Newtonian dynamics in the plane corresponding to straight and cyclic motions on the hyperelliptic curve $\mu^2 = \nu^n - 1,\ n \in \mathbb{Z}$

Appunti, tratti da [1, 2], relativi ad un paio di lezioni tenute durante il corso di:

Fisica teorica: sistemi evolutivi non lineari

AA 2005-06, 2008-09; docente: Paolo Maria Santini

1

We investigate the following two classes of Newtonian dynamical systems in the plane

$$\begin{array}{l} \ddot{\vec{x}} = \vec{F}_1^{(m)}(\vec{x}), \quad m \in \mathbb{Z} \\ \ddot{\vec{x}} = \vec{F}_2^{(m)}(\vec{x}, \dot{\vec{x}}), \quad m \in \mathbb{Z}, \end{array} \tag{1}$$

where $\vec{x} = (x, y) \in \mathbb{R}^2$, $\ddot{f} = d^2 f/dt^2$. The forces $\vec{F}_{1,2}^{(m)}$ are defined by

$$\begin{array}{l} \vec{F}_1^{(m)}(\vec{x}) \equiv -\left(Re(\alpha^2(x+iy)^m), Im(\alpha^2(x+iy)^m)\right)\right), \\ \vec{F}_2^{(m)}(\vec{x}, \dot{\vec{x}}) \equiv \dot{\vec{x}} \wedge \vec{h}_m + 2\frac{m+1}{(m-1)^2}\omega^2 \vec{x} + \left(Re(x+iy)^m, Im(x+iy)^m\right), \end{array} \tag{2}$$

where the constant vectors $\vec{h}_m,\ m \in \mathbb{Z}$ are orthogonal to the (x, y) plane, with $||\vec{h}_m|| = \frac{m+3}{m-1}\omega$, and $\omega > 0$, $a \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ are arbitrary parameters [to be checked].

The classes (1a) and (1b) originate from the following complex ODE:

$$\frac{d^2\zeta}{d\tau^2} = \zeta^m\ , \quad m \in \mathbb{Z}, m \neq -1, \quad \tau \in \mathbb{C}, \quad \zeta \equiv \zeta(\tau) \in \mathbb{C} \tag{3}$$

solved by the complex quadrature

$$\tau = \int\limits_{\zeta(0)}^{\zeta} \frac{d\zeta}{\sqrt{2(E + \frac{\zeta^n}{n})}}, \quad n \equiv m + 1, \tag{4}$$

respectively via the following two distinguished changes of dependent and independent variables

$$\begin{aligned} z(t) &= x(t) + iy(t) = \zeta(\tau), \\ \tau(t) &= \alpha t + \beta, \quad \alpha, \beta \in \mathbb{C} \end{aligned} \tag{5}$$

and (Calogero's transformation)

$$\begin{aligned} z\,(t) &= x(t) + iy(t) = \exp\left(\tfrac{2i\,\omega\, t}{n-2}\right)\, \zeta\,(\tau)\ \ , \quad (k \neq -1)\ , \\ \tau(t) &= \tfrac{\exp(i\,\omega\, t)-1}{i\,\omega}, \end{aligned} \tag{6}$$

corresponding respectively to straight and cyclic motions on the hyperelliptic curve of the quadrature (4).

We remark that the equations of the class (1a) are *locally Liouville integrable*. Indeed, from the Hamiltonian character of (3):

$$\dot{z} = \frac{\partial \mathcal{H}}{\partial \pi}, \quad \dot{\pi} = -\frac{\partial \mathcal{H}}{\partial z}, \tag{7}$$

with the complex Hamiltonian

$$\mathcal{H} = \frac{\pi^2}{2} + \frac{\zeta^n}{n}, \tag{8}$$

it follows the Hamiltonian character of (1a):

$$\begin{aligned} \dot{x} &= \tfrac{\partial H}{\partial p_x}, \quad \dot{y} = -\tfrac{\partial H}{\partial p_y}, \\ \dot{p}_x &= -\tfrac{\partial H}{\partial x}, \quad \dot{p}_y = -\tfrac{\partial H}{\partial y}, \end{aligned} \tag{9}$$

where $z = x + iy,\ \pi = p_x - ip_y$ and $H = Re\ \mathcal{H}$ [check it]. Moreover (1a) possesses two independent constants of motion: $H = Re\ \mathcal{H},\ I = Im\ \mathcal{H}$, in involution:

$$\{H, I\} = \frac{\partial H}{\partial x}\frac{\partial I}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial I}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial I}{\partial p_y} - \frac{\partial H}{\partial p_y}\frac{\partial I}{\partial y} = 0 \tag{10}$$

