Chapter 4

Geodesics of the Kerr metric

Here we will study the geodesic motion outside a Kerr black hole; therefore, we will only consider the region outside the outer horizon,

$$r \geq r_+ \tag{4.1}$$

which is the relevant region for astrophysical considerations.

Let us consider a geodesic with affine parameter λ and tangent vector

$$u^\mu = \frac{dx^\mu}{d\lambda} \equiv \dot{x}^\mu \tag{4.2}$$

in Boyer-Linquist coordinates (we remind that, in our conventions, the overdot has the meaning of derivation with respect to the affine parameter λ). The tangent vector u^μ is solution of the geodesic equation

$$u^\mu {u^\nu}_{;\mu} = 0\,. \tag{4.3}$$

The geodesic equation (4.3) is equivalent to the Euler-Lagrange equations

$$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = \frac{\partial \mathcal{L}}{\partial x^\alpha} \tag{4.4}$$

associated to the Lagrangian

$$\mathcal{L}\left(x^\mu, \dot{x}^\mu\right) = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\,. \tag{4.5}$$

One can define the *conjugate momentum* p_μ to the coordinate x^μ as

$$p_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\mu\nu}\dot{x}^\mu\,. \tag{4.6}$$

In terms of the conjugate momenta, the Euler-Lagrange equations can be written as

$$\frac{d}{d\lambda}p_\mu = \frac{\partial \mathcal{L}}{\partial x^\mu}\,. \tag{4.7}$$

Notice that, if the metric does not depend on a given coordinate x^μ, the corresponding conjugate momentum is a constant of motion. The existence of such a constant can also be seen in a simpler way. We know that, if the metric is independent by a coordinate, there is a Killing vector ξ^μ corresponding to this symmetry, and $\xi_\mu \dot{x}^\mu$ is a constant of geodesic motion; actually, this constant of motion coincides with the one arising from Euler-Lagrange equations (as we will see, for instance, in the case of t and ϕ for the Kerr metric).

Let us come back to the geodesic equation for Kerr spacetime. Equation (4.3) (or, equivalently, (4.4)) in Kerr spacetime is very complicate to solve directly. To study geodesic motion in a simple way, we would need to find a way to express the geodesic u^μ in terms of conserved quantities, as we have done in the case of Schwarzschild spacetime. For this, we need four algebraic relations involving u^μ.

As we have seen in Section 3.2, Kerr spacetime has two Killing vectors: a timelike Killing vector $k^\mu = (1,0,0,0)$ and a spacelike killing vector $m^\mu = (0,0,0,1)$. This corresponds to the fact that the metric (in Boyer-Lindquist coordinates) does not depend explicitly by t and ϕ: the spacetime is stationary and axially symmetric.

Therefore, in geodesic motion there are two conserved quantities:

$$E \equiv -k_\mu u^\mu = -g_{t\mu}u^\mu = -p_t \qquad \text{constant along geodesics} \tag{4.8}$$

$$L \equiv m_\mu u^\mu = g_{\phi\mu}u^\mu = p_\phi \quad \text{constant along geodesics}\,. \tag{4.9}$$

As explained in Section 3.2, in the case of massive particle E is the energy at infinity per mass unit while L is the angular momentum per mass unit; in the case of massless particles, E is the energy at infinity and L is the angular momentum.

Furthermore, we have the relation

$$g_{\mu\nu}u^\mu u^\nu = \kappa \tag{4.10}$$

where

$$\kappa = -1 \qquad \text{for timelike geodesics}$$

$$
\begin{aligned}
\kappa &= 1 \qquad \text{for spacelike geodesics}\\
\kappa &= 0 \qquad \text{for null geodesics}\,.
\end{aligned}
\tag{4.11}
$$

The relations (4.8), (4.9), (4.10) give us three algebraic relations involving u^μ, but they are not sufficient to to determine the *four* unknowns u^μ. This is different from the case of Schwarzschild spacetime, where we have the further condition of planarity of the orbit ($u^\theta(\lambda) \equiv 0$ if $\theta(\lambda = 0) = \pi/2$ and $u^\theta(\lambda = 0) = 0$) arising from the Euler-Lagrange equations, which have a simple form in that case.

Without this further condition, we cannot study geodesic motion using only (4.8), (4.9), (4.10). Actually, there is a further conserved quantity, the *Carter constant*, which allows to find the tangent vector u^μ through algebraic relations.

Anyway, it is possible to study geodesic motion with the only help of (4.8), (4.9), (4.10) in the particular case of *equatorial motion*, i.e. with $\theta \equiv \pi/2$. Therefore, we will first study equatorial geodesics, and then, in Section 4.2, we will consider the case of general motion and derive the Carter constant.

4.1 Equatorial geodesics

In this section we study equatorial geodesics, i.e. geodesics with

$$\theta \equiv \frac{\pi}{2}\,. \tag{4.12}$$

First of all, let us prove that such geodesics exist, i.e. that equatorial geodesics are solutions of the geodesic equation (4.3), or, equivalently, of the Euler-Lagrange equations (4.4). The θ component of (4.4) is

$$\frac{d}{d\lambda}(g_{\theta\mu}\dot{x}^\mu) = \frac{d}{d\lambda}(\Sigma\dot{\theta}) = \Sigma\ddot{\theta} + \Sigma_{,\mu}\dot{x}^\mu\dot{\theta} = \frac{1}{2}g_{\mu\nu,\theta}\dot{x}^\mu\dot{x}^\nu \tag{4.13}$$

where we have used the fact that the only non-vanishing $g_{\theta\mu}$ component in (3.1) is $g_{\theta\theta} = \Sigma$.

