Darboux-Dressing for NLS and
an introduction to the theory of rogue (anomalous) waves
(alcuni appunti del corso di Onde non lineari e solitoni)

Paolo Maria Santini

1 NLS and its integrability scheme

As we have seen in the course "Onde non lineari e solitoni", the focusing nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + 2|u|^2 u = 0 \tag{1}$$

describes quasi-monochromatic waves in weekly nonlinear media in Nature. Its constant background solution

$$u_0(x,t) = e^{2it} \tag{2}$$

describing, for instance, Stokes waves in the theory of surface waves in deep water, a state of constant light intensity in nonlinear optics, and a state of constant boson density in the theory of Bose - Einstein condensates, is unstable under monochromatic perturbations of sufficiently long wave length, and this instability is considered the main cause for the formation of rogue waves (RWs) in Nature.

The NLS equation (1) is the compatibility condition [10]

$$X_t - T_x + [X,T] = 0 \tag{3}$$

for the Zakharov - Shabat (ZS) Lax pair

$$\begin{array}{l} \Psi_x(\lambda;x,t) = X(\lambda;x,t)\Psi(\lambda;x,t), \\ \Psi_t(\lambda;x,t) = T(\lambda;x,t)\Psi(\lambda;x,t) \end{array} \tag{4}$$

where Ψ is a 2×2 matrix fundamental solution of (4) (so that the inverse matrix Ψ^{-1} exists), and

$$\begin{array}{l} X(\lambda;x,t) = -i\lambda\sigma_3 + iU(x,t), \\ T(\lambda;x,t) = 2\lambda X(\lambda;x,t) + iV(x,t), \\ U = \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} |u|^2 & iu_x \\ -i\bar{u}_x & -|u|^2 \end{pmatrix} \end{array} \tag{5}$$

EsRW 01. Verify that the NLS equation (1) is the integrability condition for (3)-(5).

Since
$$X^\dagger(\bar\lambda) = -X(\lambda), \quad T^\dagger(\bar\lambda) = -T(\lambda) \tag{6}$$
it follows that $\Psi^{-1}(\lambda)$ and $\Psi^\dagger(\bar\lambda)$ satisfy the same matrix equations
$$F_x = -FX, \quad F_t = -FT; \tag{7}$$
therefore the normalization of the fundamental solution can be chosen such that
$$\Psi^{-1}(\lambda) = \Psi^\dagger(\bar\lambda) \tag{8}$$
(we often omit to indicate the dependence on x, t, if not necessary).

Let $u_0(x,t)$ be a particular solution of NLS, and let $\Psi_0(\lambda; x, t)$ be the corresponding fundamental solution of (4). Again its normalization is chosen such that
$$\Psi_0^{-1}(\lambda) = \Psi_0^\dagger(\bar\lambda); \tag{9}$$

2 Darboux dressing [8, 9]

We look for the following relation
$$\Psi(\lambda; x, t) = \chi(\lambda; x, t)\Psi_0(\lambda; x, t) \tag{10}$$
between the matrix solutions $\Psi(\lambda; x, t)$ and $\Psi_0(\lambda; x, t)$ of (4), corresponding to the particular solutions $u(x,t)$ and $u_0(x,t)$ of NLS, where $\chi(\lambda; x, t)$ is the so-called Darboux (Dressing) matrix.

We also assume that
$$\chi(\lambda; x, t) = I + \frac{\tilde\chi(x,t)}{\lambda} + O(\lambda^{-2}), \quad |\lambda| \gg 1. \tag{11}$$
If Ψ and Ψ_0 satisfy (8) and (9), then
$$\chi^{-1}(\lambda) = \chi^\dagger(\bar\lambda). \tag{12}$$

Substituting (10) in (4) and using (11), we infer that
$$U = U_0 + [\sigma_3, \tilde\chi], \tag{13}$$
implying that
$$u(x,t) = u_0(x,t) + 2\left(\tilde\chi(x,t)\right)_{12}, \tag{14}$$
where $(M)_{12}$ is the component 12 of matrix M.

