Relativity constraints on the $O(p^2)$ two-nucleon contact Lagrangian

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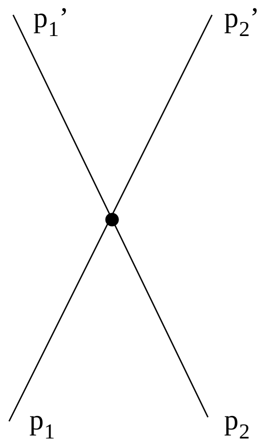
2 operators at leading order, $\mathcal{L} = -\frac{1}{2} C_S N^\dagger N N^\dagger N - \frac{1}{2} C_T N^\dagger \vec{\sigma} N N^\dagger \vec{\sigma} N$

14 operators at $O(p^2)$ [Ordóñez *et al.*: PRC53 (1996) 2086]

$$\begin{aligned}\mathcal{L} \quad = \quad & -C_1'(N^\dagger \vec{\nabla} N)^2 - 1/2 C_2'(N^\dagger \vec{\nabla} N) \cdot (\vec{\nabla} N^\dagger N) - C_3' N^\dagger N N^\dagger \vec{\nabla}^2 N \\ & -i C_4' N^\dagger \vec{\nabla} N \cdot (\vec{\nabla} N^\dagger \times \vec{\sigma} N) - i/2 C_5' N^\dagger N (\vec{\nabla} N^\dagger \cdot \vec{\sigma} \times \vec{\nabla} N) - i/2 C_6' (N^\dagger \vec{\sigma} N) \cdot (\vec{\nabla} N^\dagger \times \vec{\nabla} N) \\ & -(C_7' \delta_{ik}\delta_{jl} + C_8' \delta_{il}\delta_{kj} + C_9' \delta_{ij}\delta_{kl}) N^\dagger \sigma_k \partial_i N N^\dagger \sigma_l \partial_j N \\ & -1/2 (C_{10}' \delta_{ik}\delta_{jl} + C_{11}' \delta_{il}\delta_{kj} + C_{12}' \delta_{ij}\delta_{kl}) N^\dagger \sigma_k \partial_i N \partial_j N^\dagger \sigma_l \partial_j N \\ & -[1/2 C_{13}' (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}) + C_{14}' \delta_{ij}\delta_{kl}] \partial_i N^\dagger \sigma_k \partial_j N N^\dagger \sigma_l N + \text{h.c.}\end{aligned}$$

actually only 12 are independent [Pastore *et al.*: PRC80 (2009) 034004]



$$\begin{aligned} v^{\text{CT2}}(\mathbf{k}, \mathbf{K}) \quad = \quad & C_1 k^2 + C_2 K^2 + (C_3 k^2 + C_4 K^2)\, \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + C_6 \boldsymbol{\sigma}_1 \cdot \mathbf{k}\, \boldsymbol{\sigma}_2 \cdot \mathbf{k} \\ & + i C_5 \frac{\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2}{2} \cdot \mathbf{K} \times \mathbf{k} + C_7 \boldsymbol{\sigma}_1 \cdot \mathbf{K}\, \boldsymbol{\sigma}_2 \cdot \mathbf{K} \\ & + i C_1^* \frac{\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2}{2} \cdot \mathbf{P} \times \mathbf{k} + C_2^* (\boldsymbol{\sigma}_1 \cdot \mathbf{P}\, \boldsymbol{\sigma}_2 \cdot \mathbf{K} - \boldsymbol{\sigma}_1 \cdot \mathbf{K}\, \boldsymbol{\sigma}_2 \cdot \mathbf{P}) \\ & + (C_3^* + C_4^* \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) P^2 + C_5^* \boldsymbol{\sigma}_1 \cdot \mathbf{P}\, \boldsymbol{\sigma}_2 \cdot \mathbf{P}\end{aligned}$$

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad \mathbf{k} = \tfrac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_1' + \mathbf{p}_2'), \quad \mathbf{K} = \tfrac{1}{4}(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_1' - \mathbf{p}_2')$$

Questions

- Are these 5 extra operators really needed?
- Do they come from genuine dynamics?
- Perhaps they are generated kinematically, by boosting the center of mass interaction?
- Are the corresponding coupling constants free parameters, to be fitted from data, e.g. in the 3 nucleon system?