The right-hand side is

$$
\begin{aligned}
g_{\mu\nu,\theta}\dot{x}^\mu\dot{x}^\nu &= \Sigma_{,\theta}\left(\frac{(\dot{r})^2}{\Delta} + (\dot{\theta})^2\right) + 2\sin\theta\cos\theta(r^2+a^2)(\dot{\phi})^2\\
- \quad \frac{2Mr}{\Sigma^2}\Sigma_{,\theta}&\left(a\sin^2\theta\dot{\phi} - \dot{t}\right)^2 + \frac{4Mr}{\Sigma}\left(a\sin^2\theta\dot{\phi} - \dot{t}\right)2a\sin\theta\cos\theta\dot{\phi}
\end{aligned}
\tag{4.14}
$$

where $\Sigma_{,\theta} = -2a^2 \sin\theta \cos\theta$ and $\Sigma_{,r} = 2r$. It is easy to check that $\theta \equiv \pi/2$ is a solution of equation (4.13).

If, at $\lambda = 0$, the particles moves in the equatorial plane, $\theta(\lambda = 0) = \pi/2$ and $\dot{\theta}(\lambda = 0) = 0$; then we have a well-posed Cauchy problem of the form

$$\begin{aligned} \ddot{\theta} &= \ldots \\ \dot{\theta}(\lambda = 0) &= 0 \\ \theta(\lambda = 0) &= \frac{\pi}{2} \end{aligned} \tag{4.15}$$

which admits one and only one solution; since $\theta \equiv \pi/2$ is a solution, it is the solution. Thus, a geodesic which starts in the equatorial plane, remains in the equatorial plane.

This also happens in the Schwarzschild metric. But, while in that case it is possible to generalize the result to any orbit, thanks to the spherical symmetry, and prove that all Schwarzschild geodesics are planar, such a generalization is not possible for the Kerr metric which is axially symmetric. We only can say that geodesics starting in the equatorial plane are planar.

On the equatorial plane, $\Sigma = r^2$, therefore

$$\begin{aligned} g_{tt} &= -\left(1 - \frac{2M}{r}\right) \\ g_{t\phi} &= -\frac{2Ma}{r} \\ g_{rr} &= \frac{r^2}{\Delta} \\ g_{\phi\phi} &= r^2 + a^2 + \frac{2Ma^2}{r} \end{aligned} \tag{4.16}$$

and

$$E = -g_{t\mu}u^\mu = \left(1 - \frac{2M}{r}\right)\dot{t} + \frac{2Ma}{r}\dot{\phi} \tag{4.17}$$

$$L = g_{\phi\mu}u^\mu = -\frac{2Ma}{r}\dot{t} + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\dot{\phi}\,. \tag{4.18}$$

To solve (4.17), (4.18) for $\dot{t}$, $\dot{\phi}$ we define

$$A \equiv 1 - \frac{2M}{r}$$

$$
\begin{aligned}
B &\equiv \frac{2Ma}{r} \\
C &\equiv r^2 + a^2 + \frac{2Ma^2}{r} \qquad (4.19)
\end{aligned}
$$

so that (4.17), (4.18) can be written as

$$
\begin{aligned}
E &= A\dot{t} + B\dot{\phi} \qquad (4.20) \\
L &= -B\dot{t} + C\dot{\phi}\,. \qquad (4.21)
\end{aligned}
$$

We also have

$$
\begin{aligned}
AC + B^2 &= \left(1 - \frac{2M}{r}\right)\left(r^2 + a^2 + \frac{2Ma^2}{r}\right) + \frac{4M^2a^2}{r^2} \\
&= r^2 - 2Mr + a^2 = \Delta\,. \qquad (4.22)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
CE - BL &= [AC + B^2]\dot{t} = \Delta\dot{t} \\
AL + BE &= [AC + B^2]\dot{\phi} = \Delta\dot{\phi} \qquad (4.23)
\end{aligned}
$$

i.e.

$$
\begin{aligned}
\dot{t} &= \frac{1}{\Delta}\left[\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)E - \frac{2Ma}{r}L\right] \\
\dot{\phi} &= \frac{1}{\Delta}\left[\left(1 - \frac{2M}{r}\right)L + \frac{2Ma}{r}E\right]\,. \qquad (4.24)
\end{aligned}
$$

Equation (4.10) can be written in terms of A, B, C:

$$
\begin{aligned}
g_{\mu\nu}u^\mu u^\nu &= \kappa \\
&= -A\dot{t}^2 - 2B\dot{t}\dot{\phi} + C\dot{\phi}^2 + \frac{r^2}{\Delta}\dot{r}^2 \\
&= -[A\dot{t} + B\dot{\phi}]\dot{t} + [-B\dot{t} + C\dot{\phi}]\dot{\phi} + \frac{r^2}{\Delta}\dot{r}^2 \\
&= -E\dot{t} + L\dot{\phi} + \frac{r^2}{\Delta}\dot{r}^2 \qquad (4.25)
\end{aligned}
$$

where we have used (4.20), (4.21). Therefore,

$$
\begin{aligned}
\dot{r}^2 &= \frac{\Delta}{r^2}(E\dot{t} - L\dot{\phi} + \kappa) \\
&= \frac{1}{r^2}\left[CE^2 - 2BLE - AL^2\right] + \frac{\kappa\Delta}{r^2} \\
&= \frac{C}{r^2}(E - V_+)(E - V_-) + \frac{\kappa\Delta}{r^2} \qquad (4.26)
\end{aligned}
$$

where $V_\pm(r)$ are the solutions of the equation in E

$$CE^2 - 2BLE - AL^2 = 0\,, \tag{4.27}$$

i.e.

$$V_\pm = \frac{BL \pm \sqrt{B^2L^2 + ACL^2}}{C} = \frac{1}{C}(BL \pm |L|\sqrt{\Delta})\,. \tag{4.28}$$

Some authors write this formula without the modulus (which is equivalent to exchange the definitions of V_+ and V_- when $L < 0$), but we prefer this notation, in which $V_+ \geq V_-$ for every value of L.

