Some notes (under construction)

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1 Dynamical systems and vector fields

Here we summarise some known facts on dynamical systems (systems of ODEs).

The **dynamical system**

$$\begin{array}{l} \frac{dx^i}{dt} = u^i(\vec{x}, t), \quad i = 1, \dots, N \qquad \left(\frac{d\vec{x}}{dt} = \vec{u}(\vec{x}, t)\right). \\ \vec{u}(\vec{x}, t) = (u^1(\vec{x}, t), \dots, u^N(\vec{x}, t)) \in \mathbb{R}^N, \\ \vec{x} = (x^1, \dots, x^N) \in \mathbb{R}^N, \quad \nabla_{\vec{x}} = (\partial_{x^1}, \dots, \partial_{x^N}), \end{array} \tag{1}$$

together with the initial condition $\vec{x}(t_0) = \vec{x}_0 \in \mathbb{R}^N$, define a flow (a trajectory in the phase space $\mathbb{R}^N$), tangent to the **vector field** $\vec{u}(\vec{x}, t)$ (see Fig. 1).

The general solution of (1), depending on N arbitrary constants $\vec{c}$, is characterized by the system of nondifferential equations:

$$\varphi_j(\vec{x}, t) = c_j, \quad j = 1, \dots, N, \tag{2}$$

where the c_j's are N independent constants. Solving the system wrt $\vec{x}$, if $\frac{\partial(\varphi_1, \dots, \varphi_N)}{\partial(x_1, \dots, x_N)}$, one obtains the general solution of (1):

$$\vec{x} = \vec{X}(t, \vec{c}). \tag{3}$$

Definition 1. $I(\vec{x}, t)$ is an integral of motion of (1) iff I satisfies the linear PDE:

$$I_t + \vec{u} \cdot \nabla_{\vec{x}} I = 0 \tag{4}$$

(I is constant on the characteristic curves (integral curves) of (1)).

Definition 2. Equation (4) can be written as

$$\hat{u} I = 0, \tag{5}$$

$$\hat{u} := \partial_t + \vec{u} \cdot \nabla_{\vec{x}} = \sum_{k=1}^{N+1} u^k \partial_{x^k}, \quad u^{N+1} = 1, \ x^{N+1} = t. \tag{6}$$

where **the first order linear operator $\hat{u}$ is also called vector field** associated with the ODE (1).

The application of $\hat{u}$ to a scalar differentiable function $f(\vec{x}, t)$:

$$\hat{u} f(\vec{x}, t) \tag{7}$$

is the "directional derivative of f, at the point $(\vec{x}, t) \in \mathbb{R}^{N+1}$ of the extended phase space, in the direction of the vector $(1, \vec{u}(\vec{x}, t))$ (whose components are the coefficients of the vector field $\hat{u}$)". We are therefore identifying the extended vector field $(1, \vec{u}(\vec{x}, t)) = (1, u^1, \ldots, u^N)$ with the operator $\hat{u}$, that takes a directional derivative in the direction of $(1, \vec{u})$! One of the advantages of such identification is that $\hat{u}$ does not depend on coordinates.

Definition 3. A dynamical system (1) is Hamiltonian, and the associated vector field $\hat{u}$ is Hamiltonian, if $N = 2n$ is even and there exists a function $H(\underline{x})$ such that the ODE (1) and $\hat{u}$ can be written in the form:

$$\tfrac{dx^i}{dt} = \sum_{k=1}^{2n} J^{ik} \tfrac{\partial H}{\partial x^k} = \{x^i, H\}_{\vec{x}}, \;\; i = 1, .., 2n, \;\; (\tfrac{d\vec{x}}{dt} = J \nabla_{\vec{x}} H), \tag{8}$$

$$\hat{u} = \partial_t + \{\cdot, H\}_{\vec{x}}, \tag{9}$$

where

$$J := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}, \tag{10}$$

and the expression $\{f, g\}_{\vec{x}}$ is the Poisson bracket

$$\{f, g\}_{\vec{x}} := \sum_{a,b=1}^{2n} J^{ab} \tfrac{\partial f}{\partial x^a} \tfrac{\partial g}{\partial x^b} = (\nabla_{\vec{x}} f, J \nabla_{\vec{x}} g). \tag{11}$$

Introducing the canonical variables $(\underline{q}, \underline{p})$ as follows

$$\begin{array}{l} \vec{x} = (\underline{q}, \underline{p})^T = (q^1, ..q^n, p_1, .., p_n)^T, \\ \nabla_{\vec{x}} = (\nabla_{\underline{q}}, \nabla_{\underline{p}}) = (\partial_{q^1}, .., \partial_{q^n}, \partial_{p_1}, .., \partial_{p_n}) \end{array} \tag{12}$$

the dynamical system takes the familiar form of the Hamilton equations

$$\begin{array}{l} \tfrac{dq_i}{dt} = \tfrac{\partial H(\underline{q}, \underline{p}, t)}{\partial p_i} = \{q^i, H\}_{\underline{q}, \underline{p}}, \quad i = 1, \ldots, n = \tfrac{N}{2}, \\ \tfrac{dp_i}{dt} = -\tfrac{\partial H(\underline{q}, \underline{p}, t)}{\partial q_i} = \{p_i, H\}_{\underline{q}, \underline{p}}, \quad i = 1, \ldots, n, \end{array} \tag{13}$$

where

$$\{f,g\}_{\underline{q},\underline{p}} = \{f,g\}_{\vec{x}} = \sum_{k=1}^{n}\left(\frac{\partial f}{\partial q^k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q^k}\right) = \nabla_{\underline{q}} f \cdot \nabla_{\underline{p}} g - \nabla_{\underline{p}} f \cdot \nabla_{\underline{q}} g \quad (14)$$

and the Hamiltonian vector field (6) reads

$$\hat{u} = \partial_t + \{\cdot, H\}_{\vec{x}} = \partial_t + \{\cdot, H\}_{\underline{q},\underline{p}}. \quad (15)$$

Definition 4. The vector field $\vec{u}(\vec{x},t)$ ($\hat{u}$) is divergence - less iff $\nabla_{\vec{x}} \cdot \vec{u} = 0$.

Lemma 1.

1. A dynamical system (1) associated with a divergence-less vector field $\vec{u}$ gives rise to volume preserving flows (**prove it!**).
2. A Hamiltonian vector field is divergence-less (**check it!**), but the opposite may not be true..
3. A two-dimensional divergence-less vector field is Hamiltonian (**check it!**).

Proposition 1.

