Chapter 10

Symmetries

H. Weyl: "*Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection.*"

The solution of a physical problem can be considerably simplified if it allows some symmetries. Let us consider for example the equations of Newtonian gravity. It is easy to find a solution which is spherically symmetric, but it may be difficult to find the analytic solution for an arbitrary mass distribution.

In euclidean space a symmetry is related to an invariance with respect to some operation. For example plane symmetry implies invariance of the physical variables with respect to translations on a plane, spherically symmetric solutions are invariant with respect to translation on a sphere, and the equations of Newtonian gravity are symmetric with respect to time translations

$$t' \to t + \tau.$$

Thus, a symmetry corresponds to invariance under translations along certain lines or over certain surfaces. This definition can be applied and extended to Riemannian geometry. A solution of Einstein's equations has a symmetry if there exists an n-dimensional manifold, with $1 \leq n \leq 4$, such that the solution is invariant under translations which bring a point of this manifold into another point of the same manifold. For example, for spherically symmetric solutions the manifold is the 2-sphere, and n=2. This is a simple example, but there exhist more complicated four-dimensional symmetries. These definitions can be made more precise by introducing the notion of Killing vectors.

10.1 The Killing vectors

Consider a vector field $\vec{\xi}(x^\mu)$ defined at every point x^α of a spacetime region. $\vec{\xi}$ identifies a symmetry if an infinitesimal translation along $\vec{\xi}$ leaves the line-element unchanged, i.e.

$$\delta(ds^2) = \delta(g_{\alpha\beta}dx^a dx^b) = 0. \tag{10.1}$$

This implies that

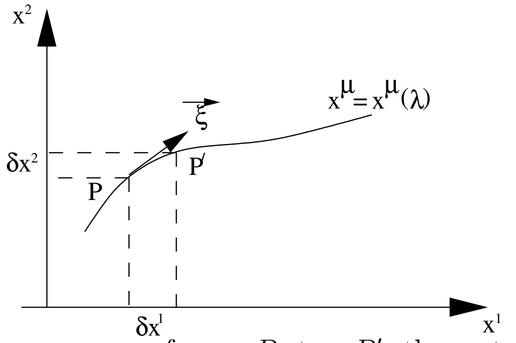
$$\delta g_{\alpha\beta}dx^a dx^b + g_{\alpha\beta}\left[\delta(dx^a)dx^b + dx^a\delta(dx^b)\right] = 0. \tag{10.2}$$

$\vec{\xi}$ is the tangent vector to some curve $x^\alpha(\lambda)$, i.e. $\xi^\alpha = \frac{\delta x^\alpha}{d\lambda}$, therefore an infinitesimal translation in the direction of $\vec{\xi}$ is an infinitesimal translation along the curve from a point $P(\lambda)$ to the point $P'(\lambda + d\lambda)$. Putting

$$\delta x^\alpha = x^\alpha(\lambda + d\lambda) - x^\alpha(\lambda) = \frac{dx^\alpha}{d\lambda} d\lambda = \xi^\alpha d\lambda\,,$$

the coordinates of $P(\lambda)$ and $P'(\lambda + d\lambda)$ are, respectively,

$$P = (x^\alpha) \qquad \text{and} \qquad P' = (x^\alpha + \delta x^\alpha).$$



$$P = (x^1, x^2)$$
$$P' = (x^1 + \delta x^1, x^2 + \delta x^2)$$

When we move from P to P' the metric components change as follows

$$\begin{aligned} g_{\alpha\beta}(P') &\simeq g_{\alpha\beta}(P) + \frac{\partial g_{\alpha\beta}}{\partial \lambda} d\lambda + ... \\ &= g_{\alpha\beta}(P) + \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\mu}{d\lambda} d\lambda + ... \\ &= g_{\alpha\beta}(P) + g_{\alpha\beta,\mu}\xi^\mu d\lambda, \end{aligned} \tag{10.3}$$

hence

$$\delta g_{\alpha\beta} = g_{\alpha\beta,\mu}\xi^\mu d\lambda. \tag{10.4}$$

Moreover, since the operators δ and d commute, we find

$$\begin{aligned} \delta(dx^a) &= d(\delta x^\alpha) = d(\xi^\alpha d\lambda) = d\xi^\alpha d\lambda \\ &= \frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu d\lambda = \xi^\alpha_{,\mu} dx^\mu d\lambda\,. \end{aligned} \tag{10.5}$$

