2.2 VECTORS

2.2.1 The traditional definition of a vector

Let us consider an n-dimensional manifold, and a generic coordinate transformation

$$x^{i'} = x^{i'}(x^j), \qquad i', j = 1, \ldots, n\,. \tag{2.16}$$

A *contravariant* vector

$$\vec{V} \to_O \{V^i\}_{i=1,\ldots n}\,, \tag{2.17}$$

where the symbol $\to_O$ indicates that $\vec{V}$ has components $\{V^\mu\}$ with respect to a given frame O, is a collection of n numbers which transform under the coordinate transformation (2.16) as follows:

$$V^{i'} = \sum_{j=1,\ldots,n} \frac{\partial x^{i'}}{\partial x^j} V^j \equiv \frac{\partial x^{i'}}{\partial x^j} V^j\,. \tag{2.18}$$

Notice that in writing the last term we have used Einstein's convention. $V^{i'}$ are the components of the vector in the new frame. If we now define the $n \times n$ matrix

$$(\Lambda^{i'}{}_j) = \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} & \ldots \\ . & . & \ldots \\ . & . & \ldots \\ . & . & \ldots \\ \frac{\partial x^{n'}}{\partial x^1} & \frac{\partial x^{n'}}{\partial x^2} & \ldots \end{pmatrix}, \tag{2.19}$$

the transformation law can be written in the general form

$$V^{i'} = \Lambda^{i'}{}_j V^j. \tag{2.20}$$

In addition, *covariant vectors* are defined as objects that transform according to the following rule

$$A_{i'} = \frac{\partial x^j}{\partial x^{i'}} A_j = \Lambda^j{}_{i'} A_j\,, \tag{2.21}$$

where $\Lambda^j{}_{i'}$ is the inverse matrix of $\Lambda^{i'}{}_j$ (see Box 2-D). This definition of a vector relies on the choice of a coordinate system. However, we know that in $\mathbb{R}^n$ the vector can be defined as a *geometrical object* (an oriented segment joining two points), without introducing a coordinate frame. We shall now show that in a *general manifold* it is possible to define a vector as a **geometrical object**, i.e. one that exists regardless of the coordinate system. Of course, once a coordinate system is given we can associate to a vector its components with respect to that sytstem and, when the frame is changed the vector components transform as in Eq. 2.18. However, the vector itself does not change.

Box 2-D

The matrices $\Lambda^{i}{}_{j'}$ and $\Lambda^{i'}{}_{j}$

Given a coordinate transformation $x^{i\prime} = x^{i'}(x^j)$, or the inverse $x^j = x^j(x^{i'})$, $i', j = 1, \ldots, n$, the matrices

$$\Lambda^{i'}{}_{j} = \frac{\partial x^{i'}}{\partial x^{j}} \tag{2.22}$$

and

$$\Lambda^{i}{}_{j'} = \frac{\partial x^{i}}{\partial x^{j'}}\,, \tag{2.23}$$

are one the inverse of the other. Indeed

$$\Lambda^{i'}{}_{k}\Lambda^{k}{}_{j'} = \frac{\partial x^{i'}}{\partial x^{k}}\frac{\partial x^{k}}{\partial x^{j'}} = \frac{\partial x^{i'}}{\partial x^{j'}} = \delta^{i'}_{j'}\,. \tag{2.24}$$

Note that: when we write $\Lambda^{i'}{}_{j}$ or $\Lambda^{i}{}_{j'}$, the first index (i.e. the one on the left) refers to the row of the matrix, the second to the column.

Box 2-E

A comment on notation

Here and in the following, we shall use indices with and without primes to refer to different coordinate frames.
Strictly speaking, Eq. 2.16 should be written as

$$x'^{i'} = x'^{i'}(x^j), \qquad i', j = 1, \ldots, n\,, \tag{2.25}$$

because the coordinate with (say) $i' = 1$ belongs to the new frame, and is then different from the coordinate with $j = 1$, belonging to the old frame. However, for brevity of notation, we will omit the primes in the coordinates, keeping only the primes in the indices.