1. Vector fields $\hat{u}$ form a Lie algebra whose Lie bracket is given by the usual commutator. Indeed (**check it!**):

   $$[\hat{u}, \hat{v}] = \hat{w}, \quad (16)$$

   where

   $$\begin{array}{ll} \hat{u} = \sum_k u^k \partial_{x^k}, & \hat{v} = \sum_k v^k \partial_{x^k}, \\ \hat{w} = \sum_k w^k \partial_{x^k}, & w^k := \hat{u} v^k - \hat{v} u^k. \end{array} \quad (17)$$

2. If the vector fields $\hat{u}_1, \hat{u}_2$

   $$\begin{array}{l} \hat{u}_j = \partial_{t_j} + \vec{u}_j \cdot \nabla_{\vec{x}}, \quad j = 1,2, \\ \vec{u}_j = (u_j^1(\vec{x},t), \dots, u_j^N(\vec{x},t)), \quad j = 1,2, \end{array} \quad (18)$$

   are Hamiltonian, with Hamiltonians H_1, H_2, then the following identity holds true (**check it !**):

   $$\begin{array}{l} [\hat{u}_1, \hat{u}_2] = \sum_{i=1}^{n}\left(\frac{\partial \mathcal{H}_{12}}{\partial q^i}\frac{\partial}{\partial p_i} - \frac{\partial \mathcal{H}_{12}}{\partial p_i}\frac{\partial}{\partial q^i}\right) = \\ (\nabla_q \mathcal{H}_{12}) \cdot \nabla_p - (\nabla_p \mathcal{H}_{12}) \cdot \nabla_q = \{\mathcal{H}_{12}, \cdot\}_x = \{\mathcal{H}_{12}, \cdot\}_{q,p}, \end{array} \quad (19)$$

   where

   $$\mathcal{H}_{12} := \{H_1, H_2\}_{q,p} - H_{2t_1} + H_{1t_2}. \quad (20)$$

1.1 Symmetries of ODEs, commutation of vector fields and involutivity

Proposition 2 The following statements are equivalent.

1. The two dynamical systems

$$\begin{array}{l} \frac{d\vec{x}}{dt_1} = \vec{u}_1(\vec{x},\vec{t}), \quad \vec{t}=(t_1,t_2), \\ \frac{d\vec{x}}{dt_2} = \vec{u}_2(\vec{x},\vec{t}) \end{array} \tag{21}$$

   commute (or the flows generated by them commute; or one dynamical system is symmetry of the other).

2. The vector fields $\hat{u}_1$ and $\hat{u}_2$ defined in (18) commute (see Fig. 2):

$$[\hat{u}_1, \hat{u}_2] = \hat{0}. \tag{22}$$

3. The following two linear PDEs

$$\hat{u}_1\psi = \hat{u}_2\psi = 0 \tag{23}$$

   are satisfied for the same eigenfunction ψ (the two ODEs share the same constant of motion ψ).

4. The following quasilinear PDEs of hydrodynamic type for the components of the vectors $\vec{u}_1(\vec{x},\vec{t}), \vec{u}_2(\vec{x},\vec{t})$ are satisfied:

$$\hat{u}_1\vec{u}_2 = \hat{u}_2\vec{u}_1 \quad \Leftrightarrow \quad \vec{u_2}_{t_1} + (\vec{u}_1\cdot\nabla)\vec{u}_2 = \vec{u_1}_{t_2} + (\vec{u}_2\cdot\nabla)\vec{u}_1. \tag{24}$$

Proposition 3. If the two commuting vector fields are Hamiltonian, then (**check it!**):

$$\mathcal{H}_{12} \equiv \{H_1, H_2\}_{q,p} - H_{2t_1} + H_{1t_2} = 0 \quad \Rightarrow \quad [\hat{u}_1, \hat{u}_2] = 0, \tag{25}$$

$$[\hat{u}_1, \hat{u}_2] = 0 \quad \Rightarrow \quad \mathcal{H}_{12} = c \quad (\mathcal{H}_{12} \text{ is a numerical constant,}) \tag{26}$$

and usually such a numerical constant is 0. In this case, if the two Hamiltonians H_1, H_2 do not depend explicitely on t, then the commutation of the two Hamiltonian vector fields is equivalent to the condition

$$\{H_1, H_2\}_{q,p} = 0, \tag{27}$$

called "involutivity" of H_1, H_2.

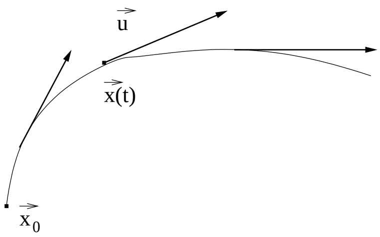


Fig. 1 The flow generated by $\vec{u}$.

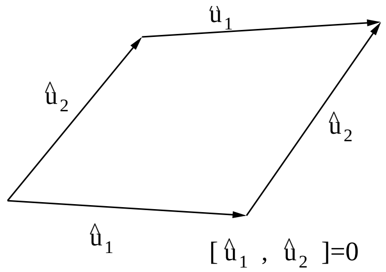


Fig. 2 Commuting vector fields.

2 Linear and quasi-linear PDEs of the first order

Consider the evolution equation in $N+1$ dimensions:

$$\psi_t + \sum_{k=1}^{N} u^k \psi_{x^k} = h. \tag{28}$$

Different cases

1. If the coefficients

$$u^k = u^k(\vec{x}, t) \in \mathbb{R}^N, \ \ h = h(\vec{x}, t) \in \mathbb{R}, \tag{29}$$

are given functions, then equation (28) is a linear first order PDE in $N+1$ dimensions for the unknown $\psi(\vec{x}, t)$.

2. If the arguments of the u^i's and h depend also on the unknown $\psi(\vec{x}, t)$:

$$u^i = u^i(\vec{x}, t, \psi), \quad h = h(\vec{x}, t, \psi), \tag{30}$$

equation (28) is a quasi-linear PDE (linear in the highest derivatives) of the first order in $N+1$ dimensions for the unknown $\psi(\vec{x}, t)$.