[check it]. But the resulting dynamics is unbounded, due to the branch point singularities associated with (4), and the 2-dimensional variety on which the motion takes place is not a torus. It will be shown that this lack of global integrability is intimately connected to the sensitive dependence on the initial data exhibited by this class.

2

Through the change of variables

$$w(\xi) = c_1\zeta(\tau), \quad \xi = c_2\tau + \xi_0, \quad \xi_0 = \int\limits_0^{c_1\zeta(0)} \frac{dw}{\sqrt{1-w^n}},$$
$$c_1 = \left(-\frac{1}{nE}\right)^{\frac{1}{n}}, \qquad c_2 = \sqrt{2}E^{\frac{n-2}{2n}} n^{-\frac{1}{n}}, \tag{11}$$

we write the quadrature (4) in its adimensional form

$$\xi = \int\limits_0^w \frac{dw}{\sqrt{1-w^n}}. \tag{12}$$

The quadrature is associated with the hyperelliptic curve defined by

$$\mu^2 = 1 - w^n, \quad n \in \mathbb{Z} \tag{13}$$

whose square root branch points are the n roots $\rho_j = e^{\frac{2\pi i(k-1)}{n}}$, $k = 1, .., n$ of unity. Let g be the genus of the curve (13); then $n = 2g + 1$, if n is odd (in this case, $w = \infty$ is also a branch point) and $n = 2(g + 1)$ if n is even.

The symmetry

$$w \to we^{\frac{2\pi i}{n}}, \quad \mu \to \mu \tag{14}$$

of the curve (13) strongly suggests to cut the w-plane through the rays

$$\gamma_j = \{\arg w = \arg \rho_j, \ |w| \geq |\rho_j|\}, \quad j = 1, .., n. \tag{15}$$

Then the cutted w-plane is mapped into the interior of the regular n-gone of side $2l$, see Fig.1, where:

$$l = \int\limits_1^\infty \frac{dw}{\sqrt{w^n - 1}}. \tag{16}$$

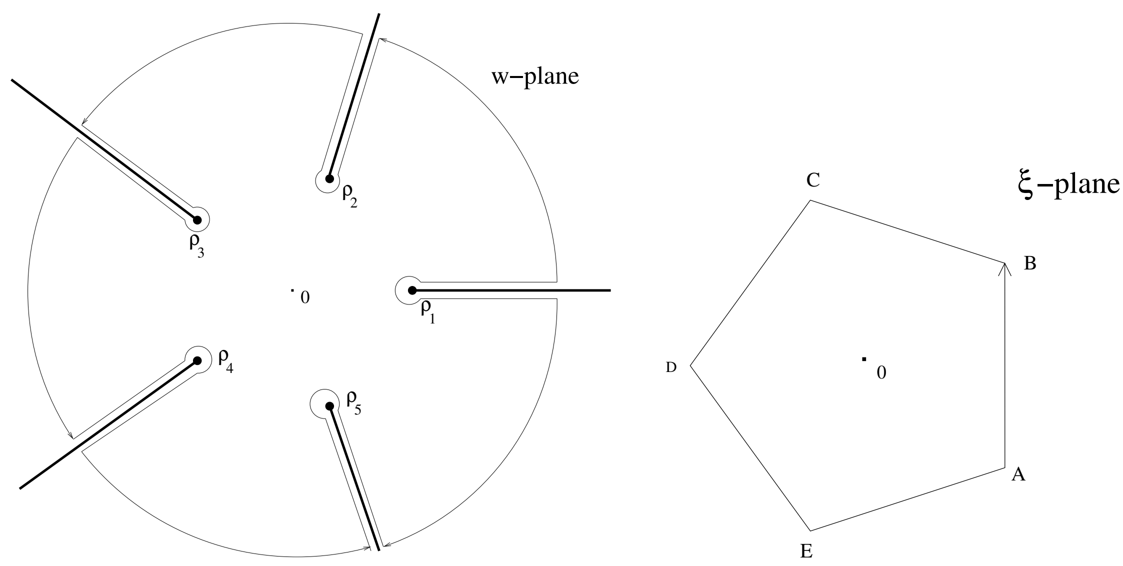

Fig. 1; case $n = 5$.

