I ESONERO 13-11-2009 - Compito A

Sia dato uno spazio-tempo descritto, nel riferimento O coordinate $x^\mu = (t, r, \theta, \phi)$, dalla metrica

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

con $\nu(r)$, $\lambda(r)$ funzioni reali della coordinata r. I simboli di Christoffel non nulli sono:

$$\Gamma^{\ t}_{rt} = \Gamma^{\ t}_{tr} = \frac{1}{2}\nu_{,r}\,, \qquad \Gamma^{\ r}_{rr} = \frac{1}{2}\lambda_{,r}$$

$$\Gamma^{\ \theta}_{\theta r} = \Gamma^{\ \theta}_{r\theta} = \Gamma^{\ \phi}_{\phi r} = \Gamma^{\ \phi}_{r\phi} = \frac{1}{r}$$

$$\Gamma^{\ r}_{\theta\theta} = -re^{-\lambda}\,, \qquad \Gamma^{\ \phi}_{\theta\phi} = \Gamma^{\ \phi}_{\phi\theta} = \cot\theta$$

oltre a

$$\Gamma^{\ r}_{tt},\ \Gamma^{\ r}_{\phi\phi},\ \Gamma^{\ \theta}_{\phi\phi}.$$

1. Calcolare $\Gamma^{\ r}_{tt}$, $\Gamma^{\ r}_{\phi\phi}$ e $\Gamma^{\ \theta}_{\phi\phi}$.

2. Dato il tensore $\begin{pmatrix} 0 \\ 2 \end{pmatrix} T$, di componenti nel riferimento O

$$T_{\mu\nu} = \begin{pmatrix} e^\nu & 0 & 0 & -e^\nu \sin^2\theta \\ 0 & e^\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix},$$

   calcolare le componenti del tensore $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, ad esso associato, $T^{\mu\nu}$.

3. Calcolare

$$T_{t\theta;\phi}\,, \qquad T_{\phi\phi;\theta}\,, \qquad T_{t\phi;\theta}\,.$$

4. Sia dato il riferimento O', di coordinate $\{x^{\alpha'}\} = (t', r', \theta', \phi')$, definito dalla trasformazione di coordinate $x^{\alpha'} = x^{\alpha'}(x^\mu)$

$$\begin{aligned} t' &= 2t \\ r' &= r \\ \theta' &= \pi - \theta \\ \phi' &= \phi\,. \end{aligned}$$

   Determinare $\Lambda^{\alpha'}_{\ \ \mu} = \frac{\partial x^{\alpha'}}{\partial x^\mu}$ e $\Lambda^{\mu}_{\ \ \alpha'} = \frac{\partial x^\mu}{\partial x^{\alpha'}}$, e calcolare le componenti del tensore $T_{\mu\nu}$ nel riferimento O'.

I ESONERO 13-11-2009 - Compito B

Sia dato uno spazio-tempo descritto, nel riferimento O coordinate $x^\mu = (t, r, \theta, \phi)$, dalla metrica

$$ds^2 = -e^{-\lambda(r)}dt^2 + e^{-\nu(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

con $\nu(r)$, $\lambda(r)$ funzioni reali della coordinata r. I simboli di Christoffel non nulli sono:

$$\Gamma^{\ r}_{rr} = -\frac{1}{2}\nu_{,r}\,, \qquad \Gamma^{\ t}_{rt} = \Gamma^{\ t}_{tr} = -\frac{1}{2}\lambda_{,r}$$
$$\Gamma^{\ \theta}_{\theta r} = \Gamma^{\ \theta}_{r\theta} = \Gamma^{\ \phi}_{\phi r} = \Gamma^{\ \phi}_{r\phi} = \frac{1}{r}$$
$$\Gamma^{\ r}_{\theta\theta} = -re^{\nu}\,, \qquad \Gamma^{\ \phi}_{\theta\phi} = \Gamma^{\ \phi}_{\phi\theta} = \cot\theta$$

oltre a

$$\Gamma^{\ \theta}_{\phi\phi},\ \Gamma^{\ r}_{tt},\ \Gamma^{\ r}_{\phi\phi}.$$

1. Calcolare $\Gamma^{\ \theta}_{\phi\phi}$, $\Gamma^{\ r}_{tt}$ e $\Gamma^{\ r}_{\phi\phi}$.

2. Dato il tensore $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ T, di componenti nel riferimento O

$$T_{\mu\nu} = \begin{pmatrix} e^{-\lambda} & 0 & 0 & -e^{-\lambda}\sin^2\theta \\ 0 & e^{-\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix},$$

   calcolare le componenti del tensore $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, ad esso associato, $T^{\mu\nu}$.