The quantity C can be rewritten as follows:

$$\begin{aligned}
\frac{(r^2+a^2)^2 - a^2\Delta}{r^2} &= \frac{1}{r^2}[(r^2+a^2)(r^2+a^2) - a^2(r^2+a^2-2Mr)] \\
&= \frac{1}{r^2}[(r^2+a^2)r^2 + 2Mra^2] = r^2 + a^2 + \frac{2Ma^2}{r} \\
&= C
\end{aligned} \tag{4.29}$$

therefore (4.26) and (4.28) can be written as

$$\dot{r}^2 = \frac{(r^2+a^2)^2 - a^2\Delta}{r^4}(E - V_+)(E - V_-) + \frac{\kappa\Delta}{r^2} \tag{4.30}$$

$$V_\pm = \frac{2MLar \pm r^2|L|\sqrt{\Delta}}{(r^2+a^2)^2 - a^2\Delta}\,. \tag{4.31}$$

Notice that since $(r^2+a^2)^2 - a^2\Delta > 0$, $C > 0$.

In the Schwarzschild limit $a \to 0$, we have

$$V_+ + V_- \propto a \to 0\ , \qquad V_+V_- \to -\frac{L^2\Delta}{r^4} \tag{4.32}$$

therefore, if we define $V \equiv -V_+V_- - \kappa\Delta/r^2$, Eqns. (4.30), (4.31) reduce to the well known form

$$\begin{aligned}
\dot{r}^2 &= E^2 - V(r) \\
V(r) &= -\frac{\kappa\Delta}{r^2} + \frac{L^2\Delta}{r^4} \\
&= \left(1 - \frac{2M}{r}\right)\left(-\kappa + \frac{L^2}{r^2}\right)
\end{aligned} \tag{4.33}$$

where we recall that $\kappa = -1$ for timelike geodesics, $\kappa = 0$ for null geodesics, $\kappa = 1$ for spacelike geodesics.

4.1.1 Null geodesics

In the case of null geodesics (4.30) becomes

$$\dot{r}^2 = \frac{(r^2+a^2)^2 - a^2\Delta}{r^4}(E - V_+)(E - V_-) \tag{4.34}$$

with

$$V_\pm = \frac{2MLar \pm |L| r^2 \sqrt{\Delta}}{(r^2+a^2)^2 - a^2\Delta} \,. \tag{4.35}$$

We have two possibilities: $La > 0$ and $La < 0$, i.e. the photon and the black hole corotating and counterrotating; notice that we always have

$$V_+ \geq V_- \,. \tag{4.36}$$

Since $\dot{r}^2$ must be positive, from eq. (4.34) we see that, being $(r^2 + a^2)^2 - a^2\Delta > 0$, null geodesics are possible for massless particle whose constant of motion E satisfies the following inequalities

$$E < V_- \qquad \text{or} \qquad E > V_+ \,. \tag{4.37}$$

Thus, the region $V_- < E < V_+$ is forbidden.

In general, we find that:

- The two curves coincide for $\Delta = 0$, i.e. at

  $$r = r_+ = M + \sqrt{M^2 - a^2} \tag{4.38}$$

  while for $r > r_+$, $\Delta > 0$ and then $V_+ > V_-$. Furthermore,

  $$V_+(r_+) = V_-(r_+) = \frac{2Mr_+La}{(r_+^2 + a^2)^2} = L\Omega_H \,, \tag{4.39}$$

  which is positive if $La > 0$, negative if $La < 0$.

- In the limit $r \to \infty$, $V_\pm \to 0$.

- If $La > 0$ (corotating orbits), the potential V_+ is definite positive; V_- (which is positive at r_+) vanishes when

  $$r\sqrt{\Delta} = 2M|a| \quad \Rightarrow \quad r^2(r^2 - 2Mr + a^2) = 4M^2a^2 \tag{4.40}$$

  which gives

  $$r^4 - 2Mr^3 + a^2r^2 - 4M^2a^2 = (r - 2M)(r^3 + a^2r + 2Ma^2) = 0; \tag{4.41}$$

  thus V_- vanishes at $r = 2M$, which is the location of the ergosphere in the equatorial plane ($r_{S_+}(\theta = \pi/2) = 2M$).

- If $La < 0$ (counterrotating orbits), the potential V_- is definite negative and V_+ (which is negative at r_+) vanishes at the ergosphere $r = r_{S_+} = 2M$.
- The study of the derivatives of $V_\pm$, which is too long to be reported here, tells us that both potentials, V_+ and V_-, have only one stationary point.

In summary, $V_+(r)$ and $V_-(r)$ have the shapes shown in Figure 4.1 where in the two panels we show the cases $La > 0$ (up) and $La < 0$ (down), respectively. Once we have drawn the curves $V_+(r), V_-(r)$

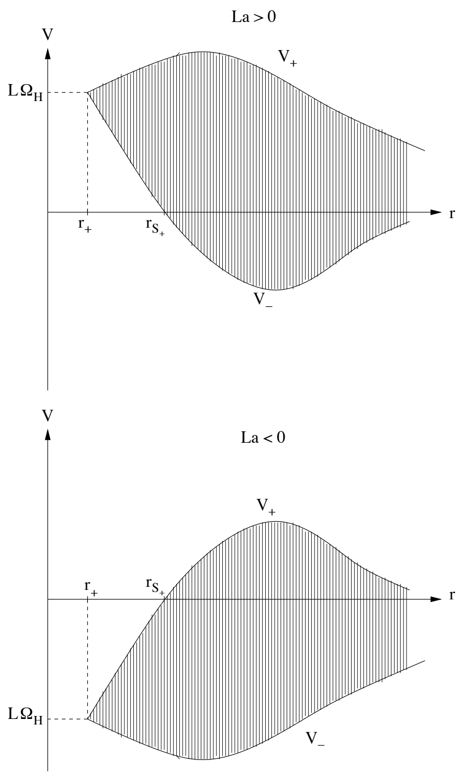

Figure 4.1: The potentials $V_+(r)$ and $V_-(r)$, for corotating ($La > 0$) and counterrotating ($La < 0$) orbits. The shadowed region is not accessible to the motion of photons or other massless particles.

for assigned values of a, M and L, we can make a qualitative study of the orbits. To this purpose it is useful to compute the radial acceleration. By differentiating eq. (4.34)

$$(\dot{r})^2 = \frac{C}{r^2}(E - V_+)(E - V_-) \tag{4.42}$$

with respect to the geodesic parameter λ, we find

$$\begin{aligned} 2\dot{r}\ddot{r} &= \left[\left(\frac{C}{r^2}\right)'(E - V_+)(E - V_-) - \frac{C}{r^2}V_+'(E - V_-)\right. \\ &\quad \left. - \frac{C}{r^2}V_-'(E - V_+)\right]\dot{r} \end{aligned} \tag{4.43}$$

i.e.

