9.6 THE BIRKHOFF THEOREM

The solution (9.24) has been found by imposing that the spacetime is static and spherically symmetric, therefore it represents the gravitational field external to a non-rotating, sperically symmetric body whose structure is time-independent. However, the Schwarzschild solution is more general, since, as shown by G.D. Birkhoff in 1923, it is the only *spherically symmetric, asymptotically flat solution to vacuum, Einstein's field equations*. Thus, to prove Birkhoff's theorem we need to relax the assumption that the metric admits a timelike, hypersurface orthogonal Killing vector field. We shall now generalize the results of Sec. 9.1, where we showed how to choose the coordinates by imposing the spherical symmetry, assuming that the metric depends on time. As in Sec. 9.1 we fill the three-dimensional space with 2-spheres, with 2-metric (see Eq. 9.8)

$$ds^2_{(2)} = a^2(x^0, x^1)(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{9.111}$$

where $a^2(x^0, x^1)$ is an unspecified function. Contrary to what we did in Sec. 9.1, we shall now retain the dependence on x^0. The basis vectors $\vec{e}_{(\theta)}$ and $\vec{e}_{(\varphi)}$ are tangent, respectively, to the coordinate lines ($\varphi = const, \theta = const$), which we choose on the 2-spheres. Then, we align the poles of all 2-spheres as explained in Sec. 9.1; in addition we choose the basis vector $\vec{e}_{(1)}$ parallel to the vector $\vec{\zeta}$ shown in figure 9.1, which joins points with fixed values of θ and φ on neighbouring spheres. In this way $\vec{e}_{(1)}$ is orthogonal, at each space point, to both $\vec{e}_{(\theta)}$ and $\vec{e}_{(\varphi)}$, and the metric of the 3-space can be written as

$$ds^2_{(3)} = g_{11}(dx^1)^2 + a^2(x^0, x^1)(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{9.112}$$

The metric of the four-dimensional spacetime therefore becomes

$$ds^2 = g_{00}(x^0, x^1)(dx^0)^2 + g_{11}(x^0, x^1)(dx^1)^2 + 2g_{01}(x^0, x^1)dx^0dx^1 + a^2(x^0, x^1)(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{9.113}$$

We now change coordinates from (x^0, x^1) to (x^0, r) where

$$r = a(x^0, x^1)$$

so that the metric becomes

$$ds^2 = g_{00}(x^0, r)(dx^0)^2 + g_{rr}(x^0, r)(dr^1)^2 + 2g_{0r}(x^0, r)dx^0dr + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{9.114}$$

We want to find a function $t(x^0, r)$ such that, if we choose it as a time-coordinate, the cross term g_{tr} in the metric vanishes and the first three terms in Eq. 9.114 can be written as

$$g_{00}(x^0, r)(dx^0)^2 + g_{rr}(x^0, r)(dr^1)^2 + 2g_{0r}(x^0, r)dx^0dr = bdt^2 + cdr^2\,, \tag{9.115}$$

where b and c are functions of (x^0, r) to be determined. Being

$$dt = \frac{\partial t}{\partial x^0}dx^0 + \frac{\partial t}{\partial r}dr\,, \tag{9.116}$$

Eq. 9.115 becomes

$$\begin{aligned} &g_{00}(dx^0)^2 + g_{rr}dr^2 + 2g_{0r}dx^0dr \\ &= b\left(\frac{\partial t}{\partial x^0}\right)^2(dx^0)^2 + b\left(\frac{\partial t}{\partial r}\right)^2 dr^2 + 2b\frac{\partial t}{\partial x^0}\frac{\partial t}{\partial r}dtdr + cdr^2\,, \end{aligned} \tag{9.117}$$

which gives

$$b\left(\frac{\partial t}{\partial x^0}\right)^2 = g_{00} \tag{9.118}$$
$$b\left(\frac{\partial t}{\partial r}\right)^2 + c = g_{rr}$$
$$b\frac{\partial t}{\partial x^0}\frac{\partial t}{\partial r} = g_{0r}\,.$$

These are three equations for the three unknown functions $t(x^0, r)$, $b(x^0, r)$ and $c(x^0, r)$, which can in principle be solved. By inverting the function $t(x^0, r)$ with respect to x^0 and by replacing the result in b and c, they become function of (t, r) and the metric can be written as

$$ds^2 = b(t, r)dt^2 + c(t, r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{9.119}$$

Since we require the spacetime to be Lorentzian, the function $b(x^0, r)$ has be negative. We know that this choice may go wrong due to the choice of coordinates, as it occurs in the Schwarzschild metric when $r < 2m$, however we will assume it for now. Furthermore, we will replace $b(t, r)$ and $c(t, r)$ with the functions $\nu(t, r)$ and $\lambda(t, r)$, where

$$b(t, r) = -e^{2\nu(t,r)}\,, \qquad c = e^{2\lambda(t,r)}\,,$$

so that, finally, the metric of a time-dependent, spherically symmetric spacetime becomes

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{9.120}$$

To prove Birkhoff's theorem we only need the components R_{tr} and $R_{\theta\theta}$ of the Ricci tensor:

$$a) R_{tr} = \frac{2}{r}\frac{\partial\lambda}{\partial t} = 0\,, \tag{9.121}$$
$$b) R_{\theta\theta} = 1 - e^{-2\lambda}\left[1 + r\frac{\partial(\nu - \lambda)}{\partial r}\right] = 0\,.$$

From Eq. (9.121a) it follows that λ must depend only on the radial coordinate r. Then from eq. (9.121b) it follows that

$$\frac{\partial\nu}{\partial r} = \frac{\partial\lambda}{\partial r} + \frac{e^{2\lambda(r)} - 1}{r}\,,$$

i.e. $\frac{\partial\nu}{\partial r}$ depends only on r. Consequently we can write

$$\nu = \nu(r) + f(t)\,, \tag{9.122}$$

and

$$ds^2 = -e^{2\nu(r)}e^{2f(t)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{9.123}$$

The term $e^{2f(t)}$ can be reabsorbed by a coordinate transformation such that

$$dt' = e^{f(t)}dt\,, \tag{9.124}$$

and the metric finally becomes

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{9.125}$$

where the prime has be suppressed for simplicity. Thus, we have shown that even if we assume that the metric of a spherically symmetric spacetime depends on time, by suitable coordinate transformations it can be made time independent. Since, as we have shown in Sec. 9.1 the only solution to the vacuum Einstein equations for the metric 9.125, which is asymptotically flat, is the Schwarzschild solution, we have then proved the Birkhoff theorem stated at the beginning of this section. An important consequence of this theorem is the following. The metric external to a spherically symmetric star is the Schwarzschild metric even when the star is collapsing, or exploding, or radially pulsating. Thus, spherically symmetric systems can never emit gravitational waves. A similar situation occurs in electrodynamics: a spherically symmetric distribution of charges and currents does not radiate.