2.2.2 A geometrical definition

In order to define vectors as geometrical objects, we need to go by steps: firstly we shall introduce the notions of *paths* and *curves* to define the tangent vectors to a curve at a given point $\mathbf{p}$. Then we shall introduce the directional derivative along a curve in $\mathbf{p}$, which will be shown to be in a ono-to-one correspondence with the vector tangent to the same curve at the same point. This will allow us to give a definition of vectors that is independent of the coordinate system.

Paths and curves

A **path** $\mathcal{C}$ is a connected series of points in a manifold. An example of path is shown in Figure 2.12.

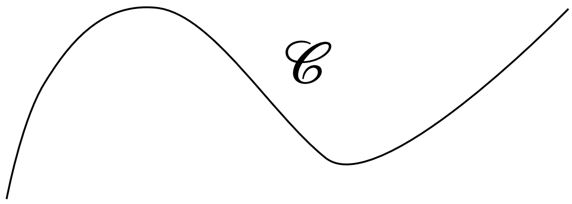

Figure 2.12 Path on a manifold.

A **curve** is a mapping from an interval $I = [a, b] \subset \mathbb{R}$ to a path,

$$\gamma \;:\; s \in [a,b] \mapsto \gamma(s) \in \mathcal{C}\,. \tag{2.26}$$

Thus, a curve γ associates a real number to each point of the path. We say that the curve

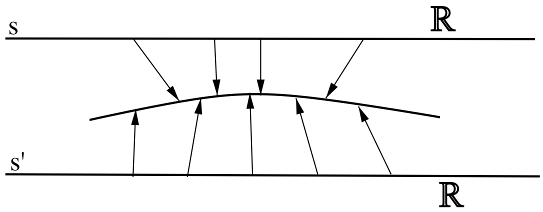

Figure 2.13 Different parametrizations of the same path.

is a *parametrization* of the path $\mathcal{C}$, and the variable $s \in [a, b]$ is called **parameter** of the curve. The path is then the image of the real interval I in the manifold.

Given a coordinate system $(x^1, \ldots, x^n)$ defined in the open set of the mainfold containing the path, we can express the curve γ as a set of n real functions $(x^1(s), \ldots, x^n(s))$

$$\gamma \;:\; s \in [a,b] \mapsto (x^1(s), x^2(s), \ldots, x^n(s))\,. \tag{2.27}$$

We say that the curve is C^k if the n functions are C^k.

If we change the parameter by a parameter transformation $s' = s'(s)$, the number associated to a given point of the path changes, i.e. the curve changes (see Fig. 2.13); therefore we get

$$\gamma' \;:\; s' \in [a', b'] \mapsto (x^1(s(s')), x^2(s(s')), \ldots, x^n(s(s'))) = (x'^1(s'), x'^2(s'), \ldots, x'^n(s'))\,, \tag{2.28}$$

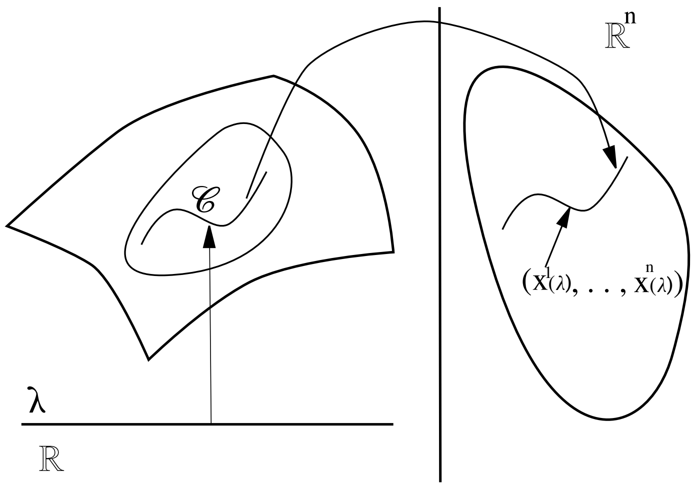

Figure 2.14 Curve on a manifold expressed as n real functions.

where x'^1, x'^2 are new functions of s'. This is a *new curve*, although the path is the same.

Box 2-F

Example

The position of a bullet shot by a gun in the 2-dimensional plane (x,z) is a path; when we associate the parameter t (time) at each point of the trajectory, we define a curve; if we change the parameter, say for instance the curvilinear abscissa, we define a new curve.

