17.4 The Schwarzschild solution for a homogeneous star

An analytic solution of the equations of stellar structure (17.60) can be obtained by considering the very simple equation of state:

\[ \epsilon = \text{const}. \]

This solution was found by K. Schwarzschild in 1916 and this is the only exact solution of eqs. (17.60) found up to the present time.

Although homogeneous stars are unrealistic (the speed of sound \( v = \left( \frac{dp}{d\epsilon} \right) \to \infty \)) they can be used as a good approximation for the core of very dense stars, and the interior Schwarzschild solution has been used as a simplified model in a variety of situations to study the effects of gravity in a regime as strong as it can ever become under the condition of hydrostatic equilibrium.

If \( \epsilon = \text{const} \)

\[ m(r) = \frac{4}{3} \pi r^3 \epsilon, \quad (17.71) \]

and from eq. (17.55) one of the metric functions is immediately found

\[ e^{2\lambda} = \left( 1 - \frac{2m(r)}{r} \right)^{-1} \quad \rightarrow \quad e^{2\lambda} = \left( 1 - \frac{8}{3} \pi r^2 \epsilon \right)^{-1}. \quad (17.72) \]

The Oppenheimer-Volkoff equations reduce to

\[ \frac{dp}{dr} = -\frac{4}{3} \pi r \frac{(\epsilon + p)(\epsilon + 3p)}{1 - \frac{8}{3} \pi r^2 \epsilon}, \quad (17.73) \]

that can be integrated to find the pressure; the integration is performed between \( r \) and the radius of the star \( r = R \), where the pressure vanishes:

\[ \log \left( \frac{\epsilon + 3p}{\epsilon + p} \right) \bigg|_{p(r)}^{0} = \frac{1}{2} \log \left( 1 - \frac{8}{3} \pi r^2 \epsilon \right) \bigg|_{r}^{R}. \quad (17.74) \]

which gives

\[ p = \epsilon \frac{(y - y_1)}{(3y_1 - y)}, \quad (17.75) \]

where

\[ y^2 = 1 - \frac{8}{3} \pi r^2 \epsilon = 1 - \frac{2m(r)}{r}, \quad \text{and} \quad y_1^2 = y^2(R) = 1 - \frac{2M}{R}, \quad (17.76) \]

and \( M \equiv M(R) \).

It is interesting to note that if we put \( r = 0 \) in eq. (17.75) we find

\[ p(r = 0) = p_0 = \epsilon \frac{1 - \sqrt{1 - \frac{2M}{R}}}{3 \sqrt{1 - \frac{2M}{R} - 1}}. \quad (17.77) \]
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If the denominator of this expression is zero the central pressure becomes infinite, and negative if it is smaller than zero. Thus, homogeneous stars can exist only if

$$3\sqrt{1 - \frac{2M}{R}} - 1 > 0 \quad \Rightarrow \quad \frac{M}{R} < \frac{4}{9},$$

or, equivalently,

$$R > \frac{9}{4}M.$$  \hspace{1cm} (17.79)

This equation sets a lower limit on the radius that a star of a given mass can have, provided $\epsilon = \text{const.}$ However, in the next section we will show that this result holds for a generic equation of state.

The radius of the star can be found by integrating eq. (17.73) from $r = 0$ (where $p = p_0$) and $r$

$$-\log\left(\frac{\epsilon + 3p}{\epsilon + p}\right) = \frac{1}{2}\log\left(1 - \frac{8\pi R^2}{3}\right) \quad \rightarrow \quad \log\left(\frac{\epsilon + 3p}{\epsilon + p}\right)^{-1} = \log\sqrt{1 - \frac{2M}{R}},$$

from which we find

$$R = \frac{2M}{1 - \frac{(\epsilon + p_0)^2}{y_1^2}}.$$  \hspace{1cm} (17.80)

Thus for any assigned value of $\epsilon$ and of $p_0$ we have a configuration of radius $R$ given by (17.81).