EsRW 02. Verify (14).

From the definition (10) we also infer that

$$\chi_x = X\chi - \chi X_0, \quad \chi_t = T\chi - \chi T_0 \tag{15}$$

and that

$$\begin{aligned} X(\lambda) &= -\chi(\lambda)\left(\partial_x - X_0(\lambda)\right)\chi^{-1}(\lambda), \\ T(\lambda) &= -\chi(\lambda)\left(\partial_t - T_0(\lambda)\right)\chi^{-1}(\lambda). \end{aligned} \tag{16}$$

We remark that the matrices X_0, T_0, X, T must depend on λ polinomially:

$$\begin{aligned} &X_0(\lambda;x,t) = -i\lambda\sigma_3 + iU_0(x,t), \quad X(\lambda;x,t) = -i\lambda\sigma_3 + iU(x,t), \\ &T_0(\lambda;x,t) = 2\lambda X_0(\lambda;x,t) + iV_0(x,t), \quad T(\lambda;x,t) = 2\lambda X(\lambda;x,t) + iV(x,t), \end{aligned} \tag{17}$$

and this will imply suitable constraints on χ.

2.0.1 Rational dependence on λ

We also assume that $\chi(\lambda)$ be a rational function of λ:

$$\chi(\lambda;x,t) = I + \sum_{m=1}^{N} \frac{A_m(x,t)}{\lambda - \lambda_m}, \quad \lambda_m \in \mathbb{C}, \tag{18}$$

implying that

$$u(x,t) = u_0(x,t) + 2\sum_{m=1}^{N} \left(A_m(x,t)\right)_{12}. \tag{19}$$

Using (12) it follows that

$$\chi(\lambda)\chi^{-1}(\lambda) = \chi(\lambda)\chi^{\dagger}(\bar\lambda) = I, \tag{20}$$

Consequently we have

$$I = \chi(\lambda)\chi^{-1}(\lambda) \sim \chi(\bar\lambda_n)\frac{A_n^{\dagger}}{\lambda - \bar\lambda_n}, \quad \lambda \sim \bar\lambda_n \tag{21}$$

implying that

$$\chi(\bar\lambda_n)A_n^{\dagger} = 0, \quad 1 \le n \le N. \tag{22}$$

It follows that the matrices $A_n, A_n^{\dagger}$ are degenerate

$$\det(A_n) = \det(A_n^{\dagger}) = 0, \quad \forall n \tag{23}$$

with the following representation

$$A_n = \boldsymbol{p^{(n)}} \cdot \boldsymbol{q^{(n)}}^T, \quad A_n^{\dagger} = \overline{\boldsymbol{q^{(n)}}} \cdot \overline{\boldsymbol{p^{(n)}}}^{\boldsymbol{T}}, \tag{24}$$

where

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \boldsymbol{v}^T = (v_1, v_2), \tag{25}$$

or, in components:

$$(A_n)_{\alpha\beta} = p^{(n)}_\alpha q^{(n)}_\beta, \quad \left(A^\dagger_n\right)_{\alpha\beta} = \overline{q^{(n)}_\alpha}\ \overline{p^{(n)}_\beta}. \tag{26}$$

Therefore the constraint (22) is satisfied if

$$\chi(\bar\lambda_n)\ \overline{\boldsymbol{q^{(n)}}} = \boldsymbol{0}. \tag{27}$$

In addition, equations (12) and (16) imply that, if X_0, T_0 have the λ-dependence indicated in (17), X, T would be singular in $\bar\lambda_n$:

$$X(\lambda) \sim -\chi(\bar\lambda_n)\left(\partial_x - X_0(\bar\lambda_n)\right)\overline{\boldsymbol{q^{(n)}}}\cdot\overline{\boldsymbol{p^{(n)}}}^{\boldsymbol{T}}(\lambda - \bar\lambda_n)^{-1}, \quad \lambda \sim \bar\lambda_n. \tag{28}$$

