction of the Hamiltonian vector field reads:

$$\begin{gathered}\hat{u}f = f_t + \Big((\nabla_{\underline{p}}H)\cdot\nabla_{\underline{q}} - (\nabla_{\underline{q}}H)\cdot\nabla_{\underline{p}}\Big)f = \\ f_t + J\begin{pmatrix}\nabla_{\underline{q}}H\\ \nabla_{\underline{p}}H\end{pmatrix}\cdot\begin{pmatrix}\nabla_{\underline{q}}f\\ \nabla_{\underline{p}}f\end{pmatrix} =: f_t + \{H,f\}_{\underline{p},\underline{q}}\end{gathered} \tag{14}$$

i.e, equation (8).

Definition 4. The vector field $\vec{u}(\vec{x},t)$ $(\hat{u})$ is divergence - less iff $\nabla_{\vec{x}}\cdot\vec{u} = 0$.

Proposition 1.

1. If the dynamical system (1) associated with a divergence-less vector field $\vec{u}$, it gives rise to volume preserving flows (**prove it!**).
2. A Hamiltonian vector field is divergence-less (**check it!**), but the opposite may not be true..
3. A two-dimensional divergence-less vector field is Hamiltonian (**check it!**).

Proposition 2.

1. Vector fields $\hat{u}$ form a Lie algebra whose Lie bracket is given by the usual commutator. Indeed (**check it!**):

$$[\hat{u},\hat{v}] = \hat{w}, \tag{15}$$

where

$$\begin{aligned}&\hat{u} = \sum_k u^k\partial_{x^k}, \quad \hat{v} = \sum_k v^k\partial_{x^k},\\ &\hat{w} = \sum_k w^k\partial_{x^k}, \quad w^k := \hat{u}v^k - \hat{v}u^k.\end{aligned} \tag{16}$$

2. If the vector fields $\hat{u}_1, \hat{u}_2$

$$\begin{aligned}&\hat{u}_j = \partial_{t_j} + \vec{u}_j\cdot\nabla_{\vec{x}}, \quad j=1,2,\\ &\vec{u}_j = (u_j^1(\vec{x},t),\dots,u_j^N(\vec{x},t)), \quad j=1,2,\end{aligned} \tag{17}$$

are Hamiltonian, with Hamiltonians H_1, H_2, then the following identity holds true (**check it !**):

$$[\hat{u}_1, \hat{u}_2] = \sum_{i=1}^{n} \left(\frac{\partial \mathcal{H}_{12}}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial \mathcal{H}_{12}}{\partial q_i} \frac{\partial}{\partial p_i} \right) = (\nabla_p \mathcal{H}_{12}) \cdot \nabla_q - (\nabla_q \mathcal{H}_{12}) \cdot \nabla_p,$$
$$\mathcal{H}_{12} := \{H_1, H_2\}_{p,q} + H_{2t_1} - H_{1t_2}. \tag{18}$$

1.1 Symmetries of ODEs and commutation of vector fields

Proposition 3 The following statements are equivalent.

1. The two dynamical systems
$$\begin{array}{l} \frac{d\vec{x}}{dt_1} = \vec{u}_1(\vec{x}, \vec{t}), \quad \vec{t} = (t_1, t_2), \\ \frac{d\vec{x}}{dt_2} = \vec{u}_2(\vec{x}, \vec{t}) \end{array} \tag{19}$$
commute (or the flows generated by them commute; or one dynamical system is symmetry of the other).

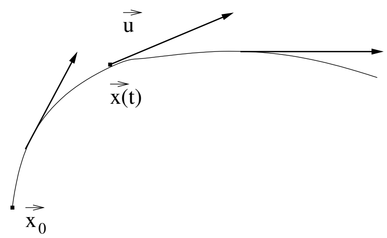
2. The vector fields $\hat{u}_1$ and $\hat{u}_2$ defined in (17) commute (see Fig. 2):
$$[\hat{u}_1, \hat{u}_2] = \hat{0}. \tag{20}$$

3. The following two linear PDEs
$$\hat{u}_1 \psi = \hat{u}_2 \psi = 0 \tag{21}$$
are satisfied for the same eigenfunction ψ (the two ODEs share the same constant of motion ψ).