3. If $h = 0$, it is a homogeneous equation in the derivatives.

2.1 Quasi-linear PDEs vs systems of ODEs

Consider the first order quasi-linear PDE in M dimensions

$$\sum_{k=1}^{M} P^k(\underline{x},\psi)\psi_{x^k} = Q(\underline{x},\psi), \quad (\underline{P}(\underline{x},\psi)\cdot\nabla_{\underline{x}}\psi = Q(\underline{x},\psi))\,. \tag{31}$$

where $\underline{x} = (x^1,\dots,x^M)$. If one of the independent variables is time $t = x^M$, the PDE becomes, more conveniently,

$$\frac{\partial\psi}{\partial t} + \vec{u}(\vec{x},t,\psi)\cdot\nabla_{\vec{x}}\psi = h(\vec{x},t,\psi) \tag{32}$$

where $\underline{x} = (\vec{x},t)$ and

$$\vec{x} = (x^1,\dots,x^N), \quad \vec{u} = (\frac{P^1}{P^M},\dots,\frac{P^N}{P^M})^T, \quad h = \frac{Q}{P^M}, \quad N \equiv M-1. \tag{33}$$

It turns out that the system of PDEs (31) (or (32)) is intimately related to the following system of M ODEs:

$$\tfrac{dx^1}{P^1} = \cdots = \tfrac{dx^M}{P^M} = \tfrac{d\psi}{Q}. \tag{34}$$

becoming

$$\begin{array}{l} \frac{d\vec{x}}{dt} = \vec{u}(\underline{x},\psi), \\ \frac{d\psi}{dt} = h(\underline{x},\psi). \end{array} \tag{35}$$

if $t = x^M$. Now we show the deep relations between equations (31) and (34).
Proposition. Any solution $\psi(\vec{x},t)$ of the PDE (31) (or (32)) defines a solution of the system of ODEs (34) (or (35)). Let $\psi(\underline{x})$ be a solution of (31), defined by the implicit equation:

$$\varphi(\underline{x},\psi) = c \tag{36}$$

(if $\frac{\partial\varphi}{\partial\psi} \neq 0$, we can solve it wrt ψ, obtaing a solution $\psi(\underline{x})$ of (31)). Equation (36) defines an *integral surface* S of (31)) in the $(M+1)$-dimensional $(\underline{x},\psi)$ space (an hypersurface of dimension M).

Since, from (36), it follows

$$\nabla_{\underline{x}}\varphi + \frac{\partial\varphi}{\partial\psi}\nabla_{\underline{x}}\psi = \underline{0}, \tag{37}$$

if $\frac{\partial\varphi}{\partial\psi} \neq 0$, we can replace $\nabla_{\underline{x}}\psi$ by $-\nabla_{\underline{x}}\varphi(\partial\varphi/\partial\psi)^{-1}$ in (31), obtaining

$$\underline{P} \cdot \nabla_{\underline{x}}\varphi + Q\frac{\partial\varphi}{\partial\psi} = (\underline{P}, Q) \cdot (\nabla_{\underline{x}}, \partial_\psi)\varphi = 0. \tag{38}$$

Since the gradient of φ: $(\nabla_{\underline{x}}, \partial/\partial_\psi)\varphi$ is normal to the integral surface S, the $(M+1)$-dimensional vector $\underline{V} = (\underline{P}, Q)$ is tangent to S at the point $(\underline{x}, \psi)$, and defines a direction on S at that point. Moving along that direction, one construct a "characteristic curve", always tangent to S. If s is the arc length parameter along the characteristic curve, then $(d\underline{x}/ds, d\psi/ds)$ is parallel to $\underline{V}$, i.e.:

$$\frac{d\underline{x}}{ds} = \mu\underline{P}, \quad \frac{d\psi}{ds} = \mu Q \tag{39}$$

for some scalar μ, implying the system of M ODEs (34) (or (35)), characterizing the characteristic curves.

More rapidly, if the solutions $\psi(\vec{x}, t)$ of (32) is assumed to be known, then $\vec{u}(\vec{x}, t, \psi(\vec{x}, t))$ is also known. Consider now the vector ODE

$$\frac{d\vec{x}}{dt} = \vec{u}(\vec{x}, t, \psi(\vec{x}, t)), \tag{40}$$

and its solution $\vec{x}(t)$. Then

$$\frac{d}{dt}\psi(\vec{x}(t), t) = \psi_t + \sum_{k=1}^{N} \psi_{x^k}\frac{dx^k}{dt} = \psi_t + \vec{u} \cdot \nabla_{\vec{x}}\psi = h. \tag{41}$$

Therefore the PDE (32) is transformed into the (more convenient) system of ODEs (35).

The opposite is also true; we show it for equations (32) and (35).

Proposition . Any solution of the system of ODEs (35) defines a solution $\psi(\vec{x}, t)$ of the PDE (32). More in general, the general solution of the system of $N+1$ ODEs (35) generates the general solution of the first order quasi-linear PDE in $N+1$ dimensions (32).

Proof. Indeed, consider a solution of (35) described implicitely by the system of equations

$$\varphi_j(\vec{x}, t, \psi) = c_j, \quad j = 1, \ldots, N+1, \tag{42}$$

We first show that a **single** equation of the above system defines a **particular** solution of the PDE (32). Take the jth equation above: $\varphi_j(\vec{x}(t), t, \psi(t)) = c_j$, wher $\vec{x}(t)$ and $\psi(t)$ are solutions of (35); then, applying to the equation d/dt, one obtains

$$\varphi_{j_t} + \vec{u} \cdot \nabla_{\vec{x}} \varphi_j + h \frac{\partial \varphi_j}{\partial \psi} = 0. \tag{43}$$

On the other hand, if $\partial\varphi_j/\partial\psi \neq 0$, we can interprete the jth equation as a way to define function $\psi(\vec{x}, t; c_j)$: $\varphi_j(\vec{x}, t, \psi(\vec{x}, t; c_j)) = c_j$, and taking partial derivatives of such equation, we obtain:

$$\nabla_{\vec{x}} \varphi_j + \frac{\partial \varphi_j}{\partial \psi} \nabla_{\vec{x}} \psi = \underline{0}, \quad \varphi_{j_t} + \frac{\partial \varphi_j}{\partial \psi} \psi_t = 0. \tag{44}$$

Using these last equations, one finally shows that equation (43) becomes (32).

Now we show how to obtain the general solution of (32). The general solution of (35) is described by the $M = N + 1$ equations

$$\varphi_j(\vec{x}, t, \psi) = c_j, \quad j = 1, \ldots, N + 1, \tag{45}$$

where the c_j's are $N + 1$ independent constants. Their independence is expressed by the relation $\phi(c_1, \ldots, c_{N+1}) = 0$ among them, where $\phi(\cdot, .., \cdot)$ is an "arbitrary" differentiable function of $N + 1$ variables. So that we can also write:

$$\phi(c_1, \ldots, c_{N+1}) = 0, \quad \Rightarrow \quad c_{N+1} = F(c_1, \ldots, c_N) \tag{46}$$

since $\partial\phi/\partial c_{N+1} \neq 0$, due to the above independence, where F is an **arbitrary** differentiable function of N arguments. Therefore:

$$\varphi_{N+1}(\vec{x}, t, \psi) = F\left(\varphi_1(\vec{x}, t, \psi), \ldots, \varphi_N(\vec{x}, t, \psi)\right). \tag{47}$$

Solving (47) wrt to $\psi = \psi(\vec{x}, t)$, one obtains the general solution of the PDE (31), given in terms of an arbitrary function F of N variables. (**Show it!**). $\square$

Find, using the method of characteristics, the general solution of $\psi_t + \psi^n \psi_x = 1, \ n = 1, 2$.

