Differentials as one-forms

The differential of a real function Φ, $d\Phi$, is the variation of the function in an *unspecified direction*, at first order in the displacement. If we specify the direction as given by an arbitrary vector $\vec{V}$, we can explicitly compute the specific variation of the function, which is a real number, i.e.[2]

$$d\Phi(\vec{V}) = V^j \frac{\partial \Phi}{\partial x^j} . \tag{2.100}$$

This expression is linear in $\vec{V}$, thus, according to the definition 2.77, *the differential of a function Φ is a one-form.* The components of $d\Phi$ are

$$d\Phi_i = d\Phi(\vec{e}_{(i)}) = e^j_{(i)} \frac{\partial \Phi}{\partial x^j} = \delta^j_i \frac{\partial \Phi}{\partial x^j} = \frac{\partial \Phi}{\partial x^i} . \tag{2.101}$$

Thus, *the components of the one-form $d\Phi$ are the components of the gradient of the function.* To hereafter, we shall omit the superscripted tilde over the differential $d\Phi$ to follow the standard notation of textbooks.

According to the definition, the differential of any coordinate x^i is the one-form dx^i such that

$$dx^i(\vec{V}) = V^j \frac{\partial x^i}{\partial x^j} = V^j \delta^i_j = V^i . \tag{2.102}$$

i.e. the one-form dx^i associates to any vector $\vec{V}$ the component V^i. Note that this is the definition of the one-forms of the coordindate basis given in Eq. 2.84, i.e. $\tilde{\omega}^{(i)}(\vec{V}) = V^i$; therefore, *the differentials dx^i are the coordinate basis one-forms*:

$$\tilde{\omega}^{(i)} = dx^i , \tag{2.103}$$

whose components are

$$\omega^{(i)}_j = dx^i(\vec{e}_{(j)}) = \delta^i_j . \tag{2.104}$$

[2]Note that the right-hand side of Eq. 2.100 coincides with that of Eq. 2.58, i.e. with the directional derivative of the function Φ along $\vec{V}$. However, while the directional derivative $d/d\lambda$ maps C^1 functions to $\mathbb{R}$, the differential of a function $d\Phi$ maps vectors to $\mathbb{R}$.