Let us consider, for the sake of concreteness, $n = 5$. Due to the Schwartz reflection principle, the 5 images of the second w-sheet, obtained by analytic continuation through the cuts $\gamma_j, \ j = 1, .., 5$, are the 5 regular pentagones in Fig. 2 (the blue ones); these images are obtained, one from another, by means of the 4 independent translations $p_i, \ i = 1, .., 4$ connecting the next nearest vertices of the central (white) pentagone:

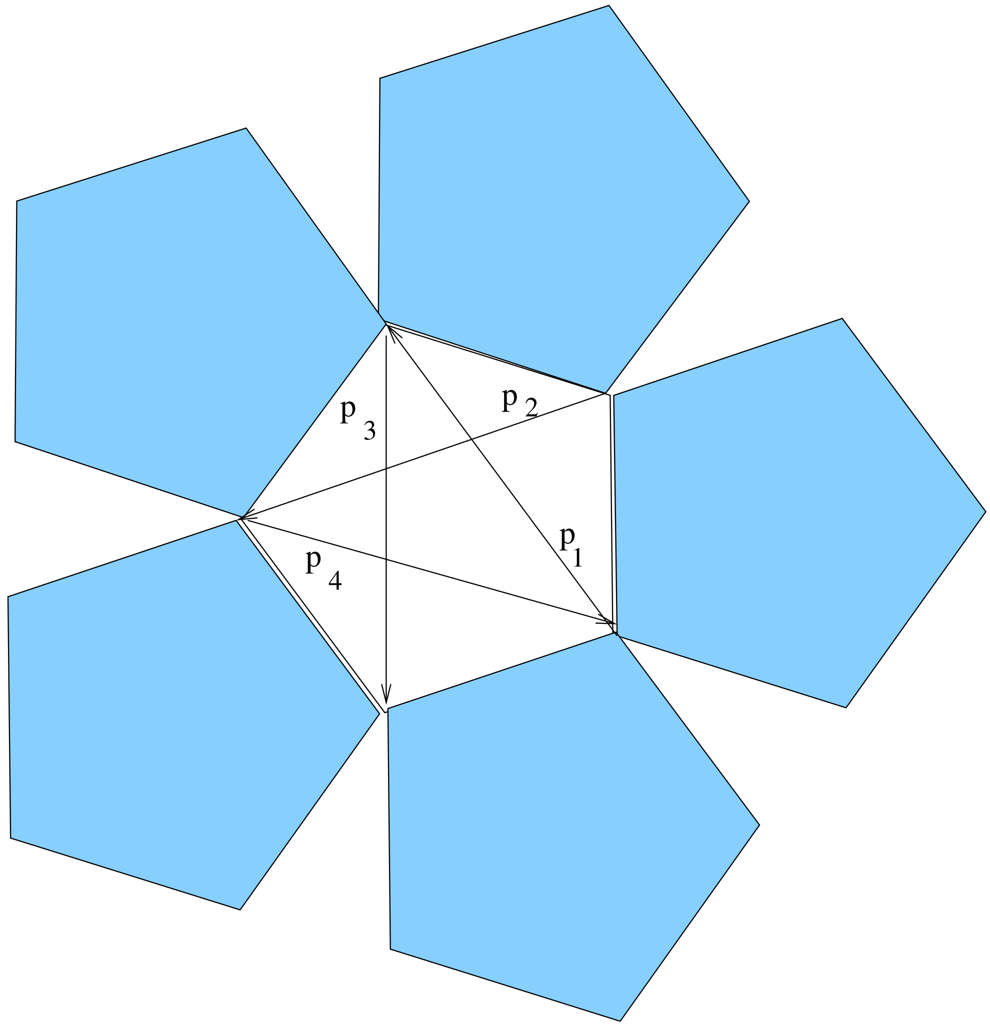

Fig. 2; case $n = 5$.