To complete the solution we need to find the metric function $\nu(r)$, which can be determined from eq. (17.63)

$$\nu - \nu_0 = -\int_0^r \frac{p_x}{[\epsilon + p(r)]} dr = -\int_1^y \frac{p_y}{[\epsilon + p(y)]} dy;$$

since

$$p_y = \epsilon \left[\frac{2y_1}{(3y_1 - y)^2}\right], \quad (\epsilon + p) = \epsilon \left[\frac{2y_1}{(3y_1 - y)}\right],$$

we find

$$\nu = \nu_0 - \int_1^y \frac{dy}{(3y_1 - y)} \quad \rightarrow \quad e^{2\nu} = e^{2\nu_0} \left(\frac{3y_1 - y}{3y_1 - 1}\right)^2.$$  \hspace{1cm} (17.83)

At the boundary of the star $y(R) = y_1$ and

$$e^{2\nu(R)} = e^{2\nu_0} \left(\frac{4y_1^2}{(3y_1 - 1)^2}\right);$$

on the other hand we know that the metric must reduce to the Schwarzschild metric, therefore it must also be

$$e^{2\nu(R)} = 1 - \frac{2M}{R} \equiv y_1^2,$$  \hspace{1cm} (17.84)

and by equating eq. (17.84) and (17.85) we find the value of the integration constant $\nu_0$

$$e^{2\nu_0} = \frac{(3y_1 - 1)^2}{4},$$
and the solution for $\nu(r)$ is completely determined

$$e^{2\nu(r)} = \frac{(3y_1 - y)^2}{4}. \quad (17.86)$$

### 17.5 Relativistic polytropes

In this section we shall generalize the Lane-Emden equation in general relativity. Following what we did in section 12.4 for the newtonian case (see eqs. 16.38), we shall solve the relativistic equations of stellar structure (17.60) assuming that the equation of state of the matter inside the star is of a polytropic form, i.e.

\[
\begin{aligned}
\frac{dm(r)}{dr} &= 4\pi r^2 \epsilon \\
\frac{dp}{dr} &= -\frac{(\epsilon + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]} \\
p &= Kc^2(\gamma),
\end{aligned}
\]

where we remind that $\epsilon$ is the relativistic energy density, and we shall also assume that $\epsilon$ and $p$ are expressible in the manner (cfr. eq. (16.40))

\[
\begin{aligned}
\gamma &= 1 + \frac{1}{n}, \\
\epsilon &= \epsilon_0 \Theta^n(r) \\
p &= K\rho_0^{1 + \frac{1}{n}} \Theta^{(n+1)}(r) = p_0 \Theta^{(n+1)}(r), \quad p_0 = K\rho_0^{1 + \frac{1}{n}},
\end{aligned}
\]

With these substitutions eqs. (17.87) become

\[
\begin{aligned}
\frac{dm(r)}{dr} &= 4\pi \epsilon_0 r^2 \Theta^n(r) \\
\frac{d\Theta(r)}{dr} &= -\left[\frac{\epsilon_0 + p_0 \Theta}{p_0(n+1)}\right] \frac{m(r) + 4\pi r^3 p_0 \Theta^{(n+1)}}{r[r - 2m(r)]}.
\end{aligned}
\]

By putting

$$\alpha_0 = \frac{\epsilon_0}{p_0},$$

these equations become

\[
\begin{aligned}
\frac{dm(r)}{dr} &= 4\pi \epsilon_0 r^2 \Theta^n(r) \\
\frac{d\Theta(r)}{dr} &= -\left[\frac{\alpha_0 + \Theta}{(n+1)}\right] \frac{m(r) + 4\pi r^3 \frac{\alpha_0}{\alpha_0} \Theta^{(n+1)}}{r[r - 2m(r)]}.
\end{aligned}
\]
As explained in section 12.7, both \( \epsilon \) and \( p \) have dimensions \([l^{-2}]\), therefore the quantity \( \sqrt{\epsilon_0} \) has dimension \([l^{-1}]\) and we can use it to rescale the radial coordinate as follows. We put

\[
\xi = r \sqrt{\epsilon_0}, \quad \text{and} \quad M = \sqrt{\epsilon_0} m
\]

and rewrite eqs. (17.91) in terms of the new variables (note that \( \xi \) and \( M \) are dimensionless quantities)

\[
\begin{align*}
\frac{dM(\xi)}{d\xi} &= 4\pi \xi^2 \Theta^n(\xi) \\
\frac{d\Theta(\xi)}{d\xi} &= -\left(\frac{\alpha_0 + \Theta}{(n+1)}\right) \frac{M(\xi) + 4\pi \xi^3 \frac{\alpha_0}{\alpha_0} \Theta^{(n+1)}}{\xi [\xi - 2M(\xi)]}.
\end{align*}
\]

We may, at this point, multiply the second equation by \( \xi^2 \), differentiate with respect to \( \xi \) and substituting \( \frac{dM(\xi)}{d\xi} \) into the resulting equation find a second order differential equation for \( \Theta \) in the Lane-Emden form (cfr. 16.43); however, the equation we would get is much more complicated than eq. (16.43), and it is much better to work with the system of two first order eqs. (17.93).

