I ESONERO 2-11-2006 - Compito A

Sia dato uno spazio-tempo descritto, nel riferimento M di coordinate $\{x^\mu\} = (u, r, \theta, \phi)$, dalla metrica

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

con $M > 0$ costante.

I simboli di Christoffel non nulli sono:

Γ^u_{uu}	Γ^r_{ur}	$\Gamma^\theta_{r\theta} = \frac{1}{r}$	$\Gamma^\phi_{r\phi} = \frac{1}{r}$
$\Gamma^u_{\theta\theta} = r$	$\Gamma^r_{uu} = \frac{M}{r^3}(r - 2M)$	$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$	$\Gamma^\phi_{\theta\phi} = \cot\theta$
$\Gamma^u_{\phi\phi} = r\sin^2\theta$	$\Gamma^r_{\theta\theta} = -r + 2M$		
	$\Gamma^r_{\phi\phi} = -(r - 2M)\sin^2\theta$		

1

Calcolare Γ^r_{ur}.

2

Dato il tensore T di rango $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, di componenti, nel riferimento M,

$$T_{\mu\nu} = \begin{pmatrix} 0 & r & 0 & 0 \\ -r & u & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix}.$$

calcolare la derivata covariante $T_{ur;r}$.

3

Calcolare le componenti del tensore $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$T^{\mu}{}_{\nu}\,.$$

4

Dato il riferimento M' di coordinate $\{x^{\alpha\prime}\} = (t', r', \theta', \phi')$, definito dalla trasformazione di coordinate

$$x^{\mu} = x^{\mu}(x^{\alpha\prime}) \quad \Rightarrow \quad \left\{ \begin{array}{lcc} u & = & t' - r' \\ r & = & r' \\ \theta & = & \theta' \\ \phi & = & \phi' \end{array} \right. ,$$

calcolare le componenti del tensore $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ T nel riferimento M':

$$T_{\alpha\prime\beta'}\,.$$

Soluzioni compito A

1.

$$
\begin{aligned}
g_{\mu\nu} &= \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \\
g^{\mu\nu} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 1-\frac{2M}{r} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix} \\
\Gamma^r_{ur} &= g^{r\alpha}\Gamma_{ur\alpha} = g^{ru}\Gamma_{uru} + g^{rr}\Gamma_{urr} \\
\Gamma_{uru} &= \frac{1}{2}(g_{uu,r} + g_{ru,u} - g_{ur,u}) = \frac{1}{2}g_{uu,r} = -\frac{M}{r^2} \\
\Gamma_{urr} &= \frac{1}{2}(g_{ur,r} + g_{rr,u} - g_{ur,r}) = 0
\end{aligned}
$$

quindi

$$\Gamma^r_{ur} = g^{ru}\Gamma_{uru} = \frac{M}{r^2}\,.$$

2.

$$
\begin{aligned}
T_{ur;r} &= T_{ur,r} - \Gamma^\alpha_{ur}T_{\alpha r} - \Gamma^\alpha_{rr}T_{u\alpha} \\
&= T_{ur,r} - \Gamma^r_{ur}T_{rr} = 1 - \frac{Mu}{r^2}\,.
\end{aligned}
$$

3.

$$
T^\mu_{\ \nu} = g^{\mu\alpha}T_{\alpha\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 1-\frac{2M}{r} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix} \begin{pmatrix} 0 & r & 0 & 0 \\ -r & u & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix}
$$

$$= \begin{pmatrix} r & -u & 0 & 0 \\ -r+2M & -r+u-\frac{2Mu}{r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin^{-2}\theta \end{pmatrix}$$

4.

$$\Lambda = (\Lambda^{\mu}{}_{\alpha'}) = \left(\frac{\partial x^{\mu}}{\partial x^{\alpha'}}\right) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_{\alpha'\beta'} = \Lambda^{\mu}{}_{\alpha'}\Lambda^{\nu}{}_{\beta'}T_{\mu\nu}\,.$$

Definendo le matrici T, T' di componenti

$$T = (T_{\mu\nu}) \qquad T' = (T_{\alpha'\beta'})$$

abbiamo l'equazione matriciale

$$\begin{aligned} T' &= \Lambda^T T \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & r' & 0 & 0 \\ -r' & t'-r' & 0 & 0 \\ 0 & 0 & r'^2 & 0 \\ 0 & 0 & 0 & r'^2 \end{pmatrix} \Lambda \\ &= \begin{pmatrix} 0 & r' & 0 & 0 \\ -r' & t'-2r' & 0 & 0 \\ 0 & 0 & r'^2 & 0 \\ 0 & 0 & 0 & r'^2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & r' & 0 & 0 \\ -r' & t'-r' & 0 & 0 \\ 0 & 0 & r'^2 & 0 \\ 0 & 0 & 0 & r'^2 \end{pmatrix}. \end{aligned}$$