Obviously these questions call for relativity
But the original Lagrangian was constructed with no consideration of Lorentz (nor of Galileian) relativity
$\Longrightarrow$ study the constraints that Lorentz symmetry imposes on the Lagrangian

Heavy baryon formalism

Based on the heavy baryon formalism [Epelbaum *et al.*: PRC 65 (2002) 044001] $\Longrightarrow$ only 7 operators in the Lagrangian

Velocity superselection rule: the (soft) interactions do not change the heavy baryon 4-velocity [Georgi: PLB 240 (1990) 447], [Jenkins and Manohar: PLB 255 (1991) 558]

$$N = e^{-imv\cdot x}\left[\frac{1+\not{v}}{2}N_v + \frac{1-\not{v}}{2}h_v\right]$$

In the 1-nucleon sector this is well understood: integrate out h_v from the theory $\Longrightarrow$ induced interactions for N_v

However:

–The same is not true for interactions with four nucleon legs

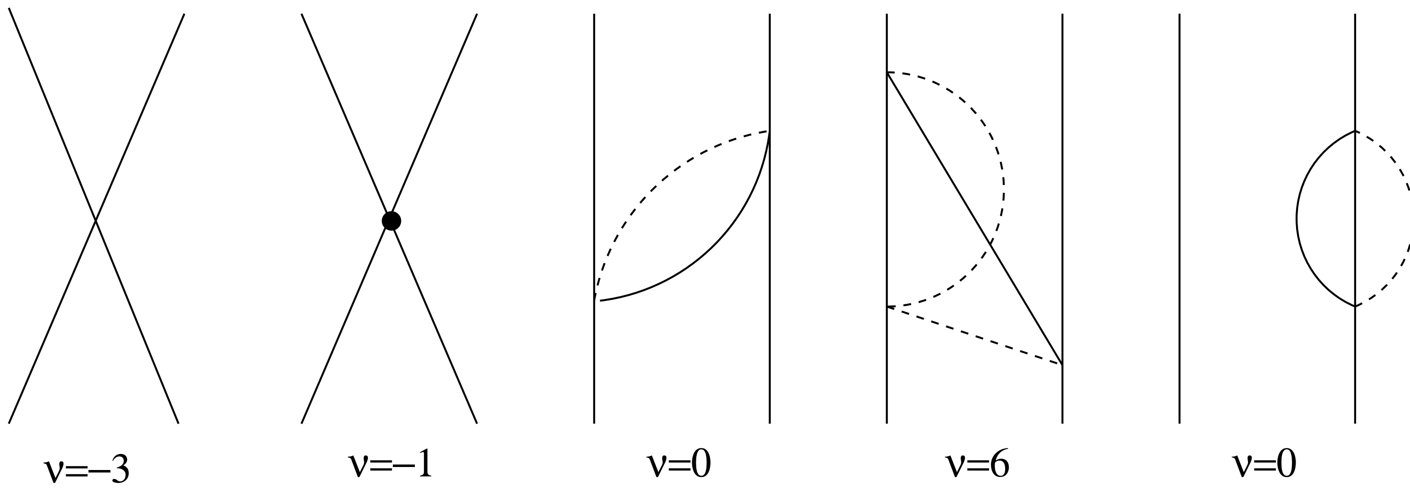
–Nothing forbids that the two nucleons have *different* 4-velocities

Relativistic theory

The general vertex is $\sim (\bar{\psi}^{(+)} + \bar{\psi}^{(-)})(\psi^{(+)} + \psi^{(-)})(\bar{\psi}^{(+)} + \bar{\psi}^{(-)})(\psi^{(+)} + \psi^{(-)})$

Claim: $\psi^{(-)}$ is irrelevant for the $O(p^2)$ Lagrangian

Consider 2PI time ordered diagrams: they are $O(p^{\nu})$, with $\nu = 3(L - c)$



At low energies, diagrams with antiparticle exchanges shrink to contact vertices. But none of them contribute to the order we are after, $\nu = -1$

Construction of the relativistic Lagrangian

Building blocks are

$$\partial_{\mu_1}...\partial_{\mu_i}[\bar{\psi}\overleftrightarrow{\partial}_{\nu_1}...\overleftrightarrow{\partial}_{\nu_j}\Gamma_1(\tau^a)\psi]\partial_{\lambda_1}...\partial_{\lambda_k}[\bar{\psi}\overleftrightarrow{\partial}_{\rho_1}...\overleftrightarrow{\partial}_{\rho_l}\Gamma_2(\tau^a)\psi]$$