2.2 Linear PDEs and vector field equations

In the linear case, equation (32) reduces to

$$\psi_t + \vec{u}(\vec{x}, t) \cdot \nabla_{\vec{x}} \psi = h_0(\vec{x}, t) + h_1(\vec{x}, t)\psi, \tag{48}$$

and solutions of (48) are in one to one correspondence with solutions of the system of ODEs:

$$\begin{array}{l} \frac{d\vec{x}}{dt} = \vec{u}(\vec{x},t), \\ \frac{d\psi}{dt} = h_0(\vec{x},t) + h_1(\vec{x},t)\psi \end{array} \tag{49}$$

which is now decoupled, and the characteristic curves, defined by (49a), do not depend on ψ .

2.2.1 Homogeneous case: the vector field equation

In the important subcase $h_0 = h_1 = 0$, we obtain the vector field equation:

$$\hat{u}\psi = \psi_t + \vec{u}(\vec{x},t) \cdot \nabla_{\vec{x}}\psi = 0, \tag{50}$$

that we are going to discuss from now on. Solutions of (50) are in one to one correspondence with solutions of the system of ODEs:

$$\begin{array}{l} \frac{d\vec{x}}{dt} = \vec{u}(\vec{x},t), \\ \frac{d\psi}{dt} = 0 \quad (\Rightarrow \quad \psi = c_{N+1}) \, . \end{array} \tag{51}$$

Indeed, from solutions of (50) one obtains solutions of (51) specializing the procedure presented in the above section. Viceversa, from the general solution of (51), we construct the general solution of (50). Indeed, the general solution of (51a) is characterized by the system of nondifferential equations

$$\phi_j(\vec{x},t) = c_j, \quad j = 1,\ldots,N \quad (\Rightarrow \quad \vec{x} = \vec{x}\,(t, c_1, \ldots, c_N))\,; \tag{52}$$

from the usual condition $c_{N+1} = F(c_1, \ldots, c_N)$ it follows that the general solution of the PDE (50) reads

$$\psi(\vec{x},t) = F\left(\phi_1(\vec{x},t), \ldots, \phi_N(\vec{x},t)\right), \tag{53}$$

where F is an arbitrary differentiale function of N arguments. It is straightforward to verify that the N functions $\phi_j(\vec{x},t), \quad j = 1, \ldots, N$ are particular solutions of (50):

$$0 = \frac{d}{dt}\phi_j(\vec{x}(t),t) = \phi_{j_t} + \vec{u} \cdot \nabla_{\vec{x}}\phi_j, \quad j = 1,\ldots,N. \tag{54}$$

The following proposition summarizes the important properties of the solutions of a vector field equation.

Proposition 5

1. Equation (50) admits N independent solutions.
   Proof. Suppose we have $N+1$ solutions
$$\psi_i(\vec{x},t), \quad i=1,\dots,N+1, \tag{55}$$
   of (50); then the corresponding system
$$\psi_{i_t}+\vec{u}\cdot\nabla_{\vec{x}}\psi_i=0, \quad i=1,\dots,N+1, \tag{56}$$
   viewed as an algebraic system of $N+1$ equations for the $N+1$ components of $(1,\vec{u})$, gives rise to a nontrivial solution iff
$$\frac{\partial(\psi_1,\dots,\psi_{N+1})}{\partial(x_1,\dots,x_N,t)}=0; \tag{57}$$
   i.e., the $N+1$ solutions $\psi_1,\dots,\psi_{N+1}$ of the PDE (48) are dependent $\square$.

2. The solution space of (50) is a ring; i.e., an arbitrary differentiable function $f(\psi_1,\dots,\psi_N)$ of solutions of (50) is a solution of (50) (**check it!**). Also, if $\{\psi_1,\dots,\psi_N\}$ are N independent solutions of (50), they are a basis in the ring, and any solution ψ of (50) can be written in the form
$$\psi=F(\psi_1,\dots,\psi_N), \tag{58}$$
   for some differentiable function F of N arguments (compare with (52)).

3. If the vector field associated with (48) is Hamiltonian, then the solution space of (50) is also a Lie algebra, whose Lie bracket is the Poisson bracket.

   Proof. If $\psi_1,\ \psi_2$ are solutions of (48) and $\hat{u}$ is Hamiltonian:
$$\hat{u}\psi_j=\psi_{j_t}+\vec{u}\cdot\nabla_{\vec{x}}\psi_j=\psi_{j_t}+\{\psi_j,H\}=0, \quad j=1,2 \tag{59}$$
   Then
$$\begin{gathered}\hat{u}\{\psi_1,\psi_2\}=\{\psi_1,\psi_2\}_t+\{\{\psi_1,\psi_2\},H\}=\{\psi_{1t},\psi_2\}+\{\psi_1,\psi_{2t}\}- \\ \{\{H,\psi_1\},\psi_2\}-\{\{\psi_2,H\},\psi_1\}= \\ \{\psi_1,\psi_{2t}+\{\psi_2,H\}\}+\{\psi_{1t}+\{\psi_1,H\},\psi_2\}=0,\end{gathered} \tag{60}$$
   having used the Jacobi identity, implying that also $\{\psi_1,\psi_2\}$ is solution of (48) $\square$.

Find, using the method of characteristics, the general solution of $x\psi_x+y\psi_y+z\psi_z=0$ and a good basis in the space of solutions.

2.3 Homogeneous version of the quasi-linear PDE (31)

If $Q = h = 0$:

$$\psi_t + \vec{u}(\vec{x}, t, \psi) \cdot \nabla_{\vec{x}}\psi = 0; \tag{61}$$

then ψ is constant on the characteristic curves defined by (35):

$$\begin{aligned} &\tfrac{d\psi}{dt} = 0, \quad \Rightarrow \quad \psi = A, \\ &\tfrac{d\vec{x}}{dt} = \vec{u}(\vec{x}, t, \psi) = \vec{u}(\vec{x}, t, A). \end{aligned} \tag{62}$$

The general solution of (62b) is characterized by the N nondifferential equations:

$$\varphi_j(\vec{x}, t, A) = \xi_j, \quad j = 1, \ldots, N \tag{63}$$

where $\vec{\xi} = (\xi_1, \ldots, \xi_N)$ is an arbitrary constant vector, and, as above, $A = F(\vec{\xi})$. Therefore the general solution of (61) is characterized by the following implicit equation

$$\psi = F(\vec{\varphi}(\vec{x}, t, \psi)) \tag{64}$$

or, equivalently, by the system

$$\begin{aligned} &\vec{\varphi}(\vec{x}, t, F(\vec{\xi})) = \vec{\xi}, \\ &\psi = F(\vec{\xi}), \end{aligned} \tag{65}$$

(where one solves (65a) wrt $\vec{\xi} = \vec{\xi}(\vec{x}, t)$ and then replace it into (65b), obtaining $\psi = F(\vec{\xi}(\vec{x}, t))$).