Thus, using eqs. (10.5) and (10.4), eq. (10.2) becomes

$$g_{\alpha\beta,\mu}\xi^\mu d\lambda dx^\alpha dx^\beta + g_{\alpha\beta}\left[\xi^\alpha_{,\mu} dx^\mu d\lambda dx^\beta + \xi^\beta_{,\gamma} dx^\gamma d\lambda dx^\alpha\right] = 0, \tag{10.6}$$

and, after relabelling the indices,

$$\left[g_{\alpha\beta,\mu}\xi^\mu + g_{\delta\beta}\xi^\delta_{,\alpha} + g_{\alpha\delta}\xi^\delta_{,\beta}\right] dx^\alpha dx^\beta d\lambda = 0. \tag{10.7}$$

In conclusion, a solution of Einstein's equations is invariant under translations along $\vec{\xi}$, if and only if

$$g_{\alpha\beta,\mu}\xi^\mu + g_{\delta\beta}\xi^\delta_{,\alpha} + g_{\alpha\delta}\xi^\delta_{,\beta} = 0. \tag{10.8}$$

In order to find the Killing vectors of a given a metric $g_{\alpha\beta}$ we need to solve eq. (10.8), which is a system of differential equations for the components of $\vec{\xi}$. If eq. (10.8) does not admit a solution, the spacetime has no symmetries. It may look like eq. (10.8) is not covariant, since it contains partial derivatives, but it is easy to show that it is equivalent to the following covariant equation (see appendix A)

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0. \tag{10.9}$$

This is the **Killing equation**.

10.1.1 Lie-derivative

The variation of a tensor under an infinitesimal translation along the direction of a vector field $\vec{\xi}$ is the **Lie-derivative** ($\vec{\xi}$ must not necessarily be a Killing vector), and it is indicated as $L_{\vec{\xi}}$. For a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor

$$L_{\vec{\xi}} T_{\alpha\beta} = T_{\alpha\beta,\mu}\xi^{\mu} + T_{\delta\beta}\xi^{\delta}_{,\alpha} + T_{\alpha\delta}\xi^{\delta}_{,\beta} \ . \tag{10.10}$$

For the metric tensor

$$L_{\vec{\xi}} g_{\alpha\beta} = g_{\alpha\beta,\mu}\xi^{\mu} + g_{\delta\beta}\xi^{\delta}_{,\alpha} + g_{\alpha\delta}\xi^{\delta}_{,\beta} = \xi_{\alpha;\beta} + \xi_{\beta;\alpha} \ ; \tag{10.11}$$

if $\vec{\xi}$ is a Killing vector the Lie-derivative of $g_{\alpha\beta}$ vanishes.

10.1.2 Killing vectors and the choice of coordinate systems

The existence of Killing vectors remarkably simplifies the problem of choosing a coordinate system appropriate to solve Einstein's equations. For instance, if we are looking for a solution which admits a timelike Killing vector $\vec{\xi}$, it is convenient to choose, at each point of the manifold, the timelike basis vector $\vec{e}_{(0)}$ aligned with $\vec{\xi}$; with this choice, the time coordinate lines coincide with the worldlines to which $\vec{\xi}$ is tangent, i.e. with the **congruence of worldlines** of $\vec{\xi}$, and the components of $\vec{\xi}$ are

$$\xi^{\alpha} = (\xi^0, 0, 0, 0) \ . \tag{10.12}$$

If we parametrize the coordinate curves associated to $\vec{\xi}$ in such a way that ξ^0 is constant or equal unity, then

$$\xi^{\alpha} = (1, 0, 0, 0) \ , \tag{10.13}$$

and from eq. (10.8) it follows that

$$\frac{\partial g_{\alpha\beta}}{\partial x^0} = 0 \ . \tag{10.14}$$

This means that **if the metric admits a timelike Killing vector, with an appropriate choice of the coordinate system it can be made independent of time.**

A similar procedure can be used if the metric admits a spacelike Killing vector. In this case, by choosing one of the spacelike basis vectors, say the vector $\vec{e}_{(1)}$, parallel to $\vec{\xi}$, and by a suitable reparametrization of the corresponding conguence of coordinate lines, one can write

$$\xi^\alpha = (0, 1, 0, 0) \; , \tag{10.15}$$

and with this choice the metric is independent of x^1, i.e. $\partial g_{\alpha\beta}/\partial x^1 = 0$.