$\Gamma_{1,2} = \mathbf{1}, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}$
Lorentz indices to be contracted with $g_{\mu\nu}$ or $\epsilon_{\mu\nu\alpha\beta}$

Notice that $\partial_\mu \sim O(p)$, while $\overleftrightarrow{\partial}_\mu \sim O(1)$
However $\overleftrightarrow{\partial}^\mu\overleftrightarrow{\partial}_\mu = \partial^2 + 4m^2$, therefore:
no two Lorentz indices can be contracted with one another inside a fermion bilinear, except with the ϵ tensor and for the (suppressed) ∂^2 acting on the whole bilinear
–indeed we can always make use of the field equations of motion to eliminate terms with $\not{\partial}\psi$ in favor of terms with less derivatives–

It is easier to think in terms of momenta instead of derivatives.
Consider the process $1 + 2 \to 3 + 4$
due to overall momentum conservation, it depends on 3 p_i, e.g.

$$p_2 - p_4 \sim p_3 - p_4 \sim O(p), \quad p_2 + p_4 \sim O(1)$$

The non-suppressed combination corresponds to $\overleftrightarrow{\partial}$ acting on the second bilinear

Although in principle one could include an arbitrary number of $\overleftrightarrow{\partial}_\mu$, at some point their Lorentz indices will have to be contracted with one-another (and then $(p_2 + p_4)^2 = 4m^2 - (p_2 - p_4)^2$) or with the ϵ tensor (and then they vanish due to antisymmetry)

We are therefore guaranteed that at each order only a finite number of spacetime derivatives appear

Better to work with $p_3 + p_1$ instead of $p_3 - p_4$ (more direct transformation properties under charge and hermitian conjugation), using

$$p_1 + p_3 = 2(p_3 - p_4) + (p_2 + p_4) - 2(p_2 - p_4)$$

Thus we can formally consider that $\overleftrightarrow{\partial}_\mu \sim O(p)$ when acting on the first bilinear.

The Lagrangian should be hermitian and invariant under C and P

	$\mathbf{1}$	γ_5	γ_μ	$\gamma_\mu\gamma_5$	$\sigma_{\mu\nu}$	$g_{\mu\nu}$	$\epsilon_{\mu\nu\lambda\rho}$	$\overleftrightarrow{\partial}_\mu$
parity	+	−	+	−	+	+	−	+
charge conjugation	+	+	−	+	−	+	+	−
hermitian conjugation	+	−	+	+	+	+	+	−

While the hermiticity condition does not impose any constraint, since one can always multiply the single bilinears by appropriate factors of i, the C and P symmetry must be enforced.

We then compiled a (redundant) list of 116 operators

$$\bar{\psi}\psi\bar{\psi}\psi$$
$$\frac{1}{16m^4}\bar{\psi}\gamma^{\mu}\overleftrightarrow{\partial}^{\nu}\overleftrightarrow{\partial}^{\alpha}\psi\bar{\psi}\gamma_{\nu}\overleftrightarrow{\partial}_{\mu}\overleftrightarrow{\partial}_{\alpha}\psi$$
$$\frac{i}{16m^5}\epsilon_{\mu\nu\alpha\beta}\bar{\psi}\gamma^{\mu}\gamma_5\overleftrightarrow{\partial}^{\gamma}\overleftrightarrow{\partial}^{\delta}\psi\bar{\psi}\overleftrightarrow{\partial}^{\nu}\overleftrightarrow{\partial}_{\gamma}\overleftrightarrow{\partial}_{\delta}\sigma^{\alpha\beta}\psi$$
$$\frac{1}{4m^2}\bar{\psi}\sigma^{\mu\nu}\overleftrightarrow{\partial}^{\alpha}\psi\bar{\psi}\sigma_{\mu\nu}\overleftrightarrow{\partial}_{\alpha}\psi$$
$$\frac{1}{16m^4}\bar{\psi}\sigma^{\mu\nu}\overleftrightarrow{\partial}^{\alpha}\overleftrightarrow{\partial}^{\beta}\psi\bar{\psi}\sigma_{\mu\nu}\overleftrightarrow{\partial}_{\alpha}\overleftrightarrow{\partial}_{\beta}\psi$$
$$\bar{\psi}\psi\partial^2\bar{\psi}\psi$$
$$i\epsilon_{\mu\nu\alpha\beta}\bar{\psi}\gamma^{\mu}\gamma_5\overleftrightarrow{\partial}^{\nu}\psi\partial^{\alpha}\bar{\psi}\gamma^{\beta}\psi$$
$$\bar{\psi}\gamma^{\mu}\gamma_5\psi\partial^2\bar{\psi}\gamma_{\mu}\gamma_5\psi$$
$$\bar{\psi}\gamma^{\mu}\gamma_5\overleftrightarrow{\partial}^{\nu}\psi\bar{\psi}\gamma_{\nu}\gamma_5\overleftrightarrow{\partial}_{\mu}\psi$$
$$\frac{i}{m}\epsilon_{\mu\nu\alpha\beta}\bar{\psi}\gamma^{\mu}\gamma_5\overleftrightarrow{\partial}^{\nu}\overleftrightarrow{\partial}_{\gamma}\psi\bar{\psi}\overleftrightarrow{\partial}^{\alpha}\sigma^{\beta\gamma}\psi$$
$$\bar{\psi}\sigma^{\mu\alpha}\overleftrightarrow{\partial}^{\beta}\psi\bar{\psi}\overleftrightarrow{\partial}_{\alpha}\sigma_{\mu\beta}\psi$$
$$\frac{1}{4m^2}\epsilon_{\mu\nu\gamma\delta}\epsilon_{\alpha\beta\rho\sigma}\bar{\psi}\sigma^{\mu\nu}\psi\partial^{\alpha}\partial^{\gamma}\bar{\psi}\overleftrightarrow{\partial}^{\beta}\overleftrightarrow{\partial}^{\delta}\sigma^{\rho\sigma}\psi$$