4. The following quasilinear PDEs of hydrodynamic type for the components of the vectors $\vec{u}_1(\vec{x}, \vec{t}), \vec{u}_2(\vec{x}, \vec{t})$ are satisfied:
$$\hat{u}_1 \vec{u}_2 = \hat{u}_2 \vec{u}_1 \quad \Leftrightarrow \quad \vec{u}_{2t_1} + (\vec{u}_1 \cdot \nabla)\vec{u}_2 = \vec{u}_{1t_2} + (\vec{u}_2 \cdot \nabla)\vec{u}_1. \tag{22}$$

Proposition 4. If the two commuting vector fields are Hamiltonian, then (**check it!**):

$$[\hat{u}_1, \hat{u}_2] = 0 \quad \Leftrightarrow \quad \mathcal{H}_{12} = \{H_1, H_2\}_{p,q} + H_{2t_1} - H_{1t_2} = 0. \tag{23}$$



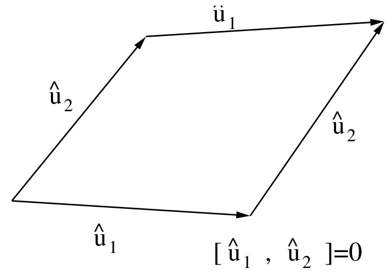


Fig. 1 The flow generated by $\vec{u}$.   Fig. 2 Commuting vector fields.

2 Linear and quasi-linear PDEs of the first order [1, 2]

Consider the evolution equation in $N+1$ dimensions:

$$\psi_t + \sum_{k=1}^{N} u^k \psi_{x^k} = h \tag{24}$$

Different cases

1. If the coefficients
$$u^k = u^k(\vec{x}, t) \in \mathbb{R}^N \tag{25}$$
are given functions, it's a linear first order PDE in $N+1$ dimensions for the unknown $\psi(\vec{x}, t)$.

2. If the arguments of the u^i's and h depend also on the unknown $\psi(\vec{x}, t)$:
$$u^i = u^i(\vec{x}, t, \psi), \quad h = h(\vec{x}, t, \psi), \tag{26}$$
it's a first order quasi-linear PDE (linear in the highest derivatives) in $N+1$ dimensions for the unknown $\psi(\vec{x}, t)$.

3. If $h = 0$, it is a homogeneous equation.

2.1 Quasi-linear PDEs

The first order quasi-linear PDE in n dimensions

$$\underline{P}(\underline{x},\psi)\cdot\nabla_{\underline{x}}\psi = Q(\underline{x},\psi) \tag{27}$$

$\underline{x} = (x^1,\ldots,x^n)$ is intimately related to the following system of n ODEs:

$$\tfrac{dx^1}{P_1} = \cdots = \tfrac{dx^n}{P_n} = \tfrac{d\psi}{Q}. \tag{28}$$

If one of the independent variables is time $t = x^n$, the PDE becomes, more conveniently,

$$\frac{\partial\psi}{\partial t} + \vec{u}(\vec{x},t,\psi)\cdot\nabla_{\vec{x}}\psi = h(\vec{x},t,\psi) \tag{29}$$

where

$$\vec{x} = (x^1,\ldots,x^N),\quad \vec{u} = (\frac{P_1}{P_n},\ldots,\frac{P_{n-1}}{P_n})^T,\quad h = \frac{Q}{P_n},\quad N = n-1, \tag{30}$$

and the ODE becomes

$$\begin{array}{l}\frac{d\vec{x}}{dt} = \vec{u}(\underline{x},\psi),\\ \frac{d\psi}{dt} = h(\underline{x},\psi).\end{array} \tag{31}$$

To show these relations, let

$$\varphi(\underline{x},\psi) = c \tag{32}$$

be the equation defining a solution of (27) (if $\frac{\partial\varphi}{\partial\psi}\neq 0$, we can solve it wrt ψ, obtaing the solution $\psi(\underline{x})$ of (27)). Equation (32) defines an integral surface S of (27)) in the $(n+1)$-dimensional $(\underline{x},\psi)$ space (an hypersurface of dimension n).

Since, from (32),

$$\nabla_{\underline{x}}\varphi + \frac{\partial\varphi}{\partial\psi}\nabla_{\underline{x}}\psi = \underline{0}, \tag{33}$$

if $\frac{\partial\varphi}{\partial\psi}\neq 0$, it follows that

$$\underline{P}\cdot\nabla_{\underline{x}}\varphi + Q\frac{\partial\varphi}{\partial\psi} = (\underline{P},Q)\cdot(\nabla_{\underline{x}},\partial_\psi)\varphi = 0. \tag{34}$$

Since the gradient of φ: $(\nabla_{\underline{x}},\partial/\partial_\psi)\varphi$ is normal to the integral surface S, the $(n+1)$-dimensional vector $\underline{V} = (\underline{P},Q)$ is tangent to S at the point $(\underline{x},\psi)$, and defines a direction on S at that point. Moving along that direction, one construct a "characteristic curve", always tangent to S. If s is the arc length parameter along the characteristic curve, then $d\underline{x}/ds$ is parallel to $\underline{V}$:

$$\frac{d\underline{x}}{ds} = \mu \underline{P}, \quad \frac{d\psi}{ds} = \mu Q \tag{35}$$

implying the system of n ODEs (28) or (31). The opposite is also true:

Proposition 4. The general solution of the system of n ODEs (28) generates the general solution of the first order quasi-linear PDE in n dimension (27).
Proof. The general solution of (31) is described by the $n = N + 1$ equations

$$\varphi_j(\underline{x}, \psi) = c_j, \quad j = 1, \dots, n, \tag{36}$$

where the c_j' are n independent constants, so that we can write

$$\phi(c_1, \dots, c_n) = 0, \quad \Rightarrow \quad c_n = F(c_1, \dots, c_{n-1}) \quad \left(\text{if } \frac{\partial \phi}{\partial c_n} \neq 0\right), \tag{37}$$

where ϕ is an **arbitrary** differentiable function of n arguments and F is an **arbitrary** differentiable function of $(n-1)$ arguments, so that:

$$\varphi_n(\underline{x}, \psi) = F\left(\varphi_1(\underline{x}, \psi), \dots, \varphi_{n-1}(\underline{x}, \psi)\right). \tag{38}$$

Solving (38) wrt to $\psi = \psi(\underline{x})$, one obtains the general solution of the PDE (27), given in terms of an arbitrary function F of $(n-1)$ variables. Indeed,

$$\frac{d\varphi_n(\vec{x}(t), t, \psi(t))}{dt} = \frac{\partial \varphi_n}{\partial t} + \vec{u} \cdot \nabla_{\vec{x}} \varphi_n + h \frac{\partial \varphi_n}{\partial \psi} = 0, \tag{39}$$

equivalent to (29), after using (34). □

2.2 Linear PDEs

In the linear case, equation (29) reduces to

$$\psi_t + \vec{u}(\vec{x}, t) \cdot \nabla_{\vec{x}} \psi = h_0(\vec{x}, t) + h_1(\vec{x}, t)\psi, \tag{40}$$

and solutions of (40) are in one to one correspondence with solutions of the system of ODEs:

$$\begin{array}{l} \frac{d\vec{x}}{dt} = \vec{u}(\vec{x}, t), \\ \frac{d\psi}{dt} = h_0(\vec{x}, t) + h_1(\vec{x}, t)\psi \end{array} \tag{41}$$

which is now decoupled, and the characteristic curves are defined by (41a).

In the important homogeneous subcase $h_0 = h_1 = 0$

$$\hat{u}\psi = \psi_t + \vec{u}(\vec{x}, t) \cdot \nabla_{\vec{x}} \psi = 0, \tag{42}$$

that we are going to discuss from now on, solutions of (42) are in one to one correspondence with solutions of the system of ODEs:

$$\begin{array}{l} \frac{d\vec{x}}{dt} = \vec{u}(\vec{x}, t), \\ \frac{d\psi}{dt} = 0 \quad (\Rightarrow \quad \psi = c_{N+1}) \,. \end{array} \tag{43}$$

Indeed, from solutions of (42) one obtains solutions of (43) specializing the above procedure. Viceversa, from the general solution of (43a), we construct the general solution of (42). Indeed, the general solution of (43a) is characterized by the system of nondifferential equations

$$\phi_j(\vec{x}, t) = c_j, \quad j = 1, \ldots, N \quad (\Rightarrow \quad \vec{x} = \vec{x}\,(t, c_1, \ldots, c_N))\,; \tag{44}$$

from the usual condition $c_{N+1} = F(c_1, \ldots, c_N)$ it follows that the general solution of the PDE (42) reads

$$\psi(\vec{x}, t) = F\left(\phi_1(\vec{x}, t), \ldots, \phi_N(\vec{x}, t)\right), \tag{45}$$

where F is an arbitrary differentiale function of N arguments (to check it, i) verify that the N functions $\phi_j(\vec{x}, t), \;\; j = 1, \ldots, N$ are particular solutions of (40); ii) verify that $N + 1$ solutions of (40) are dependent, so that N independent solutions form a basis (see Proposition 5); iii) verify that the space of solutions of (40) form a ring (see Proposition 5)).