Tangent vector to a curve

Let us consider a regular (i.e., C^1) curve γ on a differentiable manifold $\mathbf{M}$, with parameter λ, and a point $\mathbf{p} \in \mathbf{M}$ belonging to the curve. Given a coordinate system $(x^1, \ldots, x^n)$, as shown in Eq. 2.27 we can express the curve as a set of n real C^1 functions $(x^1(\lambda), \ldots, x^n(\lambda))$ (see Fig. 2.14).

The set of numbers $\left\{\frac{dx^1}{d\lambda}, \ldots, \frac{dx^n}{d\lambda}\right\}$ are the components of the **tangent vector** to γ in $\mathbf{p}$:

$$\vec{V} \to_O \left\{\frac{dx^i}{d\lambda}\right\}_{i=1,\ldots,n} . \tag{2.29}$$

One must be careful not to confuse the curve with the path. In fact a path has, at any given point, an infinite number of tangent vectors, all parallel, but with different lenghts, corresponding to the different possible parametrizations of the path. A curve, instead, has a *unique tangent vector* in any given point. Note also that there are curves that are tangent to one another in $\mathbf{p}$, and therefore have the same tangent vector (see Fig. 2.15).

Let us now consider a different coordinate system $(x^{1'}, \ldots, x^{n'})$. Since $\mathbf{M}$ is a differen-

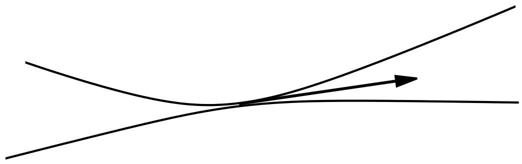

Figure 2.15 Different curves having the same tangent vector.

tiable manifold, the functions $(x^{1'}(x^1,\dots,x^n),\dots,x^{n'}(x^1,\dots,x^n))$ are regular and invertible in their domain. The components of the tangent vector in the new coordinate basis become

$$V^{i'} = \frac{dx^{i'}}{d\lambda} = \frac{\partial x^{i'}}{\partial x^j}\frac{dx^j}{d\lambda}\,. \tag{2.30}$$

For instance, if $n=2$, $x^{1'} = x^{1'}(x^1,x^2)$, $x^{2'} = x^{2'}(x^1,x^2)$. The parameter λ is unaffected, thus

$$\begin{cases} \frac{dx^{1'}}{d\lambda} = \frac{\partial x^{1'}}{\partial x^1}\frac{dx^1}{d\lambda} + \frac{\partial x^{1'}}{\partial x^2}\frac{dx^2}{d\lambda} \\ \frac{dx^{2'}}{d\lambda} = \frac{\partial x^{2'}}{\partial x^1}\frac{dx^1}{d\lambda} + \frac{\partial x^{2'}}{\partial x^2}\frac{dx^2}{d\lambda} \end{cases} \qquad \begin{pmatrix} \frac{dx^{1'}}{d\lambda} \\ \frac{dx^{2'}}{dx^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} \\ \frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} \end{pmatrix} \cdot \begin{pmatrix} \frac{dx^1}{d\lambda} \\ \frac{dx^2}{d\lambda} \end{pmatrix}. \tag{2.31}$$

As expected, this is the same transformation as in Eq. 2.20 that was used to define a contravariant vector in Sec. 2.2.1:

$$V^{i'} = \Lambda^{i'}{}_j V^j\,. \tag{2.32}$$

The definition 2.29 of vector tangent to a curve in a given point still depends on the choice of the coordinate system. In order to show that vectors are geometrical objects, i.e. objects that do not depend on the coordinate frame, we need to define the directional derivatives along a curve.

Directional derivatives along a curve

Let us consider a regular curve γ on a differentiable manifold $\mathbf{M}$, with parameter λ, and a point $\mathbf{p}\in\mathbf{M}$ belonging to the curve. Be U neighborhood of $\mathbf{p}$. Let us also consider a real, differentiable function Φ defined in U,

$$\Phi\;:\;U\to\mathbb{R}\,. \tag{2.33}$$

Given a coordinate system $(x^1,\dots,x^n)$, we can express Φ as a function on $\mathbb{R}^n$, $\Phi = \Phi(x^1,\dots,x^n)$, and the curve γ as a set of n real C^1 functions $(x^1(\lambda),\dots,x^n(\lambda))$.

