1 ONDE NON LINEARI E SOLITONI

Prof. Paolo Maria Santini

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2 Esercizi proposti (raccolta provvisoria)

2.1 Propagazione ondosa lineare e non lineare

2.1.1 Onde dispersive lineari [1, 5]

1) Given the Fourier integral representation

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}_0(k) e^{i(kx-\omega(k)t)} dk, \tag{1}$$

where $\hat{u}_0(k)$ is the Fourier transform of the initial condition $u(x,0)$:

$$\hat{u}_0(k) = \int_{\mathbb{R}} e^{-iky} u(y,0) dy, \tag{2}$$

1. show that (1) can be written in the following suggestive form:

$$u(x,t) = \int_{\mathbb{R}} S(x-y,t) u(y,0) dy, \tag{3}$$

where $S(x,t)$ is the "fundamental" solution of the PDE defined as:

$$S(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(kx-\omega(k)t)} dk. \tag{4}$$

2. If $\omega(k) = k^n$, then $S(x,t)$ is the following similarity solution of the PDE:

$$\begin{aligned} S(x,t) &= \tfrac{1}{t^{1/n}} f\left(\tfrac{x}{t^{1/n}}\right), \\ f(\xi) &= \tfrac{1}{2\pi}\int_{\mathbb{R}} e^{i(k\xi - k^n)} dk. \end{aligned} \tag{5}$$

3. Show that, if $u \in \mathbb{R}$, then:

$$\begin{aligned} \overline{\hat{u}_0(k)} &= \hat{u}_0(-k), \quad k \in \mathbb{R} \\ u(x,t) &= \tfrac{1}{\pi} Re \int_0^\infty \hat{u}_0(k) e^{i(kx-\omega(k)t)} dk \end{aligned} \tag{6}$$

If, in addition, $\hat{u}_0(k)$ can be prolongued outside the real axis, then

$$\overline{\hat{u}_0(k)} = \hat{u}_0(-\bar{k}). \tag{7}$$

(for the second of (6) we have also assumed that $\omega(k)$ is odd: $\omega(-k) = -\omega(k)$)

2) Given the following linear PDEs:

$$\begin{array}{ll} iu_t + u_{xx} = 0, & \text{free particle Schrödinger equation,} \\ u_t + u_{xxx} = 0, & \text{linearized KdV equation,} \\ u_{tt} - u_{xx} + u = 0, & \text{Klein - Gordon equation,} \end{array} \tag{8}$$

1. Construct the fundamental similarity solution (5).
2. Study the longtime behavior, for $t >> 1,\ x/t = O(1)$, of the solutions of their Cauchy problem using the stationary phase, Laplace, or saddle point methods, depending on the situation, and estimate the error.
3. Study of the relevance of exact similarity solution.

Solution:

i) Free particle Schrödinger equation:

$$\begin{array}{l} S(x,t) = \frac{1}{2\sqrt{\pi t}} e^{i(\frac{x^2}{4t} - \frac{\pi}{4})}, \\ u(x,t) = S(x,t)\left(A(\xi) + \frac{1}{t}B(\xi) + O(t^{-2})C(\xi)\right), \quad \xi = \frac{x}{2t} = O(1), \quad t >> 1 \\ A(\xi) = \hat{u}_0(\xi), \quad B(\xi) = -\frac{i}{4}A_{\xi\xi} \end{array} \tag{9}$$

ii) Linear KdV. For $x/t > 0$, the lines of constant $v(k)$ are the imaginary axis and the hyperbola ${k_R}^2 - 3{k_I}^2 + x/t = 0$. The steepest descent contour passing through the critical point $i\sqrt{\frac{x}{3t}}$ is the upper branch of the hyperbola, while the steepest descent contour passing through the critical point $-i\sqrt{\frac{x}{3t}}$ is the imaginary axis.

$$\begin{array}{l} S(x,t) = \frac{1}{(3t)^{1/3}} Ai\left(\frac{x}{(3t)^{1/3}}\right), \\ u(x,t) \sim \frac{\hat{u}_0(|x/3t|^{1/2})}{\sqrt{4\pi|3x/t|^{1/2}t}} e^{-i2|x/3t|^{3/2}t + i\pi/4} + \text{c.c.}, \quad \frac{x}{3t} = O(1) < 0, \quad t >> 1, \\ u(x,t) \sim \frac{\hat{u}_0(i|x/3t|^{1/2})}{\sqrt{12\pi|3x/t|^{1/2}t}} e^{-2|x/3t|^{3/2}t}, \quad \frac{x}{3t} = O(1) > 0, \quad t >> 1, \\ u(x,t) \sim \frac{\hat{u}_0(0)}{2\pi(3t)^{1/3}} Ai\left(\frac{x}{(3t)^{1/3}}\right) - \frac{i\hat{u}_0'(0)}{2\pi(3t)^{2/3}} Ai'\left(\frac{x}{(3t)^{1/3}}\right), \quad \frac{x}{(3t)^{1/3}} = O(1), \quad t >> 1, \\ u(x,t) \sim \frac{\hat{u}_0(0)}{2\pi} S(x,t), \quad \frac{x}{(3t)^{1/3}} = O(1), \quad t >> 1, \end{array} \tag{10}$$