But since X, T must have the λ-dependence indicated in (17) as well, it follows that the residue of the expression in (28) must be zero. Consequently, using also (27), we infer that

$$\chi(\bar\lambda_n)\left(\overline{\boldsymbol{q^{(n)}}}_x - X_0(\bar\lambda_n)\overline{\boldsymbol{q^{(n)}}}\right) = \boldsymbol{0}. \tag{29}$$

Let $\Psi_0(\bar\lambda_n)$ be a fundamental solution of

$$\Psi_{0x}(\bar\lambda_n) = X_0(\bar\lambda_n)\Psi_0(\bar\lambda_n), \tag{30}$$

and let $\boldsymbol{\xi^{(n)}}$ be a constant 2D vector; then

$$\overline{\boldsymbol{q^{(n)}}} \equiv \Psi_0(\bar\lambda_n)\ \boldsymbol{\xi^{(n)}} \tag{31}$$

solves (29), satisfying: $\overline{\boldsymbol{q^{(n)}}}_x - X_0(\bar\lambda_n)\overline{\boldsymbol{q^{(n)}}} = \boldsymbol{0}$.

Given the $\boldsymbol{q^{(n)}}$'s from (31), we observe that (27) can be rewritten as a linear inhomogeneous system for the $\boldsymbol{p^{(n)}}$'s:

$$\begin{aligned} &\sum_{m=1}^{2} B_{nm}\ \boldsymbol{p^{(m)}} = \overline{\boldsymbol{q^{(n)}}}, \quad n = 1, \dots, N, \\ &B_{nm} \equiv \frac{\sum_{\alpha=1}^{2} q^{(m)}_\alpha \overline{q^{(n)}_\alpha}}{\lambda_m - \overline{\lambda_n}}. \end{aligned} \tag{32}$$

EsRW 03. Verify that (32) define the proper $\boldsymbol{p^{(m)}}$'s.

Recapitulating, from a given solution $u_0(x,t)$ of NLS and from the corresponding fundamental solution $\Psi_0(\lambda)$ of the Lax pair (4), one constructs the $\boldsymbol{q}^{(n)}$'s and the $\boldsymbol{p}^{(n)}$'s from (31) and (32). Then $\chi(\lambda)$ is known from (18) and the new (dressed) solution $u(x,t)$ is constructed from (19) as follows

$$u(x,t) = u_0(x,t) + 2\sum_{n=1}^{N} p_1^{(n)}(x,t) q_2^{(n)}(x,t). \tag{33}$$

In the simplest case $N=1$,

$$p_1^{(1)} = \overline{q_1^{(1)}}/B_{11}, \qquad B_{11} = \frac{|\boldsymbol{q}^{(1)}|^2}{2i\mathrm{Im}\lambda_1}, \tag{34}$$

and (33) becomes

$$u(x,t) = u_0(x,t) + 4i\mathrm{Im}\lambda_1 \frac{\overline{q_1^{(1)}} q_2^{(1)}}{|\boldsymbol{q}^{(1)}|^2}. \tag{35}$$

EsRW 04. Prove (35) from (33).

If the initial solution is the unstable background (2), the corresponding fundamental solution of the Lax pair is

$$\begin{aligned} &\Psi_0(\lambda) = \tfrac{1}{\sqrt{2\mu(\mu+\lambda)}} e^{it\sigma_3} \begin{pmatrix} e^{\Theta(\lambda)} & -(\mu+\lambda)e^{-\Theta(\lambda)} \\ (\mu+\lambda)e^{\Theta(\lambda)} & e^{-\Theta(\lambda)} \end{pmatrix}, \\ &\Theta(\lambda) \equiv i\mu(x+2\lambda t), \end{aligned} \tag{36}$$

where $\sigma_3 = \mathrm{diag}(1,-1)$ is the Pauli matrix and μ, λ are complex parameters satisfying the constraint

$$\mu^2 = 1 + \lambda^2. \tag{37}$$

EsRW 05. Verify that (36) is the fundamental solution of (4) corresponding to (2), and satisfying det $\Psi_0 = 1$.

