1 ONDE NON LINEARI E SOLITONI

Prof. Paolo Maria Santini

Corso da 6 CFU della Laurea Magistrale; II Semestre, AA 2014-15

2 Esercizi proposti (raccolta provvisoria)

2.1 Propagazione ondosa lineare e non lineare

2.1.1 Onde dispersive lineari [1, 5]

1) Given the Cauchy problem

$$u_t + i\omega(-i\partial_x)u = 0, \quad u(x,0) \text{ given}, \;\; x \in \mathbb{R}, \quad t \geq 0, \tag{1}$$

1. show that the Fourier integral representation of its solution is

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}_0(k) e^{i(kx-\omega(k)t)} dk, \tag{2}$$

where $\hat{u}_0(k)$ is the Fourier transform of the initial condition $u(x,0)$:

$$\hat{u}_0(k) = \int_{\mathbb{R}} e^{-iky} u(y,0) dy. \tag{3}$$

2. Show that (2) can be written as a convolution integral, in the suggestive form:

$$u(x,t) = \int_{\mathbb{R}} S(x-y,t) u(y,0) dy, \tag{4}$$

where $S(x,t)$ is the "fundamental" solution of the PDE, defined as:

$$S(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(kx-\omega(k)t)} dk. \tag{5}$$

3. If $\omega(k) = k^n$, then $S(x,t)$ is the following similarity solution of the PDE:

$$\begin{array}{l} S(x,t) = \frac{1}{t^{1/n}} f\left(\frac{x}{t^{1/n}}\right), \\ f(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(k\xi - k^n)} dk. \end{array} \tag{6}$$

4. Show that, if $u \in \mathbb{R}$, then:

$$\begin{array}{l} \overline{\hat{u}_0(k)} = \hat{u}_0(-k), \quad k \in \mathbb{R} \\ u(x,t) = \frac{1}{\pi} Re \int\limits_0^\infty \hat{u}_0(k) e^{i(kx-\omega(k)t)} dk \end{array} \tag{7}$$

If, in addition, $\hat{u}_0(k)$ can be prolongued outside the real axis, then

$$\overline{\hat{u}_0(k)} = \hat{u}_0(-\bar{k}). \qquad (8)$$

(for the second of (7) we have also assumed that $\omega(k)$ is odd: $\omega(-k) = -\omega(k)$)

2) Given the following linear PDEs:

$$\begin{array}{l} i)\ \ iu_t + u_{xx} = 0, \quad \text{free particle Schrödinger equation}, \\ ii)\ \ u_t + u_{xxx} = 0, \quad \text{linearized KdV equation}, \\ iii)\ \ u_{tt} - u_{xx} + u = 0, \quad \text{Klein - Gordon equation}, \end{array} \qquad (9)$$

1. Construct the fundamental similarity solution (6) (only for i) and ii)).
2. Study the longtime behavior, for $t >> 1,\ x/t = O(1)$, of the solutions of their Cauchy problem using the stationary phase, Laplace, or saddle point methods, depending on the situation, and estimate the error.
3. Study of the relevance of the exact similarity solution in the longtime behavior (only for i) and ii)).

Solution:

i) Free particle Schrödinger equation:

$$\begin{array}{l} S(x,t) = \frac{1}{2\sqrt{\pi t}} e^{i(\frac{x^2}{4t} - \frac{\pi}{4})}, \\ u(x,t) = S(x,t)\left(A(\xi) + \frac{1}{t}B(\xi) + O(t^{-2})C(\xi)\right), \quad \xi = \frac{x}{2t} = O(1), \quad t >> 1 \\ A(\xi) = \hat{u}_0(\xi), \quad B(\xi) = -\frac{i}{4}A_{\xi\xi} \end{array} \qquad (10)$$

ii) Linear KdV. For $x/t > 0$, the lines of constant $v(k)$ are the imaginary axis and the hyperbola ${k_R}^2 - 3{k_I}^2 + x/t = 0$. The steepest descent contour passing through the critical point $i\sqrt{\frac{x}{3t}}$ is the upper branch of the hyperbola, while the steepest descent contour passing through the critical point $-i\sqrt{\frac{x}{3t}}$ is the imaginary axis. The asymptotics is obtained replacing the integration real line by the steepest descent contour passing through $i\sqrt{\frac{x}{3t}}$.

$$\begin{array}{l} S(x,t) = \frac{1}{(3t)^{1/3}} Ai\left(\frac{x}{(3t)^{1/3}}\right), \\ u(x,t) \sim \frac{\hat{u}_0(|x/3t|^{1/2})}{\sqrt{4\pi|3x/t|^{1/2}t}} e^{-i2|x/3t|^{3/2}t + i\pi/4} + \text{c.c.}, \quad \frac{x}{3t} = O(1) < 0, \quad t >> 1, \\ u(x,t) \sim \frac{\hat{u}_0(i|x/3t|^{1/2})}{\sqrt{12\pi|3x/t|^{1/2}t}} e^{-2|x/3t|^{3/2}t}, \quad \frac{x}{3t} = O(1) > 0, \quad t >> 1, \\ u(x,t) \sim \frac{\hat{u}_0(0)}{2\pi(3t)^{1/3}} Ai\left(\frac{x}{(3t)^{1/3}}\right) - \frac{i\hat{u}_0'(0)}{2\pi(3t)^{2/3}} Ai'\left(\frac{x}{(3t)^{1/3}}\right), \quad \frac{x}{(3t)^{1/3}} = O(1), \quad t >> 1, \\ u(x,t) \sim \frac{\hat{u}_0(0)}{2\pi} S(x,t), \quad \frac{x}{(3t)^{1/3}} = O(1), \quad t >> 1, \end{array} \qquad (11)$$

where $A_i(\xi)$ is the Airy function

$$A_i(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(k\xi - k^3)} dk, \tag{12}$$

solution of the ODE: $f(\xi)'' - \xi f(\xi) = 0$.
iii) Klein-Gordon equation. The dispersion relation is two-valued (since the PDE is second order in t):

$$\omega^{\pm}(k) = \pm\sqrt{k^2+1}; \tag{13}$$

therefore the phase velocity is greater than the light speed 1, while the group velocity is less than 1:

$$\frac{\omega}{k} = \frac{\sqrt{k^2+1}}{k} > 1, \quad \frac{d\omega}{dk} = \frac{k}{\sqrt{k^2+1}} < 1 \tag{14}$$

The Fourier representation of the real solution reads:

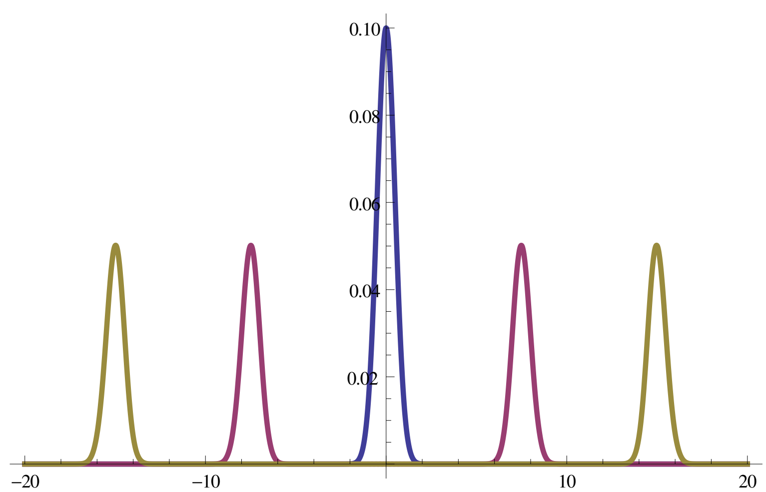
$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} A(k) e^{i(kx+\sqrt{k^2+1}t)} dk + \frac{1}{2\pi}\int_{\mathbb{R}} \overline{A(-k)} e^{i(kx-\sqrt{k^2+1}t)} dk, \tag{15}$$