3. Calcolare

$$T_{t\phi;\theta}\,, \qquad T_{t\theta;\phi}\,, \qquad T_{\phi\phi;\theta}\,.$$

4. Sia dato il riferimento O', di coordinate $\{x^{\alpha\prime}\} = (t', r', \theta', \phi')$, definito dalla trasformazione di coordinate $x^{\alpha\prime} = x^{\alpha\prime}(x^\mu)$

$$\begin{aligned} t' &= \tfrac{1}{3}t \\ r' &= r \\ \theta' &= \pi - \theta \\ \phi' &= \phi\,. \end{aligned}$$

   Determinare $\Lambda^{\alpha'}_{\ \mu} = \frac{\partial x^{\alpha'}}{\partial x^\mu}$ e $\Lambda^{\mu}_{\ \alpha'} = \frac{\partial x^\mu}{\partial x^{\alpha'}}$, e calcolare le componenti del tensore $T_{\mu\nu}$ nel riferimento O'.

Soluzioni compito A

La metrica e la metrica inversa sono

$$g_{\mu\nu} = \mathrm{diag}\left(-e^{\nu}, e^{\lambda}, r^2, r^2 \sin^2\theta\right)$$

$$g^{\mu\nu} = \mathrm{diag}\left(-e^{-\nu}, e^{-\lambda}, r^{-2}, r^{-2} \sin^{-2}\theta\right) \, .$$

1.

$$\begin{aligned}
\Gamma^r_{tt} &= \frac{1}{2} g^{r\alpha}(g_{t\alpha,t} + g_{t\alpha,t} - g_{tt,\alpha}) = -\frac{1}{2} g^{rr} g_{tt,r} = \frac{1}{2}\nu_{,r} e^{-\lambda+\nu} \\
\Gamma^r_{\phi\phi} &= \frac{1}{2} g^{r\alpha}(g_{\phi\alpha,\phi} + g_{\phi\alpha,\phi} - g_{\phi\phi,\alpha}) = -\frac{1}{2} g^{rr} g_{\phi\phi,r} = -e^{-\lambda} r \sin^2\theta \\
\Gamma^\theta_{\phi\phi} &= \frac{1}{2} g^{\theta\alpha}(g_{\phi\alpha,\phi} + g_{\phi\alpha,\phi} - g_{\phi\phi,\alpha}) = -\frac{1}{2} g^{\theta\theta} g_{\phi\phi,\theta} = -\sin\theta\cos\theta \, .
\end{aligned}$$

2.

$$T^{\mu\nu} = g^{\mu\alpha} T_{\alpha\beta} g^{\beta\nu} = \begin{pmatrix} e^{-\nu} & 0 & 0 & \frac{1}{r^2} \\ 0 & e^{-\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} .$$

3.

$$\begin{aligned}
T_{t\theta;\phi} &= T_{t\theta,\phi} - \Gamma^\alpha_{t\phi} T_{\alpha\theta} - \Gamma^\alpha_{\theta\phi} T_{t\alpha} \\
&= -\Gamma^\phi_{\theta\phi} T_{t\phi} = e^{\nu} \sin\theta\cos\theta \\
T_{\phi\phi;\theta} &= T_{\phi\phi,\theta} - \Gamma^\alpha_{\phi\theta} T_{\alpha\phi} - \Gamma^\alpha_{\phi\theta} T_{\phi\alpha} \\
&= T_{\phi\phi,\theta} - 2\Gamma^\phi_{\theta\phi} T_{\phi\phi} = 2r^2 \sin\theta\cos\theta - 2\cot\theta r^2 \sin^2\theta = 0 \\
T_{t\phi;\theta} &= T_{t\phi,\theta} - \Gamma^\alpha_{t\theta} T_{\alpha\phi} - \Gamma^\alpha_{\phi\theta} T_{t\alpha} \\
&= T_{t\phi,\theta} - \Gamma^\phi_{\phi\theta} T_{t\phi} = -2e^{\nu} \sin\theta\cos\theta + \cot\theta e^{\nu} \sin^2\theta = -e^{\nu}\sin\theta\cos\theta \, .
\end{aligned}$$

4.

$$T_{\alpha'\beta'} = \Lambda^{\mu}{}_{\alpha'} \Lambda^{\nu}{}_{\beta'} T_{\mu\nu} = \Lambda^{\mu}{}_{\alpha'} T_{\mu\nu} \Lambda^{\nu}{}_{\beta'}$$

e

$$\Lambda = (\Lambda^{\alpha'}{}_{\mu}) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

$$\Lambda^{-1} = (\Lambda^{\mu}{}_{\alpha'}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

In forma matriciale, definendo le matrici $T = (T_{\mu\nu})$, $T' = (T_{\alpha'\beta'})$,

$$\begin{aligned}
T' &= (\Lambda^T)^{-1} T \Lambda^{-1} \\
&= \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}e^{\nu} & 0 & 0 & -e^{\nu}\sin^2\theta \\ 0 & e^{\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{4}e^{\nu} & 0 & 0 & -\frac{1}{2}e^{\nu}\sin^2\theta \\ 0 & e^{\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}.
\end{aligned}$$