Another important difference with the equation of newtonian polytropes should be stressed. In the newtonian case if we assign the value of the polytropic index \( n \) and integrate the Lane-Emden equation finding \( \Theta(\xi) \) up to the stellar radius \( \xi_1 \), from this solution we can construct a family of solutions by assigning the value of \( K \) and of the central density \( \rho_0 \); no further integrations are needed and, for instance, the radius and the mass of the star can be found by eqs. (16.48) and (16.49). The situation changes in the relativistic equations, because to solve eqs. (17.93) we need to assign both \( n \) and \( \alpha_0 \), i.e. the ratio between the energy density and the pressure at \( \xi = 0 \).

In order to integrate the structure equations numerically, we need to Taylor expand the functions \( \Theta(\xi) \) and \( M(\xi) \) near the origin as in (16.55); from the newtonian expansion we know that only even powers of \( \Theta \) are needed and therefore \( M(\xi) \) will be expanded in odd powers of \( \xi \) (cfr. the first eq. 17.93), therefore we shall put

\[
\Theta(\xi) \sim 1 + \Theta_2 \xi^2 + \Theta_4 \xi^4 + O(\xi^6),
\]

\[
M(\xi) \sim m_3 \xi^3 + m_5 \xi^5 + O(\xi^7).
\]

By inserting this expansions in eqs. (17.93) we find

\[
3m_3 \xi^2 + 5m_5 \xi^4 = 4\pi \xi^2 + 4\pi n \Theta_2 \xi^4
\]

\[
2 \Theta_2 \xi + 4 \Theta_4 \xi^3 = -\frac{1}{n+1} \left\{ \left( m_3 + \frac{4\pi}{\alpha_0} \right) \left[ (1 + \alpha_0)\xi + \Theta_2 \xi^3 \right] \right\}
\]

and by equating the coefficients of the same power of \( \xi \) we find

\[
m_3 - \frac{4\pi}{3} = 0 \quad m_5 - \frac{4\pi n \Theta_2}{5} = 0
\]

\[
\Theta_2 + \frac{1 + \alpha_0}{2(n+1)} \left( m_3 + \frac{4\pi}{\alpha_0} \right) = 0 \quad \Theta_4 + \frac{\Theta_2}{2(n+1)} \left( m_3 + \frac{4\pi}{\alpha_0} \right) = 0
\]

\[
(17.94)
\]
from which we find

\[m_3 = \frac{4\pi}{3}, \quad \Theta_2 = -2\pi \frac{(1 + \alpha_0)(3 + \alpha_0)}{3\alpha_0(n + 1)}\]

\[m_5 = \frac{4\pi n\alpha}{5}, \quad \Theta_4 = -\frac{\Theta_2}{2(n + 1)} \left( m_3 + \frac{4\pi}{\alpha_0} \right).\]

With these initial conditions we can integrate the structure equations and find the value of \(\xi\) where the function \(\Theta\) vanishes so that the pressure vanishes and we are sure we have reached the boundary of the star. Be \(\xi_1\) such value and \(\Theta'_1 = \Theta'(\xi_1)\). From the second eq. (17.93) we find

\[\Theta'_1 = \frac{\alpha_0}{(n + 1)} M(\xi_1) \frac{\mathcal{M}(\xi_1)}{|\xi_1 - 2\mathcal{M}(\xi_1)|},\]

from which we find the mass of the star

\[\mathcal{M}(\xi_1) = \frac{(n + 1)\xi_1^2|\Theta'_1|}{\alpha_0 + 2\xi_1(n + 1)|\Theta'_1|}.\]  \hspace{1cm} (17.95)