If the Killing vector is null, starting from the coordinate basis vectors $\vec{e}_{(0)}, \vec{e}_{(1)}, \vec{e}_{(2)}, \vec{e}_{(3)}$, it is convenient to construct a set of new basis vectors

$$\vec{e}_{(\alpha')} = \Lambda^{\beta}_{\alpha'} \vec{e}_{(\beta)} \; , \tag{10.16}$$

such that the vector $\vec{e}_{(0')}$ is a null vector. Then, the vector $\vec{e}_{(0')}$ can be chosen to be parallel to $\vec{\xi}$ at each point of the manifold, and by a suitable reparametrization of the corresponding coordinate lines

$$\xi^\alpha = (1, 0, 0, 0) \; , \tag{10.17}$$

and the metric is independent of $x^{0'}$, i.e. $\partial g_{\alpha\beta}/\partial x^{0'} = 0$.

The map

$$f_t : \; \mathcal{M} \to \mathcal{M}$$

under which the metric is unchanged is called an *isometry*, and the Killing vector field is the generator of the isometry.

The **congruence of worldlines** of the vector $\vec{\xi}$ can be found by integrating the equations

$$\frac{\delta x^\mu}{d\lambda} = \xi^\mu(x^\alpha). \tag{10.18}$$

10.2 Examples

1) Killing vectors of flat spacetime

The Killing vectors of Minkowski's spacetime can be obtained very easily using cartesian coordinates. Since all Christoffel symbols vanish, the Killing equation becomes

$$\xi_{\alpha,\beta} + \xi_{\beta,\alpha} = 0 \; . \tag{10.19}$$

By combining the following equations

$$\xi_{\alpha,\beta\gamma} + \xi_{\beta,\alpha\gamma} = 0 \; , \qquad \xi_{\beta,\gamma\alpha} + \xi_{\gamma,\beta\alpha} = 0 \; , \qquad \xi_{\gamma,\alpha\beta} + \xi_{\alpha,\gamma\beta} = 0 \; , \tag{10.20}$$

and by using eq. (10.19) we find

$$\xi_{\alpha,\beta\gamma} = 0 \; , \tag{10.21}$$

whose general solution is

$$\xi_\alpha = c_\alpha + \epsilon_{\alpha\gamma} x^\gamma \; , \tag{10.22}$$

where c_α, $\epsilon_{\alpha\beta}$ are constants. By substituting this expression into eq. (10.19) we find

$$\epsilon_{\alpha\gamma} x^\gamma_{,\beta} + \epsilon_{\beta\gamma} x^\gamma_{,\alpha} = \epsilon_{\alpha\gamma}\delta^\gamma_\beta + \epsilon_{\beta\gamma}\delta^\gamma_\alpha = \epsilon_{\alpha\beta} + \epsilon_{\beta\alpha} = 0$$

Therefore eq. (10.22) is the solution of eq. (10.19) only if

$$\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} \, . \tag{10.23}$$

The general Killing vector field of the form (10.22) can be written as the linear combination of ten Killing vector fields $\xi^{(A)}_\alpha = \{\xi^{(1)}_\alpha, \xi^{(2)}_\alpha, \dots, \xi^{(10)}_\alpha\}$ corresponding to ten independent choices of the constants c_α, $\epsilon_{\alpha\beta}$:

$$\xi^{(A)}_\alpha = c^{(A)}_\alpha + \epsilon^{(A)}_{\alpha\gamma} x^\gamma \qquad A = 1, \dots, 10\,. \tag{10.24}$$

For instance, we can choose

$$\begin{aligned}
c^{(1)}_\alpha &= (1,0,0,0) \qquad \epsilon^{(1)}_{\alpha\beta} = 0\\
c^{(2)}_\alpha &= (0,1,0,0) \qquad \epsilon^{(2)}_{\alpha\beta} = 0\\
c^{(3)}_\alpha &= (0,0,1,0) \qquad \epsilon^{(3)}_{\alpha\beta} = 0\\
c^{(4)}_\alpha &= (0,0,0,1) \qquad \epsilon^{(4)}_{\alpha\beta} = 0\\
c^{(5)}_\alpha &= 0 \qquad \epsilon^{(5)}_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}\\
c^{(6)}_\alpha &= 0 \qquad \epsilon^{(6)}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}\\
c^{(7)}_\alpha &= 0 \qquad \epsilon^{(7)}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 \end{pmatrix}\\
c^{(8)}_\alpha &= 0 \qquad \epsilon^{(8)}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}\\
c^{(9)}_\alpha &= 0 \qquad \epsilon^{(9)}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}\\
c^{(10)}_\alpha &= 0 \qquad \epsilon^{(10)}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0 \end{pmatrix}
\end{aligned} \tag{10.25}$$

Therefore, flat spacetime admits ten linearly independent Killing vectors.