Proposition 5

1. Equation (42) admits N independent solutions.
   Proof. Suppose we have $N + 1$ solutions

   $$\psi_i(\vec{x}, t), \quad i = 1, \ldots, N + 1, \tag{46}$$

   of (42); then the corresponding system

   $$\psi_{it} + \vec{u} \cdot \nabla_{\vec{x}} \psi_i = 0, \quad i = 1, \ldots, N + 1, \tag{47}$$

viewed as an algebraic system for the N components of $\vec{u}$, gives rise to a nontrivial solution iff

$$\frac{\partial(\psi_1, \dots, \psi_{N+1})}{\partial(x_1, \dots, x_N, t)} = 0; \tag{48}$$

i.e., the $N+1$ solutions $\psi_1, \dots, \psi_{N+1}$ of the PDE (40) are dependent $\Box$.

2. The solution space of (42) is a ring; i.e., an arbitrary differentiable function $f(\psi_1, \dots, \psi_N)$ of solutions of (42) is a solution of (42) (**check it!**). Also, if $\{\psi_1, \dots, \psi_N\}$ is a basis of solutions of (42), then any solution ψ of (42) can be written in the form

$$\psi = F(\psi_1, \dots, \psi_N), \tag{49}$$

for some differentiable function F of N arguments (**check it!**).

3. If the vector field associated with (40) is Hamiltonian, then the solution space of (42) is also a Lie algebra, whose Lie bracket is the Poisson bracket.

Proof. If $\psi_1,\ \psi_2$ are solutions of (40) and $\hat{u}$ is Hamiltonian:

$$\hat{u}\psi_j = \psi_{j_t} + \vec{u} \cdot \nabla_{\vec{x}}\psi_j = \psi_{j_t} + \{H, \psi_j\} = 0, \quad j = 1,2 \tag{50}$$

Then

$$\begin{array}{l} \hat{u}\{\psi_1, \psi_2\} = \{\psi_1, \psi_2\}_t + \{H, \{\psi_1, \psi_2\}\} = \{\psi_{1t}, \psi_2\} + \{\psi_1, \psi_{2t}\} - \\ \{\psi_2, \{H, \psi_1\}\} - \{\psi_1, \{\psi_2, H\}\} = \\ \{\psi_1, \psi_{2t} + \{H, \psi_2\}\} + \{\psi_{1t} + \{H, \psi_1\}, \psi_2\} = 0, \end{array} \tag{51}$$

having used the Jacobi identity, implying that also $\{\psi_1, \psi_2\}$ is solution of (40) (see (14)) $\Box$.

2.3 Homogeneous version of the quasi-linear PDE (27)

If $Q = h = 0$:

$$\psi_t + \vec{u}(\vec{x}, t, \psi) \cdot \nabla_{\vec{x}}\psi = 0; \tag{52}$$

then ψ is constant on the characteristic curves defined by (31):

$$\begin{array}{l} \frac{d\psi}{dt} = 0, \quad \Rightarrow \quad \psi = const = F(\vec{\xi}), \\ \frac{d\vec{x}}{dt} = \vec{u}(\vec{x}, t, \psi) = \vec{u}(\vec{x}, t, f(\vec{\xi})). \end{array} \tag{53}$$

The general solution of (53b) is characterized by the $N = n-1$ nondifferential equations:

$$\varphi_j(\vec{x}, t, F(\vec{\xi})) = \xi_j, \quad j = 1, \dots, N = n-1 \tag{54}$$

where $\vec{\xi} = (\xi_1, \dots, \xi_N)$ is an arbitrary constant vector. Inverting this equation wrt $\vec{\xi}$: $\vec{\xi} = \vec{\xi}(\vec{x}, t)$, and replacing it in (53a), one obtains

$$\psi(\vec{x}, t) = F(\vec{\xi}(\vec{x}, t)), \tag{55}$$

where F is an arbitrary function of N variables. Equations (**??**) and (55) characterize the general solution of (27).

Basic example We consider, as basic example (with $\rho = \psi$), the following first order quasi-linear PDE in multidimension:

$$\rho_t + \vec{u}(\rho) \cdot \nabla_{\vec{x}} \rho = 0, \qquad \rho = \rho(\vec{x}, t), \tag{56}$$

which is nothing but the continuity equation

$$\begin{array}{l} \rho_t + \nabla_{\vec{x}} \cdot \vec{Q}(\rho) = 0, \\ \vec{u}(\rho) = \frac{\partial \vec{Q}(\rho)}{\partial \rho}, \end{array} \tag{57}$$

for some “density” ρ and some “flux vector” $\vec{Q}(\rho) = \rho \vec{v}(\rho)$. If $N = 1$ and $u^1(\rho) = \rho$, this equation reduces to the famous Hopf equation:

$$\rho_t + \rho \rho_x = 0, \tag{58}$$

the simplest prototypical model in the description of the gradient catastrophe (wave breaking) of one dimensional waves.