Basic example We consider, as basic example (with $\rho = \psi$), the following first order quasi-linear PDE in multidimension:

$$\rho_t + \vec{u}(\rho) \cdot \nabla_{\vec{x}}\rho = 0, \qquad \rho = \rho(\vec{x}, t), \tag{66}$$

which is nothing but the continuity equation

$$\begin{aligned} &\rho_t + \nabla_{\vec{x}} \cdot \vec{J}(\rho) = 0, \\ &\vec{u}(\rho) = \tfrac{\partial \vec{J}(\rho)}{\partial \rho}, \end{aligned} \tag{67}$$

for some "density" ρ and some "flux vector" $\vec{J}(\rho) = \rho\vec{v}(\rho)$. If $N = 1$ and $u^1(\rho) = \rho$, this equation reduces to the famous Riemann-Hopf equation:

$$\rho_t + \rho\rho_x = 0, \tag{68}$$

the simplest prototypical model in the description of the gradient catastrophe (wave breaking) of one dimensional waves.

From the above considerations, the solution $\rho(\vec{x}, t) = A$ of (66) is constant on the characteristic curves defined by the system of ODEs:

$$\frac{d\vec{x}}{dt} = \vec{u}(\rho(\vec{x}, t)) = \vec{u}(A). \tag{69}$$

Therefore (69) can be trivially integrated:

$$\vec{x} - \vec{u}(F(A)t = \vec{\xi} \tag{70}$$

where $\vec{\xi}$ is an arbitrary N-dimensional contant vector, describing an N-parameter family of characteristic straight lines in $\mathbb{R}^{N+1}$, and the constant value A of ρ on the straight line labelled by $\vec{\xi}$ is $A = F(\vec{\xi})$, where F is an arbitrary scalar function of N variables. Therefore the solution $\rho(\vec{x}, t)$ of (66) is characterized by the **non-differential** equations

$$\begin{aligned} &\rho(x, t) = F(\vec{\xi}), \\ &\vec{x} = \vec{u}(F(\vec{\xi}))t + \vec{\xi}. \end{aligned} \tag{71}$$

Solving (71b) wrt $\vec{\xi}$: $\vec{\xi} = \vec{\xi}(\vec{x}, t)$, and replacing it in (71a), one obtains:

$$\rho(x, t) = F(\vec{\xi}(\vec{x}, t)), \tag{72}$$

where F is an arbitrary scalar function of N variables.

Equivalently, this general solution is also characterized by the implicit nondifferential equation

$$\rho = F(\vec{x} - \vec{u}(\rho)t). \tag{73}$$

If we add, for instance, the initial condition

$$\rho(\vec{x}, 0) = \rho_0(\vec{x}), \quad \vec{x} \in \mathbb{R}^N, \tag{74}$$

then $F = \rho_0$ and

$$\rho = \rho_0(\vec{x} - \vec{u}(\rho)t) \tag{75}$$

is the equation defining implicitely the solution of the Cauchy problem

$$\rho_t + \vec{u}(\rho) \cdot \nabla_{\vec{x}}\rho = 0, \quad \rho(\vec{x}, 0) = \rho_0(\vec{x}). \tag{76}$$

3 Analytic aspects of wave-breaking in 1+1 dimensions

For the geometric aspects, see [3, 4]; for the analytic aspects, see [5] and below.

Consider the evolution of a localized one-dimensional wave according to the Hopf equation

$$\begin{array}{l} u_t + u u_x = 0. \\ u(x,0) = F(x), \quad x \in \mathbb{R} \end{array} \tag{77}$$

Such evolution is described by the implicit equation

$$\begin{array}{l} u = F(\zeta), \\ \zeta = x - F(\zeta)t, \end{array} \tag{78}$$

in which equation (78b) must be solved with respect to ζ, obtaining $\zeta = \zeta(x,t)$, and substituted in (78a), to get the solution $u = F(\zeta(x,t))$.

Such inversion is possible if the ζ-derivative of (78b) is different from zero. Therefore the **1-dimensional (movable) Singularity Manifold** (SM) of (77) is described by the equation:

$$\mathcal{S}(\zeta,t) = 1 + F_\zeta(\zeta)t = 0 \quad \Rightarrow \quad t = -\frac{1}{F_\zeta(\zeta)}. \tag{79}$$

Since

$$\begin{array}{l} \zeta_x = \frac{1}{1+tF_\zeta}, \quad \zeta_t = -\frac{F}{1+tF_\zeta}, \\ u_x = F_\zeta \zeta_x = \frac{F_\zeta}{1+tF_\zeta}, \quad u_t = F_\zeta \zeta_t = -\frac{FF_\zeta}{1+tF_\zeta} \end{array} \tag{80}$$

the wave breaks on the singularity manifold.

We are interested in the first time t_b in which the breaking of the solution occurs, corresponding to the characteristic values ξ_b such that

$$\begin{array}{l} t_b = t(\zeta_b) = \text{global min}\{t(\zeta)\} > 0 \quad \Rightarrow \\ F_{\zeta\zeta}(\zeta_b) = 0, \qquad F_\zeta(\zeta_b) < 0, \qquad F_{\zeta\zeta\zeta}(\zeta_b) > 0, \end{array} \tag{81}$$

ξ_b is an inflection point of the initial profile.