$$\ddot{r} = \frac{1}{2}\left(\frac{C}{r^2}\right)'(E - V_+)(E - V_-) - \frac{C}{2r^2}\left[V_+'(E - V_-) + V_-'(E - V_+)\right] . \tag{4.44}$$

Let us consider the radial acceleration in a point where the radial velocity $\dot{r}$ is zero, i.e. when $E = V_+$ or $E = V_-$:

$$\begin{aligned} \ddot{r} &= -\frac{C}{2r^2}V_+'(V_+ - V_-) \qquad \text{if } \; E = V_+ \\ \ddot{r} &= -\frac{C}{2r^2}V_-'(V_- - V_+) \qquad \text{if } \; E = V_- \, . \end{aligned} \tag{4.45}$$

Since

$$V_+ - V_- = \frac{2|L|r^2\sqrt{\Delta}}{(r^2 + a^2)^2 - a^2\Delta} = \frac{2|L|\sqrt{\Delta}}{C} \, , \tag{4.46}$$

we find

$$\ddot{r} = \mp\frac{|L|\sqrt{\Delta}}{r^2}V_\pm' \qquad \text{if } \; E = V_\pm \, . \tag{4.47}$$

This result shows that if, for example, $E = V_+(r_{max})$ where r_{max} is a stationary point for V_+ (i.e. $V_+'(r_{max}) = 0$), the radial acceleration vanishes; since when $E = V_+(r_{max})$ the radial velocity also vanishes, a massless particle with that value of E can be captured on a circular orbit, but the orbit is unstable, as it is the orbit at $r = 3M$ for the Schwarzschild metric.

It is possible to show that r_{max} is the solution of the equation

$$r(r - 3M)^2 - 4Ma^2 = 0 \, . \tag{4.48}$$

Note that this equation does not depend on L, thus the value of r_{max} is independent of L. The solution of (4.48) is a decreasing function of a, and, in particular,

$$\begin{aligned} r_{max} &= 3M \qquad \text{for } a = 0 \\ r_{max} &= M \qquad \text{for } a = M \\ r_{max} &= 4M \qquad \text{for } a = -M \,. \end{aligned} \tag{4.49}$$

Therefore, while for a Schwarzschild black hole the unstable circular orbit for a photon is located at $r = 3M$, for a Kerr black hole it can be located much closer to the black hole, in particular for large values of a; in the case of an extremal ($a = M$) black hole, $r_{max} = M$, which is the position of the horizon for such a black hole (see also Figure 4.2).

A photon coming from infinity with constant of motion $E > V_+(r_{max})$, falls inside the horizon, whereas if $0 < E < V_+(r_{max})$, it approaches the black hole reaching a turning point where $E = V_+(r)$ and $\dot{r} = 0$, then reverts its motion (because $\ddot{r} > 0$ there, see eq. (4.47)), and escapes free at infinity. In all these cases $E > 0$.

It remains to consider the case when $E < V_-$, and in particular to see whether negative values of E which are in principle admitted by eq. (4.37) have a physical meaning.

4.1.2 Is $E < 0$ possible?

From Figure 4.1 it appears that the constant of motion E can be, in certain cases, negative. This could appear strange, since E is, as we said, the energy of the particle at infinity; on the other hand, some geodesics never reach infinity, thus the interpretation of E as the energy at infinity cannot have a general meaning. To understand this issue, we need to consider in more detail the definition of particle energy in general relativity.

In special relativity, the energy $\mathcal{E}$ of a particle with four-momentum P^μ is $\mathcal{E} = P^0$ (remember we are using geometric units). More generally, we define the energy measured by an observer u^μ, as

$$\mathcal{E}^{(u)} = -u^\mu P_\mu \,. \tag{4.50}$$

For instance, the energy measured by the static observer $u^\mu_{st} = (1, 0, 0, 0)$ is $\mathcal{E}^{(u_{st})} = -P_0$. From this is clear that a particle cannot have a negative energy, because it would be a particle moving backwards in time. Furthermore, from a quantum-mechanical point of view, the existence of negative energy particle states would lead to an unstable vacuum, since it would be energetically favourable the creation of more and more of such particles.

The energy measured by a different observer is different, but its sign is the same: indeed, if $u^\mu = (\gamma, \gamma \mathbf{V})$ with $\gamma = (1 - V^2)^{-1/2}$, then

$$\mathcal{E}^{(u)} = -\gamma(P_0 + P_i V^i) \tag{4.51}$$

which has the same sign as $-P_0$ since $|P_i V^i| < |P_0|$. We can conclude that, in special relativity, particles must have positive energy with respect to all observers; and that if the energy is positive as measured by any observer (for instance the static observer), it is positive as measured by all observers.

In general relativity, we still define the energy of a particle measured by an observer u^μ as in eq. (4.50). Indeed, eq. (4.50) is a tensor equation which holds in a locally inertial frame, and consequently in any other frame by the principle of general covariance. For the same reason, it must be

$$\mathcal{E}^{(u)} > 0\,, \tag{4.52}$$

and it is sufficient to prove this for any observer.

Let us now consider a particle in the Kerr spacetime, located at radial infinity (where a static observer with $u^\mu_{st} = (1, 0, 0, 0)$ can always be defined). According to the definition $\mathcal{E}^{(u)} = -u^\mu P_\mu$, the energy measured by the static observer will be $\mathcal{E}^{(u_{st})} = -P_0$ and, according to eq. (3.21)[1]

$$\mathcal{E}^{(u_{st})} = -P_0 = E \tag{4.53}$$

We conclude that *for a particle coming from radial infinity*, the constant of motion E can be identified with the energy as measured by a static observer at infinity, and consequently orbits with negative values of E are not allowed.