To eliminate redundant terms we made use of

- field equations of motion
- partial integrations
- Fierz identities

In addition, we performed the nonrelativistic $O(p^2)$ reduction from

$$\psi^{(+)}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{\omega_{\mathbf{k}}} b^\alpha(\mathbf{k}) u^\alpha(\mathbf{k}) \mathrm{e}^{-ik\cdot x}$$

to

$$N(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \phi^\alpha \tilde{b}^\alpha(\mathbf{k}) \mathrm{e}^{-ik\cdot x}$$

as

$$\psi^{(+)}(x) = \begin{pmatrix} 1 + \frac{\nabla^2}{8m^2} \\ -i\frac{\boldsymbol{\sigma}\cdot\nabla}{2m} \end{pmatrix} N(x)$$

As a result, all our relativistic operators are written in the old basis

$$\bar{\psi}\psi\bar{\psi}\psi = O_S - \frac{1}{4m^2}\left(O_1 + 2O_2 + 2O_3 + 2O_5\right)$$

$$\frac{1}{16m^4}\bar{\psi}\gamma^{\mu}\overleftrightarrow{\partial}^{\nu}\overleftrightarrow{\partial}^{\alpha}\psi\bar{\psi}\gamma_{\nu}\overleftrightarrow{\partial}_{\mu}\overleftrightarrow{\partial}_{\alpha}\psi = O_S - \frac{1}{2m^2}\left(O_1 - 4O_2 - 2O_3 + O_4 + O_6\right)$$

$$\frac{1}{4m^2}\bar{\psi}\gamma^{\mu}\gamma_5\overleftrightarrow{\partial}^{\nu}\psi\bar{\psi}\gamma_{\mu}\gamma_5\overleftrightarrow{\partial}_{\nu}\psi = O_T + \frac{1}{4m^2}\left(2O_6 - O_7 - 2O_9 + 2O_{10} - 2O_{13}\right)$$

$$4m^2\bar{\psi}\gamma_5\psi\bar{\psi}\gamma_5\psi = O_7 + 2O_{10}$$

$$\vdots$$

By selecting the linearly independent combinations, we find in total 10 $O(p^2)$ operators.

This means that 2 out of the 12 original operators in the Lagrangian are not compatible with relativity

It also means that 3 additional operators, as compared to the 7 contributing in the center of mass system, need to be taken into account in a general reference frame.