At t_b, the wave breaks in the point x_b of the x-axis defined by

$$x_b = F(\zeta_b)t_b + \zeta_b. \tag{82}$$

Now we study the solution (78) near breaking:

$$x = x_b + x', \quad t = t_b + t', \quad \zeta = \zeta_b + \zeta', \tag{83}$$

where x', t', ζ' are small:

$$\zeta = x - F(\zeta)t \quad \Rightarrow \quad {\zeta'}^3 + b(t')\zeta' - \gamma X(x', t') = 0, \tag{84}$$

where

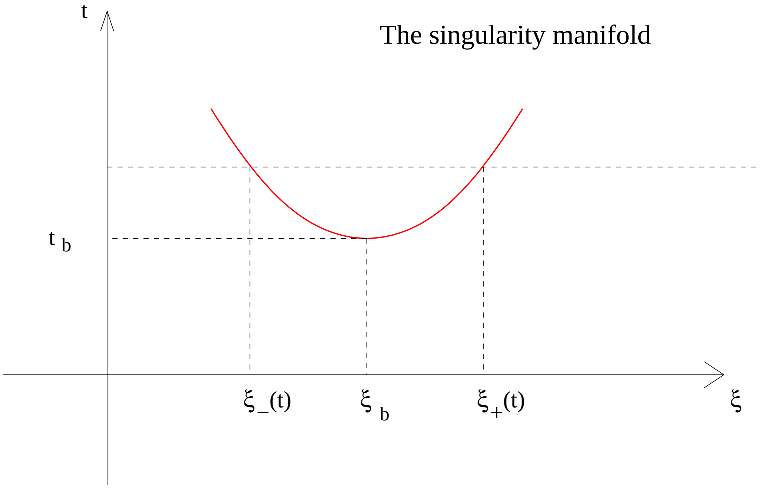
$$b(t') = \tfrac{6F_\zeta}{t_b F_{\zeta\zeta\zeta}} t' = \gamma F_\zeta t', \quad X(x', t') = x' - Ft', \quad \gamma = \tfrac{6}{t_b F_{\xi\xi\xi}} \tag{85}$$

corresponding to the maximal balance

$$|X| = O(|t'|^{3/2}), \qquad |\zeta| = O(|t'|^{1/2}). \tag{86}$$

In (85) and in the rest of this section, unless explicitely specified, F and its derivatives wrt ζ are evaluated at $\zeta = \zeta_b$. At the same order, the SM equation reads:

$$SM: \quad 0 = \mathcal{S}(\zeta', t) \sim F_\zeta t' + \frac{F_{\zeta\zeta\zeta}}{2} t_b {\zeta'}^2. \tag{87}$$



The three roots of this cubic are given by the well-known Cardano's formula:

$$\begin{aligned} {\zeta'}_0\,(x', t') &= (A_+)^{\frac{1}{3}} + (A_-)^{\frac{1}{3}}, \\ {\zeta'}_\pm\,(x', t') &= \tfrac{1}{2}\left((A_+)^{\frac{1}{3}} + (A_-)^{\frac{1}{3}}\right) \pm \tfrac{\sqrt{3}}{2} i \left((A_+)^{\frac{1}{3}} - (A_-)^{\frac{1}{3}}\right), \end{aligned} \tag{88}$$

where

$$\begin{aligned} &A_\pm = R \pm \sqrt{\Delta}, \quad \Delta = R^2 + Q^3 \\ &Q(t') = \tfrac{b(t')}{3}, \quad R(x', t') = \tfrac{\gamma}{2} X(x', t'). \end{aligned} \tag{89}$$

Once $\zeta'(x',t')$ is known from the solution of the cubic, implying also:

$$\zeta'_{x'} = \frac{1}{\mathcal{S}}, \quad \zeta'_{x'x'} = -t_b F_{\zeta\zeta\zeta} \frac{\zeta' \zeta'^2_{x'}}{\mathcal{S}}, \tag{90}$$

then the solution of the Hopf equation and its derivatives are then approximated, near breaking, by the formulae:

$$\begin{array}{l} u(x,t) \sim F(\zeta_b + \zeta'), \\ u_x(x,t) \sim F'(\zeta)\zeta'_{x'} \sim \frac{F_\zeta(\zeta_b+\zeta')}{F_\zeta t' + \frac{F_{\zeta\zeta\zeta}}{2}\zeta'^2 t_b}, \\ u_{xx}(x,t) \sim F''(\zeta)(\zeta'_{x'})^2 + F'(\zeta)\zeta'_{xx}. \end{array} \tag{91}$$

Before breaking
If $t < t_b$ ($t' < 0$), $\mathcal{S}$ and Δ are strictly positive, and only the root ξ'_0 is real; correspondingly, the real solution of (77) is single valued:

$$u \sim F(\zeta_b + {\zeta'}_0(x',t')). \tag{92}$$

and the slope u_x of the profile, finite $\forall x$, reaches its minimum at the inflection point $x_f(t')$, at wich $X = 0 \ \Rightarrow \ \xi' = 0, \xi'_{xx} = 0, u_{xx} = 0$:

$$\begin{array}{l} x_f(t') = x_b + F(\zeta_b)t' \quad (X = x - x_f(t')), \\ u(x_f(t),t) = F(\zeta_b), \quad u_x(x_f(t),t) = \frac{1}{t-t_b}, \quad u_{xx}(x_f(t),t) = 0. \end{array} \tag{93}$$

To analyse the solution in a smaller region around the inflection point, we choose

$$|X| = |x - x_f(t')| = O(|t'|^{p+1}), \quad p > 1/2. \tag{94}$$

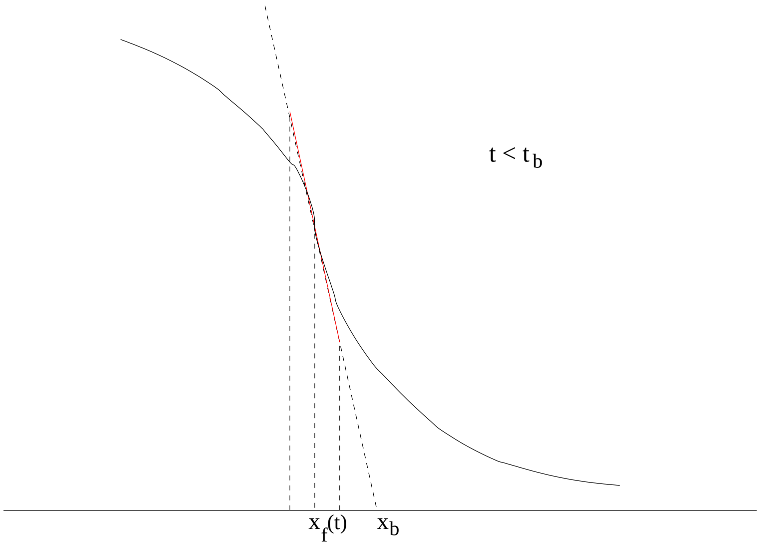
Then $\zeta' \sim |t'|^p$, $\zeta'^3 << b\zeta' \sim -\gamma X$, the solution becomes more explicit:

$$\zeta' \sim \frac{\gamma X}{b} = \frac{x' - Ft'}{F_\zeta t'}, \tag{95}$$

and the real solution of (77) reduces to the exact similarity solution of the Hopf equation:

$$u \sim F(\zeta_b + \zeta') \sim \frac{x - x_b}{t - t_b}, \tag{96}$$

describing the tangent to the profile at the inflection point.