We define the **directional derivative of** Φ in $\mathbf{p}$ along the curve γ as the real number

$$\frac{d\Phi}{d\lambda} = \frac{\partial\Phi}{\partial x^i}\frac{dx^i}{d\lambda}\,. \tag{2.34}$$

Since the function Φ is totally arbitrary, we can rewrite this expression as

$$\frac{d}{d\lambda} = \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i}\,, \tag{2.35}$$

where $\frac{d}{d\lambda}$ is the *directional derivative operator* in $\mathbf{p}$, acting on the space of the C^1 functions in U:

$$\frac{d}{d\lambda}\ :\ C^1(U) \to \mathbb{R}\,. \tag{2.36}$$

An important remark

Eq. 2.35 establishes a one-to-one relation between the directional derivative $\frac{d}{d\lambda}$ along a curve in $\mathbf{p}$, and the components of the tangent vector to the same curve in $\mathbf{p}$, $\frac{dx^i}{d\lambda}$.

Let us consider a different coordinate frame defined in U, $(x^{1'}, \dots, x^{n'})$. As discussed in the previous Section, the functions $(x^{1'}(x^1, \dots, x^n), \dots, x^{n'}(x^1, \dots, x^n))$ are regular and invertible in their domain. We can then write

$$\frac{\partial\Phi}{\partial x^{i'}}\frac{dx^{i'}}{d\lambda} = \left(\frac{\partial\Phi}{\partial x^j}\frac{\partial x^j}{\partial x^{i'}}\right)\left(\frac{\partial x^{i'}}{\partial x^k}\frac{dx^k}{d\lambda}\right) = \frac{\partial\Phi}{\partial x^i}\frac{dx^i}{d\lambda} = \frac{d\Phi}{d\lambda}\,. \tag{2.37}$$

Therefore *the value of the directional derivative of a function does not depend on the choice of the coordinate system, i.e. the directional derivative operator is a geometrical object.*

We shall now show that the space of directional derivatives along a curve on a differential manifold, satisfies the axiomatic definition of a vector space, which is the following [1].

A vector space is a set V on which two operations are defined:

1. *Vector sum*

$$(\vec{v}, \vec{w}) \to \vec{v} + \vec{w} \tag{2.38}$$

2. *Multiplication by a real number:*

$$(a, \vec{v}) \to a\vec{v} \tag{2.39}$$

(where $\vec{v}, \vec{w} \in V$, $a \in \mathbb{R}$), which satisfy the following properties:

- *Associativity and commutativity of vector sum*

$$\begin{aligned} \vec{v} + (\vec{w} + \vec{u}) &= (\vec{v} + \vec{w}) + \vec{u} && (2.40)\\ \vec{v} + \vec{w} &= \vec{w} + \vec{v}\,. \qquad \forall\, \vec{v}, \vec{w}, \vec{u} \in V\,. && (2.41)\end{aligned}$$

- *Existence of a zero vector, i.e. of an element $\vec{0} \in V$ such that*

$$\vec{v} + \vec{0} = \vec{v} \qquad \forall\, \vec{v} \in V\,. \tag{2.42}$$

- *Existence of the opposite element: for any $\vec{w} \in V$ there exists an element $\vec{v} \in V$ such that*

$$\vec{v} + \vec{w} = \vec{0}\,. \tag{2.43}$$

[1] To be precise, what we are defining here is a *real* vector space, but we will omit this specification, because in this book only real vector spaces will be considered.

- *Associativity and distributivity of multiplication by real numbers:*

$$\begin{aligned} a(b\vec{v}) &= (ab)\vec{v} \\ a(\vec{v}+\vec{w}) &= a\vec{v}+a\vec{w} \\ (a+b)\vec{v} &= a\vec{v}+b\vec{v} \qquad \forall \vec{v} \in V\,, \forall\, a,b \in \mathbb{R},. \end{aligned} \tag{2.44}$$

- *Finally, the real number* 1 *must act as an identity on vectors:*

$$1\,\vec{v} = \vec{v} \qquad \forall\,\vec{v}\,. \tag{2.45}$$

Let us now go back to directional derivatives, and consider two curves on a differential manifold **M** passing through the same point **p**. Given the coordinate system $(x^1, \ldots, x^n)$, the curves are described by the functions $x^i = x^i(\lambda)$ and $x^i = x^i(\mu)$. The directional derivatives in **p** along these curves are

$$\frac{d}{d\lambda} = \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i}, \qquad \frac{d}{d\mu} = \frac{dx^i}{d\mu}\frac{\partial}{\partial x^i}. \tag{2.46}$$

Be a a real number. We define the following two operations on the space of directional derivatives along the curves passing through **p**.