Since we look for solutions periodic in x (having in mind, for instance, Stokes water waves) and exponentially blowing or decaying in t (to describe the modulation instability), we must choose $-1 < \mu < 1$ and $\lambda \in i\mathbb{R}$, with $|\lambda| < 1$. It is therefore convenient to use the parametrization

$$\mu = \cos\phi, \quad \lambda = i\sin\phi, \quad \phi \in \mathbb{R}, \tag{38}$$

so that

$$\Theta(\lambda) = i(\cos\phi)x - (\sin 2\phi)t, \quad \mu + \lambda = e^{i\phi}. \tag{39}$$

Replacing λ by $\bar{\lambda}$ is equivalent to replacing ϕ by $-\phi$; correspondingly

$$\Theta(\bar{\lambda}) = i(\cos\phi)x + (\sin 2\phi)t \equiv \frac{ikx + \sigma t}{2}, \tag{40}$$

where

$$k = 2\mu = 2\cos\phi \tag{41}$$

plays the role of wave number and

$$\sigma = k\sqrt{4 - k^2} = 2\sin 2\phi \tag{42}$$

that of amplification factor (growing rate). Therefore

$$\Psi_0(\bar{\lambda}) = \frac{1}{\sqrt{2\mu}} e^{it\sigma_3} \begin{pmatrix} e^{\frac{ikx+\sigma t+\phi}{2}} & -e^{-\frac{ikx+\sigma t+\phi}{2}} \\ e^{\frac{ikx+\sigma t-\phi}{2}} & -e^{-\frac{ikx+\sigma t-\phi}{2}} \end{pmatrix}. \tag{43}$$

Choosing $\boldsymbol{\xi} = (\gamma, \gamma^{-1})^T$, then, from (31)

$$\overline{\boldsymbol{q}} = \Psi_0(\bar{\lambda})\ \boldsymbol{\xi} = \frac{1}{\sqrt{k}} e^{it\sigma_3} \left(\gamma \begin{pmatrix} e^{\frac{ikx+\sigma t+\phi}{2}} \\ e^{\frac{ikx+\sigma t-\phi}{2}} \end{pmatrix} + \gamma^{-1} \begin{pmatrix} -e^{-\frac{ikx+\sigma t+\phi}{2}} \\ e^{-\frac{ikx+\sigma t-\phi}{2}} \end{pmatrix} \right) \tag{44}$$

Going from the arbitrary real parameters $|\gamma|$, $\arg\gamma$ to the real parameters x_1, t_1 via

$$x_1 = -\frac{2}{k}(\arg\gamma + \pi/4), \quad t_1 = -\frac{2}{\sigma}\log|\gamma|, \tag{45}$$

we finally obtain

$$\overline{\boldsymbol{q}} = \frac{1}{\sqrt{k}} e^{it\sigma_3} \begin{pmatrix} e^{\frac{ik(x-x_1-\pi/2)+\sigma(t-t_1)+\phi}{2}} - e^{-\frac{ik(x-x_1-\pi/2)+\sigma(t-t_1)+\phi}{2}} \\ e^{\frac{ik(x-x_1-\pi/2)+\sigma(t-t_1)-\phi}{2}} + e^{-\frac{ik(x-x_1-\pi/2)+\sigma(t-t_1)-\phi}{2}} \end{pmatrix} \tag{46}$$

At last, after some algebra, (35) becomes

$$u(x,t) = \mathcal{A}(x, t; \phi, x_1, t_1, \rho), \tag{47}$$

where

$$\mathcal{A}(x, t; x_1, t_1, \rho) \equiv e^{2it+i\rho} \frac{\cosh[\sigma(t-t_1) + 2i\phi] + \sin\phi\cos[k(x-x_1)]}{\cosh[\sigma(t-t_1)] - \sin\phi\cos[k(x-x_1)]} \tag{48}$$

is the Akhmediev breather [1, 2], exact solution of NLS for all values of the real parameters ϕ, x_1, t_1, ρ, and k, σ are defined in (41), (42).