where $A_i(\xi)$ is the Airy function

$$A_i(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(k\xi - k^3)} dk, \tag{11}$$

solution of the ODE: $f(\xi)'' - \xi f(\xi) = 0$.
iii) Klein-Gordon equation. The dispersion relation is two-valued (since the PDE is second order in t):

$$\omega^{\pm}(k) = \pm\sqrt{k^2+1}; \tag{12}$$

therefore the phase velocity is greater than the light speed 1, while the group velocity is less than 1:

$$\frac{\omega}{k} = \frac{\sqrt{k^2+1}}{k} > 1, \quad \frac{d\omega}{dk} = \frac{k}{\sqrt{k^2+1}} < 1 \tag{13}$$

The Fourier representation of the real solution reads:

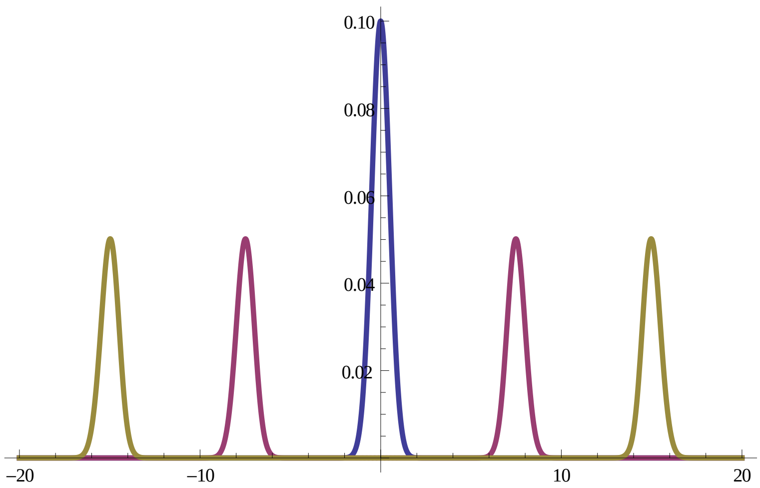
$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} A(k) e^{i(kx+\sqrt{k^2+1}t)} dk + \frac{1}{2\pi}\int_{\mathbb{R}} \overline{A(-k)} e^{i(kx-\sqrt{k^2+1}t)} dk, \tag{14}$$

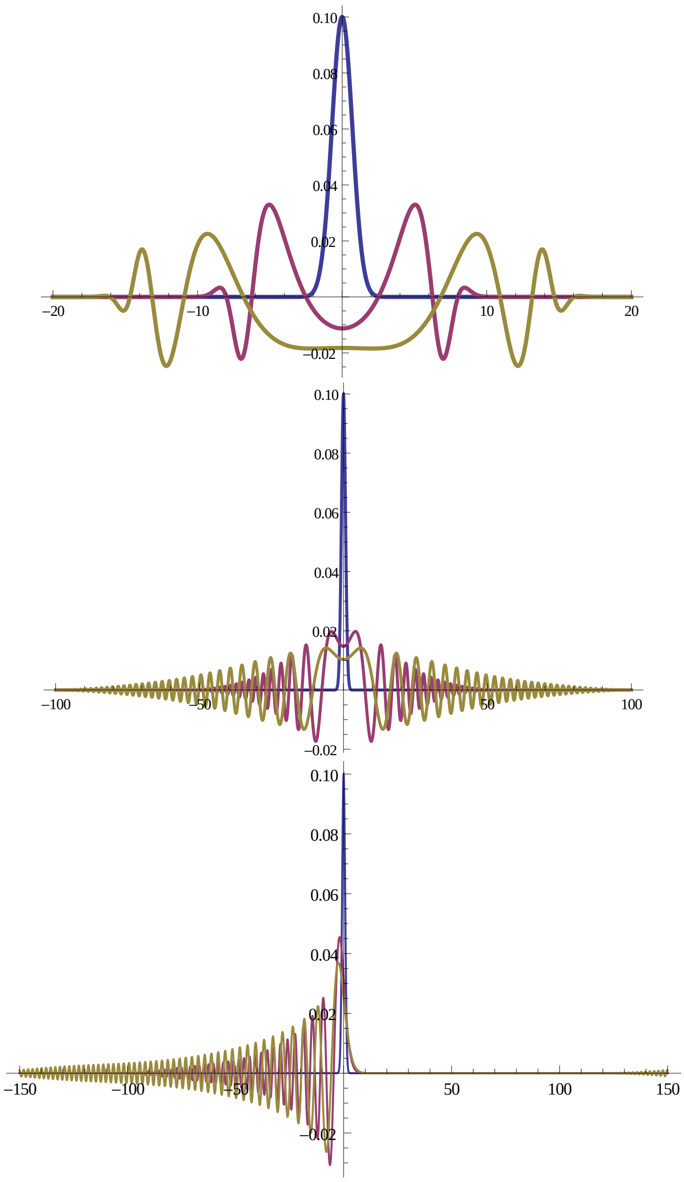
where

$$A(k) = \frac{1}{2}\left(\hat{u}_0(k) - i\frac{\hat{u}_{0t}(k)}{\sqrt{k^2+1}}\right). \tag{15}$$

For $x/t < 1$ (inside the light cone) and $t >> 1$:

$$u \sim \frac{1}{\sqrt{2\pi t}}\left(1 - \left(\frac{x}{t}\right)^2\right)^{-3/4} A\left(-\frac{x}{\sqrt{t^2-x^2}}\right) e^{i\sqrt{t^2-x^2}+i\pi/4} + c.c. \tag{16}$$





Figures 1. Three time steps ($t = 0$, $t = T/2$, $t = T$) of the evolution of a gaussian initial condition according to, respectively, the wave, the Klein-Gordon, the linear Schrödinger, and the linear KdV equations (numerical solution).