- *Sum* of two directional derivatives

$$\frac{d}{d\lambda} + \frac{d}{d\mu} \equiv \left(\frac{dx^i}{d\lambda} + \frac{dx^i}{d\mu}\right)\frac{\partial}{\partial x^i}. \tag{2.47}$$

  The numbers $\left(\frac{dx^i}{d\lambda} + \frac{dx^i}{d\mu}\right)$ are the components of a new vector, which is tangent to some curve through **p**. Therefore, there must exist a curve with a parameter, say, s, such that in **p**

$$\frac{dx^i}{ds} = \left(\frac{dx^i}{d\lambda} + \frac{dx^i}{d\mu}\right), \qquad \text{and} \qquad \frac{d}{ds} = \frac{dx^i}{ds}\frac{\partial}{\partial x^i} = \frac{d}{d\lambda} + \frac{d}{d\mu}\,. \tag{2.48}$$

- *Product* of the directional derivative $\frac{d}{d\lambda}$ with the real number a

$$a\frac{d}{d\lambda} \equiv \left(a\frac{dx^i}{d\lambda}\right)\frac{\partial}{\partial x^i}\,. \tag{2.49}$$

  The numbers $\left(a\frac{dx^i}{d\lambda}\right)$ are the components of a new vector, which is certainly tangent to some curve in **p**. Therefore, there must exist a curve with parameter, say, s', such that in **p**

$$\frac{dx^i}{ds'} = \left(a\frac{dx^i}{d\lambda}\right), \qquad \text{and} \qquad \frac{d}{ds'} = \frac{dx^i}{ds'}\frac{\partial}{\partial x^i} = a\frac{d}{d\lambda}\,. \tag{2.50}$$

It is easy to verify that the operations of sum and multiplication by a real number defined in Eqs. 2.47 and 2.49, respectively, satisfy the above properties. For instance:

- Commutativity of the sum:

$$\frac{d}{d\lambda} + \frac{d}{d\mu} = \left(\frac{dx^i}{d\lambda} + \frac{dx^i}{d\mu}\right)\frac{\partial}{\partial x^i} = \left(\frac{dx^i}{d\mu} + \frac{dx^i}{d\lambda}\right)\frac{\partial}{\partial x^i} = \frac{d}{d\mu} + \frac{d}{d\lambda}\,. \tag{2.51}$$

- Associativity of multiplication by real numbers:

$$\begin{aligned} a\left(b\frac{d}{d\lambda}\right) &= a\left(\left(b\frac{dx^i}{d\lambda}\right)\frac{\partial}{\partial x^i}\right) \\ &= \left(a\left(b\frac{dx^i}{d\lambda}\right)\right)\frac{\partial}{\partial x^i} = \left(ab\frac{dx^i}{d\lambda}\right)\frac{\partial}{\partial x^i} = ab\frac{d}{d\lambda}\,. \end{aligned} \tag{2.52}$$

- Distributivity of multiplication by real numbers:

$$\begin{aligned} a\left(\frac{d}{d\lambda}+\frac{d}{d\mu}\right) &= a\left(\frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i}+a\frac{dx^i}{d\mu}\frac{\partial}{\partial x^i}\right) = \left(a\frac{dx^i}{d\lambda}+a\frac{dx^i}{d\mu}\right)\frac{\partial}{\partial x^i} \\ &= \left(a\frac{dx^i}{d\lambda}\right)\frac{\partial}{\partial x^i}+\left(a\frac{dx^i}{d\mu}\right)\frac{\partial}{\partial x^i} = a\frac{d}{d\lambda}+a\frac{d}{d\mu}\,. \end{aligned} \tag{2.53}$$

- The zero element is the vector tangent to the curve $x^\mu \equiv const$, which is simply the point $\mathbf{p}$.
- The opposite of the vector $\vec{v}$ tangent to a given curve is obtained by changing sign to the parametrization

$$\lambda \to -\lambda\,. \tag{2.54}$$

The proof of the remaining properties is analogous. *Therefore, the set of all directional derivatives on a point of a manifold form a vector space.* We call this space the **tangent space in p to the manifold M**, $\mathbf{T_p}$. The directional derivative operator $\frac{d}{d\lambda} \in \mathbf{T_p}$ is then a **vector**.