The symmetries generated by the Killing vectors with $A = 1, \dots, 4$ are spacetime translations; the symmetries generated by the Killing vectors with $A = 5, 6, 7$ are Lorentz's boosts; the symmetries generated by the Killing vectors with $A = 8, 9, 10$ are space rotations.

2) Killing vectors of a spherical surface

Let us consider a sphere of unit radius

$$ds^2 = d\theta^2 + sin^2\theta d\varphi^2 = (dx^1)^2 + sin^2 x^1 (dx^2)^2\ . \tag{10.26}$$

Eq. (10.8)

$$g_{\alpha\beta,\mu}\xi^{\mu} + g_{\delta\beta}\xi^{\delta}_{,\alpha} + g_{\alpha\delta}\xi^{\delta}_{,\beta} = 0$$

gives

$$\begin{array}{lll} 1) \quad \alpha = \beta = 1 & 2g_{\delta 1}\xi^{\delta}_{,1} = 0 \rightarrow \xi^{1}_{,1} = 0 \\ 2) \quad \alpha = 1, \beta = 2 & g_{\delta 2}\xi^{\delta}_{,1} + g_{1\delta}\xi^{\delta}_{,2} = 0 \rightarrow \xi^{1}_{,2} + \sin^2\theta\xi^{2}_{,1} = 0 \\ 3) \quad \alpha = \beta = 2 & g_{22,\mu}\xi^{\mu} + 2g_{\delta 2}\xi^{\delta}_{,2} = 0 \rightarrow \cos\theta\xi^{1} + \sin\theta\xi^{2}_{,2} = 0. \end{array} \tag{10.27}$$

The general solution is

$$\xi^1 = Asin(\varphi + a), \qquad \xi^2 = Acos(\varphi + a)cot\theta + b. \tag{10.28}$$

Therefore a spherical surface admits three linearly independent Killing vectors, associated to the choice of the integration constants (A, a, b).

10.3 Conserved quantities in geodesic motion

Killing vectors are important because they are associated to conserved quantities, which may be hidden by an unsuitable coordinate choice.

Let us consider a massive particle moving along a geodesic of a spacetime which admits a Killing vector $\vec{\xi}$. The geodesic equations written in terms of the particle four-velocity $\vec{U} = \frac{\delta x^{\alpha}}{d\tau}$ read

$$\frac{dU^{\alpha}}{d\tau} + \Gamma^{\alpha}{}_{\beta\nu}U^{\beta}U^{\nu} = 0. \tag{10.29}$$

By contracting eq. (10.29) with $\vec{\xi}$ we find

$$\xi_{\alpha}\left[\frac{dU^{\alpha}}{d\tau} + \Gamma^{\alpha}{}_{\beta\nu}U^{\beta}U^{\nu}\right] = \frac{d(\xi_{\alpha}U^{\alpha})}{d\tau} - U^{\alpha}\frac{d\xi_{\alpha}}{d\tau} + \Gamma^{\alpha}{}_{\beta\nu}U^{\beta}U^{\nu}\xi_{\alpha}\ . \tag{10.30}$$

Since

$$U^{\alpha}\frac{d\xi_{\alpha}}{d\tau} = U^{\beta}\frac{d\xi_{\beta}}{d\tau} = U^{\beta}\frac{\partial\xi_{\beta}}{\partial x^{\nu}}\frac{\delta x^{\nu}}{d\tau} = U^{\beta}U^{\nu}\frac{\partial\xi_{\beta}}{\partial x^{\nu}}\ , \tag{10.31}$$

eq. (10.30) becomes

$$\frac{d(\xi_{\alpha}U^{\alpha})}{d\tau} - U^{\beta}U^{\nu}\left[\frac{\partial\xi_{\beta}}{\partial x^{\nu}} - \Gamma^{\alpha}{}_{\beta\nu}\xi_{\alpha}\right] = 0\ , \tag{10.32}$$

i.e.

$$\frac{d(\xi_{\alpha}U^{\alpha})}{d\tau} - U^{\beta}U^{\nu}\xi_{\beta;\nu} = 0\ . \tag{10.33}$$

Since $\xi_{\beta;\nu}$ is antisymmetric in β and ν, while $U^{\beta}U^{\nu}$ is symmetric, the term $U^{\beta}U^{\nu}\xi_{\beta;\nu}$ vanishes, and eq. (10.33) finally becomes

$$\frac{d(\xi_{\alpha}U^{\alpha})}{d\tau} = 0 \qquad \rightarrow \qquad \xi_{\alpha}U^{\alpha} = const\ , \tag{10.34}$$

i.e. the quantity $(\xi_\alpha U^\alpha)$ is a constant of the particle motion. Thus, for every Killing vector there exists an associated conserved quantity.