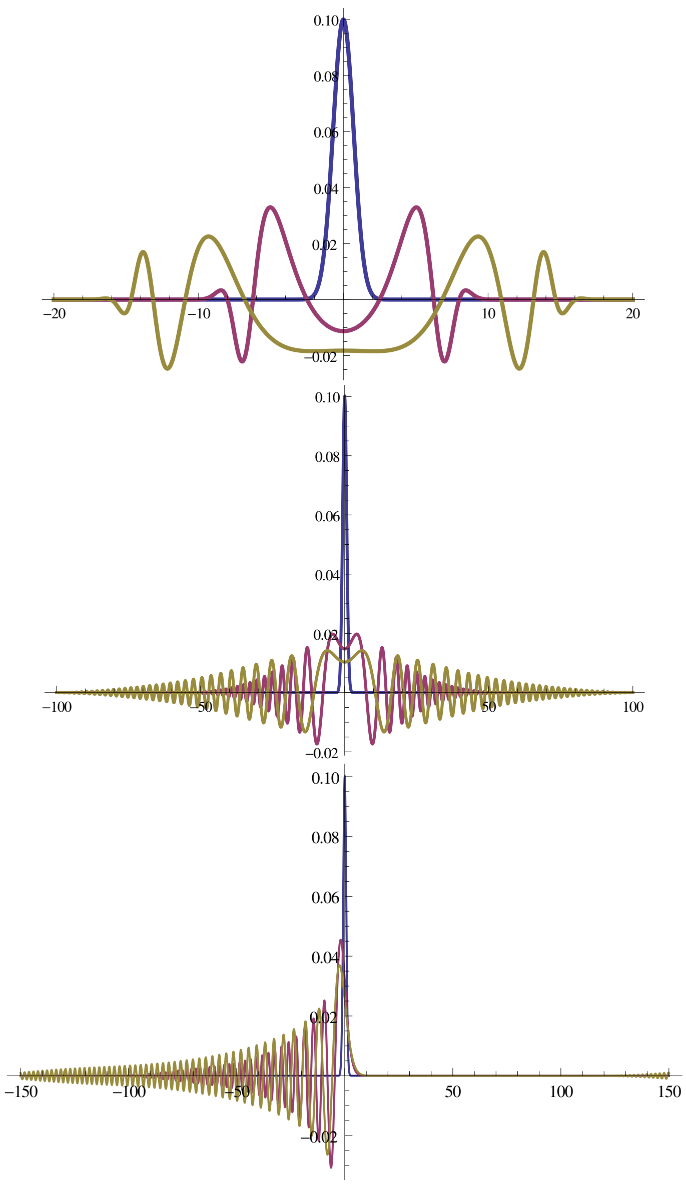
where

$$A(k) = \frac{1}{2}\left(\hat{u}_0(k) - i\frac{\hat{u}_{0t}(k)}{\sqrt{k^2+1}}\right). \tag{16}$$

For $|x/t| < 1$ (inside the light cone) and $t >> 1$:

$$u \sim \frac{1}{\sqrt{2\pi t}}\left(1 - \left(\frac{x}{t}\right)^2\right)^{-3/4} A\left(-\frac{x}{\sqrt{t^2-x^2}}\right) e^{i\sqrt{t^2-x^2}+i\pi/4} + c.c. \tag{17}$$





Figures 1. Three time steps ($t = 0$, $t = T/2$, $t = T$) of the evolution of a gaussian initial condition according to, respectively, the wave, the Klein-Gordon, the linear Schrödinger, and the linear KdV equations (numerical solution).

3) Study the longtime behavior, for $t >> 1,\ x/t = O(1)$, of the Fourier integral

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}_0(k) e^{i(kx-\omega(k)t)} dk \tag{18}$$

under the hypothesis that there exists a unique stationary phase point $k_0(x/t) \in \mathbb{R}$, and that $\omega''(k_0) = 0,\ \omega'''(k_0) \neq 0$.

4) Given the linear PDE $\mathcal{P}(\partial_t, \nabla_{\vec{x}})u(\vec{x},t) = 0,\ \vec{x} \in \mathbb{R}^n,\ t \in \mathbb{R}$ in $(n+1)$ dimensions, with $u \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$,
i) show that the solution of its Cauchy problem:

$$\mathcal{P}(\partial_t, \nabla_{\vec{x}})u(\vec{x},t) = 0, \quad u(\vec{x},0) = u_0(\vec{x}) \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n) \tag{19}$$

is given by the Fourier integral:

$$\begin{array}{l} u(\vec{x},t) = \int_{\mathbb{R}^n} \hat{u}_0(\vec{k}) e^{i(\vec{k}\cdot\vec{x}-\omega(\vec{k})t)} \frac{d\vec{k}}{(2\pi)^n} \\ \hat{u}_0(\vec{k}) = \int_{\mathbb{R}^n} u_0(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} d\vec{x} \end{array} \tag{20}$$

where $\omega(\vec{k})$ is obtained solving the equation $\mathcal{P}(-i\omega, i\vec{k}) = 0$ wrt ω.
ii) Show that, under the hypothesis that the vector equation for $\vec{k}$

$$\frac{\vec{x}}{t} = \nabla_{\vec{k}}\omega(\vec{k}) \tag{21}$$

admits a unique real solution $\vec{k}_0 = \vec{k}_0(\vec{x}/t) \in \mathbb{R}^n$, the extension of the stationary phase method for multiple integrals gives the following longtime behavior:

$$\begin{array}{l} u \sim \left(\frac{1}{2\pi t}\right)^{n/2} \left(det\left(\frac{\partial^2\omega(\vec{k}_0)}{\partial k_i \partial k_j}\right)\right)^{-1/2} \hat{u}_0(\vec{k}_0) e^{i(\vec{k}_0\cdot\vec{x}-\omega(\vec{k}_0)t+m\frac{\pi}{4})}, \\ m \equiv -\sum\limits_{j=1}^{n} sgn(\lambda_j) \end{array} \tag{22}$$

where $\lambda_j,\ j = 1,..,n$ are the (real) eigenvalues of symmetric matrix $\left(\frac{\partial^2\omega(\vec{k}_0)}{\partial k_i \partial k_j}\right)$.

5) Let $\Gamma(z)$ be the Euler Γ function:

$$\Gamma(z) = \int\limits_0^\infty e^{-t} t^{z-1} dt, \quad Re\ z > 0. \tag{23}$$

i) Show that it is the generalization of the factorial: $\Gamma(n+1) = n!,\ n \in \mathbb{N}$.
ii) Use the Laplace method to construct the Stirling formula:

$$n! = n^n e^{-n}\sqrt{2\pi n}\left(1 + O(n^{-1})\right), \quad n >> 1. \tag{24}$$

2.1.2 Onde iperboliche e la catastrofe del gradiente[17, 9, 7, 19, 8]

1) Show that the following linear PDE for the field $\rho(x,t)$:

$$\rho_t + c(x,t)\rho_x + a(x,t)\rho = b(x,t) \tag{25}$$

is equivalent to the system of two ODEs for the fields $(\tilde{\rho}(t), \tilde{x}(t))$:

$$\begin{array}{l} \frac{d\tilde{\rho}}{dt} + a(\tilde{x},t)\rho = b(\tilde{x},t), \\ \frac{d\tilde{x}}{dt} = c(\tilde{x},t). \end{array} \tag{26}$$

2) Find the general solution of the following linear PDEs:

$$\begin{array}{l} u_t + t^2 u_x + xu = 0, \quad \left(u = F(x - t^3/3)e^{-(t^4/12 + t(x - t^3/3))}\right), \\ i\gamma u_t + y u_x - x u_y = 0, \quad (...), \\ y u_x - x u_y = 0, \quad \left(u = F(x^2 + y^2)\right), \\ y u_x + x u_y = 0, \quad \left(u = F(x^2 - y^2)\right), \\ x u_x + y u_y = 0, \quad \left(u = F(y/x)\right), \\ x u_x - y u_y = 0, \quad \left(u = F(xy)\right), \\ x u_x + y u_y = x^2, \quad \left(u = x^2/2 + F(y/x)\right), \\ x u_x + y u_y = u, \quad \left(u = xF(y/x)\right), \\ x u_x + y u_y + z u_z = 0, \quad \left(u = F(y/x, z/x)\right), \\ g_y u_x - g_x u_y = 0, \quad g(x,y) \text{ given}, \quad (u = F(g(x,y))) \end{array} \tag{27}$$

3) Find the general solution of the following quasi-linear PDEs:

$$\begin{array}{l} i) \;\; u_t + c(u)u_x = 0, \quad u = F(x - c(u)t), \\ ii) \;\; u_t + c(u)u_x = 1, \\ c(u) = u \quad \Rightarrow \quad u = t + F(x - ut + t^2/2), \\ c(u) = u^2 \quad \Rightarrow \quad u = t + F(x - u^2 t + ut^2 - t^3/3) \end{array} \tag{28}$$

4) Given the two Cauchy problems for the Hopf equation:

$$\begin{array}{l} u_t + u u_x = 0, \quad u = u(x,t), \quad x \in \mathbb{R}, \; t \geq 0, \\ i) \;\; u(x,0) = e^{-x^2}, \\ ii) \;\; u(x,0) = (x^2+1)^{-1}, \end{array} \tag{29}$$

i) draw the 1-parameter family of characteristic curves; ii) find the first characteristic parameter ζ_b and the first breaking point (x_b, t_b).
A. i) $\zeta_b = 1/\sqrt{2}, \; t_b = \sqrt{e/2}, \; x_b = \sqrt{2}$. ii) $\zeta_b = 1/\sqrt{3}, \; t_b = 8\sqrt{3}/9, \; x_b = \sqrt{3}$.