Referring to figure 4.1, let us now consider a massless particle, say a photon, that moves in the ergoregion, i.e. between r_+ and r_{S_+}. A static observer cannot exist in this region, therefore we need to consider a different observer, for instance the ZAMO,

$$u^\mu_{ZAMO} = C(1, 0, 0, \Omega) \tag{4.54}$$

[1]Note that eq. (4.53) holds for a massless particle; in the case of a massive particle, where E is the energy at infinity per mass unit, we have to multiply E with the particle mass.

where (see eq. (3.29))

$$\Omega = \frac{2Mar}{(r^2+a^2)^2 - a^2\Delta} \tag{4.55}$$

and C is found by imposing $g_{\mu\nu}u^\mu_{ZAMO}u^\nu_{ZAMO} = -1$. We have $C > 0$, otherwise the ZAMO would move backwards in time.

The energy of the photon with respect to the ZAMO is

$$\mathcal{E}^{ZAMO} = -P_\mu u^\mu_{ZAMO} = C(E - \Omega L)\,. \tag{4.56}$$

Thus, the requirement $\mathcal{E}^{ZAMO} > 0$ is equivalent to

$$E > \Omega L\,. \tag{4.57}$$

Notice that, by comparing (4.55) with (4.35) we find that

$$V_- < \Omega L < V_+\,. \tag{4.58}$$

Therefore, the geodesics with $E > V_+$ satisfy (4.57), have positive energies with respect to the ZAMO, and are then allowed. The geodesics with $E < V_-$, instead, do not satisfy (4.57), have negative energies with respect to the ZAMO, and are then forbidden.

In the case $La < 0$ (counterrotating particles), V_+ is negative in the ergoregion; thus, the requirement

$$E > V_+ \tag{4.59}$$

(necessary and sufficient to have a particle with positive energy $\mathcal{E}$) allows negative values of the constant of motion E. This is not a contradiction, because it is only at infinity that E represents the physical energy of the particle; the geodesics we are considering never reach infinity.

We can conclude that the constant E (which is the energy at infinity) must always be larger than V_+

$$E > V_+(r) \tag{4.60}$$

and this guarantees that the energy of the particle, as measured by a local observer, is positive; in the counterrotating case $La < 0$, E can be negative still preserving (4.60), because inside the ergosphere $V_+ < 0$.

4.1.3 The Penrose process

A short comment about notation: in this section we will use a slightly different notation for the constants of motion E, L, which have been defined in Section 3.2 to be the energy at infinity and angular momentum *per unit mass* in the case of massive particles, and to be the energy at infinity and angular momentum in the case of massless particles, namely,

$$E = -k^\mu u_\mu \,, \qquad L = m^\mu u_\mu \,. \tag{4.61}$$

Here (only in this Section) we define E, L to be the energy at infinity and the angular momentum, both for massive and massless particles, i.e.

$$E = -k^\mu P_\mu \,, \qquad L = m^\mu P_\mu \,. \tag{4.62}$$

This means that we have multiplied E and L, in the case of massive particles, with the mass m.

After this premise, let us consider an interesting consequence of the possibility of having particles moving with negative E: it is possible to extract rotational energy from a Kerr black hole, through a mechanism called *Penrose process.*

Let us assume that the black hole has $a > 0$. The same reasoning can be repeated in the case of $a < 0$, getting the same conclusions.

We shoot a massive particle with energy E and angular momentum L from infinity radially towards the black hole in the equatorial plane. Its four-momentum, in covariant components, is

$$P_\mu = (-E, P_r, 0, L) \,. \tag{4.63}$$

As the particle approaches the black hole, P_μ changes, but

$$P_\mu k^\mu = -E \quad \text{and} \quad P_\mu m^\mu = L \tag{4.64}$$

remain constant.

When the particle is in the ergoregion, it decays in two particles, with momenta $P_{1\,\mu}$, $P_{2\,\mu}$ and constants of motion given by

$$\begin{aligned} P_{1\,\mu}k^\mu &= -E_1 \quad , \quad P_{1\,\mu}m^\mu = L_1 \\ P_{2\,\mu}k^\mu &= -E_2 \quad , \quad P_{2\,\mu}m^\mu = L_2 \,. \end{aligned} \tag{4.65}$$

The four-momentum is conserved in the decay process:

$$P_\mu = P_{1\,\mu} + P_{2\,\mu} \,; \tag{4.66}$$

contracting with $-k^\mu$ and with m^μ we find

$$\begin{aligned} E &= E_1 + E_2 \\ L &= L_1 + L_2 \,. \end{aligned} \tag{4.67}$$

We also assume that the particle 1 is massless (i.e. a photon), and that $\dot{r}_1 < 0$ and $\dot{r}_2 > 0$: the particle 1 falls in the black hole, while the particle 2 comes back to infinity.

The photon 1 never goes outside the ergosphere, thus we can require $E_1 < 0$, if $L_1 < 0$. Then, it must be

$$V_-(r_+)(= V_+(r_+)) < E_1 < 0 \tag{4.68}$$

and the particle 2 reaches infinity with

$$\begin{aligned} E_2 &= E - E_1 > E \\ L_2 &= L - L_1 > L \,. \end{aligned} \tag{4.69}$$

In this way, at the end we have at infinity a particle more energetic than the initial one. We say we have *extracted* rotational energy from the black hole because the photon 1, with $E_1 < 0, L_1 < 0$, reduces the mass-energy M of the black hole and its angular momentum $J = Ma$ to the values M_{fin}, J_{fin} respectively:

$$M_{fin} = M + E_1 < M \tag{4.70}$$

$$J_{fin} = J + L_1 < J \,. \tag{4.71}$$

The expressions (4.70), (4.71) could appear strange, after we have stressed so much that the constant E is not really the energy of the particle, but only its energy at infinity. To prove (4.70), we note that the total mass-energy of the system is

$$P^0_{tot} = \int_V d^3x(-g)(T^{00} + t^{00}) \tag{4.72}$$

where V is the volume of a $t = const.$ three-surface enclosing the black hole and the space up to very large values of r. If we neglect the gravitational field generated by the particle, t^{00} is only due to the black hole, and it contributes to the total mass-energy with M (as we have shown in Section 2.3):

$$P^0_{tot} = \int_V d^3x(-g)T^{00}_{particle} + M \,. \tag{4.73}$$

If we compute the integral at a time when the particle is far away from the black hole, then the spacetime is flat where $T^{00}_{particle} \neq 0$; the stress-energy tensor for a point particle with energy E, in Minkowskian coordinates, has

$$T^{00}_{particle} = E\delta^3(\mathbf{x} - \mathbf{x}(t))\,. \tag{4.74}$$

Integrating in V we get

$$P^0_{tot} = E + M \tag{4.75}$$

computed when the initial particle is still far away from the black hole.