3) Study the longtime behavior, for $t >> 1,\ x/t = O(1)$, of the Fourier integral

$$u(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{u}_0(k) e^{i(kx-\omega(k)t)} dk \tag{17}$$

under the hypothesis that there exists a unique stationary phase point $k_0(x/t) \in \mathbb{R}$, and that $\omega''(k_0) = 0,\ \omega'''(k_0) \neq 0$.

4) Given the linear PDE $\mathcal{P}(\partial_t, \nabla_{\vec{x}})u(\vec{x},t) = 0,\ \vec{x} \in \mathbb{R}^n,\ t \in \mathbb{R}$ in $(n+1)$ dimensions, with $u \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$,
i) show that the solution of its Cauchy problem:

$$\mathcal{P}(\partial_t, \nabla_{\vec{x}})u(\vec{x},t) = 0, \quad u(\vec{x},0) = u_0(\vec{x}) \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n) \tag{18}$$

is given by the Fourier integral:

$$\begin{array}{l} u(\vec{x},t) = \int_{\mathbb{R}^n} \hat{u}_0(\vec{k}) e^{i(\vec{k}\cdot\vec{x}-\omega(\vec{k})t)} \frac{d\vec{k}}{(2\pi)^n} \\ \hat{u}_0(\vec{k}) = \int_{\mathbb{R}^n} u_0(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} d\vec{x} \end{array} \tag{19}$$

ii) Show that, under the hypothesis that the vector equation for $\vec{k}$

$$\frac{\vec{x}}{t} = \nabla_{\vec{k}}\omega(\vec{k}) \tag{20}$$

admits a unique real solution $\vec{k}_0 = \vec{k}_0(\vec{x}/t) \in \mathbb{R}^n$, the extension of the stationary phase method for multiple integrals gives the following longtime behavior:

$$\begin{array}{l} u \sim \left(\frac{1}{2\pi t}\right)^{n/2} \left(det \left(\frac{\partial^2 \omega(\vec{k}_0)}{\partial k_i \partial k_j} \right) \right)^{-1/2} \hat{u}_0(\vec{k}_0) e^{i(\vec{k}_0\cdot\vec{x}-\omega(\vec{k}_0)t + m\frac{\pi}{4})}, \\ m \equiv -\sum\limits_{j=1}^{n} sgn(\lambda_j) \end{array} \tag{21}$$

where $\lambda_j,\ j = 1,..,n$ are the (real) eigenvalues of matrix $\left(\frac{\partial^2 \omega(\vec{k}_0)}{\partial k_i \partial k_j}\right)$.

5) Let $\Gamma(z)$ be the Euler Γ function:

$$\Gamma(z) = \int\limits_0^\infty e^{-t} t^{z-1} dt, \quad Re\ z > 0. \tag{22}$$

i) Show that it is the generalization of the factorial: $\Gamma(n+1) = n!,\ n \in \mathbb{N}$.
ii) Use the Laplace method to construct the Stirling formula:

$$n! = n^n e^{-n} \sqrt{2\pi n} \left(1 + O(n^{-1})\right), \quad n >> 1. \tag{23}$$

2.1.2 Onde iperboliche e la catastrofe del gradiente[15, 9, 7, 17, 8]

1) Show that the following linear PDE:

$$\rho_t + c(x,t)\rho_x + a(x,t)\rho = b(x,t) \tag{24}$$

is equivalent to the system of two ODEs

$$\begin{array}{l} \frac{d\rho}{dt} + a(x,t)\rho = b(x,t), \\ \frac{dx}{dt} = c(x,t) \end{array} \tag{25}$$

on the characteristic curve $dx/dt = c(x,t)$.

2) Find the general solution of the following linear PDEs:

$$\begin{array}{l} u_t + t^2 u_x + xu = 0, \quad \left(u = F(x - t^3/3)e^{-(t^4/12 + t(x - t^3/3))}\right), \\ xu_x + yu_y = 0, \quad (u = F(y/x)), \\ xu_x + yu_y = x^2, \quad \left(u = x^2/2 + F(y/x)\right), \\ xu_x + yu_y = u, \quad (u = xF(y/x)), \\ xu_x + yu_y + zu_z = 0, \quad (u = F(y/x, z/x)), \\ g_y u_x - g_x u_y = 0, \quad g(x,y) \text{ given}, \quad (u = F(g(x,y))) \end{array} \tag{26}$$