From the above considerations, the solution $\rho(\vec{x}, t)$ of (56) is constant on the characteristic curves defined by the system of ODEs:

$$\frac{d\vec{x}}{dt} = \vec{u}(\rho(\vec{x}, t)) = co\vec{n}st. \tag{59}$$

Therefore (59) can be trivially integrated:

$$\vec{\varphi} = \vec{x} - \vec{u}(F(\vec{\xi}))t = \vec{\xi} \tag{60}$$

where $\vec{\xi}$ is an arbitrary N-dimensional contant vector, describing an N-parameter family of characteristic straight lines in $\mathbb{R}^{N+1}$, and the constant value of ρ on the straight line labelled by $\vec{\xi}$ is $F(\vec{\xi})$, where F is an arbitrary scalar function of N variables. Therefore the solution $\rho(\vec{x},t)$ of (56) is characterized by the **non-differential** equations

$$\begin{array}{l} \rho(x,t) = F(\vec{\xi}), \\ \vec{x} = \vec{u}(F(\vec{\xi}))t + \vec{\xi}. \end{array} \qquad (61)$$

Solving (61b) wrt $\vec{\xi}$: $\vec{\xi} = \vec{\xi}(\vec{x},t)$, and replacing it in (61a), one obtains:

$$\rho(x,t) = F(\vec{\xi}(\vec{x},t)), \qquad (62)$$

where F is an arbitrary scalar function of N variables.

Equivalently, this general solution is also characterized by the implicit nondifferential equation

$$\rho = F(\vec{x} - \vec{u}(\rho)t). \qquad (63)$$

If we add, for instance, the initial condition

$$\rho(\vec{x},0) = \rho_0(\vec{x}), \quad \vec{x} \in \mathbb{R}^N, \qquad (64)$$

then $F = \rho_0$ and:

$$\rho = \rho_0(\vec{x} - \vec{u}(\rho)t). \qquad (65)$$

3 Analytic, geometric and universal aspects of wave-breaking in 1+1 dimensions

For the geometric aspects, see [3, 4]; for the analytic aspects, see [5] and below.

Consider the evolution of a localized one-dimensional wave according to the Hopf equation

$$\begin{array}{l} u_t + uu_x = 0. \\ u(x,0) = F(x), \quad x \in \mathbb{R} \end{array} \qquad (66)$$

Such evolution is described by the implicit equation

$$\begin{array}{l} u = F(\xi), \\ \xi = x - F(\xi)t, \end{array} \qquad (67)$$

in which equation (67b) must be solved with respect to ξ, obtaining $\xi = \xi(x,t)$, and substituted in (67a), to get the solution $u = F(\xi(x,t))$.

Such inversion is possible if the ξ-derivative of (67b) is different from zero. Therefore the **1-dimensional (movable) Singularity Manifold** (SM) of (66) is described by the equation:

$$\mathcal{S}(\xi,t) = 1 + F_\xi(\xi)t = 0 \quad \Rightarrow \quad t = -\frac{1}{F_\xi(\xi)}. \tag{68}$$

Since

$$u_x = \frac{F_\xi}{1+tF_\xi}, \tag{69}$$

the wave breaks on the singularity manifold.

We are interested in the first time t_b in which the breaking of the solution occurs, corresponding to the characteristic values ξ_b such that

$$\begin{array}{l} t_b = t(\xi_b) = \text{global min}\{t(\xi)\} > 0 \quad \Rightarrow \\ F_{\xi\xi}(\xi_b) = 0, \quad\quad F_\xi(\xi_b) < 0, \quad\quad F_{\xi\xi\xi}(\xi_b) > 0, \end{array} \tag{70}$$

ξ_b is an inflection point of the initial profile.

At t_b, the wave breaks in the point x_b of the x-axis defined by

$$x_b = F(\xi_b)t_b + \xi_b. \tag{71}$$

Now we study the solution (67) near breaking:

$$x = x_b + x', \quad t = t_b + t', \quad \xi = \xi_b + \xi', \tag{72}$$

where x', t', ξ' are small:

$$\xi = x - F(\xi)t \quad \Rightarrow \quad {\xi'}^3 + b(t')\xi' - \gamma X(x',t') = 0, \tag{73}$$