On the other hand, this quantity is conserved, because $[(-g)(T^{0\mu} + t^{0\mu})]_{,\mu} = 0$ implies $P^0_{tot\,,0} = 0$, since we neglect the outgoing gravitational flux generated by the particle. Therefore, repeating the computation when the particle 2 has reached infinity, when the mass of the black hole is M_{fin},

$$P^0_{tot} = E_2 + M_{fin}\,. \tag{4.76}$$

This proves the relation (4.70). A similar proof holds for the angular momentum.

4.1.4 The innermost stable circular orbit for timelike geodesics

The study of timelike geodesics is much more complicate, because equation (4.26), which becomes when $\kappa = -1$

$$\dot{r}^2 = \frac{C}{r^2}(E - V_+)(E - V_-) - \frac{\Delta}{r^2}\,, \tag{4.77}$$

does not allow a simple qualitative study as in the case of null geodesics. Therefore, here we only report on some results one finds by a detailed study of these equations.

A very relevant quantity (with astrophysical interest) is the location of the innermost stable circular orbit (ISCO), which, in the Schwarzschild case, is at $r = 6M$. In Kerr spacetime, the expression for r_{ISCO} is quite complicate, but its qualitative behaviour is simple: there are two solutions

$$r^{\pm}_{ISCO}(a)\,, \tag{4.78}$$

one corresponding to corotating orbits, one to counterrotating orbits. For $a = 0$, obviously the two solutions coincide to $6M$; by increasing $|a|$, the ISCO moves closer to the black hole for corotating orbits, and far away from the black hole for counterrotating orbits. When $a = \pm M$, the ISCO, in the corotating case, coincides with the horizon, at $r = M$. This behaviour is very similar to that we have already seen in the case of unstable circular orbits for null geodesics.

In Figure 4.2 we show (in the case $a \geq 0$) the locations of the last stable and unstable circular orbits for timelike geodesics, and of the unstable circular orbit for null geodesics. This figure is taken from the article where these orbits have been studied (*J. Bardeen, W. H. Press, S. A. Teukolsky, Astrophys. J. 178, 347, 1972*).

4.1.5 The 3^{rd} Kepler law

Let us consider a circular timelike geodesic in the equatorial plane. We remind that the Lagrangian (4.5) is

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \tag{4.79}$$

and the r Euler-Lagrange equation is

$$\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r} \,. \tag{4.80}$$

Being $g_{r\mu} = 0$ if $\mu \neq r$, we have

$$\frac{d}{d\lambda}(g_{rr}\dot{r}) = \frac{1}{2} g_{\mu\nu,r} \dot{x}^\mu \dot{x}^\nu \,. \tag{4.81}$$

In the case of a circular geodesic, $\dot{r} = \ddot{r} = 0$, and this equation reduces to

$$g_{tt,r}\dot{t}^2 + 2g_{t\phi,r}\dot{t}\dot{\phi} + g_{\phi\phi,r}\dot{\phi}^2 = 0 \,. \tag{4.82}$$

The angular velocity is $\omega = \dot{\phi}/\dot{t}$, thus

$$g_{\phi\phi,r}\omega^2 + 2g_{t\phi,r}\omega + g_{tt,r} = 0 \,. \tag{4.83}$$

We remind that

$$\begin{aligned} g_{tt} &= -\left(1 - \frac{2M}{r}\right) \\ g_{t\phi} &= -\frac{2Ma}{r} \\ g_{\phi\phi} &= r^2 + a^2 + \frac{2Ma^2}{r} \,, \end{aligned} \tag{4.84}$$

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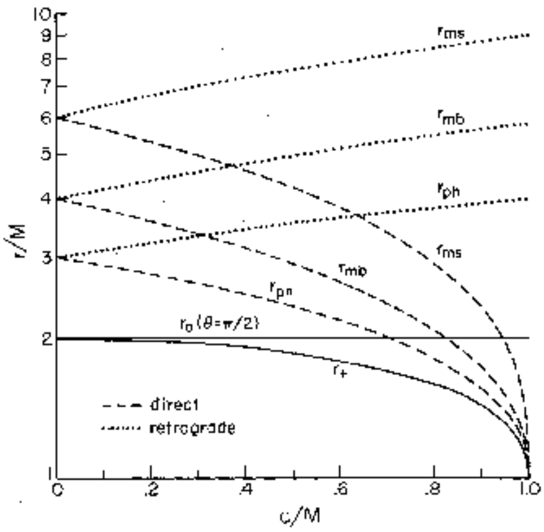

FIG. 1.—Radii of circular, equatorial orbits around a rotating black hole of mass M, as functions of the hole's specific angular momentum a. Dashed and dotted curves (for direct and retrograde orbits) plot the Boyer-Lindquist coordinate radius of the innermost stable (ms), innermost bound (mb), and photon (ph) orbits. Solid curves indicate the event horizon (r_+) and the equatorial boundary of the ergosphere (r_0).