EsRW 06. Complete the algebraic steps to get (47),(48) from (46) and (35). Why is it possible to insert in (48) the factor $\exp(i\rho)$, $\rho \in \mathbb{R}$?

This solution is exponentially localized in time over the background u_0, and changes it by the multiplicative phase factor $e^{4i\phi}$:

$$\mathcal{A}(x,t;\phi,x_1,t_1,\rho) \to e^{2it+i(\rho\pm 2\phi)}, \quad \text{as} \quad t \to \pm\infty; \tag{49}$$

in addition, its modulus takes its maximum at the point (x_1, t_1)

$$|\mathcal{A}(x_1,t_1;\phi,x_1,t_1,\rho)| = 1 + 2\sin\phi. \tag{50}$$

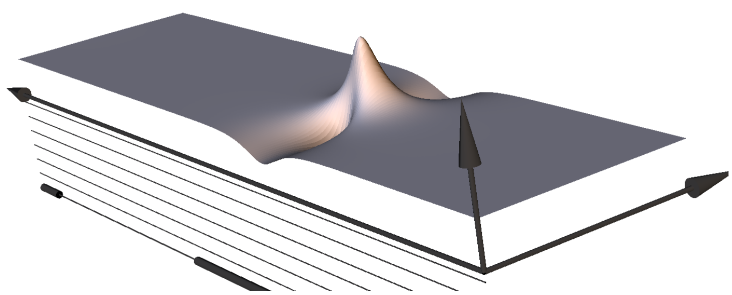


Figure 1. 3D plot of the modulus of (48).

3 A simple Cauchy problem

We now study the Cauchy problem for (1) on the segment $[0, L]$, with periodic boundary conditions, and we consider, as initial condition, a generic, smooth, periodic, zero average, small perturbation of the background solution (2):

$$u(x,0) = 1 + \varepsilon v(x), \quad v(x+L) = v(x), \quad 0 < \varepsilon \ll 1, \tag{51}$$

It is well-known that, in this Cauchy problem, the Modulation Instability (MI) is due to the fact that, expanding the initial perturbation in Fourier components:

$$v(x) = \sum_{j\geq 1} \left(c_j e^{ik_j x} + c_{-j} e^{-ik_j x} \right), \quad k_j = \frac{2\pi}{L} j, \quad |c_j| = O(1), \tag{52}$$

and defining $N \in \mathbb{N}^+$ as $N = \lfloor L/\pi \rfloor$, the first N modes $\pm k_j,\ 1 \le j \le N$, are unstable, since they give rise to exponentially growing and decaying waves of amplitudes $O(\varepsilon e^{\pm \sigma_j t})$, where the growing rates σ_j are defined by

$$\sigma_j = k_j \sqrt{4 - k_j^2} > 0, \tag{53}$$

while the remaining modes give rise to oscillations of amplitude $O(\varepsilon e^{\pm i\omega_j t})$, where $\omega_j = k_j \sqrt{k_j^2 - 4}$, and therefore are stable.

EsRW 7. Show it.

We have in mind the following *qualitative recurrence scenario* for finite N. The exponentially growing waves become $O(1)$ at times of $O({\sigma_j}^{-1} |\log\ \varepsilon|)$, when one enters the nonlinear stage of MI. In this second time interval one expects the generation of a transient, $O(1)$, coherent structure, described by a soliton - like solution of NLS over the unstable background (2), the so-called RW. Such a RW will have an internal structure, due to the nonlinear interaction between the N unstable Fourier modes, fully described by the integrable NLS theory. Due again to MI, such a coherent RW is expected to be destroyed in a finite time interval, and one enters the third asymptotic stage, characterized, like the first one, by the background plus an $O(\varepsilon)$ perturbation, and described again by the NLS theory linearized around the background. This second linearized stage is expected, due again to MI, to give rise to the formation of a second nonlinear stage of MI. This procedure should iterate forever, in the integrable NLS model, giving rise to the generation of an infinite sequence of RWs. Therefore one is expected to be dealing with the following basic deterministic issues. For a given generic initial condition of the type (51), how to predict: 1) the "generation time" of the first RW; 2) the "recurrence times" measuring the time intervals between two consecutive RWs; the analytic form of this deterministic sequence of RWs.

