## I ESONERO 13-11-2007 - Compito A

Sia dato uno spazio-tempo descritto, nel riferimento $M$ di coordinate $\{x^\mu\} = (u, v, r, \phi)$, dalla metrica

$$ds^2 = -2uvdudv + (u+v)^2 dr^2 + r^2 d\phi^2 \,.$$

I simboli di Christoffel non nulli sono

$$\Gamma^{\ r}_{ur} = \frac{1}{u+v} \quad \Gamma^{\ v}_{vv} = \frac{1}{v} \quad \Gamma^{\ r}_{vr} = \frac{1}{u+v}$$

$$\Gamma^{\ v}_{rr} = \frac{u+v}{uv} \quad \Gamma^{\ \phi}_{r\phi} = \frac{1}{r} \quad \Gamma^{\ u}_{rr} = \frac{u+v}{uv}$$

$$\Gamma^{\ u}_{uu} \neq 0 \qquad\qquad \Gamma^{\ r}_{\phi\phi} \neq 0 \,.$$

**1**

Calcolare $\Gamma^{\ u}_{uu}$ e $\Gamma^{\ r}_{\phi\phi}$.

**2**

Dato il tensore $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $T$, di componenti nel riferimento $M$

$$T^\mu_{\ \nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(v)\cos(u) & 0 \end{pmatrix},$$

calcolare le componenti del tensore $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, ad esso associato, $T^{\mu\nu}$.

**3**

Calcolare

$$T^{\phi}{}_{u;r} \quad ; \quad T^{r}{}_{r;\phi}\,.$$

**4**

Sia dato il riferimento $M'$, di coordinate $\{x^{\alpha'}\} = (t', z', r', \phi')$, definito dalla trasformazione di coordinate $x^\mu = x^\mu(x^{\alpha'})$

$$\begin{aligned} u &= t' + z' \\ v &= t' - z' \\ r &= r' \\ \phi &= \phi'\,. \end{aligned}$$

Esprimere la metrica nel riferimento $M'$.

**5**

Calcolare le componenti del tensore $T^{\mu\nu}$ nel riferimento $M'$.

**SOLUZIONI**

La metrica e la metrica inversa sono

$$g_{\mu\nu} = \begin{pmatrix} 0 & -uv & 0 & 0 \\ -uv & 0 & (u+v)^2 & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{uv} & 0 & 0 \\ -\frac{1}{uv} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{(u+v)^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2} \end{pmatrix} .$$

**1**

$$\begin{aligned} \Gamma^u_{uu} &= \frac{1}{2} g^{u\alpha}(g_{\alpha u,u} + g_{u\alpha,u} - g_{uu,\alpha}) = g^{uv} g_{uv,u} \\ &= \left(-\frac{1}{uv}\right)(-v) = \frac{1}{u} \\ \Gamma^r_{\phi\phi} &= \frac{1}{2} g^{r\alpha}(g_{\alpha\phi,\phi} + g_{\phi\alpha,\phi} - g_{\phi\phi,\alpha}) = -\frac{1}{2} g^{rr} g_{\phi\phi,r} \\ &= -\frac{1}{2}\frac{1}{(u+v)^2} 2r = -\frac{r}{(u+v)^2} . \end{aligned}$$

**2**

$$\begin{aligned} T^{\mu\nu} &= T^{\mu}_{\ \alpha} g^{\alpha\nu} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(v)\cos(u) & 0 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{uv} & 0 & 0 \\ -\frac{1}{uv} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{(u+v)^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\cos(v)\cos(u)}{(u+v)^2} & 0 \end{pmatrix}.$$

**3**

$$\begin{aligned} T^{\phi}{}_{u;r} &= T^{\phi}{}_{u,r} + \Gamma^{\phi}_{\alpha r} T^{\alpha}{}_{u} - \Gamma^{\alpha}_{ur} T^{\phi}{}_{\alpha} \\ &= -\Gamma^{r}_{ur} T^{\phi}{}_{r} = -\frac{\cos(v)\cos(u)}{u+v} \\ T^{r}{}_{r;\phi} &= T^{r}{}_{r,\phi} + \Gamma^{r}_{\alpha\phi} T^{\alpha}{}_{r} - \Gamma^{\alpha}_{r\phi} T^{r}{}_{\alpha} \\ &= \Gamma^{r}_{\phi\phi} T^{\phi}{}_{r} = -\frac{r\cos(v)\cos(u)}{(u+v)^2}. \end{aligned}$$