A basis for the space of directional derivatives

In any coordinate system $(x^1, \ldots, x^n)$ there are special curves, the *coordinate lines*. Along these lines one of the coordinates is taken as parameter, while the others are constant (think for example to the grid of cartesian coordinates). The directional derivatives along these lines are

$$\frac{d}{dx^i} = \frac{\partial x^k}{\partial x^i}\frac{\partial}{\partial x^k} = \delta^k_i\frac{\partial}{\partial x^k} = \frac{\partial}{\partial x^i}\,. \tag{2.55}$$

Thus, the operator of directional derivative along the coordinate lines coincides with the operator of partial derivative. Since, as shown in Eq. 2.35,

$$\frac{d}{d\lambda} = \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i}\,, \tag{2.56}$$

the generic directional derivative $\frac{d}{d\lambda}$ is a linear combination of the directional derivatives along the coordinate lines $\frac{\partial}{\partial x^i}$; therefore, these form a basis of the tangent space $\mathbf{T_p}$, called the **coordinate basis** associated with the coordinate system $(x^1, \ldots, x^n)$. The quantities $\{\frac{dx^i}{d\lambda}\}$ are the *components* of the vector $\frac{d}{d\lambda}$ in this basis.

Vectors as geometrical objects

As previously remarked, Eq. 2.35 establishes a one-to-one correspondence between the directional derivatives along the curves through $\mathbf{p}$ and the tangent vectors to the same curves in $\mathbf{p}$. Therefore, the tangent space $\mathbf{T_p}$ is also the space of the tangent vectors to the curves in **p**. Since the directional derivative is independent of the choice of the coordinate system, this correspondence shows that vectors are *geometrical objects*, i.e.

$$\vec{V} = \frac{d}{d\lambda} \,. \tag{2.57}$$

In a coordinate system $(x^1, \ldots, x^n)$ we can express this vector in the corresponding coordinate basis using Eq. 2.56:

$$\vec{V} = \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i} = V^i \frac{\partial}{\partial x^i} \tag{2.58}$$

where $V^i = \frac{dx^i}{d\lambda}$ are the components of $\vec{V}$ in the coordinate basis $\{\frac{\partial}{\partial x^i}\}$.

If we now apply $\frac{d}{d\lambda}$, i.e. $\vec{V}$, to a generic function Φ we find

$$\frac{d\Phi}{d\lambda} = \vec{V}(\Phi) = V^i \frac{\partial \Phi}{\partial x^i} \,, \tag{2.59}$$

and this is the directional derivative of Φ along $\vec{V}$.

Thus, *vectors map functions to real numbers.*

Note that this mapping is **linear**; indeed, given a function $\Phi = a\Phi_1 + b\Phi_2$, with Φ_1, Φ_2 functions on U and a, b real numbers, from the linearity of the partial differentiation operator and from Eq. 2.59 it follows

$$V^i \frac{\partial \Phi}{\partial x^i} = aV^i \frac{\partial \Phi_1}{\partial x^i} + bV^i \frac{\partial \Phi_2}{\partial x^i} = a\vec{V}(\Phi_1) + b\vec{V}(\Phi_2) \,, \tag{2.60}$$

i.e

$$\vec{V}(a\Phi_1 + b\Phi_2) = a\vec{V}(\Phi_1) + b\vec{V}(\Phi_2) \quad \forall \Phi_1 \,, \Phi_2 \text{ functions on } U \,, \quad \forall\, a, b \in \mathbb{R} \,. \tag{2.61}$$

In conclusion, we have shown that *a vector is a linear map which associates to any function Φ the real number $V^i \frac{\partial \Phi}{\partial x^i}$.*

It should be stressed that vectors *do not belong to the manifold* **M**: they belong to the tangent space to **M** in **p**, $\mathbf{T_p}$. If the manifold is $\mathbb{R}^n$ this distinction may be overlooked, because the tangent space (at any point) coincides with $\mathbb{R}^n$, but for a general manifold **M** the two spaces are different. Indeed, a generic manifold is *not* a vector space. For example, if the manifold is a sphere, we can not define the vectors as "arrows" on the sphere: they lie in the tangent space, which is the plane tangent to the sphere at a given point. For more general manifolds it is not easy to visualize $\mathbf{T_p}$. In any event $\mathbf{T_p}$ *has the same dimensions as the manifold* **M**.