We first consider the case in which the initial perturbation (51), (52) excites only the unstable modes (the sum in (52) goes up to N). Then, for $|t| \le O(1)$:

$$\begin{array}{l} u(x,t) = e^{2it}\left(1 + \sum\limits_{j=1}^{N} \left(\frac{|\alpha_j|}{\sin 2\phi_j} e^{\sigma_j t + i\phi_j} \cos[k_j(x - X_j)] + \right.\right. \\ \left.\left. \frac{|\beta_j|}{\sin 2\phi_j} e^{-\sigma_j t - i\phi_j} \cos[k_j(x - X_j^-)]\right)\right) + O(\varepsilon^2), \end{array} \tag{54}$$

where

$$\begin{array}{l}\alpha_j = e^{-i\phi_j}\overline{c_j} - e^{i\phi_j}c_{-j}, \quad \beta_j = e^{i\phi_j}\overline{c_{-j}} - e^{-i\phi_j}c_j, \\ X_j = \frac{\arg(\alpha_j)+\pi/2}{k_j}, \quad X^-_j = \frac{-\arg(\beta_j)+\pi/2}{k_j}, \quad j=1,\dots,N, \\ \sigma_j = 2\sin(2\phi_j), \quad k_j = 2\cos\phi_j \quad \Leftrightarrow \quad \phi_j = \arccos(k_j/2), \quad j=1,\dots,N.\end{array} \tag{55}$$

EsRW 8. Get equations (54), (55).

Therefore *the initial datum splits into exponentially growing and decaying waves, respectively the α- and β-waves, each one carrying half of the information encoded into the initial datum.* At $t = O(|\log\varepsilon|)$, the exponentially growing α-waves become $O(1)$ and the solution is described by an exact NLS solution matching with the asymptotic formula

$$u(x,t) \sim e^{2it}\sum_{j=1}^{N}\left(1 + \frac{|\alpha_j|}{\sin 2\phi_j}e^{\sigma_j t + i\phi_j}\cos[k_j(x-X_j)]\right), \tag{56}$$

obtained evaluating (54) in the intermediate region $1 \ll t \ll O(|\log\varepsilon|)$.

From now on we concentrate on the simplest case, choosing $N = 1$. Therefore we are looking, in the nonlinear region $t = O(|\log\varepsilon|)$, for an exact 1-mode, x-periodic, transient solution of NLS, matching with (56) for $N = 1$ in the overlapping region $1 \ll t \ll O(|\log\varepsilon|)$. The natural candidate for such a solution is the Akhmediev breather (48). Evaluating (48) in the overlapping region $1 \ll t \ll O(|\log\varepsilon|)$ and imposing a good matching with (56) with $N = 1$, one fixes all the free parameters in (48) as follows

$$\begin{array}{l}k = k_1,\ \sigma = \sigma_1,\ \phi = \phi_1,\ \rho = 2\phi_1, \\ x_1 = X_1,\ t_1 = T_1 \equiv \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^2}{2|\alpha_1|}\right) = O(\sigma_1^{-1}|\log\varepsilon|)\end{array} \tag{57}$$

EsRW 9. Verify the matching formulas (57).

Therefore *the first RW appears in the finite t-interval $|t - T_1| \le O(1)$, and is described by the Akhmediev breather solution of NLS:*

$$u(x,t) = \mathcal{A}\Big(x, t; \phi_1, X_1, T_1, 2\phi_1\Big) + O(\varepsilon), \tag{58}$$

whose parameters are expressed in terms of the initial data through elementary functions. It is important to remark that the first RW contains informations only on half of the initial data (the half encoded in the parameter α_1: the α_1-wave), and that the modulus of the first RW generated by the initial condition (51),(52) acquires its maximum at $t = T_1$ in the point $x = X_1$, mod L; and the value of this maximum is

$$|u(X_1, T_1)| = 1 + 2\sin\phi_1 < 1 + \sqrt{3} \sim 2.732. \tag{59}$$

This upper bound, 2.732 times the background amplitude, is consequence of the formula $\sin\phi_1 = \sqrt{1 - (\pi/L)^2}$, $\pi < L < 2\pi$, and is obtained when $L \to 2\pi$.

