QUINTA ESERCITAZIONE

Consideriamo lo spazio-tempo descritto, nel riferimento O, dalle coordinate $\{x^\mu\} = (t, r, \theta, \phi)$, con metrica

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{1}$$

dove $M > 0$ è una costante. Sia dato il vettore $\vec{V}$ di componenti

$$V^\mu = \left(\sqrt{1 - \frac{2M}{r}}, r, 0, 0\right) . \tag{2}$$

Calcolare simboli di Christoffel.

Calcolare

$$V^t{}_{;t}\,; \quad V^t{}_{;r}\,; \quad V^\theta{}_{;r}\,; \quad V^\theta{}_{;\theta}\,. \tag{3}$$

I simboli di Christoffel possono essere calcolati dalle formule

$$\Gamma^{\ \mu}_{\alpha\beta} = g^{\mu\nu}\Gamma_{\alpha\beta\,\nu} \tag{4}$$

dove abbiamo definito per comodità di calcolo i simboli di Christoffel con indici bassi

$$\Gamma_{\alpha\beta\,\nu} \equiv \frac{1}{2}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu})\,. \tag{5}$$

Le derivate non nulle delle componenti del tensore metrico sono

$$\begin{aligned} &g_{tt,r} = -\frac{2M}{r^2} \qquad g_{rr,r} = -\frac{2M}{r^2}\left(1-\frac{2M}{r}\right)^{-2} \qquad g_{\theta\theta,r} = 2r \\ &g_{\phi\phi,r} = 2r\sin^2\theta \qquad g_{\phi\phi,\theta} = 2r^2\sin\theta\cos\theta \end{aligned} \tag{6}$$

quindi i simboli di Christoffel non nulli con indici bassi sono

$$\begin{aligned} \Gamma_{tt\,r} &= \frac{1}{2}(g_{tr,t} + g_{tr,t} - g_{tt,r}) = -\frac{1}{2}g_{tt,r} = \frac{M}{r^2} \\ \Gamma_{tr\,t} = \Gamma_{rt\,t} &= \frac{1}{2}(g_{tt,r} + g_{rt,t} - g_{tr,t}) = \frac{1}{2}g_{tt,r} = -\frac{M}{r^2} \\ \Gamma_{rr\,r} &= \frac{1}{2}(g_{rr,r} + g_{rr,r} - g_{rr,r}) = \frac{1}{2}g_{rr,r} = -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-2} \\ \Gamma_{\theta\theta,r} &= -\frac{1}{2}g_{\theta\theta,r} = -r \\ \Gamma_{\theta r,\theta} = \Gamma_{r\theta,\theta} &= \frac{1}{2}g_{\theta\theta,r} = r \\ \Gamma_{\phi\phi,r} &= -\frac{1}{2}g_{\phi\phi,r} = -r\sin^2\theta \\ \Gamma_{\phi r,\phi} = \Gamma_{r\phi\phi} &= \frac{1}{2}g_{\phi\phi,r} = r\sin^2\theta \\ \Gamma_{\phi\phi,\theta} &= -\frac{1}{2}g_{\phi\phi,\theta} = -r^2\sin\theta\cos\theta \\ \Gamma_{\phi\theta,\phi} = \Gamma_{\theta\phi\phi} &= \frac{1}{2}g_{\phi\phi,\theta} = r^2\sin\theta\cos\theta\,. \end{aligned} \tag{7}$$

Contraendo con la metrica inversa

$$g^{\mu\nu} = \mathrm{diag}\left(-\left(1-\frac{2M}{r}\right)^{-1}, \left(1-\frac{2M}{r}\right), r^{-2}, r^{-2}\sin^{-2}\theta\right)\,, \tag{8}$$

si trova l'insieme dei simboli di Christoffel non nulli:

$$\begin{aligned} \Gamma^{\ r}_{tt} &= \frac{M}{r^2}\left(1-\frac{2M}{r}\right) \\ \Gamma^{\ t}_{rt} = \Gamma^{\ t}_{tr} &= \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1} \end{aligned}$$

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\begin{aligned}
\Gamma_{rr}^{\ r} &= -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1} \\
\Gamma_{\theta\theta}^{\ r} &= -(r-2M) \\
\Gamma_{\phi\phi}^{\ r} &= -(r-2M)\sin^2\theta \\
\Gamma_{\theta r}^{\ \theta} = \Gamma_{r\theta}^{\ \theta} &= \frac{1}{r} \\
\Gamma_{\phi r}^{\ \phi} = \Gamma_{r\phi}^{\ \phi} &= \frac{1}{r} \\
\Gamma_{\phi\phi}^{\ \theta} &= -\sin\theta\cos\theta \\
\Gamma_{\phi\theta}^{\ \phi} &= \cot\theta\,.
\end{aligned} \tag{9}
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Si ha

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\begin{aligned}
V^t_{\ ;t} &= V^t_{\ ,t} + \Gamma_{t\mu}^{\ t}V^\mu = \Gamma_{tr}^{\ t}V^r = \frac{M}{r\left(1-\frac{2M}{r}\right)} \\
V^t_{\ ;r} &= V^t_{\ ,r} + \Gamma_{r\mu}^{\ t}V^\mu = V^t_{\ ,r} + \Gamma_{rt}^{\ t}V^t = \frac{2M}{r^2\sqrt{1-\frac{2M}{r}}} \\
V^\theta_{\ ;r} &= V^\theta_{\ ,r} + \Gamma_{r\mu}^{\ \theta}V^\mu = 0 \\
V^\theta_{\ ;\theta} &= V^\theta_{\ ,\theta} + \Gamma_{\theta\mu}^{\ \theta}V^\mu = \Gamma_{\theta r}^{\ \theta}V^r = 1\,.
\end{aligned} \tag{10}
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