The analytic expression of the slope of the profile:

$$u_x \sim -\left(|t-t_b| + \frac{F_{\zeta\zeta\zeta}}{2} t_b^4 \left(\frac{x-x_f(t')}{t-t_b}\right)^2\right)^{-1}. \tag{97}$$

$u_x \sim (t-t_b)^{-1}$ in this very thin region; but $u_x = O(1)$ in the region $|x - x_f(t)| = O(|t'|)$.

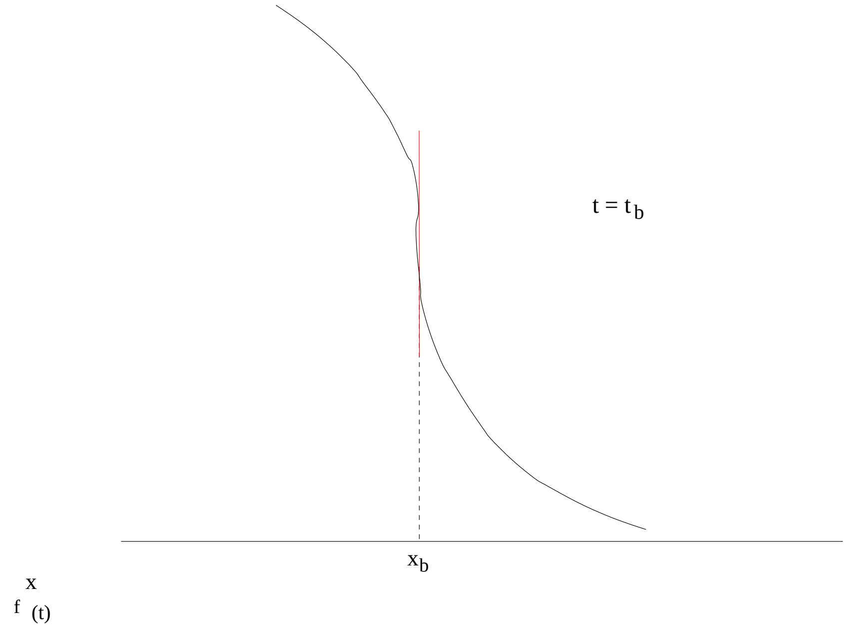
At breaking
In the limit $t \uparrow t_b$,
i) the inflection point reaches the breaking point: $x_f(t) \to x_b$, and the tangent to the inflection point becomes the vertical line $x = x_b$.
ii) the solution of the cubic simplifies:

$$\zeta' = \sqrt[3]{\gamma(x-x_b)}, \tag{98}$$

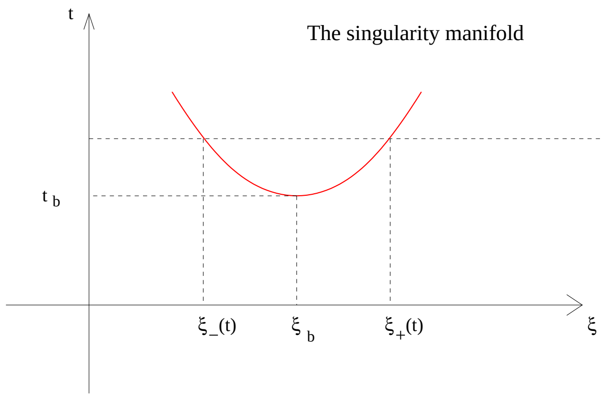
and, correspondingly,

$$u \sim F\left(\zeta_b + \sqrt[3]{\gamma(x-x_b)}\right), \quad u_x \sim \frac{\gamma^{1/3}}{3} \frac{F\left(\zeta_b + \sqrt[3]{\gamma(x-x_b)}\right)}{(x-x_b)^{2/3}}, \tag{99}$$

describing the typical vertical inflection at $t = t_b$, in the neighborhood of x_b:



After breaking



The line $t = const,\ t > t_b$ intersects the SM in the two points

$$\zeta_{\pm}(t) = \zeta_b \pm \zeta_M, \quad \zeta_M = \sqrt{\tfrac{\gamma |F_\zeta(\zeta_b)|(t-t_b)}{3}} = \tfrac{1}{t_b}\sqrt{\tfrac{2}{F_{\zeta\zeta\zeta}}(t-t_b)}. \tag{100}$$

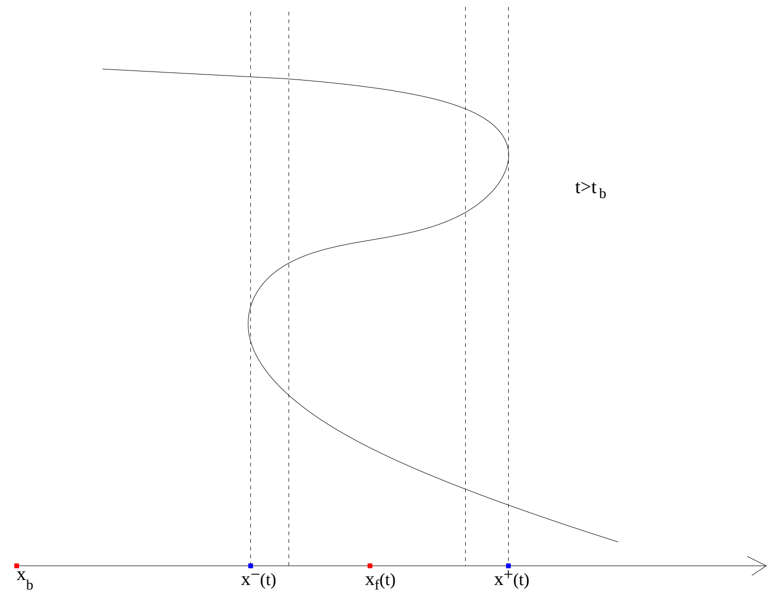
Corresponding points on the physical axes:

$$x^{\pm}(t') = x_b + F(t-t_b) \pm \tfrac{2}{3t_b^2}\sqrt{\tfrac{2}{F_{\zeta\zeta\zeta}}}(t-t_b)^{3/2}. \tag{101}$$

are obtained by the condition $\Delta = 0$, defining the singularity manifold in physical space-time.

