RELATIVITÀ GENERALE - SCRITTO 17-2-2014

Sia dato uno spazio-tempo descritto, nel riferimento O di coordinate $\{x^\mu\} = (t, x, y, z)$, dalla metrica

$$ds^2 = -e^{-t} dt^2 + x dx^2 + dy^2 + 2xy dy dz + z dz^2 \, .$$

I simboli di Christoffel non nulli sono

$$\Gamma^t_{tt} = \tfrac{1}{2}\,, \qquad \Gamma^x_{xx} = \tfrac{1}{2x}\,, \qquad \Gamma^x_{yz} = -\tfrac{1}{2}\tfrac{y}{x}\,,$$

$$\Gamma^y_{yy} = \tfrac{x^2 y}{(x^2y^2 - z)}\,, \quad \Gamma^y_{yx} = \tfrac{1}{2}\tfrac{xy^2}{(x^2y^2-z)}\,, \qquad \Gamma^y_{xz} \neq 0\,,$$

$$\Gamma^z_{xy} = -\tfrac{1}{2}\tfrac{y}{(x^2y^2-z)}\,, \quad \Gamma^z_{xz} = \tfrac{1}{2}\tfrac{xy^2}{(x^2y^2-z)}\,, \quad \Gamma^z_{yy} = -\tfrac{x}{(x^2y^2-z)}\,,$$

1. Calcolare il simbolo di Christoffel Γ^y_{xz}.
2. Dato il tensore T, di componenti nel riferimento O

$$T_{\mu\nu} = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ z & 0 & 0 & 0 \\ 0 & 0 & 0 & xy \end{pmatrix} .$$

   calcolare le componenti $T^\mu{}_\nu$.
3. Calcolare $T_{yt;x}$.
4. Sia dato il riferimento O', di coordinate $\{x^{\alpha'}\} = (u, r, \theta, v)$, definito dalla trasformazione di coordinate

$$\begin{array}{rcl} t &=& (u+v)/2 \\ x &=& r\cos\theta \\ y &=& r\sin\theta \\ z &=& (u-v)/2 \end{array} ,$$

   calcolare le componenti del tensore $T_{\mu'\nu'}$ nel riferimento O'.

Soluzione

La metrica e la metrica inversa sono

$$g_{\alpha\beta} = \begin{pmatrix} -e^{t} & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 1 & xy \\ 0 & 0 & xy & z \end{pmatrix},$$

$$g^{\alpha\beta} = \begin{pmatrix} -e^{-t} & 0 & 0 & 0 \\ 0 & \frac{1}{x} & 0 & 0 \\ 0 & 0 & -\frac{z}{(x^2y^2-z)} & \frac{xy}{(x^2y^2-z)} \\ 0 & 0 & \frac{xy}{(x^2y^2-z)} & -\frac{1}{(x^2y^2-z)} \end{pmatrix}.$$

1.

$$\begin{aligned} \Gamma^{y}_{xz} &= \frac{1}{2}g^{yk}\left(g_{kx,z} + g_{kz,x} - g_{xz,k}\right) \\ &= \frac{1}{2}g^{yy}\left(g_{yx,z} + g_{yz,x} - g_{xz,y}\right) + \frac{1}{2}g^{yz}\left(g_{zx,z} + g_{zz,x} - g_{xz,z}\right) \\ &= \frac{1}{2}g^{yy}g_{yz,x} = -\frac{zy}{2(x^2y^2 - z)}\,. \end{aligned}$$

2.

$$T^{\mu}_{\ \nu} = g^{\mu\alpha}T_{\alpha\nu} = \begin{pmatrix} -te^{-t} & 0 & 0 & 0 \\ 0 & \frac{1}{x} & 0 & 0 \\ -\frac{z^2}{(x^2y^2-z)} & 0 & 0 & \frac{x^2y^2}{(x^2y^2-z)} \\ \frac{xyz}{(x^2y^2-z)} & 0 & 0 & -\frac{xy}{(x^2y^2-z)} \end{pmatrix}.$$

3.

$$T_{yt;x} = T_{yt,x} - \Gamma^{k}_{yx}T_{kt} - \Gamma^{k}_{tx}T_{yk} = -\Gamma^{y}_{yx}T_{yt} = \frac{xy^2z}{2(z - x^2y^2)}\,.$$

4. Detta $\Lambda = (\Lambda^{\mu}{}_{\alpha'}) = \left(\frac{\partial x^{\mu}}{\partial x^{\alpha'}}\right)$ la matrice del cambiamento di coordinate, data da

$$\Lambda = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & \cos\theta & -r\sin\theta & 0 \\ 0 & \sin\theta & r\cos\theta & 0 \\ 1/2 & 0 & 0 & -1/2 \end{pmatrix},$$

si ha

$$T_{\alpha'\beta'} = \Lambda^{\mu}{}_{\alpha'}\Lambda^{\nu}{}_{\beta'}T_{\mu\nu}\,,$$

quindi definendo le matrici $T = (T_{\mu\nu})$, $T' = (T_{\alpha'\beta'})$, e detta Λ^T la trasposta di Λ

$$\begin{aligned}
T' &= \Lambda^T T \Lambda \\
&= \Lambda^T \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ z & 0 & 0 & 0 \\ 0 & 0 & 0 & xy \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & \cos\theta & -r\sin\theta & 0 \\ 0 & \sin\theta & r\cos\theta & 0 \\ 1/2 & 0 & 0 & -1/2 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -r\sin\theta & r\cos\theta & 0 \\ 1/2 & 0 & 0 & -1/2 \end{pmatrix} \begin{pmatrix} \frac{1}{2}t & 0 & 0 & \frac{1}{2}t \\ 0 & \cos\theta & -r\sin\theta & 0 \\ z/2 & 0 & 0 & z/2 \\ xy/2 & 0 & 0 & -xy/2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4}(t+xy) & 0 & 0 & \frac{1}{4}(t-xy) \\ \frac{1}{2}z\sin\theta & \cos^2\theta & -r\cos\theta\sin\theta & \frac{1}{2}z\sin\theta \\ \frac{1}{2}rz\cos\theta & -r\cos\theta\sin\theta & r^2\sin^2\theta & \frac{1}{2}rz\cos\theta \\ \frac{1}{4}(t-xy) & 0 & 0 & \frac{1}{4}(t+xy) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4}[\frac{u+v}{2} + r^2\;TT] & 0 & 0 & \frac{1}{4}[\frac{u+v}{2} - r^2\;TT] \\ \frac{u-v}{4}\sin\theta & \cos^2\theta & -rTT & \frac{u-v}{4}\sin\theta \\ \frac{u-v}{4}r\cos\theta & -rTT & r^2\sin^2\theta & \frac{u-v}{4}r\cos\theta \\ \frac{1}{4}[\frac{u+v}{2} - r^2\;TT] & 0 & 0 & \frac{1}{4}[\frac{u+v}{2} + r^2\;TT] \end{pmatrix},$$

dove abbiamo posto

$$TT = \cos\theta\sin\theta\,.$$