3) Find the general solution of the following quasi-linear PDEs:

$$\begin{array}{l} i) \;\; u_t + c(u)u_x = 0, \quad u = F(x - c(u)t), \\ ii) \;\; u_t + c(u)u_x = 1, \\ c(u) = u \quad \Rightarrow \quad u = t + F(x - ut + t^2/2), \\ c(u) = u^2 \quad \Rightarrow \quad u = t + F(x - u^2 t + ut^2 - t^3/3) \end{array} \tag{27}$$

4) Given the two Cauchy problems for the Hopf equation:

$$\begin{array}{l} u_t + uu_x = 0, \quad u = u(x,t), \quad x \in \mathbb{R}, \; t \geq 0, \\ i) \;\; u(x,0) = e^{-x^2}, \\ ii) \;\; u(x,0) = (x^2+1)^{-1}, \end{array} \tag{28}$$

i) draw the 1-parameter family of characteristic curves; ii) find the first characteristic parameter ζ_b and the first breaking point (x_b, t_b).

5) Compression and rarefaction waves Consider the Cauchy problem:

$$\begin{array}{l} u_t + uu_x = 0, \quad u = u(x,t), \quad x \in \mathbb{R}, \; t \geq 0, \\ u(x,0) = a_2 H(-1-x) + a_1 H(x-1) + H(1-x^2)\left(\frac{a_1+a_2}{2} - \frac{a_2-a_1}{2}x\right), \end{array} \tag{29}$$

in the two cases

$$\begin{array}{l} i)\ \ a_2 > a_1 > 0, \\ ii)\ \ a_1 > a_2 > 0. \end{array} \tag{30}$$

Solve it explicitely, draw the characteristic curves and show that they describe respectively a compression and a rarefaction wave. Indicate if there is wave breaking and, if so, find ζ_b and (x_b, t_b).

6) Consider the Cauchy problem

$$\begin{array}{l} u_t + uu_x = 0, \\ u(x,0) = f(x), \end{array} \tag{31}$$

where f describes a single bump, and study the behavior of the solution near breaking.

7) Given the following system of PDEs, establish if they are hyperbolic and, if so, write them in characteristic form.

i) The wave equation $u_{tt} - c^2 u_{xx} = 0$.
ii) The Klein - Gordon equation $u_{tt} - c^2 u_{xx} + u = 0$.
iii) The system

$$\begin{array}{l} u_t + c(u,v)u_x = 0, \\ v_t + c(u,v)v_x = u \end{array} \tag{32}$$

iv) The system

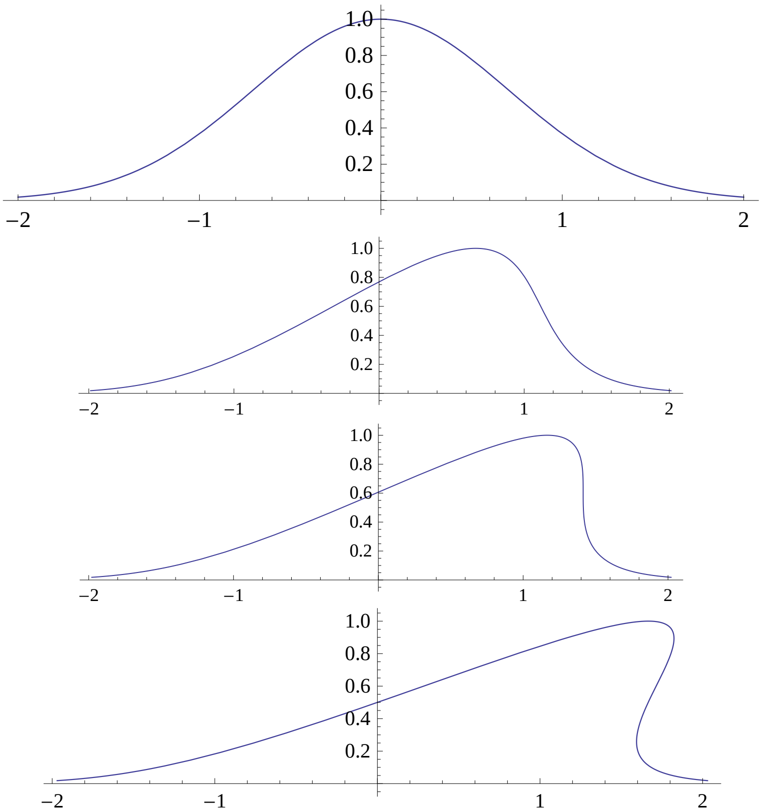
$$\begin{array}{l} u_t + c(u)u_x = 0, \\ v_t + c(u)v_x + c'(u)vu_x = 0 \end{array} \tag{33}$$

v) The gas dynamics equations

$$\begin{array}{l} \rho_t + u\rho_x + \rho u_x = 0, \\ u_t + uu_x + \frac{p_x}{\rho} = 0, \\ S_t + uS_x = 0, \end{array} \tag{34}$$

where $p = p(\rho, S)$.

8) Find the Riemann invariants of the wave equation $u_{tt} - c^2 u_{xx} = 0$ and of the gas dynamics equations (34) (under the constant entropy S hypothesis).