5) Compression and rarefaction waves.
Consider the Cauchy problem:

$$\begin{array}{l} u_t + uu_x = 0, \quad u = u(x,t), \quad x \in \mathbb{R}, \ t \geq 0, \\ u(x,0) = a_2 H(-1-x) + a_1 H(x-1) + H(1-x^2)\left(\frac{a_1+a_2}{2} - \frac{a_2-a_1}{2}x\right), \end{array} \tag{30}$$

in the two cases

$$\begin{array}{l} i) \ \ a_2 > a_1 > 0, \ \ \text{compression wave}, \\ ii) \ \ a_1 > a_2 > 0 \ \ \text{rarefaction wave}. \end{array} \tag{31}$$

Solve it explicitely, draw the characteristic curves and show that they describe respectively a compression and a rarefaction wave. Indicate if there is wave breaking and, if so, find ζ_b and (x_b, t_b).
A. For the compression wave:

$$u(x,t) = \left\{ \begin{array}{cc} a_2, & x < a_2 t - 1, \\ -\frac{a_2-a_1}{2} \frac{x - \frac{a_2+a_1}{2}t}{1 - \frac{a_2-a_1}{2}t} + \frac{a_2+a_1}{2}, & -1 + a_2 t < x < 1 + a_1 t, \\ a_1, & x > 1 + a_1 t. \end{array} \right. \tag{32}$$

There is wave breaking:

$$t_b = \frac{2}{a_2 - a_1}, \quad x_b = \frac{a_1 + a_2}{a_2 - a_1}, \quad |\zeta_b| < 1 \tag{33}$$

6) Consider the Cauchy problem

$$\begin{array}{l} u_t + uu_x = 0, \\ u(x,0) = f(x), \end{array} \tag{34}$$

where f describes a single bump, and study analytically the behavior of the solution near breaking (immediately before, at, and immediately after breaking).
A. See section 3 of Appunti 1.

7) More on rarefaction waves.
i) Show that the solution of the Cauchy problem

$$u_t + uu_x = 0, \quad u(x,0) = a_2 H(-x) + a_1 H(x), \quad a_2 < a_1 \tag{35}$$

is given by

$$u = \left\{ \begin{array}{ll} a_2, & x < a_2 t, \\ x/t, & a_2 t < x < a_1 t, \\ a_1, & x > a_1 t \end{array} \right. \tag{36}$$

Hint. Observe that this Cauchy problem can be viewd as the $l \to 0$ limit of that of the previous problem, in which the interval $(-1,1)$ is replaced by the interval $(-l,l)$. But there are other ways of doing it ...
ii) Show that the solution of the Cauchy problem

$$u_t + c(u)u_x = 0, \quad u(x,0) = a_2 H(-x) + a_1 H(x), \quad a_2 < a_1 \tag{37}$$

is given by

$$u = \begin{cases} a_2, & x < c(a_2)t, \\ A(x/t), & c(a_2)t < x < c(a_1)t, \\ a_1, & x > c(a_1)t \end{cases} \tag{38}$$

where $A(\xi)$ is the inverse of function $c(u)$.

8) Given the following system of PDEs, establish if they are hyperbolic and, if so, write them in characteristic form.

i) The wave equation $u_{tt} - c^2 u_{xx} = 0$.
ii) The Klein - Gordon equation $u_{tt} - c^2 u_{xx} + u = 0$.
iii) The system

$$\begin{array}{l} u_t + c(u,v)u_x = 0, \\ v_t + c(u,v)v_x = u \end{array} \tag{39}$$

iv) The system

$$\begin{array}{l} u_t + c(u)u_x = 0, \\ v_t + c(u)v_x + c'(u)vu_x = 0 \end{array} \tag{40}$$

v) The gas dynamics equations

$$\begin{array}{l} \rho_t + u\rho_x + \rho u_x = 0, \\ u_t + uu_x + \frac{p_x}{\rho} = 0, \\ S_t + uS_x = 0, \end{array} \tag{41}$$

where $p = p(\rho, S)$.
R. i)

$$\begin{array}{l} \frac{d}{dt}(w - cv) = 0, \quad \frac{dx}{dt} = c, \quad \Rightarrow \quad w - cv = A(x - ct), \\ \frac{d}{dt}(w + cv) = 0, \quad \frac{dx}{dt} = -c, \quad \Rightarrow \quad w + cv = B(x + ct), \\ v \equiv u_x, \ w \equiv u_t \end{array} \tag{42}$$

implying the well-known result $u = f(x - ct) + g(x + ct)$, with

$$f'(\cdot) = -\frac{1}{2c}A(\cdot), \quad g'(\cdot) = \frac{1}{2c}B(\cdot). \tag{43}$$

ii)

$$\begin{aligned} &\varphi_t - c\varphi_x + u = 0, \\ &u_t + cu_x - \varphi = 0, \\ &\varphi \equiv u_t + cu_x. \end{aligned} \tag{44}$$

iii) it is already in characteristic form, with the single characteristic $dx/dt = c(u,v)$ and two different characteristic forms (two different eigenvectors $(1,0)$ and $(0,1)$).

iv) The first equation is in characteristic form for the single field u; the second one cannot be put in characteristic form; therefore the system is not hyperbolic. Nevertheless it can be solved solving first the first equation, hyperbolic, on the characteristic $dx/dt = c(u)$, and then solving the second one on that characteristic (do it!).

v) Rewrite (41) in the form

$$\begin{aligned} &p_t + up_x + \rho a^2 u_x = 0, \\ &u_t + uu_x + \tfrac{p_x}{\rho} = 0, \\ &S_t + uS_x = 0, \end{aligned} \tag{45}$$

where $a^2(\rho) = \partial p/\partial \rho > 0$, obtaining the following eigenvalues and eigenvectors:

$$\begin{aligned} &c_0 = u \ (gas\ speed), \ \underline{L}_0 = (0,0,1), \\ &c_{\pm} u \pm a \ (sound\ speeds), \ \underline{L}_{\pm} = (1, \pm a\rho, 0). \end{aligned} \tag{46}$$

Therefore the system in characteristic for reads:

$$\begin{aligned} &\tfrac{dp}{dt} \pm \rho a \tfrac{du}{dt} = 0, \quad \tfrac{dx}{dt} = u \pm a, \\ &\tfrac{dS}{dt}, \quad \tfrac{dx}{dt} = u. \end{aligned} \tag{47}$$

9) Show that i) the Riemann invariants of the wave equation $u_{tt} - c^2 u_{xx} = 0$, $c > 0$ are given by $r_{\pm} = w \mp cv$, where $v = u_x$ and $w = u_t$, so that the PDE is written as the system of ODEs in characteristic form:

$$\frac{dr_{\pm}}{dt} = 0, \quad \frac{dx}{dt} = \pm c. \tag{48}$$

ii) The Riemann invariants of the gas dynamics equations (41) (under the constant entropy S hypothesis) are given by