No surprise: in the instant form the interactions affect the Hamiltonian *and* the boost operator

Hamiltonian dynamics

What does it mean relativistic covariance?
Not simply that the interactions commute with the boost generators!

$$[J^i, J^j] = i\epsilon^{ijk} J^k, \quad [K^i, K^j] = -i\epsilon^{ijk} J^k, \quad [J^i, K^j] = i\epsilon^{ijk} K^k, \quad [P^\mu, P^\nu] = 0,$$
$$[K^i, P^j] = -i\delta^{ij} H, \quad [J^i, P^j] = i\epsilon^{ijk} P^k, \quad [K^i, H] = -iP^i, \quad [J^i, H] = 0.$$

$$H = H_0 + H_I, \quad \vec{K} = \vec{K}_0 + \vec{K}_I \quad \Longrightarrow \left[\vec{K}_0, H_I\right] + \left[\vec{K}_I, H_0\right] + \left[\vec{K}_I, H_I\right] = 0$$

in our case

$$\begin{aligned} K_j &= \int d^3\mathbf{x} \left[t\theta^{0j} - x_j \theta^{00}\right] \\ &= -\frac{im}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \{ b_\alpha^\dagger(\mathbf{k}) \overleftrightarrow{\nabla}_\mathbf{k}\, b_\alpha(\mathbf{k}) + d_\alpha^\dagger(\mathbf{k}) \overleftrightarrow{\nabla}_\mathbf{k}\, d_\alpha(\mathbf{k}) \\ &\quad + \frac{m}{\omega_\mathbf{k}} [b_\alpha^\dagger(\mathbf{k}) b_\beta(\mathbf{k}) u^{\dagger\alpha}(\mathbf{k}) \overleftrightarrow{\nabla}_\mathbf{k}\, u^\beta(\mathbf{k}) - d_\beta^\dagger(\mathbf{k}) d_\alpha(\mathbf{k}) v^{\dagger\alpha}(\mathbf{k}) \overleftrightarrow{\nabla}_\mathbf{k}\, v^\beta(\mathbf{k})] \} \end{aligned}$$

then

$$\left[\vec{K}_0, b_\alpha(\mathbf{k})\right] = i\omega_\mathbf{k} \vec{\nabla}_\mathbf{k} b_\alpha(\mathbf{k}) + \frac{1}{2(m+\omega_\mathbf{k})} \vec{k} \times \vec{\sigma}_{\alpha\beta} b_\beta(\mathbf{k})$$

Take

$$O_1 + 2O_2 = \vec{\nabla}(N^\dagger N) \cdot \vec{\nabla}(N^\dagger N)$$

$$H_I = \int d^3\mathbf{x} \int \prod_i \frac{d^3\mathbf{k_i}}{(2\pi)^3} \tilde{b}^\dagger_\alpha(\mathbf{k}_1)\tilde{b}_\alpha(\mathbf{k}_2)\tilde{b}^\dagger_\beta(\mathbf{k}_3)\tilde{b}_\beta(\mathbf{k}_4)(\mathbf{k}_2 - \mathbf{k}_1)\cdot(\mathbf{k}_4 - \mathbf{k}_3)\mathrm{e}^{i(k_1 - k_2 + k_3 - k_4)\cdot x}$$

Then

$$\left[\vec{K}_0, H_I\right] = \left[H_0, \vec{K}_I\right] + \left[H_I, \vec{K}_I\right]$$

with

$$\vec{K}_I = \vec{W} + \delta\vec{W}, \quad \vec{W} = \int d^3\mathbf{x}\vec{x}\,(O_1 + 2O_2)$$

should be verified, using

$$\left[\vec{K}_0, \tilde{b}_\alpha(\mathbf{k})\right] \sim im\vec{\nabla}_\mathbf{k}\tilde{b}_\alpha(\mathbf{k}) + O(p)$$

Poincaré covariance has to be verified order by order in the low momentum expansion

Conclusions

- The two-derivatives, two-nucleon contact Lagrangian, in the pure nucleon sector (no sources), with all constraints imposed by Poincaré covariance, contains 10 operators
- The ensuing **P**-dependent NN potential,

$$v_{\mathbf{P}}(\mathbf{k},\mathbf{K},\mathbf{P}) = iC_1^* \frac{\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2}{2} \cdot \mathbf{P} \times \mathbf{k} + C_2^* (\boldsymbol{\sigma}_1 \cdot \mathbf{P}\, \boldsymbol{\sigma}_2 \cdot \mathbf{K} - \boldsymbol{\sigma}_1 \cdot \mathbf{K}\, \boldsymbol{\sigma}_2 \cdot \mathbf{P}) + (C_3^* + C_4^* \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) P^2 + C_5^* \boldsymbol{\sigma}_1 \cdot \mathbf{P}\, \boldsymbol{\sigma}_2 \cdot \mathbf{P}$$

  contains 3 free low-energy constants, to be fitted from data, while the 2 others are given in terms of C_S and C_T.

- This is established at the Lagrangian level. In practical applications, whether this is called a 3-nucleon force effect or a relativistic effect depends on the framework.