Eq. (10.34) can be written as follows:

$$g_{\alpha\mu}\xi^\mu U^\alpha = const \; . \tag{10.35}$$

Let us now assume that $\vec{\xi}$ is a timelike Killing vector. In section 10.1.2 we have shown that the coordinate system can be chosen in such a way that $\xi^\mu = \{1, 0, 0, 0\}$, in which case eq. (10.35) becomes

$$g_{\alpha 0}\xi^0 U^\alpha = const \qquad \to \qquad g_{\alpha 0} U^\alpha = const \; . \tag{10.36}$$

If the metric is asymptotically flat, as it is for instance when the gravitational field is generated by a distribution of matter confined in a finite region of space, at infinity $g_{\alpha\beta}$ reduces to the Minkowski metric $\eta_{\alpha\beta}$, and eq. (10.36) becomes

$$\eta_{00} U^0 = const \qquad \to \qquad U^0 = const \; . \tag{10.37}$$

Since in flat spacetime the energy-momentum vector of a massive particle is $p^\alpha = mcU^\alpha = \{E/c, mv^i\gamma\}$, the previous equation becomes

$$\frac{E}{c} = const \; , \tag{10.38}$$

i.e. at infinity the conservation law associated to a timelike Killing vector reduces to the energy conservation for the particle motion. For this reason we say that, when the metric admits a timelike Killing vector, eq. (10.34) expresses the energy conservation for the particle motion along the geodesic.

If the Killing vector is spacelike, by choosing the coordinate system such that, say, $\xi^\mu = \{0, 1, 0, 0\}$, eq. (10.34) reduces to

$$g_{\alpha 1}\xi^1 U^\alpha = const \qquad \to \qquad g_{\alpha 1} U^\alpha = const \; .$$

At infinity this equation becomes

$$\eta_{11} U^1 = const \qquad \to \qquad \frac{p^1}{mc} = const \; ,$$

showing that the component of the energy-momentum vector along the x^1 direction is constant; thus, when the metric admit a spacelike Killing vector eq. (10.34) expresses momentum conservation along the geodesic motion.

If the particle is massless, the geodesic equation cannot be parametrized with the proper time. In this case the particle worldline has to be parametrized using an affine parameter λ such that the geodesic equation takes the form (10.29), and the particle four-velocity is $U^\alpha = \frac{dx^\alpha}{d\lambda}$. The derivation of the constants of motion associated to a spacetime symmetry, i.e. to a Killing vector, is similar as for massive particles, reminding that by a suitable choice of the parameter along the geodesic $p^\alpha = \{E, p^i\}$.

It should be mentioned that in Riemannian spaces there may exist conservation laws which cannot be traced back to the presence of a symmetry, and therefore to the existence of a Killing vector field.

10.4 Killing vectors and conservation laws

In Chapter 7 we have shown that the stress-energy tensor satisfies the "conservation law"

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{10.39}$$

and we have shown that in general this is not a genuine conservation law. If the spacetime admits a Killing vector, then

$$(\xi_\mu T^{\mu\nu})_{;\nu} = \xi_{\mu;\nu} T^{\mu\nu} + \xi_\mu T^{\mu\nu}{}_{;\nu} = 0. \tag{10.40}$$

Indeed, the second term vanishes because of eq. (10.39) and the first vanishes because $\xi_{\mu;\nu}$ is antisymmetric in μ an ν, whereas $T^{\mu\nu}$ is symmetric.

Since there is a contraction on the index μ, the quantity $(\xi_\mu T^{\mu\nu})$ is a vector, and according to eq. (8.69)

$$V^\nu{}_{;\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left(\sqrt{-g} V^\nu \right) , \tag{10.41}$$

therefore eq. (10.40) is equivalent to

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left[\sqrt{-g} \, (\xi_\mu T^{\mu\nu}) \right] = 0 , \tag{10.42}$$

which expresses the conservation of the following quantity and accordingly, a conserved quantity can be defined as

$$T = \int_{(x^0 = const)} \sqrt{-g} \left(\xi_\mu T^{\mu 0} \right) dx^1 dx^2 dx^3 , \tag{10.43}$$

as shown in Chapter 7.