EsRW 10. Verify (59).

We also notice that the position $x = X_1$ of the maximum of the RW coincides with the position of the maximum of the growing sinusoidal wave of the linearized theory; this is due to the absence of nonlinear interactions with other unstable modes, if the simplest case $N = 1$.

Similar considerations can be made to study this Cauchy problem at negative times, obtaining the following result.
The first RW appearing at negative times, in the finite t-interval $|t + T_1^-| \leq O(1)$, is described again by the Akhmediev solution of NLS, but with different parameters:

$$\begin{aligned} &u(x,t) = \mathcal{A}\Big(x, t; \phi_1, X_1^-, -T_1^-, -2\phi_1\Big) + O(\varepsilon), \\ &T_1^- = \tfrac{1}{\sigma_1}\log\left(\tfrac{\sigma_1^2}{2|\beta_1|}\right). \end{aligned} \tag{60}$$

It is important to remark that this RW contains informations only on the second half of the initial data (the half encoded in the parameter β_1: the β_1-wave), and that the modulus of the first RW generated by the initial condition (51),(52) acquires its maximum at $t = -T_1^- < 0$ in the point $x = X_1^-$. Since NLS is invariant under t-translations, one infers the following exact RW recurrence, in the case of a single unstable mode.

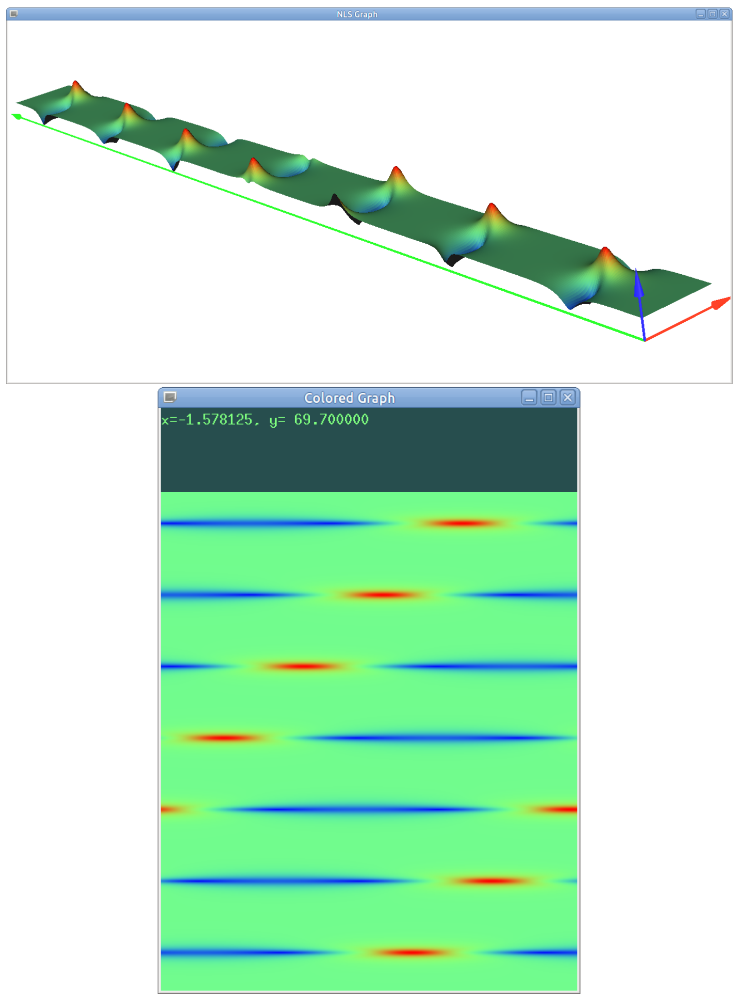
The solution of the x-periodic Cauchy problem describes, in the case of one unstable mode $\pm k_1$, an exact recurrence of Akhmediev breathers, whose parameters, changing at each appearance, are expressed in terms of the initial data via elementary functions. T_1 is the first appearance time of the RW (the time at which the RW achieves the maximum of its modulus), X_1, is the position of such a maximum, $1 + 2\sin\phi_1$ is the value of the maximum,