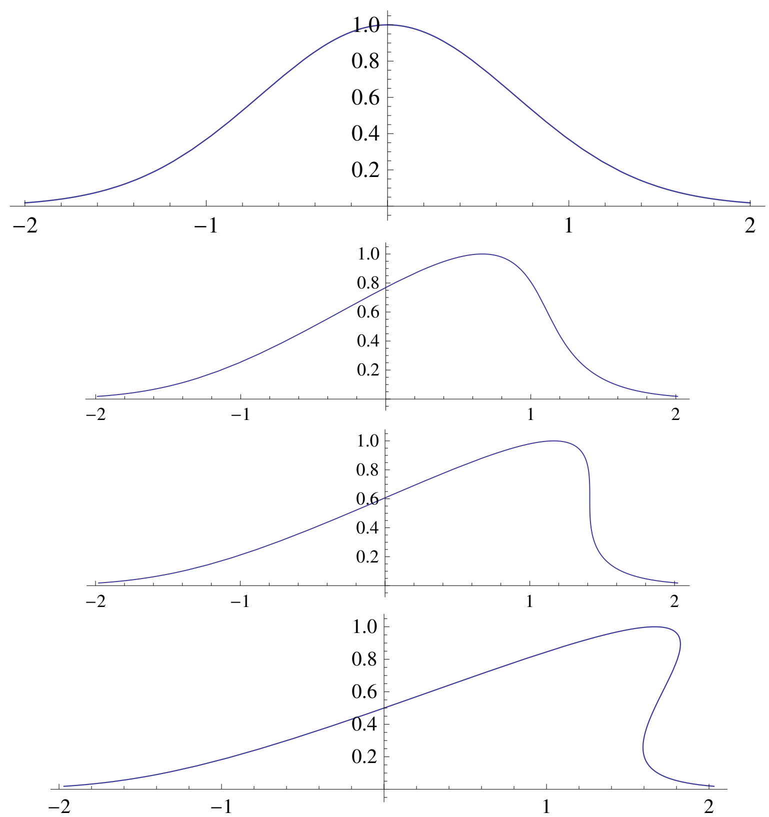
$$r_{\pm} = \int^{\rho} \frac{a(\rho')}{\rho'} d\rho' \pm u, \tag{49}$$

where $a^2(\rho) = p'(\rho) > 0$, so that the system (47) decouples as follows:

$$\frac{dr_\pm}{dt} = 0, \quad \frac{dx}{dt} = u \pm a(\rho). \tag{50}$$

Show that, for an adiabatic process ($p = \kappa \rho^\gamma$),

$$\begin{aligned} &a^2 = \kappa\gamma\rho^{\gamma-1}, \\ &r_\pm = \tfrac{2\sqrt{\kappa\gamma}}{\gamma-1}\rho^{\frac{\gamma-1}{2}} \pm u. \end{aligned} \tag{51}$$



Figures 2. The evolution of a gaussian according to the Hopf equation (through the numerical inversion of the analytic solution).

2.1.3 Regolarizzazione dissipativa e l'equazione di Burgers. Regolarizzazione dispersiva e l'equazione KdV; funzioni ellittiche

1) Regularize the compression wave of problem 5) of section 2.1.2

2) What happens if we look for discontinuous solutions of $u_t + uu_x = 0$ in the form $u = H(s(t) - x)u^-(x,t) + H(x - s(t))u^+(x,t)$, where $H(x)$ is the Heaviside step function and $u^{\pm}(x,t)$ are smooth functions?

3) Consider the Cauchy problem

$$\begin{aligned} &u_t + uu_x = 0, \\ &u(x,0) = f(x), \end{aligned} \tag{52}$$

where $f(x)$ describes a single bump, and study the behavior of the regularized (shock) solution near breaking.
A. See section 4 of Appunti 1.

4) Given the Cauchy problem

$$\begin{aligned} &u_t + c(u)u_x = 0, \quad c(u) = Q'(u), \\ &u(x,0) = f(x), \end{aligned} \tag{53}$$

where $f(x)$ describes a single bump,
i) construct the shock condition

$$\dot{s} = \frac{Q(u_2) - Q(u_1)}{u_2 - u_1} \tag{54}$$

and show that it is equivalent of placing the vertical shock to cut equal area lobi of the three valued solution.
ii) Show that, if $c(u) = u, \ Q(u) = u^2/2$, the shock equations involving $s(t), \eta_1(t), \eta_2(t)$ can be reformulated as cutting equal area lobi on the initial profile:

$$\int\limits_{\eta_1}^{\eta_2} f(\eta)d\eta = \frac{1}{2}(\eta_1 - \eta_2)(f(\eta_1) + f(\eta_2)) \tag{55}$$

5) Given the Burgers equation $u_t + uu_x = \nu u_{xx}$, i) find its traveling wave solution satisfying the boundary conditions $u(x,t) \to u_{\pm}, \ x \to \pm\infty$, where $u_{\pm}$ are constants, and discuss the shock structure. ii) Find its similarity solutions.

6) Show that the one dimensional Newton equation $\ddot{x} = -dV(x)/dx$ is integrated to the quadrature $t - t_0 = \int\limits_0^x \frac{dy}{\sqrt{2(E-V(y))}}$, where $E = \dot{x}^2/2 + V(x)$ is the constant energy.

7) Study the inversion, in the complex plane, of the quadrature

$$w(z) = \int_0^z \frac{dt}{\sqrt{1-t^2}}, \tag{56}$$

that arises in the solution of the harmonic oscillator equation, providing an alternative definition of the sine function, and infer the basic properties of $\sin w$ from such inversion:
i) $\sin w$ is entire in $\mathbb{C}$; ii) $\sin w$ has simple zeroes in $w = n\pi,\ n \in \mathbb{N}$, iii) $\sin w$ is odd and satisfies the following periodicity properties:

$$\sin(w+2\pi) = \sin w; \quad \sin(w+\pi) = -\sin w. \tag{57}$$

8) i) Study the inversion, in the complex plane, of the quadrature

$$w(z,\kappa) = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}}, \quad 0 < \kappa < 1 \tag{58}$$

defining the elliptic sine function $z = sn(w,\kappa)$ [15]. Show, in particular, that i) $w(z,\kappa))$ maps the half plane Im $z > 0$ into the rectangle of the w complex plane of vertices $-K(\kappa), K(\kappa), K(\kappa) + iK'(\kappa), -K(\kappa) + iK'(\kappa)$, where

$$\begin{array}{l} K(\kappa) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}}, \\ K'(\kappa) = K(\kappa'), \quad {\kappa'}^2 = 1-\kappa^2. \end{array} \tag{59}$$

ii) using the Schwarz reflection principle [15], show that the elliptic sine function can be analytically extended into the whole complex w plane as a meromorphic doubly periodic function with periods $4K(\kappa)$ and $2iK'(\kappa)$:

$$sn(w+4K(\kappa),\kappa) = sn(w,\kappa), \quad sn(w+2iK'(\kappa),\kappa) = sn(w,\kappa), \tag{60}$$

possessing the simple zeroes $2mK(\kappa)+i2nK'(\kappa)$ and the simple poles $2mK(\kappa)+i(2n+1)K'(\kappa)$, for $m,n \in \mathbb{Z}$. iii) Show the additional properties

$$\begin{array}{l} sn(w) = \overline{sn(2K-\bar{w}))}, \\ sn(w+2K(\kappa),\kappa) = -sn(w,\kappa), \quad sn(-w,\kappa) = -sn(w,\kappa). \end{array} \tag{61}$$

9) Basic properties of elliptic functions Having defined as elliptic function a doubly periodic complex function $f(z)$ of complex variable z, with the two independent periods $2\omega_1, 2\omega_2 \in \mathbb{C}$:

$$f(z + 2m_1\omega_1 + 2m_2\omega_2) = f(z), \quad m_1, m_2 \in \mathbb{Z}, \tag{62}$$

let Π_{00} the fundamental parallelogramme generated by the two periods. Show that i) if $f(z)$ is entire, then it is constant. ii) The order of its poles inside Π_{00} is ≥ 2. iii) The order of its poles inside Π_{00} is equal to the number ν of the zeroes (counted with their moltiplicity) of $(f(z) - A)$, with $A \in \mathbb{C}$, inside Π_{00}. Verify these properties for $sn(w)$.

10) Other elliptic functions.
The definition

$$x = sn(u, \kappa) = \sin \varphi(u, \kappa) \tag{63}$$

suggests the introduction of other elliptic functions:

$$\begin{array}{l} cn(u, \kappa) \equiv \cos \varphi(u, \kappa), \\ dn(u, \kappa) \equiv \sqrt{1 - \kappa^2 \sin^2 \varphi(u, \kappa)}. \end{array} \tag{64}$$

Show that

$$sn^2 u + cn^2 u = 1, \quad dn^2 u + \kappa^2 sn^2 u = 1 \tag{65}$$

and that

$$\begin{array}{l} \frac{d\varphi(u)}{du} = \left(\frac{du}{d\varphi}\right)^{-1} = \sqrt{1 - \kappa^2 \sin^2 \varphi(u)} = dn(u), \\ \frac{dsn(u)}{du} = \frac{d \sin \varphi(u)}{du} = \cos \varphi(u) \frac{d\varphi(u)}{du} = cn(u)dn(u), \\ \frac{dcn(u)}{du} = -sn(u)dn(u), \\ \frac{ddn(u)}{du} = -\kappa^2 sn(u)dn(u). \end{array} \tag{66}$$