In classical mechanics energy is conserved when the hamiltonian is independent of time; thus, conservation of energy is associated to a symmetry with respect to time translations. In section 10.1.2 we have shown that if a metric admits a timelike Killing vector, with a suitable choice of coordinates it can me made time independent (where now "time" indicates more generally the x^0-coordinate). Thus, in this case it is natural to interpret the quantity defined in eq. (10.43) as a conserved energy.

In a similar way, when the metric addmits a spacelike Killing vector, the associated conserved quantities are indicated as "momentum" or "angular momentum", although this is more a matter of definition.

It should be stressed that the energy of a gravitational system can be defined in a non ambiguous way only if there exists a timelike Killing vector field.

10.5 Hypersurface orthogonal vector fields

Given a vector field $\vec{V}$ it identifies a **congruence of worldlines**, i.e. the set of curves to which the vector is tangent at any point of the considered region. If there exists a family of surfaces $f(x^\mu) = const$ such that, at each point, the worldlines of the congruence are perpendicular to that surface, $\vec{V}$ is said to be **hypersurface orthogonal**. This is equivalent to require that $\vec{V}$ is orthogonal to all vectors $\vec{t}$ tangent to the hypersurface, i.e.

$$\vec{t} \cdot \vec{V} = 0 \quad \rightarrow \quad t^\alpha V^\beta g_{\alpha\beta} = 0 \ . \tag{10.44}$$

We shall now show that, as consequence, $\vec{V}$ is parallel to the gradient of f. As described in Chapter 3, section 5, the gradient of a function $f(x^\mu)$ is a one-form

$$\tilde{d}f \rightarrow (\frac{\partial f}{\partial x^0}, \frac{\partial f}{\partial x^1}, ... \frac{\partial f}{\partial x^n}) = \{f_{,\alpha}\}. \tag{10.45}$$

When we say that $\vec{V}$ is parallel to $\tilde{d}f$ we mean that the one-form dual to $\vec{V}$, i.e. $\tilde{V} \rightarrow \{g_{\alpha\beta}V^\beta \equiv V_\alpha\}$ satisfies the equation

$$V_\alpha = \lambda f_{,\alpha} \ , \tag{10.46}$$

where λ is a function of the coordinates $\{x^\mu\}$. This equation is equivalent to eq. (10.44). Indeed, given any curve $x^\alpha(s)$ lying on the hypersurface, and being $t^\alpha = dx^\alpha/ds$ its tangent vector, since $f(x^\mu) = const$ the directional derivative of $f(x^\mu)$ along the curve vanishes, i.e.

$$\frac{df}{ds} = \frac{\partial f}{\partial x^\alpha}\frac{dx^\alpha}{ds} = f_{,\alpha}t^\alpha = \lambda^{-1}V_\alpha t^\alpha = 0 \ , \tag{10.47}$$

i.e. eq. (10.44).

If (10.46) is satisfied, it follows that

$$\begin{aligned} V_{\alpha;\beta} - V_{\beta;\alpha} &= (\lambda f_{,\alpha})_{;\beta} - (\lambda f_{,\beta})_{;\alpha} \\ &= \lambda (f_{,\alpha;\beta} - f_{,\beta;\alpha}) + f_{,\alpha}\lambda_{;\beta} - f_{,\beta}\lambda_{;\alpha} = \\ &= \lambda (f_{,\alpha,\beta} - f_{,\beta,\alpha} - \Gamma^\mu{}_{\beta\alpha}f_{,\mu} + \Gamma^\mu{}_{\alpha\beta}f_{,\mu}) + f_{,\alpha}\lambda_{,\beta} - f_{,\beta}\lambda_{,\alpha} \\ &= V_\alpha\frac{\lambda_{,\beta}}{\lambda} - V_\beta\frac{\lambda_{,\alpha}}{\lambda} \ , \end{aligned} \tag{10.48}$$

i.e.

$$V_{\alpha;\beta} - V_{\beta;\alpha} = V_\alpha\frac{\lambda_{,\beta}}{\lambda} - V_\beta\frac{\lambda_{,\alpha}}{\lambda} \ . \tag{10.49}$$

If we now define the following quantity, which is said *rotation*

$$\omega^\delta = \frac{1}{2}\epsilon^{\delta\alpha\beta\mu}V_{[\alpha;\beta]}V_\mu \ , \tag{10.50}$$