**4**

$$\begin{aligned} du &= dt' + dz' \\ dv &= dt' - dz' \\ dr &= dr' \\ d\phi &= d\phi' \end{aligned} \qquad \Rightarrow \qquad dudv = (dt')^2 - (dz')^2$$

quindi

$$\begin{aligned} ds^2 &= -2uvdudv + (u+v)^2 dr^2 + r^2 d\phi^2 \\ &= -2\left((t')^2 - (z')^2\right)\left((dt')^2 - (dz')^2\right) + 4(t')^2(dr')^2 + (r')^2(d\phi')^2 . \end{aligned}$$

**5**

$$\Lambda = (\Lambda^{\mu}{}_{\alpha'}) = \left(\frac{\partial x^{\mu}}{\partial x^{\alpha'}}\right) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Lambda^{-1} = (\Lambda^{\alpha'}{}_{\mu}) = \left(\frac{\partial x^{\alpha\prime}}{\partial x^{\mu}}\right) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Si ha

$$T^{\alpha\prime\beta\prime} = \Lambda^{\alpha'}_{\ \mu} T^{\mu\nu} \Lambda^{\beta\prime}_{\ \nu}$$

quindi definendo le matrici di componenti $T = (T^{\mu\nu})$, $T' = (T^{\alpha\prime\beta\prime})$, possiamo scrivere

$$\begin{aligned} T' &= \Lambda^{-1} T (\Lambda^{-1})^T \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\cos(v)\cos(u)}{(u+v)^2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\cos(t'-z')\cos(t'+z')}{4(t')^2} & 0 \end{pmatrix}. \end{aligned}$$

## I ESONERO 13-11-2007 - Compito B

Sia dato uno spazio-tempo descritto, nel riferimento $M$ di coordinate $\{x^\mu\} = (u, v, r, \phi)$, dalla metrica

$$ds^2 = 2uvdudv + (u-v)^2 dr^2 + r^2 d\phi^2 \,.$$

I simboli di Christoffel non nulli sono

$$\Gamma^{\ r}_{vr} = \frac{1}{v-u} \quad \Gamma^{\ u}_{uu} = \frac{1}{u} \quad \Gamma^{\ r}_{ur} = \frac{1}{u-v}$$

$$\Gamma^{\ u}_{rr} = \frac{u-v}{uv} \quad \Gamma^{\ \phi}_{r\phi} = \frac{1}{r} \quad \Gamma^{\ v}_{rr} = \frac{v-u}{uv}$$

$$\Gamma^{\ v}_{vv} \neq 0 \qquad\qquad \Gamma^{\ r}_{\phi\phi} \neq 0 \,.$$

**1**

Calcolare $\Gamma^{\ v}_{vv}$ e $\Gamma^{\ r}_{\phi\phi}$.

**2**

Dato il tensore $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $T$, di componenti nel riferimento $M$

$$T_\mu^{\ \nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin(v)\sin(u) \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

calcolare le componenti del tensore $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, ad esso associato, $T_{\mu\nu}$.

**3**

Calcolare

$$T_r{}^r{}_{;\phi} \quad ; \quad T_u{}^\phi{}_{;r} \,.$$

**4**

Sia dato il riferimento $M'$, di coordinate $\{x^{\alpha'}\} = (t', z', r', \phi')$, definito dalla trasformazione di coordinate $x^\mu = x^\mu(x^{\alpha'})$

$$\begin{aligned} u &= -t' + z' \\ v &= t' + z' \\ r &= r' \\ \phi &= \phi' \,. \end{aligned}$$

Esprimere la metrica nel riferimento $M'$.