11) Elliptic integral of second type
Introduced the elliptic integral of second type:

$$E(\varphi, \kappa) = \int_0^{\varphi} \sqrt{1 - \kappa^2 \sin^2 \varphi} d\varphi = \int_0^x \sqrt{\frac{1 - \kappa^2 t^2}{1 - t^2}} dt \tag{67}$$

where the second integral follows from the change of variables $x = \sin \varphi$, and the complete elliptic integral of second type:

$$E(\kappa) \equiv E(\pi/2, \kappa) = \int_0^{\pi/2} \sqrt{1 - \kappa^2 \sin^2 \varphi} d\varphi = \int_0^1 \sqrt{\frac{1 - \kappa^2 t^2}{1 - t^2}} dt, \tag{68}$$

i) show that

$$E(\varphi, \kappa) = \int_0^u dn^2(u) du \tag{69}$$

ii) show that sn^2 and dn^2 are periodic of period $2K(\kappa)$, and show that their average over that period are:

$$\overline{dn^2} = \frac{E(\kappa)}{K(\kappa)}, \quad \overline{sn^2} = \frac{1}{\kappa^2}\frac{K(\kappa) - E(\kappa)}{K(\kappa)}. \tag{70}$$

12) Show that

$$\begin{array}{l} K(\kappa) \to \infty, \ E(\kappa) \to 1, \quad \text{as } \kappa \to 1, \\ K(\kappa) = \frac{\pi}{2} + \frac{\pi}{8}\kappa^2 + O(\kappa^4), \quad E(\kappa) = \frac{\pi}{2} - \frac{\pi}{8}\kappa^2 + O(\kappa^4), \quad \kappa << 1. \end{array} \tag{71}$$

13) Rectification of the ellipse.
Given the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0, \tag{72}$$

show that the length of the ellipse arc having as boundary points $P(0)$ and $P(\varphi)$ is

$$s = aE(\varphi, \kappa^2), \quad \kappa \equiv \frac{\sqrt{a^2 - b^2}}{a}, \tag{73}$$

and infer that the ellipse length is $4aE(\kappa)$.

14) The simple pendulum.
i) Show that the general solution of the simple pendulum equation $\ddot{\theta} + \frac{g}{L}\sin\theta = 0$ is expressed in terms of the elliptic sine function in the following way:

$$\theta(t) = 2\sin^{-1}\left(\kappa sn\left(\omega(t - t_0), \kappa\right)\right), \quad \omega = \sqrt{\frac{g}{L}}, \tag{74}$$

and the period of oscillations is

$$T = \frac{4K(\kappa)}{\omega} \tag{75}$$

where $sn(z, \kappa)$ is the Jacobi elliptic sine function, $\kappa = \sqrt{\frac{1+E/\omega^2}{2}}$, and $E = \frac{\dot{\theta}^2}{2} - \omega^2\cos\theta$ is the constant energy of the system.
ii) Show that, if $\bar{\theta}$ is the inversion angle, then:

$$\begin{array}{l} \cos\bar{\theta} = -\frac{E}{\omega^2}, \quad \kappa^2 = \sin^2\frac{\bar{\theta}}{2}, \\ \theta(t) \sim \bar{\theta} - \omega^2\kappa\sqrt{1-\kappa^2}(t - \frac{T}{4})^2, \quad t \sim \frac{T}{4}. \end{array} \tag{76}$$

iii) Show that, in the case of small oscillations $|\bar{\theta}| << 1$, one has $\kappa << 1$, and

$$
\begin{array}{l}
\omega(t-t_0) \sim \int\limits_0^x \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} x \quad \Rightarrow \quad sn(\omega(t-t_0)) \sim \sin(\omega(t-t_0)), \\
K(\kappa) \sim \int\limits_0^1 \frac{dt}{\sqrt{1-t^2}} = \pi/2, \\
T(\kappa) \sim \frac{2\pi}{\omega}, \\
\theta \sim 2\sin^{-1}(\kappa \sin(\omega(t-t_0))) \sim 2\kappa \sin(\omega(t-t_0)) = \bar{\theta}\sin(\omega(t-t_0)), \\
\kappa \sim \bar{\theta}/2.
\end{array}
\tag{77}
$$

15) Construct the traveling wave solution $u = U(\zeta)$, $\zeta = x - ct - x_0$ of the Korteweg - de Vries equation $u_t + uu_x + \varepsilon^2 u_{xxx} = 0$, through the quadrature

$$
\frac{1}{\sqrt{3}} \frac{\zeta}{\varepsilon} = \int\limits_\gamma^U \frac{dU}{\sqrt{P(U)}}, \tag{78}
$$

where $P(U) = -(U-\alpha)(U-\beta)(U-\gamma)$, α, β, γ are three real arbitrary constants, with $\alpha \leq \beta \leq \gamma$, and

$$
c = \frac{\alpha + \beta + \gamma}{3}, \tag{79}
$$

and show that the solution can be written in terms of the elliptic sine as:

$$
\begin{array}{l}
U = \gamma - (\gamma - \beta) sn^2 \left(\sqrt{\frac{\gamma-\alpha}{12}} \frac{\zeta}{\varepsilon}, \kappa \right), \\
\kappa = \sqrt{\frac{\gamma-\beta}{\gamma-\alpha}}.
\end{array}
\tag{80}
$$

12) Show that, if $\beta \to \alpha$ (the case of two coinciding roots), then $\kappa \to 1$, and

$$
sn(u, \kappa) \to \tanh u. \tag{81}
$$

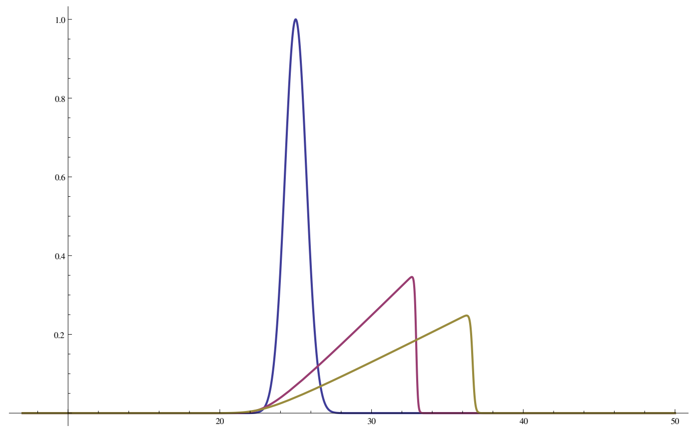
Consequently, the travelling wave solution of KdV reduces to

$$
U = \gamma - (\gamma - \alpha) \tanh^2 \left(\sqrt{\frac{\gamma - \alpha}{12}} \frac{\zeta}{\varepsilon} \right) = \alpha + \frac{\gamma - \alpha}{\cosh^2 \left(\sqrt{\frac{\gamma-\alpha}{12}} \frac{\zeta}{\varepsilon} \right)} \tag{82}
$$

If, in addition, $\alpha = 0$, then the travelling wave solution reduces to the so-called 1-soliton solution of KdV

$$
U = \frac{3c}{\cosh^2 \left(\frac{\sqrt{c}}{2} \frac{x-ct-x_0}{\varepsilon} \right)}, \tag{83}
$$

an exponentially localized travelling wave whose velocity is proportional to the amplitude and inversely proportional to the $\sqrt{width}$.



Figures 3. Three time steps ($t = 0$, $t = T/2$, $t = T$) of the evolution of a gaussian initial condition according to the Burgers equation with small dissipation (numerical solution).

2.2 La propagazione ondosa in Natura, il metodo multiscala e le equazioni modello [4, 8, 1, 3, 16]

1) Consider the two anharmonic oscillators

$$\begin{array}{l} \ddot{q} + q - \frac{\epsilon}{6} q^3 = 0, \quad \text{Hamiltonian cubic pendulum}, \quad 0 < \epsilon << 1, \\ \ddot{q} + q + \epsilon \dot{q}^3 = 0, \quad \text{with nonlinear friction} \end{array} \tag{84}$$

with the same initial conditions

$$q(0) = 1, \quad \dot{q}(0) = 0. \tag{85}$$

Use the multiscale method to show that

$$\begin{array}{l} q(t) = \cos\left(t - \frac{1}{16}\varepsilon t\right) + O(\varepsilon), \\ q(t) = \left(1 + \frac{3}{4}\varepsilon t\right)^{-1/2} \cos t + O(\varepsilon) \end{array} \tag{86}$$

2) Use the multiscale method to construct the solution

$$q(t) = \frac{a_0 e^{\varepsilon t/2}}{\sqrt{1 + \left(\frac{a_0}{2}\right)^2 (e^{\varepsilon t} - 1)}} \cos(t + \phi_0) + O(\varepsilon) \tag{87}$$

of the Van Der Pol oscillator

$$\ddot{q} + q - \varepsilon(1 - q^2)\dot{q} = 0, \tag{88}$$

and show that

$$q(t) \to 2\cos(t + \phi_0), \quad t \to \infty, \tag{89}$$

i.e., the solution tends to a limiting cycle (at O(ε), the circle of radius 2).