Figure 4.2: Geodesics of Massive particles in Kerr metric, from Bardeen *et al.*, Astrophys. J. 178, 347, 1972.

then

$$2\left(r - \frac{Ma^2}{r^2}\right)\omega^2 + \frac{4Ma}{r^2}\omega - \frac{2M}{r^2} = 0\,. \tag{4.85}$$

The equation

$$(r^3 - Ma^2)\omega^2 + 2Ma\omega - M = 0 \tag{4.86}$$

has discriminant

$$M^2a^2 + M(r^3 - Ma^2) = Mr^3 \tag{4.87}$$

and solutions

$$\begin{aligned}
\omega_\pm &= \frac{-Ma \pm \sqrt{Mr^3}}{r^3 - Ma^2} = \pm\sqrt{M}\,\frac{r^{3/2} \mp a\sqrt{M}}{r^3 - Ma^2} \\
&= \pm\sqrt{M}\frac{r^{3/2} \mp a\sqrt{M}}{(r^{3/2} + a\sqrt{M})(r^{3/2} - a\sqrt{M})} \\
&= \pm\frac{\sqrt{M}}{r^{3/2} \pm a\sqrt{M}}\,.
\end{aligned} \tag{4.88}$$

This is the relation between angular velocity and radius of circular orbits, and reduces, in Schwarzschild limit $a = 0$, to

$$\omega_\pm = \pm\sqrt{\frac{M}{r^3}}\,, \tag{4.89}$$

which is Kepler's 3^{rd} law.

4.2 General geodesic motion: the Carter constant

To study generic geodesics in Kerr spacetime, we need to apply the *Hamilton-Jacobi approach*, which allows to identify a further constant of motion (not related to any isometry of the metric).

To apply the Hamilton-Jacobi approach, we first have to express our system with an Hamiltonian description. Given the Lagrangian of the system

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{4.90}$$

we have defined the conjugate momenta[2]

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\mu\nu}\dot{x}^\nu \,. \tag{4.91}$$

By inverting (4.91), we can express $\dot{x}^\mu$ in terms of the conjugate momenta:

$$\dot{x}^\mu = g^{\mu\nu} p_\nu \,. \tag{4.92}$$

The *Hamiltonian* is a functional of the coordinate functions $x^\mu(\lambda)$ and of their conjugate momenta $p_\mu(\lambda)$, defined as

$$H(x^\mu, p_\nu) = p_\mu \, \dot{x}^\mu(p_\nu) - \mathcal{L}\left(x^\mu, \dot{x}^\mu(p_\nu)\right) \,. \tag{4.93}$$

In our case, we have

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \,. \tag{4.94}$$

The geodesics equation is equivalent to the Euler-Lagrange equations for the Lagrangian functional, which are equivalent to the Hamilton equations for the Hamiltonian functional:

$$\begin{aligned} \dot{x}^\mu &= \frac{\partial H}{\partial p_\mu} \\ \dot{p}_\mu &= -\frac{\partial H}{\partial x^\mu} \,. \end{aligned} \tag{4.95}$$

To solve equations (4.95) is as difficult as to solve the Euler-Lagrange equations. Anyway, it is possible to apply the Hamilton-Jacobi approach, which here we briefly recall.

In the Hamilton-Jacobi approach, we look for a function of the coordinates and of the curve parameter λ,

$$S = S(x^\mu, \lambda) \tag{4.96}$$

which is solution of the *Hamilton-Jacobi equation*

$$H\left(x^\mu, \frac{\partial S}{\partial x^\mu}\right) + \frac{\partial S}{\partial \lambda} = 0 \,. \tag{4.97}$$

In general such a solution will depend on four integration constants.

It can be shown that, if S is a solution of the Hamilton-Jacobi equation, then

$$\frac{\partial S}{\partial x^\mu} = p_\mu \,. \tag{4.98}$$

[2] Not to be confused with the four-momentum of the particle, which we denote with P^μ.

Therefore, once we have solved (4.97), we have the expressions of the conjugate momenta (and then of $\dot{x}^\mu$) in terms of four constants, which allow to write the solutions of the geodesic equation in a closed form, in terms of integrals.

In general, it is more difficult to solve (4.97), which is a partial differential equation, than to solve the geodesic equation, but in this case we can find such a solution.

First of all, we can use what we already know, i.e.

$$\begin{aligned} H &= \frac{1}{2}g^{\mu\nu}p_\mu p_\nu = \frac{1}{2}\kappa \\ p_t &= -E \quad \text{constant} \\ p_\phi &= L \qquad \text{constant}\,. \end{aligned} \tag{4.99}$$

These conditions require that

$$S = -\frac{1}{2}\kappa\lambda - Et + L\phi + S^{(r\theta)}(r,\theta) \tag{4.100}$$

where $S^{(r\theta)}$ is a function of r and θ to be determined.

Furthermore, we look for a separable solution, by making the ansatz

$$S = -\frac{1}{2}\kappa\lambda - Et + L\phi + S^{(r)}(r) + S^{(\theta)}(\theta)\,. \tag{4.101}$$

Substituting (4.101) into the Hamilton-Jacobi equation (4.97), and using the expression (3.16) for the inverse metric, we find

$$\begin{aligned} &-\kappa + \frac{\Delta}{\Sigma}\left(\frac{dS^{(r)}}{dr}\right)^2 + \frac{1}{\Sigma}\left(\frac{dS^{(\theta)}}{d\theta}\right)^2 \\ &-\frac{1}{\Delta}\left[r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta\right]E^2 + \frac{4Mra}{\Sigma\Delta}EL + \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta}L^2 = 0\,. \end{aligned} \tag{4.102}$$

Using the relation (3.12)

$$(r^2+a^2) + \frac{2Mra^2}{\Sigma}\sin^2\theta = \frac{1}{\Sigma}\left[(r^2+a^2)^2 - a^2\sin^2\theta\Delta\right] \tag{4.103}$$

and multiplying by $\Sigma = r^2 + a^2\cos^2\theta$, we get

$$-\kappa(r^2 + a^2\cos^2\theta) + \Delta\left(\frac{dS^{(r)}}{dr}\right)^2 + \left(\frac{dS^{(\theta)}}{d\theta}\right)^2$$

$$-\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]E^2 + \frac{4Mra}{\Delta}EL + \left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right)L^2 = 0 \tag{4.104}$$

i.e.