where

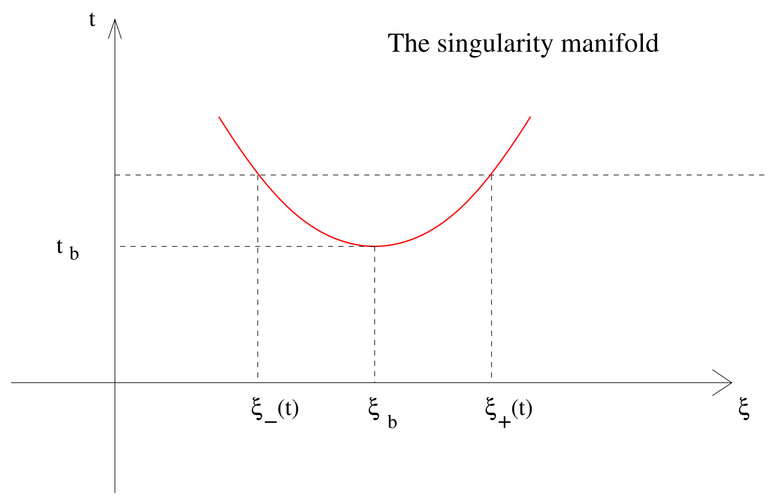
$$b(t') = \tfrac{6F_\xi}{t_b F_{\xi\xi\xi}}t', \quad X(x',t') = x' - F(\xi_b)t', \quad \gamma = \tfrac{6}{t_b F_{\xi\xi\xi}} \tag{74}$$

corresponding to the maximal balance

$$|X| = O(|t'|^{3/2}), \quad\quad |z| = O(|t'|^{1/2}). \tag{75}$$

At the same order, the SM equation reads:

$$SM: \quad 0 = \mathcal{S}(\xi,t) \sim F_\xi(\xi_b)t' + \frac{F_{\xi\xi\xi}(\xi_b)}{2}t_b{\xi'}^2. \tag{76}$$



The three roots of this cubic are given by the well-known Cardano's formula:

$$\begin{aligned} &{\xi'}_0\,(x',y',t') = (S_+)^{\frac{1}{3}} + (S_-)^{\frac{1}{3}}, \\ &{\xi'}_\pm\,(x',y',t') = \tfrac{1}{2}\left((S_+)^{\frac{1}{3}} + (S_-)^{\frac{1}{3}}\right) \pm \tfrac{\sqrt{3}}{2} i \left((S_+)^{\frac{1}{3}} - (S_-)^{\frac{1}{3}}\right), \end{aligned} \tag{77}$$

where

$$\begin{aligned} &S_\pm = R \pm \sqrt{\Delta}, \quad \Delta = R^2 + Q^3 \\ &Q(t') = \tfrac{b(t')}{3}, \quad R(x',t') = \tfrac{\gamma}{2} X(x',t'). \end{aligned} \tag{78}$$

Once $\xi(x',t')$ is known from the solution of the cubic, implying also:

$$\xi'_{x'} = \frac{1}{\mathcal{S}}, \quad \xi'_{x'x'} = -t_b F_{\xi\xi\xi} \frac{\xi' {\xi'}^2_{x'}}{\mathcal{S}}, \tag{79}$$

then the solution of the Hopf equation and its derivatives are then approximated, near breaking, by the formulae:

$$\begin{aligned} &u(x,t) \sim F(\xi_b + \xi'), \\ &u_x(x,t) \sim F'(\xi)\xi'_{x'} \sim \frac{F_\xi(\xi_b+\xi')}{G_\xi t' + \frac{G_{\xi\xi\xi}}{2}{\xi'}^2 t_b}, \\ &u_{xx}(x,t) \sim F''(\xi)(\xi'_{x'})^2 + F'(\xi)\xi'_{xx}. \end{aligned} \tag{80}$$

Before breaking

If $t < t_b$ ($t' < 0$), $\mathcal{S}$ and Δ are strictly positive, and only the root ξ'_0 is real; correspondingly, the real solution of (66) is single valued:

$$u \sim F(\xi_b + {\xi'}_0(x',t')). \tag{81}$$

and the slope u_x of the profile, finite $\forall x$, reaches its minimum at the inflection point $x_f(t')$, at wich $X = 0 \;\Rightarrow\; \xi' = 0, \xi'_{xx} = 0, u_{xx} = 0$:

$$\begin{array}{l} x_f(t') = x_b + F(\xi_b)t' \quad (X = x - x_f(t')), \\ u(x_f(t), t) = F(\xi_b), \quad u_x(x_f(t), t) = \frac{1}{t - t_b}, \quad u_{xx}(x_f(t), t) = 0. \end{array} \tag{82}$$