**5**

Calcolare le componenti del tensore $T_{\mu\nu}$ nel riferimento $M'$.

## SOLUZIONI

La metrica e la metrica inversa sono

$$g_{\mu\nu} = \begin{pmatrix} 0 & uv & 0 & 0 \\ uv & 0 & (u-v)^2 & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{uv} & 0 & 0 \\ \frac{1}{uv} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{(u-v)^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2} \end{pmatrix} .$$

**1**

$$\begin{aligned} \Gamma^v_{vv} &= \frac{1}{2} g^{v\alpha}(g_{\alpha v,v} + g_{v\alpha,v} - g_{vv,\alpha}) = g^{vu} g_{vu,v} \\ &= \frac{1}{uv} u = \frac{1}{v} \\ \Gamma^r_{\phi\phi} &= \frac{1}{2} g^{r\alpha}(g_{\alpha\phi,\phi} + g_{\phi\alpha,\phi} - g_{\phi\phi,\alpha}) = -\frac{1}{2} g^{rr} g_{\phi\phi,r} \\ &= -\frac{1}{2}\frac{1}{(u-v)^2} 2r = -\frac{r}{(u-v)^2} . \end{aligned}$$

**2**

$$\begin{aligned} T_{\mu\nu} &= T_\mu{}^\alpha g_{\alpha\nu} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin(v)\sin(u) \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & uv & 0 & 0 \\ uv & 0 & (u-v)^2 & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix} \end{aligned}$$

$$
= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2 \sin(v)\sin(u) \\ 0 & 0 & 0 & 0 \end{pmatrix} .
$$

**3**

$$
\begin{aligned}
T_u{}^{\phi}{}_{;r} &= T_u{}^{\phi}{}_{,r} + \Gamma^{\phi}_{\alpha r} T_u{}^{\alpha} - \Gamma^{\alpha}_{ur} T_{\alpha}{}^{\phi} \\
&= -\Gamma^{r}_{ur} T_r{}^{\phi} = \frac{\sin(v)\sin(u)}{v-u} \\
T_r{}^{r}{}_{;\phi} &= T_r{}^{r}{}_{,\phi} + \Gamma^{r}_{\alpha\phi} T_r{}^{\alpha} - \Gamma^{\alpha}_{r\phi} T_{\alpha}{}^{r} \\
&= \Gamma^{r}_{\phi\phi} T_r{}^{\phi} = -\frac{r\sin(v)\sin(u)}{(u-v)^2} .
\end{aligned}
$$

**4**

$$
\begin{aligned}
du &= -dt' + dz' \\
dv &= dt' + dz' \\
dr &= dr' \\
d\phi &= d\phi'
\end{aligned}
\qquad \Rightarrow \qquad dudv = -(dt')^2 + (dz')^2
$$

quindi

$$
\begin{aligned}
ds^2 &= 2uvdudv + (u-v)^2 dr^2 + r^2 d\phi^2 \\
&= 2\left(-(t')^2 + (z')^2\right)\left(-(dt')^2 + (dz')^2\right) + 4(t')^2(dr')^2 + (r')^2(d\phi')^2 .
\end{aligned}
$$

**5**

$$
\Lambda = (\Lambda^{\mu}{}_{\alpha'}) = \left(\frac{\partial x^{\mu}}{\partial x^{\alpha'}}\right) = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ,
$$

$$\Lambda^{-1} = (\Lambda^{\alpha'}{}_{\mu}) = \left(\frac{\partial x^{\alpha\prime}}{\partial x^{\mu}}\right) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Si ha

$$T_{\alpha\prime\beta\prime} = \Lambda^{\mu}{}_{\alpha\prime} T_{\mu\nu} \Lambda^{\nu}{}_{\beta\prime}$$

quindi definendo le matrici di componenti $T = (T_{\mu\nu})$, $T' = (T_{\alpha\prime\beta\prime})$, possiamo scrivere

$$\begin{aligned} T' &= \Lambda^T T \Lambda \\ &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2 \sin(v) \sin(u) \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2 \sin(t' + z') \sin(-t' + z') \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$