Once we know the function \(\Theta(\xi)\), from eqs. (17.88) we know how the energy density and the pressure are distributed inside the star and we can compute the function \(\mathcal{M}(\xi)\)

\[\mathcal{M}(\xi) = \int\xi^4 4\pi\epsilon_0 \xi^2 \Theta^n(\xi') \, d\xi',\]

and the metric function \(e^{2\lambda}\)

\[e^{2\lambda} = \frac{1}{1 - \frac{2\mathcal{M}(\xi)}{\xi}}.\]

The remaining metric function \(e^{2\nu}\) can be found from eq. (17.63) which now becomes

\[\nu = \int \frac{p_\xi}{(\epsilon + p)} \, d\xi + \nu_0 = \int \frac{(n + 1)\Theta_0}{\Theta + \Theta_0} \, d\xi + \nu_0 = \ln \left[ \frac{\alpha_0 + 1}{\alpha_0 + \Theta(\xi)} \right]^{(n + 1)} + \nu_0,\]  \hspace{1cm} (17.96)

At the surface of the star the metric must reduce to the Schwarzschild metric and therefore

\[e^{2\nu_0} \cdot \left[ \frac{(\alpha_0 + 1)}{\alpha_0 + \Theta} \right]^{2(n + 1)} = 1 - \frac{2\mathcal{M}(\xi)}{\xi}\]

which, using eq. (17.95), gives

\[e^{2\nu_0} = \left[ \frac{\alpha_0}{\alpha_0 + 1} \right]^{2(n + 1)} \cdot \frac{\alpha_0}{\alpha_0 + 2\xi_1(n + 1)|\Theta'_1|}\]

\hspace{1cm} (17.97)

Thus,

\[e^{2\nu(\xi)} = \left[ \frac{\alpha_0}{\alpha_0 + 1} \right]^{2(n + 1)} \cdot \frac{\alpha_0}{\alpha_0 + 2\xi_1(n + 1)|\Theta'_1|} \cdot \left[ \frac{\alpha_0 + 1}{\alpha_0 + \Theta(\xi)} \right]^{2(n + 1)}\]  \hspace{1cm} (17.98)

and the solution is finally complete.
17.6 Buchdhal’s theorem

A theorem proved by Buchdhal in 1959 establishes that the result obtained in the section 17.4 about the maximum value that the ratio $M/R$ in a star of constant energy-density can reach, i.e. $M/R < 4/9$, is much more general. The theorem is based on the only assumption that the star is static, and that the energy density is positive, and monotonically decreasing function of the radial coordinate, i.e.

$$\epsilon \geq 0, \quad \frac{de}{dr} \leq 0.$$ 

No assumption is made on the equation of state that relates $\epsilon$ and the pressure $p$. The relevant equations are

$$\begin{align*}
G_{rr} &= 8\pi T_{rr} \\
G_{\theta\theta} &= 8\pi T_{\theta\theta} \\
\text{(1)} &- e^{-2\lambda} \left[\frac{2}{r} + \frac{2}{r}\nu_r\right] + \frac{2}{r}\nu_r = 8\pi p e^{2\lambda}, \\
\text{(2)} &- r^2 e^{-2\lambda} \left[\nu_{rr} + \nu_r^2 + \frac{\nu_r}{r} - \nu_r \lambda_r - \lambda r\right] = 8\pi r^2 p,
\end{align*}$$

(17.99)

By taking the following combination of eqs. (17.99)

$$r^2 e^{-2\lambda}EQ.(1) - EQ.(2) = 0,$$

(17.100)

we find

$$\frac{d}{dr} \left[ e^{-\lambda} \frac{d}{r} \frac{d}{dr} (\epsilon^r) \right] = e^{\nu + \lambda} \frac{d}{dr} \left[ \frac{m(r)}{r^3} \right],$$

(17.101)

where, as usual, $m(r) = 4\pi \int_0^r \epsilon r^2 dr'$. For any $r$ we can always define a density $\bar{\epsilon}_r$ such that

$$m(r) = \frac{4}{3} \pi \bar{\epsilon}_r r^3,$$

(17.102)

and since $\epsilon$ is a monotonically decreasing function of $r$, $\bar{\epsilon}_r$, and consequently $\frac{m(r)}{r^3}$, are also monotonically decreasing. Hence

$$\frac{d}{dr} \left[ e^{-\lambda} \frac{d}{r} \frac{d}{dr} \left( \epsilon^r \right) \right] \leq 0.$$ 