$$\Delta T = T_1 + T_1^- = \frac{2}{\sigma_1}\log\left(\frac{\sigma_1^2}{2\varepsilon\sqrt{|\alpha_1\beta_1|}}\right), \tag{61}$$

is the recurrence time (the time interval between two consecutive RW appearances),

$$\Delta X_n = X_1 - X_1^- = \frac{\arg(\alpha_1 \beta_1)}{k_1}, \quad mod\ L \tag{62}$$

is the x-shift of the position of the maxima in the recurrence. In addition, after each appearance, the RW changes the background by the multiplicative phase factor $\exp(4i\phi_1)$ *(see Figures 2 and 3).* This exact RW recurrence, an interesting example of Fermi-Pasta-Ulam recurrence [3], and it is in good agreement with a nonlinear optics experiment [6]



Figures 2 and 3. 3D plot and density plot of $|u(x,t)|$ describing the RW recurrence of one unstable mode.

EsRW 11. Verify this recurrence.

The deterministic aspects of the RW dynamics for a finite number of unstable modes has been recently completely clarified in [7]. The matched asymptotics expansions techniques used in the case of one unstable mode are not adequate anymore, and techniques involving the finite gap method (a nonlinear analogue of the Fourier series method for linear PDEs) have been used.

References

[1] N. N. Akhmediev and V.I. Korneev, Theor. Math. Phys. 69,1089 (1986).

[2] N. N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin, Exact first order solutions of the Nonlinear Schödinger equation, Theor. Math. Phys, 72, 809 (1987)

[3] G. Gallavotti (Ed.), "The Fermi-Pasta-Ulam Problem: A Status Report", *Lecture Notes in Physics*, Vol. 728, Springer, Berlin Heidelberg, 2008; doi:10.1007/978-3-540-72995-2.

[4] P. G. Grinevich and P. M. Santini: "The finite gap method and the analytic description of the exact rogue wave recurrence in the periodic NLS Cauchy problem. 1", Nonlinearity **31**, 5258-5308 (2018). doi.org/10.1088/1361-6544/aaddcf

[5] P. G. Grinevich and P. M. Santini: "The exact rogue wave recurrence in the NLS periodic setting via matched asymptotic expansions, for 1 and 2 unstable modes", Phys. Lett. A **382** (2018) 973-979. https://doi.org/10.1016/j.physleta.2018.02.014

[6] D. Pierangeli, M. Flammini, L. Zhang, G. Marcucci, A. J. Agranat, P. G. Grinevich, P. M. Santini, C. Conti, and E. DelRe, Observation of exact Fermi-Pasta-Ulam-Tsingou recurrence and its exact dynamics, Phys. Rev. X **8**, 041017 (2018). DOI:https://doi.org/10.1103/PhysRevX.8.041017

[7] P. G. Grinevich and P. M. Santini: The finite gap method and the periodic NLS Cauchy problem of the anomalous waves, for a finite number of unstable modes, arXiv:1810.09247, RUSS MATH SURV, 2019, 74, DOI:10.1070/RM9863 (english version in press).

[8] V. B. Matveev and M. A. Salle, *Darboux transformations and solitons*, (Berlin, Heidelberg: Springer Series in Nonlinear Dynamics, Springer-Verlag), 1991.

[9] V. E. Zakharov and A. A. Gelash, On the nonlinear stage of Modulation Instability, PRL **111**, 054101 (2013).

[10] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Sov. Phys. JETP 34, 1, 62-69, 1972.