Figures 2. The evolution of a gaussian according to the Hopf equation (numerical inversion of the analytic solution).

2.1.3 Il problema della regolarizzazione

1) Regularize the compression wave of problem 5) of section 2.1.2

2) What happens if we look for discontinuous solutions of $u_t + uu_x = 0$ in the form $u = H(s(t) - x)u^-(x,t) + H(x - s(t))u^+(x,t)$, where $H(x)$ is the Heaviside step function and $u^\pm(x,t)$ are smooth functions?

3) Consider the Cauchy problem

$$\begin{aligned} &u_t + uu_x = 0, \\ &u(x,0) = f(x), \end{aligned} \tag{35}$$

where $f(x)$ describes a single bump, and study the behavior of the regularized (shock) solution near breaking.

4) Given the Cauchy problem

$$\begin{aligned} & u_t + c(u)u_x = 0, \quad c(u) = Q'(u), \\ & u(x,0) = f(x), \end{aligned} \tag{36}$$

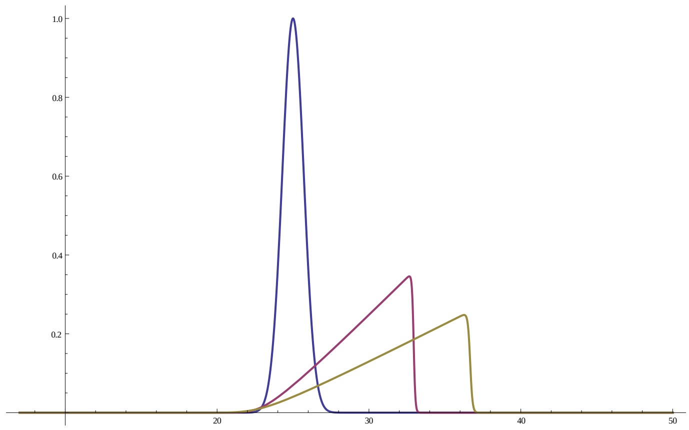
where $f(x)$ describes a single bump,
i) show that the shock condition

$$\dot{s} = \frac{Q(u_2) - Q(u_1)}{u_2 - u_1} \tag{37}$$

is equivalent of placing the shock in order to cut equal area lobi of the three valued solution.
ii) Show that, if $c(u) = u$, $Q(u) = u^2/2$, the shock equations involving $s(t), \eta_1(t), \eta_2(t)$ can be reformulated as cutting equal area lobi on the initial profile:

$$\int_{\eta_1}^{\eta_2} f(\eta)d\eta = \frac{1}{2}(\eta_1 - \eta_2)(f(\eta_1) + f(\eta_2)) \tag{38}$$

5) Given the Burgers equation $u_t + uu_x = \nu u_{xx}$, i) find its traveling wave solution satisfying the boundary conditions $u(x,t) \to u_\pm$, $x \to \pm\infty$, where $u_\pm$ are constants, and discuss the shock structure. ii) Find its similarity solutions.



Figures 3. Three time steps ($t = 0$, $t = T/2$, $t = T$) of the evolution of a gaussian initial condition according to the Burgers equation with small dissipation (numerical solution).

2.2 La propagazione ondosa in Natura, il metodo multiscala e le equazioni modello [4, 8, 1, 3, 14]

1) Consider the two anharmonic oscillators

$$\begin{array}{l} \ddot{q}+q-\frac{\epsilon}{6}q^3=0, \quad \text{cubic pendulum}, \quad 0<\epsilon<<1, \\ \ddot{q}+q+\epsilon\dot{q}^3=0, \quad \text{with nonlinear friction} \end{array} \tag{39}$$

with the same initial conditions

$$q(0)=1, \quad \dot{q}(0)=0. \tag{40}$$

Use the multiscale method to show that

$$\begin{array}{l} q(t)=\cos\left(t-\frac{1}{16}\varepsilon t\right)+O(\varepsilon), \\ q(t)=\left(1+\frac{3}{4}\varepsilon t\right)^{-1/2}\cos t+O(\varepsilon) \end{array} \tag{41}$$

2) Use the multiscale method to construct the solution

$$q(t)=\frac{a_0 e^{\varepsilon t/2}}{\sqrt{1+\left(\frac{a_0}{2}\right)^2\left(e^{\varepsilon t}-1\right)}}\cos(t+\phi_0)+O(\varepsilon) \tag{42}$$

of the Van Der Pol oscillator

$$\ddot{q}+q-\varepsilon(1-q^2)\dot{q}=0, \tag{43}$$

and show that

$$q(t)\to 2\cos(t+\phi_0), \quad t\to\infty. \tag{44}$$

3) Derive the dKP(3,1) equation from the equations of Acoustics, under the hypothesis of i) weak nonlinearity and ii) quasi one-dimensionality.

4) Derive the KdV equation (see [1, 3]) in the context of surface water wave in $(1+1)$ dimensions, under the hypothesis of i) small amplitudes and ii) shallow water ($kh<<1$, where k is the wave number and h is the depth of the fluid). Derive the KP equation (see [2, 3]) in the context of surface water waves in $(2+1)$ dimensions, under the hypothesis of i) small amplitudes, ii) shallow water, and iii) quasi one-dimensionality. Show that, neglecting dispersion, one obtains the dKP(2,1) equation.