3) Derive the dKP(3,1) equation $(u_t + uu_x)_x + u_{yy} + u_{zz} = 0$ from the equations of Acoustics, under the hypothesis of i) weak nonlinearity and ii) quasi one-dimensionality.

4) Derive the KdV equation (see [1, 3]) in the context of surface water wave in $(1 + 1)$ dimensions, under the hypothesis of i) small amplitudes and ii) shallow water ($kh << 1$, where k is the wave number and h is the depth of the fluid). Derive the KP equation (see [2, 3]) in the context of surface water waves in $(2 + 1)$ dimensions, under the hypothesis of i) small amplitudes, ii) shallow water, and iii) quasi one-dimensionality. Show that, neglecting dispersion, one obtains the dKP(2,1) equation.

5) Derive (see [3]) the NLS equation in the context of surface water waves in $(1 + 1)$ dimensions, under the hypothesis of i) small amplitude ($a << \lambda$) and ii) quasi monocromatic waves in sufficiently deep water. Derive its multidimensional generalization in the context of surface water waves in $(2 + 1)$ dimensions, under the hypothesis of

6) Derive (see [20]) the NLS equation in the framework of Langmuir waves in a plasma, described by the system of equations:

$$n_t + (nv)_x = 0, \quad v_t + vv_x = \phi_x - n_x/n, \quad \phi_{xx} = n - 1,$$

with boundary onditions $n \to 1, \ v \to 0, \ \phi \to 0$ as $|x| \to \infty$, where n is the electron density, v is the electron velocity and ϕ is the electrostatic potential in dimensionless variables, expanding the fields around the equilibrium solution:

$$n = 1 + \varepsilon n_1 + \varepsilon^2 n_2 + O(\varepsilon^3), \quad v = \varepsilon v_1 + O(\varepsilon^2), \quad \phi = \varepsilon\phi_1 + O(\varepsilon^2).$$

7) Derive (see [8]) the NLS equation in nonlinear optics, for a homogeneous and isothropic dielectric.

2.3 La teoria dei solitoni

1) Analyticity projectors. Show that the operators

$$P^{\pm} f(\lambda) := \pm \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(\lambda')}{\lambda' - (\lambda \pm i\varepsilon)} d\lambda. \tag{90}$$

are analyticity projectors on the real line; i.e., they map a Holder function $f(\lambda),\ \lambda \in \mathbb{R}$ decaying at ∞ sufficiently fast into functions analytic in the upper and lower halves of the complex λ plane respectively. ii) Show, in particular, that

$$(P^+)^2 = P^+,\ (P^-)^2 = P^-,\ P^+P^- = P^-P^+ = 0,\ P^+ + P^- = 1. \tag{91}$$

2) Given a Holder function $f(\lambda)$ for $\lambda \in \mathbb{R}$ decaying at ∞ sufficiently fast, a polynomial $P(\lambda)$, a set of complex numbers $\{k_j^+,\ R_j^+,\ j = 1, \ldots, N^+,$ $k_j^-,\ R_j^-,\ j = 1, \ldots, N^-\}$, where Im $k_j^+ > 0$ and Im $k_j^- < 0$, show that the unique solution of the Riemann problem

$$\psi^+(\lambda) - \psi^-(\lambda) = f(\lambda), \quad \lambda \in \mathbb{R} \tag{92}$$

where $\psi^{\pm}(\lambda)$ are analytic in the upper and lower halves of the complex λ plane respectively, except for the simple poles $k_j^{\pm}$'s with residues $R_j^{\pm}$'s, and $\psi^{\pm}(\lambda) \to P(\lambda),\ |\lambda| >> 1$, is

$$\psi^{\pm}(\lambda) = P(\lambda) + \sum_{j=1}^{N^+} \frac{R_j^+}{\lambda - k_j^+} + \sum_{j=1}^{N^-} \frac{R_j^-}{\lambda - k_j^-} \pm P^{\pm} f(\lambda). \tag{93}$$

3) Let $u(x) = -A\delta(x - x_0),\ A \in \mathbb{R}$, be the potential of the Schrödinger equation $[-\partial_x^2 + u(x)]\psi = k^2\psi$. Evaluate explicitly: i) the eigenfunctions of the continuous spectrum and the coefficients $a(k), b(k), R(k), T(k)$; ii) the discrete spectrum p_j, the corresponding eigenfunctions and the norming constants b_j. Show that the existence of discrete spectrum depends on the sign of A.

4) Assume $u(x) = O(\varepsilon),\ \varepsilon << 1$, and construct the first two terms of the ε - expansion of the eigenfunctions and of the spectral data.

5) *Scattering problem.* Study the scattering problem described by the Schrödinger equation

$$-\psi''(x, k) + u(x)\psi(x, k) = k^2\psi(x, k), \quad x \in \mathbb{R}, \quad k > 0,$$

where $\psi(x,k)$, the eigenfunction of the continuous spectrum of the Schrödinger operator $-d^2/dx^2+V(x)$, represents the wave function of a particle beam scattered by the localized potential $u(x)$ e $E=k^2>0$ is the energy of the beam (the continuous spectrum $\sigma_c=\{E>0\}$), with the following boundary conditions:

$$\psi(x,k)\sim\ R(k)e^{-ikx}+e^{ikx},\ x\sim-\infty;\qquad \psi(x,k)\sim\ T(k)e^{ikx},\ x\sim\infty$$

describing an incoming beam of particles of wave number k and intensity 1, partially reflected and transmitted through the potential ($R(k)$ e $T(k)$ are respectively the reflection and transmission coefficients).

i) Observe that the function $\phi(x,k)=\psi(x,k)/T(k)$ satisfies a simpler scattering problem:

$$\phi''(x,k)+k^2\phi(x,k)=u(x)\phi(x,k),\quad x\in\mathbb{R},,\qquad k>0$$

$$\phi(x,k)\sim\frac{R(k)}{T(k)}e^{-ikx}+\frac{e^{ikx}}{T(k)},\ x\sim-\infty;\quad \phi(x,k)\sim\ e^{ikx},\ x\sim\infty$$

and use the advanced Green function of the operator d^2/dx^2+k^2 to rewrite such a problem as a Volterra integral equation [5], obtaining:

$$\phi(x,k)=e^{ikx}-\int_x^\infty dy\frac{\sin k(x-y)}{k}u(y)\phi(y,k)$$

and the following integral representations for the reflection and transmission coefficients:

$$\frac{1}{T(k)}=1-\int_{\mathbb{R}}dk\frac{e^{-iky}}{2ik}u(y)\phi(y,k),\quad \frac{R(k)}{T(k)}=\int_{\mathbb{R}}dk\frac{e^{iky}}{2ik}u(y)\phi(y,k).$$

Such an integral equation, equivalent to the Schrödinger differential equation + boundary conditions, is the most convenient formulation of the problem to extract informations.

ii) Use the method of successive approximations to study the properties of ϕ in the following way.

a) Rerwrite the integral equation for the unknown $f(x,k)=\phi(x,k)e^{-ikx}$, such that $f\sim 1,\ x\to\infty$:

$$f(x,k)=1+\int_x^\infty\frac{e^{2ik(y-x)}-1}{2ik}u(y)f(y,k)dy$$

and look for the solution as a Neumann series:

$$f(x,k) = \sum_{i=0}^{\infty} h_i(x,k), \quad h_0 = 1, \tag{94}$$

obtaining the recursion relation:

$$h_{j+1}(x,k) = \int\limits_x^{\infty} \frac{e^{2ik(y-x)} - 1}{2ik} u(y) h_j(y,k) dy, \quad j \geq 0. \tag{95}$$

b) From the inequality: $|e^{2ik(y-x)} - 1|/|2ik| \leq 1$, valid for Im $k \geq 0$, $k \neq 0$, show that

$$|h_{j+1}(x,k)| \leq \frac{1}{|k|} \int\limits_x^{\infty} |u(y)||h_j(y,k)|dy, \tag{96}$$

and then that:

$$\begin{aligned} &|h_n(x,k)| \leq \frac{1}{n!}\left(\frac{A(x)}{|k|}\right)^n \leq \frac{1}{n!}\left(\frac{A(-\infty)}{|k|}\right)^n, \\ &A(x) := \int\limits_x^{\infty} |V(y)|dy. \end{aligned} \tag{97}$$