(17.103)

It should be noted that the minimum value of $\bar{\epsilon}_r$ is attained at the boundary, where

$$M = \frac{4}{3} \pi \bar{\epsilon}_R R^3,$$

(17.104)

where $\bar{\epsilon}_R = \epsilon_{min}$. Inside the star $\bar{\epsilon}_r \geq \epsilon_{min}$, and consequently

$$\frac{4}{3} \pi \bar{\epsilon}_r R^3 \geq \frac{4}{3} \pi \epsilon_{min} R^3 \quad \rightarrow \quad m(r) \geq \frac{M}{R^3} r^3,$$

(17.105)

i.e. $m(r)$ is always bigger than what it would be if the density would be constant and equal to $\epsilon_{min}$. From eq. (17.103) it follows that

$$\frac{e^{-\lambda} \frac{d}{r} \frac{d}{dr} (\epsilon^r)}{\epsilon^r} \geq \frac{e^{-\lambda} \frac{d}{r} \frac{d}{dr} (\epsilon^r)}{\epsilon^r} \bigg|_{r=R}.$$ 

(17.106)
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When \( r = R \) the metric reduces to the Schwarzschild metric, therefore \( e^{2\nu} \big|_{r=R} = e^{-2\lambda} \big|_{r=R} = 1 - \frac{2M}{R} \) and eq. (17.106) gives

\[
e^{-\lambda} \frac{d(e^{\nu})}{dr} \geq \frac{M}{R^3} \quad \rightarrow \quad \frac{de^{\nu}}{dr} \geq re^{\lambda} \frac{M}{R^3}.
\]

(17.107)

By integrating eq. (17.107) between 0 and \( R \) we find

\[
e^{\nu}(R) - e^{\nu}(0) \geq \frac{M}{R^3} \int_0^R re^{\lambda} dr,
\]

(17.108)

and since \( e^{-2\lambda} = 1 - \frac{2m(r)}{r} \)

\[
e^{\nu}(0) \leq \sqrt{1 - \frac{2M}{R} - \frac{M}{R^3} \int_0^R \sqrt{1 - \frac{2m(r)}{r}} r dr},
\]

(17.109)

We want to establish an upper boundary for \( e^{\nu}(0) \), and therefore we need to determine when the right hand side of eq. (17.109) attains its maximum value. From eq. (17.105) we know that \( m(r) \geq \frac{M}{R^3} r^3 \), and consequently

\[
\sqrt{1 - \frac{2m(r)}{r}} \leq \sqrt{1 - \frac{2M}{R} - \frac{M}{R^3} r^2}, \quad \rightarrow \quad \int_0^R \frac{r dr}{\sqrt{1 - \frac{2m(r)}{r}}} \geq \int_0^R \frac{r dr}{\sqrt{1 - \frac{2M}{R} - \frac{1}{2}}},
\]

Thus the maximum value of the right hand side of eq. (17.109) is

\[
R.H.S._{\text{max}} = \sqrt{1 - \frac{2M}{R} - \frac{M}{R^3} \int_0^R \sqrt{1 - \frac{2m(r)}{r}} r dr} = \frac{3}{2} \sqrt{1 - \frac{2M}{R} - \frac{1}{2}},
\]

(17.111)

and consequently

\[
e^{\nu}(0) \leq \frac{3}{2} \sqrt{1 - \frac{2M}{R} - \frac{1}{2}},
\]

(17.112)

We shall now use the second hypothesis of the theorem, i.e. the condition that the metric is static. A static spacetime admits a timelike Killing vector, which must remain timelike in the interior of the star, i.e.

\[
\vec{\xi} \cdot \vec{\xi} = g_{00}(\xi^0)^2 < 0 \quad \rightarrow \quad g_{00} = -e^{2\nu} < 0 \quad \rightarrow \quad e^{2\nu} > 0.
\]

(17.113)

It follows from eq. (17.112) that

\[
\frac{3}{2} \sqrt{1 - \frac{2M}{R} - \frac{1}{2}} > 0,
\]

(17.114)

and finally

\[
\frac{M}{R} < \frac{4}{9},
\]

(17.115)

Q.E.D.