Therefore the 4 complex numbers $p_i,\ i = 1, .., 4$ are a complete set of independent periods of the inverse function $w = w(\xi)$:

$$
\begin{aligned}
&w\,(\xi + p) = w(\xi),\\
&p = \sum_{k=1}^{4} n_k p_k, \quad n_k \in \mathbb{Z}.
\end{aligned}
\tag{17}
$$

In general, he multi-periodicity property implies immediately that one can restrict the dynamics to the union ot two regular n-gones with periodicity conditions at the parallel edges of the two n-gones (the edges indicated by the same letter in Fig. 3). The resulting space is topologically equivalent to a single $2(n-1)$-gone with periodicity conditions at parallel edges.

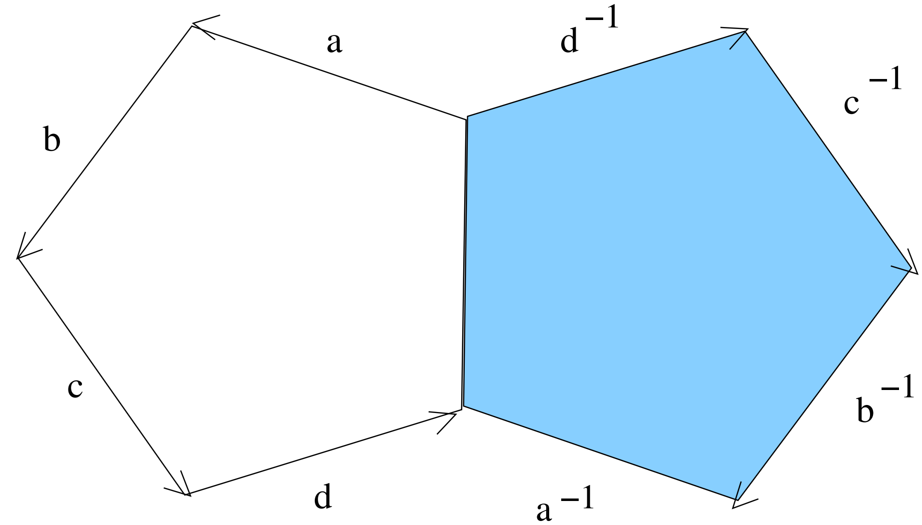

Fig. 3; case $n = 5$.

The regular polygones constructed by analytic continuation through the cuts, using the Schwartz reflection principle, generically do not superimpose, and the Riemann surface defined by $w = w(\xi)$ is an infinite covering of the ξ-plane with branch points, located at the vertices of the n-gones, whose projections to the ξ-plane are everywhere dense. Near a generic branch point ξ_b, w is unbounded, exhibiting the following behavior:

$$w \sim c(\xi - \xi_b)^{-\frac{2}{n-2}}, \quad \xi \sim \xi_b. \tag{18}$$

There are however few important exceptions to this picture, for $n = 3, 4, 6$, corresponding respectively to the equilater triangle, the square and the regular exhagone, which are well-known to cover exactly the plane.

Two initially close rectilinear trajectories are shown in Fig. 5; when a branch point happens to be inside the strip drawn by them, the two trajectories separate. It is important to remark that, since the trajectories cover ergodically the space, no matter how close the two trajectories are initially, sooner or later this bifurcation will take place, and it will repeat over and over. So, the straight motion exhibits sensitive dependence on the initial data.

