$V^\alpha{}_{;\beta}$ are the components of a tensor field

Let us define the following quantity:

$$\nabla\vec{V} = V^\alpha{}_{;\beta}\ \vec{e}_{(\alpha)} \otimes \tilde{\omega}^{(\beta)}\,. \tag{3.14}$$

We shall now show that $\nabla\vec{V}$ is a $\binom{1}{1}$ tensor whith components $V^\alpha{}_{;\beta}$.

We remind that a $\binom{1}{1}$ tensor, say $\mathbf{F} = F^\alpha{}_\beta\ \vec{e}_{(\alpha)} \otimes \tilde{\omega}^{(\beta)}$, maps vectors to vectors. Indeed, if we apply $\mathbf{F}$ to a generic vector $\vec{V}$, remembering that $\tilde{\omega}^{(\beta)}(\vec{V}) = V^\beta$, we get

$$F(\ \ ,\vec{V}) = F^\alpha{}_\beta \vec{e}_{(\alpha)}\ \tilde{\omega}^{(\beta)}(\vec{V}) = F^\alpha{}_\beta V^\beta\ \vec{e}_{(\alpha)}\,. \tag{3.15}$$

Thus, the result of this operation is a vector with components $F^\alpha = F^\alpha{}_\beta V^\beta$.

Let us now consider a curve on the manifold $x^\mu(\lambda)$, passing through the point $\mathbf{p}$, with tangent vector $t^\mu = \frac{dx^\mu}{d\lambda}$, and let us apply $\nabla\vec{V}$ to $\vec{t}$

$$\nabla\vec{V}(\vec{t}) = V^\alpha{}_{;\beta}\ \vec{e}_{(\alpha)}\ \tilde{\omega}^{(\beta)}(\vec{t}) = V^\alpha{}_{;\beta}\ t^\beta\ \vec{e}_{(\alpha)}\,. \tag{3.16}$$

The quantities $V^\alpha{}_{;\beta}t^\beta$ are the components of the directional derivative of the vector $\vec{V}$ along the curve; indeed, using Eq. 3.12 we find

$$\frac{d\vec{V}}{d\lambda} = \frac{\partial\vec{V}}{\partial x^\beta}\frac{dx^\beta}{d\lambda} = V^\alpha{}_{;\beta}t^\beta\ \vec{e}_{(\alpha)}\,. \tag{3.17}$$

The directional derivative of a vector field along a curve

$$\frac{d\vec{V}}{d\lambda} = \lim_{\Delta\lambda\to 0}\frac{\vec{V}(\lambda+\Delta\lambda)-\vec{V}(\lambda)}{\Delta\lambda} \tag{3.18}$$

is a vector, because it is the difference of two vectors (which we can compute after having defined a connection) divided by the real number $\Delta\lambda$. Therefore

$$\nabla\vec{V}(\vec{t}) = V^\alpha{}_{;\beta}t^\beta\ \vec{e}_{(\alpha)} = \frac{d\vec{V}}{d\lambda}\,. \tag{3.19}$$

We also denote the covariant derivative of $\vec{V}$ along $\vec{t}$ as $\nabla_{\vec{t}}\vec{V} \equiv \nabla\vec{V}(\vec{t})$, and, when $\vec{t}$ is a basis vector, $\nabla_\mu\vec{V} \equiv \nabla_{\vec{e}_{(\mu)}}\vec{V}$.

Thus, $\nabla\vec{V}$ maps the vector $\vec{t}$ to the vector $\frac{d\vec{V}}{d\lambda}$, i.e. it is a $\binom{1}{1}$ tensor field, called **covariant derivative of the vector** $\vec{V}$. Its components are

$$(\nabla\vec{V})^\alpha{}_\beta \equiv \nabla_\beta V^\alpha \equiv V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} + V^\mu\Gamma^\alpha_{\beta\mu}\,. \tag{3.20}$$

Since in a LIF Christoffel's symbols vanish, from Eq. 3.11 it follows that

$$V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} \Longrightarrow \frac{\partial\vec{V}}{\partial x^\beta} = V^\alpha{}_{,\beta}\ \vec{e}_{(\alpha)}\,. \tag{3.21}$$

Thus, in a locally inertial frame covariant and ordinary derivatives coincide.