Therefore the Neumann series representing the solution is absolutely and uniformely convergent for Im $k \geq 0$, $k \neq 0$, if $u(x) \in L_1(\mathbb{R})$. Under these conditions, the solution exists unique, and it is analytic in the upper half of the complex k plane. Analogously one can prove that $1/T(k)$ is analytic in the upper half of the complex k plane. Under more stringent conditions on u, one could show, in a similar manner, that the eigenfunction is also continuous on the real k axes, where the physics takes place.
c) Let $k_j,\ j = 1,..,N$ be the zeroes of the function $1/T(k)$ in the upper half of the complex k plane (the poles of the transmission coefficient). Then, since $\lambda_j = E_j = k_j^2 \in \mathbb{R}$, it follows that a) k_j is purely imaginary: $k_j = ip_j,\ p_j > 0,\ j = 1,..,N$, b) the functions $\phi(x,k_j),\ j = 1,..,N$ are exponentially localized:

$$\phi_j(x) := \phi(x,k_j) = O(e^{-p_j|x|}), \quad |x| \to \infty, \quad j = 1,..N$$

and then they are eigenfunctions of the Schrödinger operator in $L_2(\mathbb{R})$:

$$-\phi_j''(x) + u(x)\phi_j(x) = -p_j^2\phi_j(x), \quad x \in \mathcal{R}$$

corresponding to negative eigenvalues $\lambda_j = E_j = -p_j^2 < 0$ of the energy (the discrete spectrum: $\sigma_p = \{-p_j^2\}_1^N$). Summarizing: $\sigma = \sigma_p \cup \sigma_c = \{-p_j^2\}_1^N \cup \mathbb{R}^+$.

d) Show that the set of $\lambda_j = -p_j^2,\ j = 1, .., N$ bounded from below.
Hint. Take the scalar product of the eigenfunction ϕ_j, normalized to 1, with the Schrödinger equation, obtaining:

$$\lambda_j - (\phi_j, u\phi_j) = (\phi_j', \phi_j') \geq 0 \quad \Rightarrow \quad |\lambda_j| \leq -(\phi_j, V\phi_j) \leq |(\phi_j, u\phi_j)| \leq ||u||_\infty.$$

e) Show that, if $u(x) = u_0\delta(x - x_0)$, the integral equation admits the solution

$$\phi(x, k) = e^{ikx} - u_0 H(x_0 - x)\frac{\sin k(x - x_0)}{k}e^{ikx_0}.$$

Then:

$$\phi(x, k) = \frac{2ik - u_0}{2ik}e^{ikx} + \frac{u_0 e^{2ikx_0}}{2ik}e^{-ikx}, \quad x < x_0$$

$$T(k) = \frac{2ik}{2ik - u_0}, \qquad R(k) = \frac{u_0 e^{2ikx_0}}{2ik - u_0}.$$

Found $\phi(x, k)$, at last reconstruct $\psi(x, k) = \frac{2ik}{2ik - u_0}\phi(x, k)$.
f) Verify that the solution we found for $k \in \mathbb{R}$, if extended outside the real k axis, diverges always at + or - infinity, unless $k = -iu_0/2 \in i\mathbb{R}^+$. Therefore, if the potential is positive ($u_0 > 0$), no eigenfunctions exist in $L_2(\mathbb{R})$; if, instead, the potential is negative, then there exists one and only one $L_2(\mathbb{R})$ eigenfunction $\psi_1(x) := \phi(x, i|u_0|/2) \in L_2(\mathbb{R})$:

$$\psi_1(x) = H(x - x_0)e^{-\frac{|u_0|}{2}x} + H(x_0 - x)e^{\frac{|u_0|}{2}x}$$

corresponding to the negative energy $E_1 = k_1^2 = -u_0^2/4$, and describing a bound state (a localized quantum particle): $\sigma_p = \{E_1\}$.
g) If $u(x) = \epsilon v(x),\ \epsilon << 1$, show that:

$$\phi(x, k) = e^{ikx} - \epsilon\int_x^\infty dy\frac{\sin k(x - y)}{k}v(y)e^{iky} + O(\epsilon^2),$$

$$T(k) = 1 + \frac{\epsilon}{2ik}\int_{\mathcal{R}} dx v(x) + O(\epsilon^2), \quad R(k) = \frac{\epsilon}{2ik}\int_{\mathcal{R}} dx v(x)e^{-2ikx} + O(\epsilon^2)$$

6) Using the above strategy, study the scattering problem

$$\phi''(x, k) + k^2\phi(x, k) = u(x)\phi(x, k), \quad x \in \mathcal{R}, \qquad \phi(x, k) \sim\ e^{-ikx},\ x \sim -\infty$$

showing that, in this case, it is convenient to use the retarded Green function of the operator $d^2/dx^2 + k^2$.

7) Let $\varphi(x,k)$ and $\psi(x,k)$ be the Jost eigenfunctions of the Schrödinger operator satisfying the boundary conditions:

$$\varphi(x,k) \sim e^{-ikx}, \quad x \to -\infty, \qquad \psi(x,k) \sim e^{-ikx}, \quad x \to \infty \tag{98}$$

i) Write the integral equations satisfied by them; ii) show that $\varphi(x,k)e^{ikx}$ and $\psi(x,k)e^{ikx}$ are analytic respectively in the upper and lower halves of the k plane; iii) show that

$$-2i\frac{d}{dx}[k(\psi(x,k)e^{ikx} - 1)] \to u(x), \quad |k| \gg 1. \tag{99}$$

8) Let k_0 be a zero of $a(k) = 1/T(k)$, where $T(k)$ is the transmission coefficient of the Schrödinger equation. i) Show that k_0 belongs to the discrete spectrum (therefore $k_0 = ip,\ p > 0$) and, correspondingly, that $\varphi(x,k_0) \in L^2(\mathbb{R})$, with the asymptotics

$$\varphi(x,k_0) \sim e^{px}, \quad x \sim -\infty, \quad \varphi(x,k_0) \sim be^{-px}, \quad x \sim \infty \tag{100}$$

where $b \in \mathbb{R}$.
ii) Show that the zeroes $k_0 = ip$ of $a(k)$ are simple, and that $iba'(ip) > 0$.
A. For i), use the Wronskian relation $W(\varphi, \bar{\psi}) = 2ika(k)$ to infer that $\varphi(x,k_0) = b\overline{\psi(x,k_0)} = b\psi(x,-k_0)$.

9) *Inverse Problem.* Using the analyticity properties of $\varphi(x,k), \psi(x,k), a(k)$, together with their asymptotics for large k, i) rewrite the scattering equation

$$\varphi(x,k) = a(k)\psi(x,k) + b(k)\psi(x,-k), \quad k \in \mathbb{R} \tag{101}$$

for the Schrödinger operator as a linear Riemann - Hilbert problem on the real k axis, for a given set of scattering data. ii) Express the solution of such a linear RH problem in terms of integral equations for the eigenfunctions, and iii) reconstruct the potential u(x) in terms of the scattering data.

10) t *- evolution of the scattering data.* Obtain the t evolution of the scattering data if u evolves according to KdV.

11) Construct the 2-soliton solution of KdV and study the interaction of the two solitons.

2.4 Equazioni non lineari integrabili di tipo idrodinamico e la rottura di onde multidimensionali

2.4.1 Campi vettoriali commutanti ed equazioni integrabili di tipo idrodinamico

2.4.2 Trasformata spettrale per campi vettoriali

2.4.3 Come si rompono onde quasi - unidimensionali in Natura

1) Given the dKP_n equation [12]:

$$\begin{array}{l} (u_t + uu_x)_x + \Delta_\perp u = 0, \quad u = u(x, \vec{y}, t), \quad \vec{y} = (y_1, \dots, y_{n-1}) \\ \Delta_\perp = \sum\limits_{i=1}^{n-1} \partial^2_{y_i}, \quad n \geq 2, \end{array} \tag{102}$$

i) show that it is invariant under motions on the associated paraboloid

$$x + \frac{1}{4t} \sum_{i=1}^{n-1} y_i^2 = \xi. \tag{103}$$

ii) Use such invariance to look for particular solutions in the form

$$u = v(\xi, t), \quad \xi = x + \frac{1}{4t} \sum_{i=1}^{n-1} y_i^2, \tag{104}$$

obtaining the exact (but implicit) solution

$$u = \begin{cases} t^{-\frac{n-1}{2}} F\left(x + \frac{1}{4t} \sum\limits_{i=1}^{n-1} y_i^2 - \frac{2ut}{3-n}\right), & n \neq 3, \\ t^{-1} F\left(x + \frac{1}{4t} \sum\limits_{i=1}^{n-1} y_i^2 - u\, t \ln t\right), & n = 3, \end{cases} \tag{105}$$

where F is an arbitrary function of a single variable. What kind of wave is described by this solution?