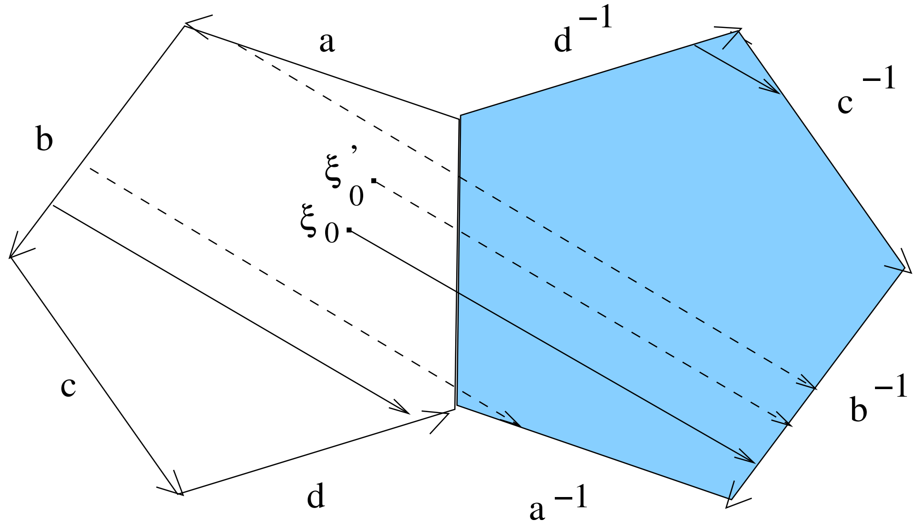

Fig. 5; case $n = 5$.

Interval Exchange Map

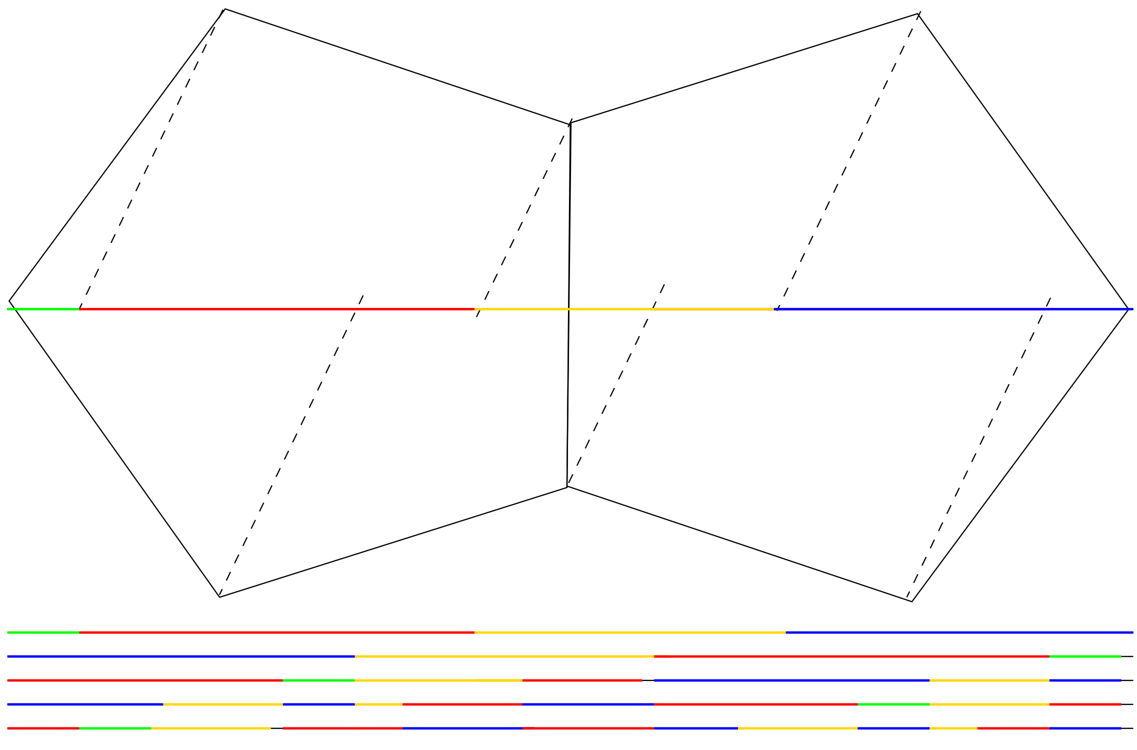

Fig. 6; the interval exchange map, case $n = 5$.