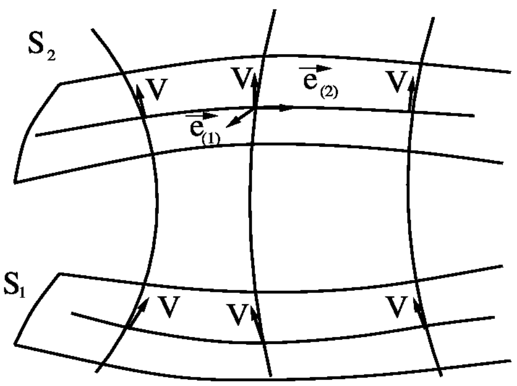
using the definition of the antisymmetric unit pseudotensor $\epsilon^{\delta\alpha\beta\mu}$ given in Appendix B, it follows that

$$\omega^\delta = 0. \tag{10.51}$$

Then, if the vector field $\vec{V}$ is hypersurface horthogonal, (10.51) is satisfied. Actually, (10.51) is a necessary and sufficient condition for $\vec{V}$ to be hypersurface horthogonal; this result is the *Frobenius theorem.*

10.5.1 Hypersurface-orthogonal vector fields and the choice of coordinate systems

The existence of a hypersurface-orthogonal vector field allows to choose a coordinate frame such that the metric has a much simpler form. Let us consider, for the sake of simplicity, a three-dimensional spacetime (x^0, x^1, x^2).



Be S_1 and S_2 two surfaces of the family $f(x^\mu) = cost$, to which the vector field $\vec{V}$ is orthogonal. As an example, we shall assume that $\vec{V}$ is timelike, but a similar procedure can be used if $\vec{V}$ is spacelike. If $\vec{V}$ is timelike, it is convenient to choose the basis vector $\vec{e}_{(0)}$ parallel to $\vec{V}$, and the remaining basis vectors as the tangent vectors to some curves lying on the surface, so that

$$g_{00} = g(\vec{e}_{(0)}, \vec{e}_{(0)}) = \vec{e}_{(0)} \cdot \vec{e}_{(0)} \neq 0 \qquad (10.52)$$
$$g_{0i} = g(\vec{e}_{(0)}, \vec{e}_{(i)}) = 0, \qquad i = 1, 2.$$

Thus, with this choice, the metric becomes

$$ds^2 = g_{00}(dx^0)^2 + g_{ik}(dx^i)(dx^k), \qquad i, k = 1, 2\ . \qquad (10.53)$$

The generalization of this example to the four-dimensional spacetime, in which case the surface S is a hypersurface, is straightforward.

In general, given a timelike vector field $\vec{V}$, we can always choose a coordinate frame such that $\vec{e}_{(0)}$ is parallel to $\vec{V}$, so that in this frame

$$V^\alpha(x^\mu) = (V^0(x^\mu), 0, 0, 0)\,. \qquad (10.54)$$

Such coordinate system is said **comoving**. If, in addition, $\vec{V}$ is hypersurface-horthogonal, then $g_{0i} = 0$ and, as a consequence, the one-form associated to $\vec{V}$ also has the form

$$V_\alpha(x^\mu) = (V_0(x^\mu), 0, 0, 0)\,, \qquad (10.55)$$

since $V_i = g_{i\mu}V^\mu = g_{i0}V^0 + g_{ik}V^k = 0$.

10.6 Appendix A

We want to show that eq. (10.8) is equivalent to eq. (10.9).

$$\begin{aligned}\xi_{\alpha;\beta} &= (g_{\alpha\mu}\xi^{\mu})_{;\beta} \\ &= g_{\alpha\mu}\xi^{\mu}_{;\beta} = g_{\alpha\mu}\left(\xi^{\mu}_{,\beta} + \Gamma^{\mu}{}_{\delta\beta}\xi^{\delta}\right),\end{aligned} \tag{10.56}$$

hence

$$\begin{aligned}\xi_{\alpha;\beta} + \xi_{\beta;\alpha} &= g_{\alpha\mu}\left(\xi^{\mu}_{,\beta} + \Gamma^{\mu}{}_{\delta\beta}\xi^{\delta}\right) \\ &+ g_{\beta\mu}\left(\xi^{\mu}_{,\alpha} + \Gamma^{\mu}{}_{\alpha\delta}\xi^{\delta}\right) \\ &= g_{\alpha\mu}\xi^{\mu}_{,\beta} + g_{\beta\mu}\xi^{\mu}_{,\alpha} + \left(g_{\alpha\mu}\Gamma^{\mu}{}_{\delta\beta} + g_{\beta\mu}\Gamma^{\mu}{}_{\alpha\delta}\right)\xi^{\delta}.\end{aligned} \tag{10.57}$$