We shall denote the vectors of the coordinate basis (associated with the coordinate system $\{x^i\}$) as

$$\vec{e}_{(i)} \equiv \frac{\partial}{\partial x^i} \tag{2.62}$$

(see e.g. Fig 2.17). To hereafter, we shall enclose within ( ) the indices that indicate the vector of a given basis, not to be confused with the index that indicates the components of the vector. The only exception is the operator of partial derivative, $\frac{\partial}{\partial x^i}$. For instance $e^1_{(2)}$ indicates the component 1 of the basis vector $\vec{e}_{(2)}$.

Any vector $\vec{A}$ at a point **p**, can be expressed as a linear combination of the basis vectors

$$\vec{A} = A^i \vec{e}_{(i)} \,, \quad \text{i.e. } \vec{A} \to_O \{A^i\} \tag{2.63}$$

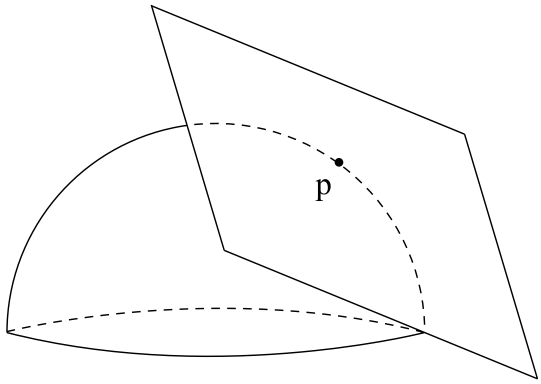

Figure 2.16 Tangent space $\mathbf{T_p}$ to a manifold $\mathbf{M}$.

where the numbers A^i are the components of $\vec{A}$ with respect to the chosen basis. If we make a coordinate transformation, the new set of coordinates $(x^{1'}, x^{2'}, \ldots, x^{n'})$ defines a new coordinate basis $\{\vec{e}_{(i')}\} \equiv \frac{\partial}{\partial x^{i'}}$. Expanding the vector $\vec{A}$ in the new basis,

$$\vec{A} = A^{j'}\vec{e}_{(j')}, \tag{2.64}$$

where $A^{j'}$ are the components of $\vec{A}$ with respect to the new basis $\vec{e}_{(j')}$. Since the vector $\vec{A}$ is a geometrical object, i.e. it is independent on the coordinate frame, the following equality must hold

$$A^i\vec{e}_{(i)} = A^{i'}\vec{e}_{(i')}. \tag{2.65}$$

From Eq. 2.32 we know how to express $A^{i'}$ as functions of the components of $\vec{A}$ in the old basis, i.e. $A^{i'} = \Lambda^{i'}{}_j A^j$, and replacing these expressions into Eq. 2.65 we find

$$A^i\vec{e}_{(i)} = \Lambda^{i'}{}_j A^j\vec{e}_{(i')} \tag{2.66}$$

where $\Lambda^{i'}{}_j = \frac{\partial x^{i'}}{\partial x^j}$. By relabelling the dummy indices this equation can be written as

$$\left[\vec{e}_{(j)} - \Lambda^{i'}{}_j\vec{e}_{(i')}\right] A^j = 0\,. \tag{2.67}$$

Since Eq. 2.67 must be satisfied for any non-vanishing vector $\vec{A}$, the term in square brakets must vanish, i.e.