5) Derive (see [3]) the NLS equation in the context of surface water waves in $(1+1)$ dimensions, under the hypothesis of i) small amplitude ($a<<\lambda$)

and ii) quasi monocromatic waves in sufficiently deep water. Derive its multidimensional generalization in the context of surface water waves in $(2+1)$ dimensions, under the hypothesis of

6) Derive (see [18]) the NLS equation in the framework of Langmuir waves in a plasma, described by the system of equations:

$$n_t + (nv)_x = 0, \quad v_t + vv_x = \phi_x - n_x/n, \quad \phi_{xx} = n - 1,$$

with boundary onditions $n \to 1,\ v \to 0,\ \phi \to 0$ as $|x| \to \infty$, where n is the electron density, v is the electron velocity and ϕ is the electrostatic potential in dimensionless variables, expanding the fields around the equilibrium solution:

$$n = 1 + \varepsilon n_1 + \varepsilon^2 n_2 + O(\varepsilon^3), \quad v = \varepsilon v_1 + O(\varepsilon^2), \quad \phi = \varepsilon \phi_1 + O(\varepsilon^2).$$

7) Derive (see [8]) the NLS equation in nonlinear optics, for a homogeneous and isotropous dielectric.

2.3 La teoria dei solitoni

1) Analyticity projectors. Show that the operators

$$P^{\pm} f(\lambda) := \pm \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(\lambda')}{\lambda' - (\lambda \pm i\varepsilon)} d\lambda. \tag{45}$$

are analyticity projectors on the real line; i.e., they map a Holder function $f(\lambda),\ \lambda \in \mathbb{R}$ decaying at ∞ sufficiently fast into functions analytic in the upper and lower halves of the complex λ plane respectively. ii) Show, in particular, that

$$(P^+)^2 = P^+,\ (P^-)^2 = P^-,\ P^+P^- = P^-P^+ = 0,\ P^+ + P^- = 1. \tag{46}$$

2) Given an Holder function $f(\lambda)$ for $\lambda \in \mathbb{R}$ decaying at ∞ sufficiently fast, a polynomial $P(\lambda)$, a set of complex numbers $\{k_j^+,\ R_j^+,\ j = 1, \dots, N^+,$ $k_j^-,\ R_j^-,\ j = 1, \dots, N^-\}$, where Im $k_j^+ > 0$ and Im $k_j^- < 0$, show that the unique solution of the Riemann problem

$$\psi^+(\lambda) - \psi^-(\lambda) = f(\lambda), \quad \lambda \in \mathbb{R} \tag{47}$$

where $\psi^{\pm}(\lambda)$ are analytic in the upper and lower halves of the complex λ plane respectively, except for the simple poles $k_j^{\pm}$'s with residues $R_j^{\pm}$'s, and $\psi^{\pm}(\lambda) \to P(\lambda),\ |\lambda| >> 1$, is

$$\psi^{\pm}(\lambda) = P(\lambda) + \sum_{j=1}^{N^+} \frac{R_j^+}{\lambda - k_j^+} + \sum_{j=1}^{N^-} \frac{R_j^-}{\lambda - k_j^-} \pm P^{\pm} f(\lambda). \tag{48}$$

3) Let $u(x) = -A\delta(x - x_0),\ A \in \mathbb{R}$, be the potential of the Schrödinger equation $[-\partial_x^2 + u(x)]\psi = k^2\psi$. Evaluate explicitly: i) the eigenfunctions of the continuous spectrum and the coefficients $a(k), b(k), R(k), T(k)$; ii) the discrete spectrum p_j, the corresponding eigenfunctions and the norming constants b_j. Show that the existence of discrete spectrum depends on the sign of A.

4) Assume $u(x) = O(\varepsilon),\ \varepsilon << 1$, and construct the first two terms of the ε - expansion of the eigenfunctions and of the spectral data.

5) *Scattering problem.* Study the scattering problem described by the Schrödinger equation

$$-\psi''(x,k) + u(x)\psi(x,k) = k^2\psi(x,k), \quad x \in \mathbb{R}, \quad k > 0,$$

where $\psi(x,k)$, the eigenfunction of the continuous spectrum of the Schrödinger operator $-d^2/dx^2 + V(x)$, represents the wave function of a particle beam scattered by the localized potential $u(x)$ e $E = k^2 > 0$ is the energy of the beam (the continuous spectrum $\sigma_c = \{E > 0\}$), with the following boundary conditions:

$$\psi(x,k) \sim\ R(k)e^{-ikx} + e^{ikx},\ x \sim -\infty; \qquad \psi(x,k) \sim\ T(k)e^{ikx},\ x \sim \infty$$

describing an incoming beam of particles of wave number k and intensity 1, partially reflected and transmitted through the potential ($R(k)$ e $T(k)$ are respectively the reflection and transmission coefficients).
i) Observe that the function $\phi(x,k) = \psi(x,k)/T(k)$ satisfies a simpler scattering problem:

$$\phi''(x,k) + k^2\phi(x,k) = u(x)\phi(x,k), \quad x \in \mathbb{R},, \quad k > 0$$

$$\phi(x,k) \sim \frac{R(k)}{T(k)}e^{-ikx} + \frac{e^{ikx}}{T(k)},\ x \sim -\infty; \quad \phi(x,k) \sim\ e^{ikx},\ x \sim \infty$$

and use the advanced Green function of the operator $d^2/dx^2 + k^2$ to rewrite such a problem as a Volterra integral equation [5], obtaining:

$$\phi(x,k) = e^{ikx} - \int_x^\infty dy \frac{\sin k(x-y)}{k} u(y)\phi(y,k)$$

and the following integral representations for the reflection and trnsmission coefficients:

$$\frac{1}{T(k)} = 1 - \int_{\mathbb{R}} dk \frac{e^{-iky}}{2ik} u(y)\phi(y,k), \quad \frac{R(k)}{T(k)} = \int_{\mathbb{R}} dk \frac{e^{iky}}{2ik} u(y)\phi(y,k).$$

Such an integral equation, equivalent to the Schrödinger differential equation + boundary conditions, is the most convenient formulation of the problem to extract informations.

ii) Use the method of successive approximations to study the properties of ϕ in the following way.

a) Rerwrite the integral equation for the unknown $f(x,k) = \phi(x,k)e^{-ikx}$, such that $f \sim 1, \ x \to \infty$:

$$f(x,k) = 1 + \int_x^\infty \frac{e^{2ik(y-x)} - 1}{2ik} u(y) f(y,k) dy$$

and look for the solution as a Neumann series:

$$f(x,k) = \sum_{i=0}^{\infty} h_i(x,k), \quad h_0 = 1, \tag{49}$$

obtaining the recursion relation:

$$h_{j+1}(x,k) = \int_x^\infty \frac{e^{2ik(y-x)} - 1}{2ik} u(y) h_j(y,k) dy, \quad j \geq 0. \tag{50}$$

b) From the inequality: $|e^{2ik(y-x)} - 1|/|2ik| \leq 1$, valid for $\text{Im}\, k \geq 0,\ k \neq 0$, show that

$$|h_{j+1}(x,k)| \leq \frac{1}{|k|} \int_x^\infty |u(y)||h_j(y,k)| dy, \tag{51}$$

and then that:

$$\begin{aligned} &|h_n(x,k)| \leq \frac{1}{n!}\left(\frac{A(x)}{|k|}\right)^n \leq \frac{1}{n!}\left(\frac{A(-\infty)}{|k|}\right)^n, \\ &A(x) := \int_x^\infty |V(y)| dy. \end{aligned} \tag{52}$$

Therefore the Neumann series representing the solution is absolutely and uniformely convergent for Im $k \geq 0,\ k \neq 0$, if $u(x) \in L_1(\mathbb{R})$. Under these conditions, the solution exists unique, and it is analytic in the upper half of the complex k plane. Analogously one can prove that $1/T(k)$ is analytic in the upper half of the complex k plane. Under more stringent conditions on u, one could show, in a similar manner, that the eigenfunction is also continuous on the real k axes, where the physics takes place.

c) Let $k_j,\ j = 1, .., N$ be the zeroes of the function $1/T(k)$ in the upper half of the complex k plane (the poles of the transmission coefficient). Then, since $\lambda_j = E_j = k_j^2 \in \mathbb{R}$, it follows that a) k_j is purely imaginary: $k_j = ip_j,\ p_j > 0,\ j = 1, .., N$, b) the functions $\phi(x, k_j),\ j = 1, .., N$ are exponentially localized:

$$\phi_j(x) := \phi(x, k_j) = O(e^{-p_j|x|}), \quad |x| \to \infty, \quad j = 1, ..N$$

and then they are eigenfunctions of the Schrödinger operator in $L_2(\mathbb{R})$:

$$-\phi_j''(x) + u(x)\phi_j(x) = -p_j^2\phi_j(x), \quad x \in \mathcal{R}$$

corresponding to negative eigenvalues $\lambda_j = E_j = -p_j^2 < 0$ of the energy (the discrete spectrum: $\sigma_p = \{-p_j^2\}_1^N$). Summarizing: $\sigma = \sigma_p \cup \sigma_c = \{-p_j^2\}_1^N \cup \mathbb{R}^+$.

d) Show that the set of $\lambda_j = -p_j^2,\ j = 1, .., N$ bounded from below.