The term in parenthesis can be written as

$$\begin{aligned}&\frac{1}{2}\left[g_{\alpha\mu}g^{\mu\sigma}\left(g_{\delta\sigma,\beta} + g_{\sigma\beta,\delta} - g_{\delta\beta,\sigma}\right) + g_{\beta\mu}g^{\mu\sigma}\left(g_{\alpha\sigma,\delta} + g_{\sigma\delta,\alpha} - g_{\alpha\delta,\sigma}\right)\right] \\ &= \frac{1}{2}\left[\delta^{\sigma}_{\alpha}\left(g_{\delta\sigma,\beta} + g_{\sigma\beta,\delta} - g_{\delta\beta,\sigma}\right) + \delta^{\sigma}_{\beta}\left(g_{\alpha\sigma,\delta} + g_{\sigma\delta,\alpha} - g_{\alpha\delta,\sigma}\right)\right] \\ &= \frac{1}{2}\left[g_{\delta\alpha,\beta} + g_{\alpha\beta,\delta} - g_{\delta\beta,\alpha} + g_{\alpha\beta,\delta} + g_{\beta\delta,\alpha} - g_{\alpha\delta,\beta}\right] \\ &= g_{\alpha\beta,\delta},\end{aligned} \tag{10.58}$$

and eq. (10.57) becomes

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = g_{\alpha\mu}\xi^{\mu}_{,\beta} + g_{\beta\mu}\xi^{\mu}_{,\alpha} + g_{\alpha\beta,\delta}\xi^{\delta} \tag{10.59}$$

which coincides with eq. (10.8).

10.7 Appendix B: The Levi-Civita completely antisymmetric pseudotensor

We define the *Levi-Civita symbol* (also said *Levi-Civita tensor density*), $e_{\alpha\beta\gamma\delta}$, as an object whose components change sign under interchange of any pair of indices, and whose non-zero components are ± 1. Since it is completely antisymmetric, all the components with two equal indices are zero, and the only non-vanishing components are those for which all four indices are different. We set

$$e_{0123} = 1. \tag{10.60}$$

Under general coordinate transformations, $e_{\alpha\beta\gamma\delta}$ does not transform as a tensor; indeed, under the transformation $x^{\alpha} \to x^{\alpha'}$,

$$\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\frac{\partial x^{\beta}}{\partial x^{\beta'}}\frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\frac{\partial x^{\delta}}{\partial x^{\delta'}}\, e_{\alpha\beta\gamma\delta} = J\, e_{\alpha'\beta'\gamma'\delta'} \tag{10.61}$$

where J is defined (see Chapter 7) as

$$J \equiv \det\left(\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\right) \tag{10.62}$$

and we have used the definiton of determinant.

We now define the *Levi-Civita pseudo-tensor* as

$$\epsilon_{\alpha\beta\gamma\delta} \equiv \sqrt{-g}\, e_{\alpha\beta\gamma\delta}\,. \tag{10.63}$$

Since, from (8.26), for a coordinate transformation $x^\alpha \to x^{\alpha\prime}$

$$|J| = \frac{\sqrt{-g'}}{\sqrt{-g}}\,, \tag{10.64}$$

then

$$\epsilon_{\alpha\beta\gamma\delta} \to \epsilon_{\alpha\prime\beta\prime\gamma\prime\delta\prime} = \mathrm{sign}(J)\,\frac{\partial x^\alpha}{\partial x^{\alpha\prime}}\frac{\partial x^\beta}{\partial x^{\beta\prime}}\frac{\partial x^\gamma}{\partial x^{\gamma\prime}}\frac{\partial x^\delta}{\partial x^{\delta\prime}}\,\epsilon_{\alpha\beta\gamma\delta}\,. \tag{10.65}$$

Thus, $\epsilon_{\alpha\beta\gamma\delta}$ is not a tensor but a pseudo-tensor, because it transforms as a tensor times the sign of the Jacobian of the transformation. It transforms as a tensor only under a subset of the general coordinate transformations, i.e. that with $\mathrm{sign}(J) = +1$.

Warning: do not confuse the Levi-Civita symbol, $e_{\alpha\beta\gamma\delta}$, with the Levi-Civita pseudo-tensor, $\epsilon_{\alpha\beta\gamma\delta}$.