$$\begin{aligned} &\Delta\left(\frac{dS^{(r)}}{dr}\right)^2 - \kappa r^2 - \frac{(r^2+a^2)^2}{\Delta}E^2 + \frac{4Mra}{\Delta}EL - \frac{a^2}{\Delta}L^2 \\ = \; &-\left(\frac{dS^{(\theta)}}{d\theta}\right)^2 + \kappa a^2\cos^2\theta - a^2\sin^2\theta E^2 - \frac{1}{\sin^2\theta}L^2 \,. \end{aligned} \tag{4.105}$$

We rearrange equation (4.105) by adding to both sides the constant quantity $a^2E^2 + L^2$:

$$\begin{aligned} &\Delta\left(\frac{dS^{(r)}}{dr}\right)^2 - \kappa r^2 - \frac{(r^2+a^2)^2}{\Delta}E^2 + \frac{4Mra}{\Delta}EL - \frac{a^2}{\Delta}L^2 + a^2E^2 + L^2 \\ = \; &-\left(\frac{dS^{(\theta)}}{d\theta}\right)^2 + \kappa a^2\cos^2\theta + a^2\cos^2\theta E^2 - \frac{\cos^2\theta}{\sin^2\theta}L^2 \,. \end{aligned} \tag{4.106}$$

In equation (4.106), the left-hand side does not depend on θ, and it is equal to the right-hand side which does not depend on r; therefore, this quantity must be a constant $\mathcal{C}$:

$$\begin{aligned} &\left(\frac{dS^{(\theta)}}{d\theta}\right)^2 - \cos^2\theta\left[(\kappa+E^2)a^2 - \frac{1}{\sin^2\theta}L^2\right] = \mathcal{C} \\ &\Delta\left(\frac{dS^{(r)}}{dr}\right)^2 - \kappa r^2 - \frac{(r^2+a^2)^2}{\Delta}E^2 + \frac{4Mra}{\Delta}EL - \frac{a^2}{\Delta}L^2 + E^2a^2 + L^2 \\ = \; &\Delta\left(\frac{dS^{(r)}}{dr}\right)^2 - \kappa r^2 + (L-aE)^2 - \frac{1}{\Delta}\left[E(r^2+a^2) - La\right]^2 = -\mathcal{C} \,. \end{aligned} \tag{4.107}$$

Notice that in rearranging the terms between the last two lines, we have used, for the LE terms, the relation

$$-2aLE + 2aLE\frac{r^2+a^2}{\Delta} = \frac{4aMr}{\Delta}LE \,. \tag{4.108}$$

If we define the functions $R(r)$ and $\Theta(\theta)$ as

$$
\begin{aligned}
\Theta(\theta) &\equiv \mathcal{C} + \cos^2\theta \left[(\kappa + E^2)a^2 - \frac{1}{\sin^2\theta} L^2\right] \\
R(r) &\equiv \Delta\left[-\mathcal{C} + \kappa r^2 - (L - aE)^2\right] + \left[E(r^2 + a^2) - La\right]^2 \,,
\end{aligned}
\tag{4.109}
$$

then

$$
\begin{aligned}
\left(\frac{dS^{(\theta)}}{d\theta}\right)^2 &= \Theta \\
\left(\frac{dS^{(r)}}{dr}\right)^2 &= \frac{R}{\Delta^2}
\end{aligned}
\tag{4.110}
$$

and the solution of the Hamilton-Jacobi equation has the form

$$
S = -\frac{1}{2}\kappa\lambda - Et + L\phi + \int \frac{\sqrt{R}}{\Delta} dr + \int \sqrt{\Theta} d\theta \,. \tag{4.111}
$$

Once we have the solution of the Hamilton-Jacobi equations, depending on four constants $(\kappa, E, L, \mathcal{C})$, it is possible to solve the geodesic motion. Indeed, from (4.98) we know the expressions of the conjugate momenta

$$
\begin{aligned}
p_\theta^2 &= (\Sigma\dot{\theta})^2 = \Theta(\theta) \\
p_r^2 &= \left(\frac{\Sigma}{\Delta}\dot{r}\right)^2 = \frac{R(r)}{\Delta^2}
\end{aligned}
\tag{4.112}
$$

therefore

$$
\begin{aligned}
\dot{\theta} &= \pm\frac{1}{\Sigma}\sqrt{\Theta} \\
\dot{r} &= \pm\frac{1}{\Sigma}\sqrt{R}
\end{aligned}
\tag{4.113}
$$

which can be solved by simple integration. The constant $\mathcal{C}$ is called Carter constant, and characterize, together with E and L, the geodesic motion. Notice that differently from E and L, the Carter constant is not related to any isometry of the spacetime.

To integrate equations (4.113), it is convenient to define a new variable λ' to parametrize the geodesic, called “Mino time”:

$$
d\lambda = \Sigma d\lambda' = (r^2 + a^2\cos^2\theta)d\lambda' \,. \tag{4.114}
$$

Then we have

$$\begin{aligned}\frac{d\theta}{d\lambda'} &= \pm\sqrt{\Theta(\theta)} \\ \frac{dr}{d\lambda'} &= \pm\sqrt{R(r)}\,, \end{aligned} \tag{4.115}$$

thus the equations for r and θ, once expressed in the Mino time, are decoupled. An important consequence of this property is that for bound orbits, which satisfy $R(r) > 0$ for $r_1 < r < r_2$ and $\Theta(\theta) > 0$ for $\theta_1 < \theta < \theta_2 (= \pi - \theta_1)$, the functions $r(\lambda')$ and $\theta(\lambda')$ Are:

$$\begin{aligned} r(\lambda') &= r(\lambda' + n\Lambda_r) \\ \theta(\lambda') &= \theta(\lambda' + k\Lambda_\theta) \end{aligned} \tag{4.116}$$

with n, k integers and periods

$$\begin{aligned} \Lambda_r &= 2\int_{r_1}^{r_2} \frac{dr}{\sqrt{R}} \\ \Lambda_\theta &= 2\int_{\theta_1}^{\theta_2} \frac{d\theta}{\sqrt{\Theta}}\,. \end{aligned} \tag{4.117}$$

The orbits, still, are very complicate, since the periods in r and θ are different, and t, ϕ are not periodic at all: a generic bound orbit does not resemble to an ellipse.