If $|\zeta'| \leq \zeta_M(t)$ ($x^-(t) \leq x \leq x^+(t)$), $\Delta = Q^3 + R^2 \leq 0$ and all the three roots of the cubic are real, and the solution of (77) becomes three-valued:

$$u_0(x,t) = F(\zeta_b + \zeta'_0(x',t')), \quad u_\pm(x,t) = F(\zeta_b + \zeta'_\pm(x',t')). \tag{102}$$



4 Discontinuous shock wave regularization

Now we replace, after breaking, the regular, but multivalued solution, by the discontinuous, but single valued one, whose discontinuity is characterized, for $t \geq t_b$, by the following system of three equations [3]:

$$s = \zeta_1 + F(\zeta_1)t = \zeta_2 + F(\zeta_2)t, \tag{103}$$

$$\dot{s} = \tfrac{F(\zeta_1)+F(\zeta_2)}{2}, \tag{104}$$

with initial conditions

$$s(t_b) = x_b, \quad \zeta_1(t_b) = \zeta_2(t_b) = \zeta_b, \tag{105}$$

where $s(t)$ is the position of the shock wave front and $\zeta_1(t)$, $\zeta_2(t)$ are two of the three parameters of the characteristic curves meeting at $t > t_b$ ($\zeta_1(t)$

is the maximum and $\zeta_2(t)$ is the minimum of these three parameters). We remark that (104) is the equal area condition, corresponding to cutting the three valud profile with a vertical line removing equal area lobi. Solving equations (103) we obtain, in principle, $\zeta_{1,2}$ as functions of s and t: $\zeta_{1,2}(s,t)$. Then (104) is an ODE for $s(t)$, whose solution yields the three functions $s(t)$, $\zeta_{1,2}(s(t),t)$ characterizing the discontinuous shock solution of (77) for $t > t_b$:

$$\begin{array}{l} u = F(\zeta), \quad x = \zeta + F(\zeta)t, \quad x \neq s(t), \\ u = F(\zeta_1), \quad x \downarrow s(t); \quad u = F(\zeta_2), \quad x \uparrow s(t), \end{array} \tag{106}$$

while, for $0 \leq t \leq t_b$, the solution is that constructed in the previous section:

$$u = F(\zeta), \quad x = \zeta + F(\zeta)t, \quad x \in \mathbb{R}. \tag{107}$$

To get a more explicit result, let us investigate the system (103), (104) immediately after breaking; then equations simplify to

$$\begin{array}{l} {\zeta'_{1,2}}^3 + \gamma F_\zeta t' \zeta'_{1,2} = \gamma(s' - ft'), \\ \dot{s}' = F + \frac{F_\zeta}{2}(\zeta_1 + \zeta_2), \end{array} \tag{108}$$

where F, F_ζ are evaluated at ζ_b, and

$$\begin{array}{ll} \zeta'_{1,2}(t') = \zeta_{1,2}(t) - \zeta_b, & \zeta'_{1,2}(0) = \zeta_b, \\ s'(t') = s(t) - x_b, & s'(0) = 0. \end{array} \tag{109}$$

Therefore $\zeta_{1,2}$ are explicit functions of s', t', defined by:

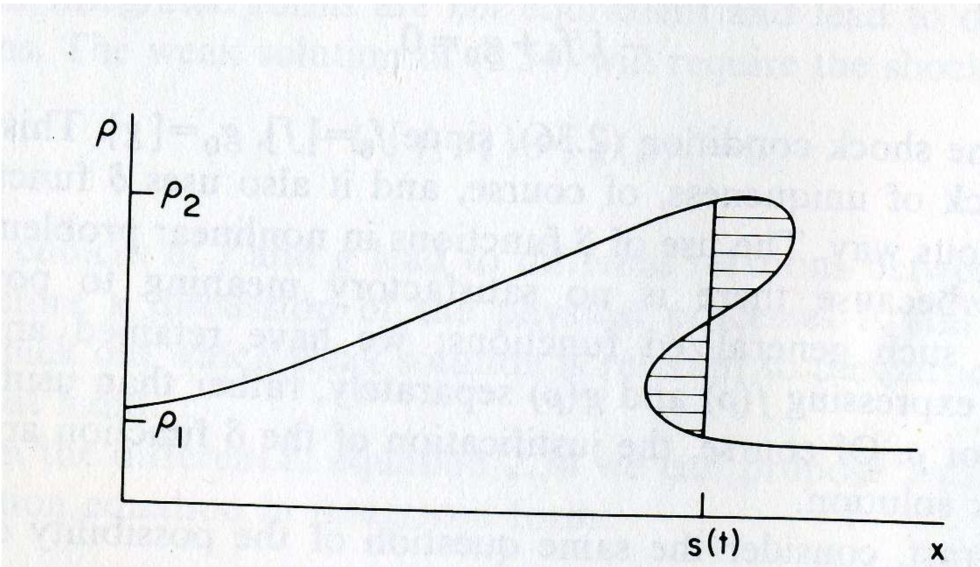
$$\zeta_1(s',t') = \zeta'_0(s',t'), \quad \zeta'_2(s',t') = \zeta_+(s',t'); \tag{110}$$

Substituting (110) into (104), one gets the ODE $\dot{s}' = [F(\zeta_1(s',t')) + F(\zeta_2(s',t'))]/2$ whose solution yields $s'(t')$ and, consequently, $\zeta_{1,2}(t')$. Taking account of (86), we simplify further the problem setting:

$$s'(t') - Ft' = \beta {t'}^{\frac{3}{2}}, \quad \zeta_{1,2}(t') = \alpha_{1,2} {t'}^{\frac{1}{2}}, \tag{111}$$

then the problem is reduced to that of finding the constants $\alpha_{1,2}$ and β from the algebraic system

$$\begin{array}{l} {\alpha_{1,2}}^3 + \gamma F_\zeta \alpha_{1,2} = \gamma\beta, \\ \beta = \frac{F_\zeta}{2}(\alpha_1 + \alpha_2). \end{array} \tag{112}$$



References

[1] R. Courant and D. Hilbert, *Methods of Mathematical Physics*; Vol. II: *Partial Differential equations*, by R. Courant, Interscience Publishers, J. Wiley and sons, New York, 1962.

[2] F. B. Hildebrand, *Advanced Calculus for Applications*; Prentice-Hall, NJ, 1976.

[3] J. B. Whitham, *Linear and Nonlinear Waves*, Wiley, NY, 1974.

[4] Appunti, a cura di G. Angilella, del corso di Dottorato: "Onde non lineari; metodi perturbativi ed esatti", tenuto da P.M.Santini a Catania, nell'AA 95-96 (pagine 1-9) http://www.ct.infn.it/ angilell/corsidott.pdf

[5] P.M.Santini: "The wave breaking"; terza parte del corso: "The transition from regular to irregular motion as travel on Riemann surfaces", NEEDS 2007-School, L'Ametlla de Mar, June 16-17, 2007.
http://www.roma1.infn.it/people/santini/Didattica2/break.pdf (Username: needs Passwd: soliton2007)