$$\vec{e}_{(j)} = \Lambda^{i'}{}_j\vec{e}_{(i')}\,. \tag{2.68}$$

Multiplying both members by $\Lambda^j{}_{k'}$ and remembering that $\Lambda^j{}_{k'}\Lambda^{i'}{}_j = \delta^{i'}_{j'}$ (see Eq. 2.24), we find

$$\Lambda^j{}_{k'}\vec{e}_{(j)} = \Lambda^j{}_{k'}\Lambda^{i'}{}_j\vec{e}_{(i')} = \delta^{i'}_{k'}\vec{e}_{(i')}\,, \tag{2.69}$$

i.e.

$$\vec{e}_{(k')} = \Lambda^j{}_{k'}\vec{e}_{(j)}. \tag{2.70}$$

It should be noted that we do not need to choose necessarily a coordinate basis. We may

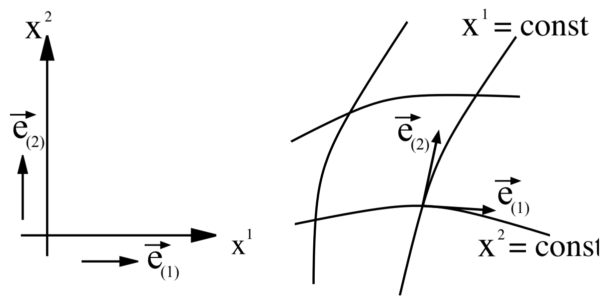

Figure 2.17 Coordinate basis of the tangent space

choose a set of independent basis vectors that are not tangent to the coordinate lines. In this case the matrix which transforms one basis to another has to be assigned, since it can not be written in terms of partial derivatives of a coordinate transformation.

Box 2-G

Let us consider the 4-dimensional flat spacetime of Special Relativity, restricted to the $(x-y)$ plane, where we choose the coordinates $(ct, x, y) \equiv (x^0, x^1, x^2)$. The coordinate basis is the set of vectors

$$\begin{aligned} \frac{\partial}{\partial x^0} &= \vec{e}_{(0)} \rightarrow_O (1,0,0)\,, \\ \frac{\partial}{\partial x^1} &= \vec{e}_{(1)} \rightarrow_O (0,1,0)\,, \\ \frac{\partial}{\partial x^2} &= \vec{e}_{(2)} \rightarrow_O (0,0,1)\,, \end{aligned} \tag{2.71}$$

or, in a compact form

$$e^{\beta}_{(\alpha)} = \delta^{\beta}_{\alpha} \tag{2.72}$$

(the superscript β now indicates the β-component of the α-th vector). Let us consider the coordinate transformation $(x^0, x, y) \rightarrow (x^0, r, \theta)$

$$\begin{cases} x^0 = x^{0'} \\ x^1 = r\cos\theta \\ x^2 = r\sin\theta, \end{cases}$$

i.e. $x^{0'} = x^0$, $x^{1'} = r$, $x^{2'} = \theta$. The basis vectors transform according to Eq. 2.70, i.e.

$$\vec{e}_{(\mu')} = \Lambda^{\alpha}{}_{\mu'}\vec{e}_{(\alpha)}\,, \qquad \text{where} \qquad \Lambda^{\alpha}{}_{\mu'} = \frac{\partial x^{\alpha}}{\partial x^{\mu'}}. \tag{2.73}$$

The matrix $\Lambda^{\alpha}{}_{\mu'}$ written for the coordinate transformation 2.73 is

$$\Lambda^{\alpha}{}_{\mu'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -r\sin\theta \\ 0 & \sin\theta & r\cos\theta \end{pmatrix}. \tag{2.74}$$

Remember that the first index (α in this case) indicates the row of the matrix, and the second (μ') indicates the column. The new coordinate basis therefore is

$$\begin{cases} \vec{e}_{(0')} = \Lambda^{\alpha}{}_{0'}\ \vec{e}_{(\alpha)} = \vec{e}_{(0)} \\ \vec{e}_{(1')} \equiv \vec{e}_{(r)} = \Lambda^{\alpha}{}_{1'}\ \vec{e}_{(\alpha)} = \cos\theta\vec{e}_{(1)} + \sin\theta\vec{e}_{(2)} \\ \vec{e}_{(2')} \equiv \vec{e}_{(\theta)} = \Lambda^{\alpha}{}_{2'}\ \vec{e}_{(\alpha)} = -r\sin\theta\vec{e}_{(1)} + r\cos\theta\vec{e}_{(2)}\ . \end{cases} \tag{2.75}$$