2) Write the Fourier representation of the solution of the Cauchy problem for the linearized dKP equation:

$$\begin{array}{l} u_{xt} + \Delta_\perp u = 0, \\ u(x, \vec{y}, 0) = u_0(x, \vec{y}) \end{array} \tag{106}$$

and show that, for $t >> 1$, the solution reads

$$u(x, \vec{y}, t) \sim t^{-\frac{n-1}{2}} G\left(x + \frac{1}{4t} \sum_{i=1}^{n-1} y_i^2, \frac{\vec{y}}{2t}\right), \tag{107}$$

where

$$G(\xi,\vec{\eta}) := 2^{-n}\pi^{-\frac{n+1}{2}} \int_{\mathbb{R}} d\lambda |\lambda|^{\frac{n-1}{2}} \hat{u}_0(\lambda,\lambda\vec{\eta}) e^{i\lambda\xi - i\frac{\pi}{4}(n-1)\,\mathrm{sign}\,\lambda}, \tag{108}$$

in the space-time region

$$(x-\xi)/t,\ y_i/t = O(1), \quad i=1,\dots,n, \tag{109}$$

on the paraboloid (103). Outside the paraboloid, the solution decays faster. Hint. Write u as

$$\begin{array}{l} u(x,y,t) = \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ikx} I(k,\vec{y},t), \\ I(k,\vec{y},t) \equiv \int_{\mathbb{R}^{n-1}} \frac{d\vec{\kappa}}{(2\pi)^{n-1}} \hat{u}_0(k,\vec{\kappa}) e^{i(\vec{\kappa}\cdot\vec{y} - \frac{|\vec{\kappa}|^2}{k})t} \end{array} \tag{110}$$

and use, for the multiple integral I, the results of exercise 4) of 2.1.1 .

3) Comparing equations (107) and (111) and doing some matching, infer that the longtime behavior of the solution, in the nonlinear regime, reads

$$u \sim \begin{cases} t^{-\frac{n-1}{2}} G\left(x + \frac{1}{4t}\sum_{i=1}^{n-1} y_i^2 - \frac{2ut}{3-n}\right), & n=1,2, \\ t^{-1} G\left(x + \frac{1}{4t}\sum_{i=1}^{n-1} y_i^2 - u\, t\ln t\right), & n=3, \end{cases} \tag{111}$$

for $n \leq 3$, and coincides with (107) for $n > 4$. It follows that small and localized initial data evolving according to dKP_n break in the longtime regime iff $n \leq 3$.

4) Consider the exact (but implicit) family of solutions

$$u = \frac{1}{\sqrt{t}} A(x + \frac{y^2}{4t} - 2ut) \tag{112}$$

of dKP_2, where A is an arbitrary differentiable function of a single variable. Show that, if A is a localized function, it describes a solution constant on the parabolic wave front $x + \frac{y^2}{4t} = const.$, and breaking at time

$$t_b = \left(\frac{\tau_b}{2}\right)^2 \tag{113}$$

in all points of the parabola

$$x + \frac{y^2}{4t} = \xi_b, \qquad \xi_b := X_b + A(X_b)\tau_b, \tag{114}$$

where

$$\tau_b = -\frac{1}{A'(X_b)} \tag{115}$$

and X_b is a global max of function $A'(\cdot)$ such that $A'(X_b) < 0$.

5) It is known that solutions of dKP in 2+1 dimensions depend on x through the combination $(x - 2ut)$ [11]. This suggests to look for solutions of dKP in the form [11, 14]

$$u = F(\zeta, y, t), \quad x = 2F(\zeta, y, t)t + \zeta. \tag{116}$$

Show that (116) is compatible with the class of solutions of the previous exercise, and verify that F satisfies, at breaking, the conditions

$$F_{\zeta t} = F_\zeta^2, \quad F_y^2 = FF_{\zeta t} - F_\zeta F_t \tag{117}$$

derived in [11, 14] for a generic solution.

6) Consider the above exact (but implicit) solutions

$$u = \frac{1}{\sqrt{t}} A(x + \frac{y^2}{4t} - 2ut) \tag{118}$$

of dKP_2. Show that, for $t \gg 1$, $x/t, y/t = O(1), x + \frac{y^2}{4t} - 2ut = O(1)$, the formula

$$u \sim \frac{1}{\sqrt{t}} F\left(x + \frac{y^2}{4t} - 2ut, \frac{y}{2t}\right) \tag{119}$$

describes the longtime behavior of solutions of dKP_2.

7) Use the longtime formula derived in the previous problem to study analytically how a localized two dimensional initial bump of dKP_2 breaks. Suggestion: proceed as in the case of the Hopf equation.

References

[1] M. J. Ablowitz, Nonlinear Dispersive Waves, Asymptotic Analysis and Solitons, Cambridge Texts in Applied Mathematics (No. 47), 2011.

[2] M. J. Ablowitz and P. A. Clarkson, *Solitons, nonlinear evolution equations and Inverse Scattering*, London Math. Society Lecture Note Series, vol. 194, Cambridge University Press, Cambridge (1991).

[3] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM Philadelphia, 1981.

[4] Appunti del Corso di Dottorato "Onde non lineari. Metodi perturbativi ed esatti" tenuto da P. M. Santini, a cura di G. Angilella. Università di Catania, AA 1995-96. http://www.angilella.it/teaching/nlw/corsidott.pdf

[5] C. Bernardini, O. Ragnisco, P. M. Santini, *Metodi Matematici della Fisica*, Carocci Editore, Roma, 2002.

[6] F. Calogero and A. Degasperis, Spectral Transform and Solitons I, North-Holland Publishing Company, 1982.

[7] R. Courant and D. Hilbert, *Methods of Mathematical Physics*; Vol. II: *Partial Differential equations*, by R. Courant, Interscience Publishers, J. Wiley and sons, New York, 1962.

[8] Dispense del Corso di "Onde non lineari e Solitoni" tenuto da A. Degasperis, a cura di G. Ferrari e D. Dell'Arciprete. Università di Roma "La Sapienza", AA 2006-07.

[9] F. B. Hildebrand, *Advanced Calculus for Applications*; Prentice-Hall, NJ, 1976.

[10] S. V. Manakov and P. M. Santini: "The Cauchy problem on the plane for the dispersionless Kadomtsev-Petviashvili equation"; JETP Letters, **83**, No 10, 462-466 (2006). http://arXiv:nlin.SI/0604016.

[11] S. V. Manakov and P. M. Santini: "On the solutions of the dKP equation: nonlinear Riemann Hilbert problem, longtime behaviour, implicit solutions and wave breaking", J.Phys.A: Math.Theor. **41** (2008) 055204. (arXiv:0707.1802 (2007))

[12] S. V. Manakov and P. M. Santini: "Solvable vector nonlinear Riemann problems, exact implicit solutions of dispersionless PDEs and wave breaking", J. Phys. A: Math. Theor. 44 (2011) 345203 (19pp), doi:10.1088/1751-8113/44/34/345203. arXiv:1011.2619.

[13] S. V. Manakov and P. M. Santini: "On the dispersionless Kadomtsev-Petviashvili equation in n+1 dimensions: exact solutions, the Cauchy problem for small initial data and wave breaking", J. Phys. A: Math. Theor. 44 (2011) 405203 (15pp). (arXiv:1001.2134).

[14] S. V. Manakov and P. M. Santini: "Wave breaking in solutions of the dispersionless Kadomtsev-Petviashvili equation at finite time", Theor. Math. Phys. **172** (2) 1118-1126 (2012).

[15] A. I. Markusevich, *Elementi di teoria delle funzioni analitiche*, Edizioni Mir (Mosca) - Editori Riuniti (Roma), 1988.

[16] F. Santucci: "Onde debolmente non lineari e quasi - unidimensionali in Natura; la rottura di tali onde ed il problema della loro regolarizzazione", Tesi della Laurea Specialistica, AA 2011-12.

[17] P.M.Santini: "appunti 1" (alcuni appunti di lezioni del corso).

[18] M. Tabor, *Chaos and integrability in nonlinear dynamics*, J. Wiley and sons.

[19] J. B. Whitham, *Linear and Nonlinear Waves*, Wiley, NY, 1974.

[20] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, H. C. Morris, *Solitons and Nonlinear Wave Equations*, Academic Press, London, 1982.