To analyse the solution in a smaller region around the inflection point, we choose

$$|X| = |x - x_f(t')| = O(|t'|^{p+\frac{1}{2}}), \quad p > 1, \tag{83}$$

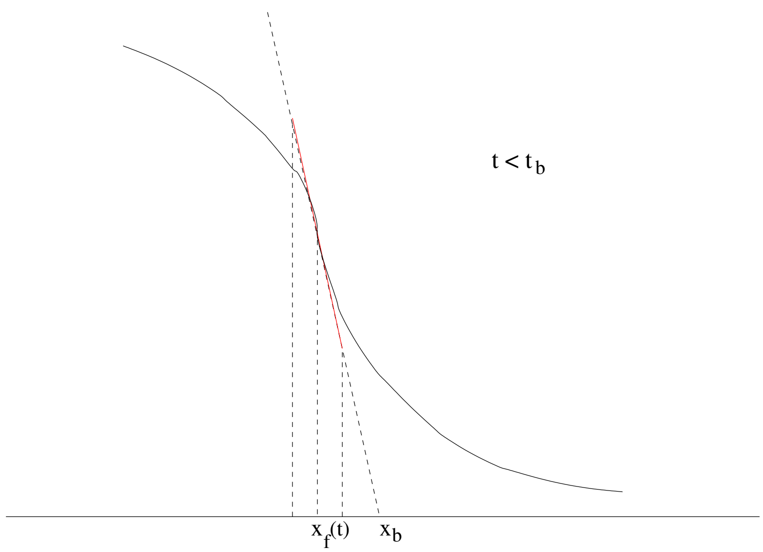
Then ${\xi'}^3 << b\xi' \sim -\gamma X$ and the solution becomes more explicit:

$$\xi' \sim \frac{\gamma X}{b} = \frac{x' - F(\xi_b)t'}{F_\xi(\xi_b)t'}, \tag{84}$$

and the real solution of (66) reduces to the
exact similarity solution of the Hopf equation:

$$u \sim F(\xi_b + \xi') \sim \frac{x - x_b}{t - t_b}, \tag{85}$$

describing the tangent to the profile at the inflection point.



The analytic expression of the slope of the profile:

$$u_x \sim -\left(|t - t_b| + \frac{F_{\xi\xi\xi}(\xi_b)}{2} t_b^2 \left(\frac{x - x_f(t')}{t - t_b}\right)^2\right)^{-1}. \tag{86}$$

$u_x \sim (t - t_b)^{-1}$ in this very thin region; but $u_x = O(1)$ in the region $|x - x_f(t)| = O(|t'|)$.

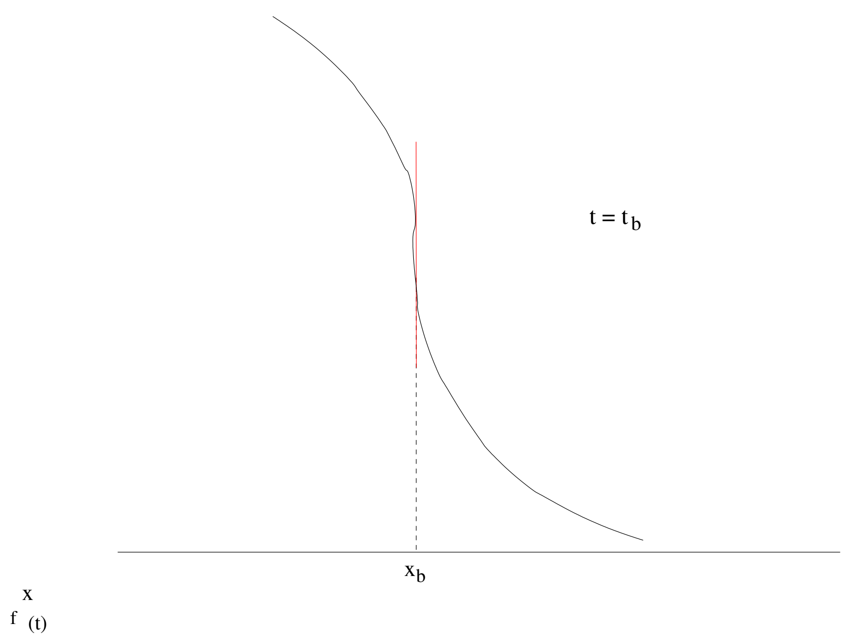
At breaking
In the limit $t \uparrow t_b$,
i) the inflection point reaches the breaking point: $x_f(t) \to x_b$, and the tangent to the inflection point becomes the vertical line $x = x_b$.
ii) the solution of the cubic simplifies:

$$\xi' = \sqrt[3]{\gamma(x - x_b)}, \tag{87}$$

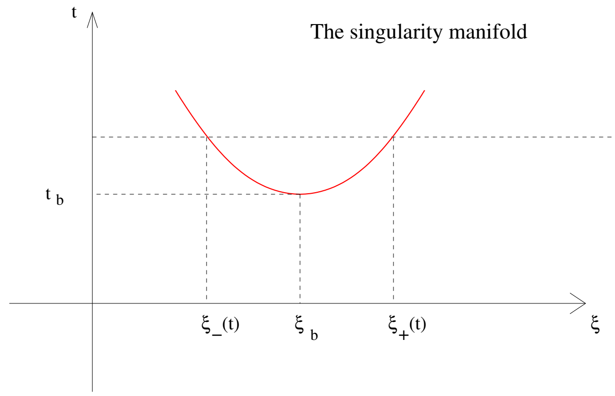
and, correspondingly,

$$u \sim F\left(\xi_b + \sqrt[3]{\gamma(x - x_b)}\right), \quad u_x \sim \frac{\gamma}{3} \frac{F\left(\xi_b + \sqrt[3]{\gamma(x - x_b)}\right)}{(x - x_b)^{2/3}}, \tag{88}$$

describing the typical vertical inflection at $t = t_b$, in the neighborhood of x_b:



After breaking



The line $t = const,\ t > t_b$ intersects the SM in the two points

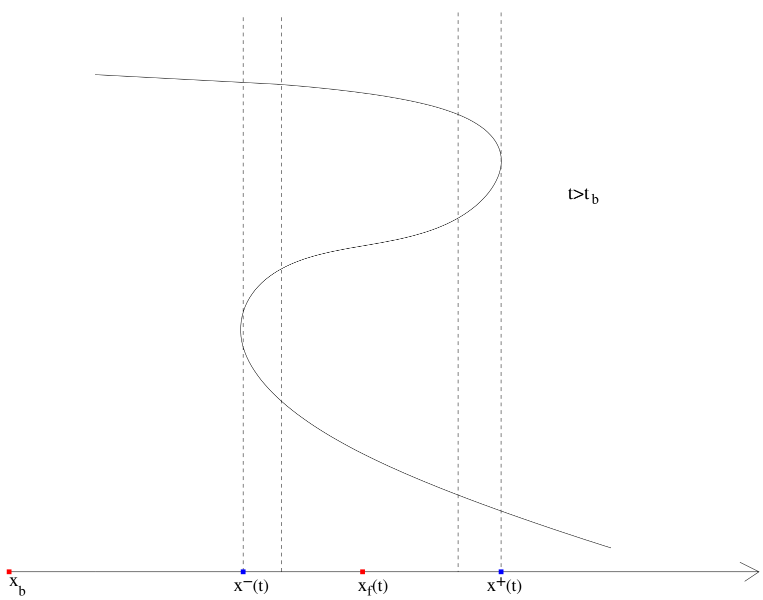
$$\xi_{\pm}(t) = \xi_b \pm \xi_M, \quad \xi_M = \sqrt{\tfrac{\gamma |F_\xi(\xi_b)|(t-t_b)}{3}}. \tag{89}$$

Corresponding points on the physical axes:

$$x^{\pm}(t') = x_f(t) \pm x_M, \quad x_M = 2\sqrt{\tfrac{\gamma |F_\xi(\xi_b)|^3}{27}}(t-t_b)^{3/2} \tag{90}$$

If $|\xi'| < \xi_M(t)$ $(x^-(t) < x < x^+(t))$, $\Delta = Q^3 + R^2 < 0$ and all the three roots of the cubic are real, and the solution of (66) becomes three-valued:

$$u_0(x,t) = F(\xi_b + {\xi'}_0(x',t')), \quad u_{\pm}(x,t) = F(\xi_b + {\xi'}_{\pm}(x',t')). \tag{91}$$



At the end points $\xi_\pm$ $(x^\pm)$ of the segments, $\Delta = 0$ and the two real roots ξ_+ and ξ_- coincide (branch points):

$$\xi_+(x^+) = \xi_-(x^+) = \xi_M, \quad \xi_+(x^-) = \xi_-(x^-) = -\xi_M, \tag{92}$$

Expanding the cubic around these branch points, the solution exhibits the universal square root behavior:

$$\begin{array}{l} u_\pm \sim F\left(\xi_b + Z \pm \frac{\sqrt{\gamma(x-x_-)}}{(3b)^{1/4}},\right),\ x > x_-,\ |x - x_-| << O(|t'|^{3/2}), \\ u_\pm \sim F\left(\xi_b - Z \pm \frac{\sqrt{\gamma(x_+-x)}}{(3b)^{1/4}},\right),\ x < x_+,\ |x - x_+| << O(|t'|^{3/2}), \\ Z \equiv \sqrt{\frac{\gamma}{3}|F_\xi|(t-t_b)}. \end{array} \tag{93}$$

The movable singularity manifold presents several universal features. Correspondingly, also the solution of the Cauchy problem for the Hopf equation presents universality features near the singularity manifold

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