Hint. Take the scalar product of the eigenfunction ϕ_j, normalized to 1, with the Schrödinger equation, obtaining:

$$\lambda_j - (\phi_j, u\phi_j) = (\phi_j', \phi_j') \geq 0 \quad \Rightarrow \quad |\lambda_j| \leq -(\phi_j, V\phi_j) \leq |(\phi_j, u\phi_j)| \leq ||u||_\infty.$$

e) Show that, if $u(x) = u_0\delta(x - x_0)$, the integral equation admits the solution

$$\phi(x, k) = e^{ikx} - u_0 H(x_0 - x)\frac{\sin k(x - x_0)}{k}e^{ikx_0}.$$

Then:

$$\phi(x, k) = \frac{2ik - u_0}{2ik}e^{ikx} + \frac{u_0 e^{2ikx_0}}{2ik}e^{-ikx}, \quad x < x_0$$

$$T(k) = \frac{2ik}{2ik - u_0}, \qquad R(k) = \frac{u_0 e^{2ikx_0}}{2ik - u_0}.$$

Found $\phi(x, k)$, at last reconstruct $\psi(x, k) = \frac{2ik}{2ik - u_0}\phi(x, k)$.

f) Verify that the solution we found for $k \in \mathbb{R}$, if extended outside the real k axis, diverges always at + or - infinity, unless $k = -iu_0/2 \in i\mathbb{R}^+$. Therefore, if the potential is positive ($u_0 > 0$), no eigenfunctions exist in $L_2(\mathbb{R})$; if, instead, the potential is negative, then there exists one and only one $L_2(\mathbb{R})$ eigenfunction $\psi_1(x) := \phi(x, i|u_0|/2) \in L_2(\mathbb{R})$:

$$\psi_1(x) = H(x - x_0)e^{-\frac{|u_0|}{2}x} + H(x_0 - x)e^{\frac{|u_0|}{2}x}$$

corresponding to the negative energy $E_1 = k_1^2 = -u_0^2/4$, and describing a bound state (a localized quantum particle): $\sigma_p = \{E_1\}$.
g) If $u(x) = \epsilon v(x),\ \epsilon << 1$, show that:

$$\phi(x,k) = e^{ikx} - \epsilon \int_x^\infty dy \frac{\sin k(x-y)}{k} v(y) e^{iky} + O(\epsilon^2),$$

$$T(k) = 1 + \frac{\epsilon}{2ik}\int_{\mathcal{R}} dx v(x) + O(\epsilon^2), \quad R(k) = \frac{\epsilon}{2ik}\int_{\mathcal{R}} dx v(x) e^{-2ikx} + O(\epsilon^2)$$

6) Using the above strategy, study the scattering problem

$$\phi''(x,k) + k^2\phi(x,k) = u(x)\phi(x,k), \quad x \in \mathcal{R}, \qquad \phi(x,k) \sim\ e^{-ikx},\ x \sim -\infty$$

showing that, in this case, it is convenient to use the retarded Green function of the operator $d^2/dx^2 + k^2$.

2.4 Equazioni non lineari integrabili di tipo idrodinamico e la rottura di onde multidimensionali

2.4.1 Campi vettoriali commutanti ed equazioni integrabili di tipo idrodinamico

2.4.2 Trasformata spettrale per campi vettoriali

2.4.3 Come si rompono onde quasi - unidimensionali in Natura

1) Given the dKP_n equation:

$$\begin{aligned} &(u_t + uu_x)_x + \Delta_\perp u = 0, \quad u = u(x, \vec{y}, t), \quad \vec{y} = (y_1, \ldots, y_{n-1}) \\ &\Delta_\perp = \sum_{i=1}^{n-1} \partial_{y_i}^2, \quad n \geq 2, \end{aligned} \tag{53}$$

i) show that it is invariant under motions on the associated paraboloid

$$x + \frac{1}{4t}\sum_{i=1}^{n-1} y_i^2 = \xi. \tag{54}$$

ii) Use such invariance to look for particular solutions in the form

$$u = v(\xi, t), \quad \xi = x + \frac{1}{4t}\sum_{i=1}^{n-1} y_i^2, \tag{55}$$

obtaining the exact (but implicit) solution

$$u = \begin{cases} t^{-\frac{n-1}{2}} F\left(x + \frac{1}{4t}\sum_{i=1}^{n-1} y_i^2 - \frac{2ut}{3-n}\right), & n \neq 3, \\ t^{-1} F\left(x + \frac{1}{4t}\sum_{i=1}^{n-1} y_i^2 - u\, t \ln t\right), & n = 3, \end{cases} \tag{56}$$

where F is an arbitrary function of a single variable. What kind of wave is described by this solution?

2) Write the Fourier representation of the solution of the Cauchy problem for the linearized dKP equation:

$$\begin{array}{l} u_{xt} + \Delta_\perp u = 0, \\ u(x, \vec{y}, 0) = u_0(x, \vec{y}) \end{array} \tag{57}$$

and show that, for $t >> 1$, the solution reads

$$u(x, \vec{y}, t) \sim t^{-\frac{n-1}{2}} G\left(x + \frac{1}{4t}\sum_{i=1}^{n-1} y_i^2, \frac{\vec{y}}{2t}\right), \tag{58}$$

where

$$G(\xi, \vec{\eta}) := 2^{-n}\pi^{-\frac{n+1}{2}} \int_{\mathbb{R}} d\lambda |\lambda|^{\frac{n-1}{2}} \hat{u}_0(\lambda, \lambda\vec{\eta}) e^{i\lambda\xi - i\frac{\pi}{4}(n-1)\,\mathrm{sign}\,\lambda}, \tag{59}$$

in the space-time region

$$(x - \xi)/t, \; y_i/t = O(1), \quad i = 1, \ldots, n, \tag{60}$$

on the paraboloid (54). Outside the paraboloid, the solution decays faster.

3)

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