To study further the dynamical properties, we introduce as Poincaré Section (PS) an orizontal line separating our polygonal domain into two separate parts. Then the rectilinear motion becomes the so-called "Interval Exchange Map" (IEM), whose ergodicity is known, and for which the following notion of Lyapunov exponent can be introduced. Let $\xi_0 \in PS$ and consider N iterations of ξ_0 according to the IEM. Let $\mathcal{N}(I, \xi_0, N)$ be the number of iterates of ξ_0 belonging to a subset I of the PS. Then the following result holds:

$$\frac{\mathcal{N}(I, \xi_0, N)}{N} = \mu(I) \left(1 + O\left(\frac{1}{N^{1-\lambda}} \right) \right), \; 0 < \lambda < 1, \; N >> 1. \qquad (19)$$

It follows that:
1) the IEM is ergodic, since the fraction $\frac{\mathcal{N}}{N}$ of iterates falling in I is equal, in the limit $N \to \infty$, to the measure $\mu(I)$ of I (the analogue, in the continuous

case, of the fact that the time spent in a domain of phase space is proportional to the measure of such domain).
2) Since in the integrable case of a quasi-periodic motion on a torus $\lambda = 0$, λ is strictly connected to the asymptotic divergence of two close orbits and can be interpreted as a sort of Lyapunov exponent.

Cyclic motions

In the case of cyclic trajectories (see Fig. 6), the theory says that, due to the transformation (6), if the radius of the evolutionary circle is sufficiently small, then the motion is isochronous with period $T = 2\pi/\omega$. Preliminary numerical experiments (with accuracy 10^{-17}) show that generic initial data lead to periodic motions if n is small, with a period which is a multiple of T, eventually very large (we found examples with period $10^7 T$ and more!). This means that, although inside the evolutionary circle there are infinitely many branch points, the ones which are accessed by the dynamics are finite and the sheets visited is also finite. For sufficiently large n, the typical motion is instead aperiodic, with sensitive dependence on the initial data. Although a phenomenological explanation of this result has been given [1], no rigorous proof of it is available at the moment (see [1] and [2] for more details).

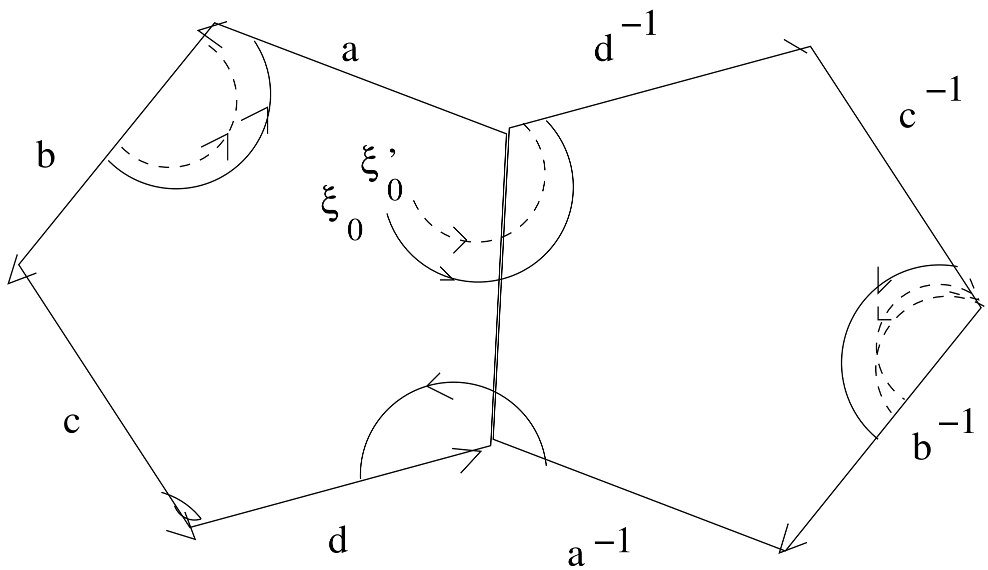

Fig. 6; case $n = 5$.

References

[1] P. G. Grinevich and P. M. Santini: "Newtonian dynamics in the plane corresponding to straight and cyclic motions on the hyperelliptic curve $\mu^2 = \nu^n - 1$, $n \in \mathbb{Z}$: ergodicity, isochrony, periodicity and fractals"; Physica D **232** (2007) 22-32. http://arXiv:nlin.CD/0607031.

[2] P.M.Santini: "The anharmonic oscillator"; seconda parte del corso: "The transition from regular to irregular motion as travel on Riemann surfaces", NEEDS 2007-School, L'Ametlla de Mar, June 16-17, 2007. http://www.roma1.infn.it/people/santini/Didattica2/break.pdf (Username: needs Passwd: soliton2007)