It should be noted that, since \( \frac{M}{R} < \frac{4}{9} \), it follows that \( R > \frac{9}{4} M \), and since the Schwarzschild radius is \( R_S = 2M \) this means that a star cannot have radius smaller than the Schwarzschild radius.
17.7 A necessary condition for the stability of a compact star

A solution of the TOV equations (17.60) which satisfies the appropriate boundary conditions discussed in section 17.3.1, describes a stellar configuration in hydrostatic equilibrium. This equilibrium can, in principle, be either stable or unstable. In this section we will study the conditions for stability.

Let us consider a sequence of equilibrium configurations obtained by integrating the TOV equations for an assigned EOS, with different values of the central energy density $\epsilon_0$. The gravitational mass will thus be a function of $\epsilon_0$: $M = M(\epsilon_0)$.

Let us consider the profile $M(\epsilon_0)$ given in figure 17.4. Each point of this curve identifies an equilibrium configuration. Given a star in the equilibrium configuration $A$ if a small radial perturbation reduces its central energy density to a value, say, $\epsilon_{01}$, the new (non-equilibrium) configuration will be represented by point $A_1$ (because the mass of the star does not change). Point $A_1$ is above the curve, therefore the perturbed star has a mass which is larger than the mass that the equilibrium configuration corresponding to $\epsilon_{01}$ would have. Consequently, the star is off equilibrium because gravity exceeds pressure, and the star will contract, so that its central density increases and it can return to the equilibrium configuration $A$.

In a similar way, if a perturbation increases the central energy density to $\epsilon_{02}$, the new configuration is represented by a point $A_2$ below the curve. The star in $A_2$ has mass smaller than that of the equilibrium configuration corresponding to $\epsilon_{02}$. In this case gravity is weaker than pressure, and the star will expand to return to the equilibrium configuration. Thus, the equilibrium in $A$ is stable.

We can conclude that if $A$ is a stable equilibrium configuration, in $A$

$$\frac{dM}{d\epsilon_0} > 0,$$

(17.116)
Conversely, a similar discussion about the point $B$ where

$$\frac{dM}{d\epsilon_0} < 0,$$  

shows that a displacement to $B_1$, brings the star to a configuration where gravity is weaker than pressure, so that the star expands further reducing the central density. Similarly, a displacement to $B_2$ brings the star to a configuration where gravity exceeds pressure, so that the star contracts, and the central density further increases: the equilibrium in $B$ is unstable.

In figure 17.4, the branch on the left of the maximum $C$ corresponds to stable configurations, that on the right to unstable configurations. The point $C$ is the configuration of maximum mass.

An example is the case of Newtonian polytropes: the function $M(\epsilon_0)$, given in eq.(16.49), which we rewrite here for simplicity

$$M = 4\pi \xi_1^2 |\Theta'(\xi_1)| \left[ \frac{(n + 1)K}{4\pi G} \right]^\frac{2}{n} \cdot \rho_0^\frac{2-n}{n},$$  

shows that $M$ is an increasing function of the central density $\rho_0$ for $n < 3$, it is stationary for $n = 3$, and decreasing for $n > 3$; therefore the star is stable only if $n < 3$.

For a realistic equation of state, the curve $M(\epsilon_0)$ has a profile similar to that shown in figure 17.5. The stable branch on the left of point $A$ represents white dwarf configurations,

while the stable branch $BC$ represents neutron star configurations.

If we consider the stellar mass as a function of the radius, we find that since

$$\frac{dM}{dR} = \frac{dM}{d\epsilon_0} \cdot \frac{d\epsilon_0}{dR};$$  

Figure 17.5: Masses of equilibrium stellar configurations vs. central densities. The stable branches corresponding to white dwarfs and neutron stars are explicitly shown.
the stability criterion (17.116) is satisfied if

a) \( \frac{dR}{d\epsilon} > 0 \) and \( \frac{dM}{dR} > 0 \)

or if

b) \( \frac{dR}{d\epsilon} < 0 \) and \( \frac{dM}{dR} < 0 \).

In general, both for white dwarfs and for neutron stars the radius of the star decreases as the central density increases, therefore the stable branches of the function \( M(R) \) are those for which

\[
\frac{dM}{dR} < 0. \tag{17.120}
\]

17.7.1 Is the condition \( \frac{dM}{d\epsilon_0} > 0 \) sufficient to say that a star is stable?

The question in the heading of this subsection can be rephrased as follows: if \( \frac{dM}{d\epsilon_0} > 0 \), can we say that the star is stable?