Sostituendo nelle nuove coordinate,

$$T_{\alpha'\beta'} = \begin{pmatrix} \frac{1}{4}e^{\nu(r')} & 0 & 0 & -\frac{1}{2}e^{\nu(r')}\sin^2\theta' \\ 0 & e^{\lambda(r')} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r'^2\sin^2\theta' \end{pmatrix}.$$

Soluzioni compito B

La metrica e la metrica inversa sono

$$g_{\mu\nu} = \text{diag}\left(-e^{-\lambda}, e^{-\nu}, r^2, r^2 \sin^2\theta\right)$$

$$g^{\mu\nu} = \text{diag}\left(-e^{\lambda}, e^{\nu}, r^{-2}, r^{-2} \sin^{-2}\theta\right) .$$

1.

$$\begin{aligned}
\Gamma^r_{tt} &= \frac{1}{2} g^{r\alpha}(g_{t\alpha,t} + g_{t\alpha,t} - g_{tt,\alpha}) = -\frac{1}{2} g^{rr} g_{tt,r} = -\frac{1}{2}\lambda_{,r} e^{-\lambda+\nu} \\
\Gamma^r_{\phi\phi} &= \frac{1}{2} g^{r\alpha}(g_{\phi\alpha,\phi} + g_{\phi\alpha,\phi} - g_{\phi\phi,\alpha}) = -\frac{1}{2} g^{rr} g_{\phi\phi,r} = -e^{\nu} r \sin^2\theta \\
\Gamma^\theta_{\phi\phi} &= \frac{1}{2} g^{\theta\alpha}(g_{\phi\alpha,\phi} + g_{\phi\alpha,\phi} - g_{\phi\phi,\alpha}) = -\frac{1}{2} g^{\theta\theta} g_{\phi\phi,\theta} = -\sin\theta\cos\theta .
\end{aligned}$$

2.

$$T^{\mu\nu} = g^{\mu\alpha} T_{\alpha\beta} g^{\beta\nu} = \begin{pmatrix} e^{\lambda} & 0 & 0 & \frac{1}{r^2} \\ 0 & e^{\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix} .$$

3.

$$\begin{aligned}
T_{t\theta;\phi} &= T_{t\theta,\phi} - \Gamma^\alpha_{t\phi} T_{\alpha\theta} - \Gamma^\alpha_{\theta\phi} T_{t\alpha} \\
&= -\Gamma^\phi_{\theta\phi} T_{t\phi} = e^{-\lambda} \sin\theta\cos\theta \\
T_{\phi\phi;\theta} &= T_{\phi\phi,\theta} - \Gamma^\alpha_{\phi\theta} T_{\alpha\phi} - \Gamma^\alpha_{\phi\theta} T_{\phi\alpha} \\
&= T_{\phi\phi,\theta} - 2\Gamma^\phi_{\theta\phi} T_{\phi\phi} = 2r^2 \sin\theta\cos\theta - 2\cot\theta r^2 \sin^2\theta = 0 \\
T_{t\phi;\theta} &= T_{t\phi,\theta} - \Gamma^\alpha_{t\theta} T_{\alpha\phi} - \Gamma^\alpha_{\phi\theta} T_{t\alpha} \\
&= T_{t\phi,\theta} - \Gamma^\phi_{\phi\theta} T_{t\phi} = -2e^{-\lambda} \sin\theta\cos\theta + \cot\theta e^{-\lambda} \sin^2\theta = -e^{-\lambda} \sin\theta\cos\theta .
\end{aligned}$$

4.

$$T_{\alpha'\beta'} = \Lambda^{\mu}{}_{\alpha'} \Lambda^{\nu}{}_{\beta'} T_{\mu\nu} = \Lambda^{\mu}{}_{\alpha'} T_{\mu\nu} \Lambda^{\nu}{}_{\beta'}$$

e

$$\Lambda = (\Lambda^{\alpha'}{}_{\mu}) = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

$$\Lambda^{-1} = (\Lambda^{\mu}{}_{\alpha'}) = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

In forma matriciale, definendo le matrici $T = (T_{\mu\nu})$, $T' = (T_{\alpha'\beta'})$,

$$
\begin{aligned}
T' &= (\Lambda^T)^{-1} T \Lambda^{-1} \\
&= \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3e^{-\lambda} & 0 & 0 & -e^{-\lambda}\sin^2\theta \\ 0 & e^{-\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \\
&= \begin{pmatrix} 9e^{-\lambda} & 0 & 0 & -3e^{-\lambda}\sin^2\theta \\ 0 & e^{-\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}.
\end{aligned}
$$

Sostituendo nelle nuove coordinate,

$$
T_{\alpha'\beta'} = \begin{pmatrix} 9e^{-\lambda(r')} & 0 & 0 & -3e^{-\lambda(r')}\sin^2\theta' \\ 0 & e^{-\nu(r')} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r'^2\sin^2\theta' \end{pmatrix}.
$$