The answer is No, and the reason can be understood by considering the theory of radial perturbations of stars. Since this interesting development is outside the scopes of this book, we shall just sketch the main results of the theory and give the basic notions to understand them.

A star has an infinite set of radial proper oscillation modes, labelled by an index \( n = 0, 1, 2, \ldots \); when the star oscillates in a mode, each fluid element is displaced from the equilibrium position by a radial displacement \( \xi(t, r) \). For the \( n_{th} \) mode \( \xi \) has the form

\[
\xi_n(r, t) = u_n(r)e^{i\omega_n t} \tag{17.121}
\]

where \( \omega_n \) is the mode frequency and \( u(r) \) its amplitude. The mode number \( n \) corresponds to the number of nodes that \( u(r) \) has inside the star: \( n = 0 \) for zero nodes, \( n = 1 \) for 1 node etc. The mode frequencies are ordered:

\[
\omega_0^2 < \omega_1^2 < \omega_2^2 < \ldots, \tag{17.122}
\]

and the mode corresponding to \( \omega_0 \) is said the fundamental mode.

If \( \omega_n^2 > 0 \), the fluid element oscillates about the equilibrium position and the mode is stable; conversely, if \( \omega_n^2 < 0 \) the radial displacement grows exponentially and the mode is unstable.

A stable fundamental mode corresponds to a global oscillation of the star, which is expanding or contracting all at the same time; indeed, this is also called the “breathing mode”. When the star contracts the central density increases and \( \xi(t, r) < 0 \) throughout the star, when it expands the central density decreases and \( \xi(t, r) < 0 \). The previous discussion about stability clearly applies to this case.

However the star may oscillate in a different mode with \( n > 0 \). Since in this case \( u(r) \) has one node inside the star, we may have a situation in which near the origin \( \xi(t, r) < 0 \) and the central density increases, but in some other region of the star \( \xi(t, r) > 0 \) and in that region the density would decrease. Thus, the previous discussion about stability would not be applicable, and we need to appeal to the theory of radial pulsation to understand what is
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going on. The theory states the following: suppose we compute a sequence of stellar models (with the same EOS) differing for the value of $\epsilon_0$, and for each model we compute the mass and the frequency of the various radial modes. Knowing $M(\epsilon_0)$ along the sequence, we can compute $\frac{dM}{d\epsilon_0}$. If for some value of $\epsilon_0$ we find that there is an extremal point, i.e.

$$\frac{dM}{d\epsilon_0} = 0,$$

then for that same $\epsilon_c$ the square of the frequency of one of the modes must cross the real axis and change sign, therefore in that point

$$\omega_i^2 = 0.$$  \hspace{1cm} (17.124)

This means the the $i$th-mode becomes unstable.

In general, the $n = 0$ mode (which is the one with lowest frequency) is the first to become unstable.

Now suppose that we have the curve shown in figure 17.6 and suppose that for $\epsilon_0 < \epsilon_A$ the fundamental mode has frequency such that $\omega_0^2 > 0$, i.e. it is stable. $A$ is an extremal point, therefore in $A \omega_0^2 = 0$, and all configurations belonging to the branch $AB$ will be unstable because their fundamental mode will have $\omega_0^2 < 0$.

If we increase the density we reach the second extremal point $B$. Here two things may happen:

1) $\omega_0^2$ changes sign again becoming positive. In this case the star corresponding to $B$ and all configurations of the branch $BC$ would be stable. This is the situation we have described in the previous section.

or

2) $\omega_0^2$ remains negative (i.e. the fundamental mode remains unstable) and the frequency of the $n = 1$ radial mode changes sign, so that also the $n = 1$ mode become unstable. In this case all configurations on the branch $BC$ would be unstable.

Figure 17.6: Masses of equilibrium stellar configurations vs. central densities. Though in the $BC$ branch $\frac{dM}{d\epsilon_0} > 0$, that branch may correspond to unstable configuration, as explained in the text.
This example clearly illustrates that the fact that \( \frac{dH}{d\omega} > 0 \) does not provide a sufficient condition for stability.