Non-relativistic bound states

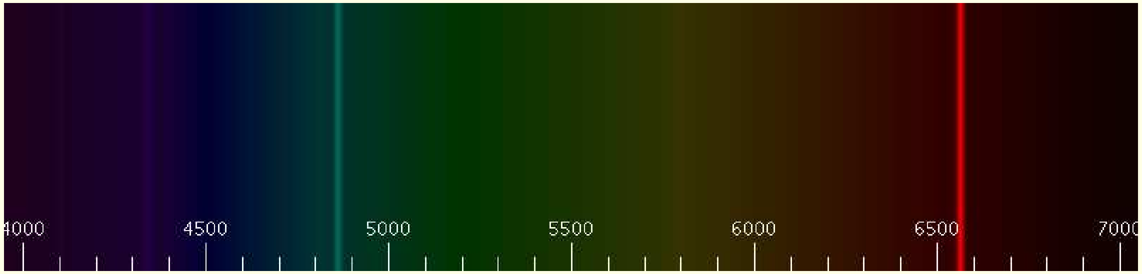
- the long way back from the Bethe–Salpeter to the Schrödinger equation -

Antonio Vairo

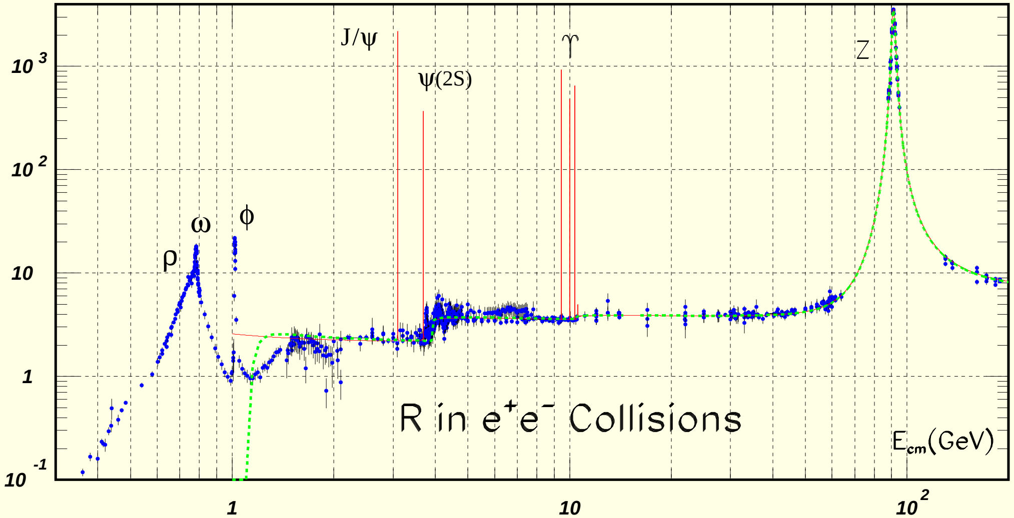
Technische Universität München

Matter is made of bound states

- Electromagnetic bound states: atoms, molecules, ...



- Strong-interaction bound states: hadrons, nuclei, ...
  (At low T and ρ, confinement only allows for bound states!)



... many of them non-relativistic

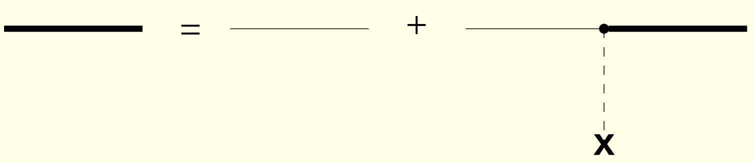
- atoms, molecules, ...
- baryonium, pionium, ...
- quarkonium (charmonium, bottomonium, top-antitop pairs, ...)

Non-relativistic quantum theory of bound states

With the development of the non-relativistic quantum theory into a relativistic quantum field theory, the description of the bound state followed the same pattern:

- 1926 Schrödinger equation: $\left(\frac{\mathbf{p}^2}{2m} + V\right)\phi = E\phi$

$$\begin{cases} g = g_0 + g_0(-iV)g \\ g_0 = \frac{i}{E - \mathbf{p}^2/(2m)} \end{cases}$$



- 1927 Pauli equation: $\left(\frac{(\mathbf{p} - e\mathbf{A})^2}{2m} + V - \frac{\boldsymbol{\sigma}\cdot e\mathbf{B}}{2m}\right)\phi = E\phi$

The relevant scales of the non-relativistic bound state dynamics are

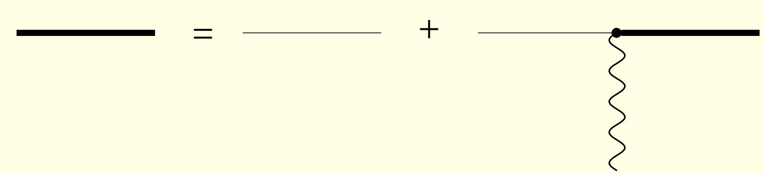
- $E \sim \frac{\mathbf{p}^2}{2m} \sim V \sim mv^2$,
- $p \sim 1/r \sim mv$;

a crucial observation: if v(elocity)$\ll 1$, then $m \gg mv \gg mv^2$.

Relativistic quantum theory of bound states

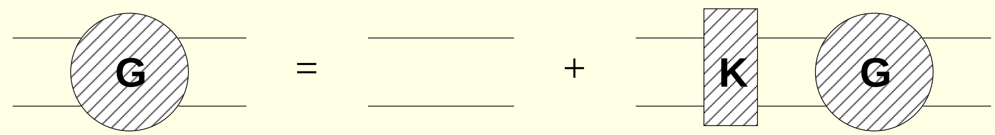
- 1928 Dirac equation: $(i\not{D} - m)\,\psi = 0$

$$\begin{cases} g^D = g_0^D + g_0^D(-ie\not{A})g^D \\ g_0^D = \frac{i}{\not{p} - m} \end{cases}$$

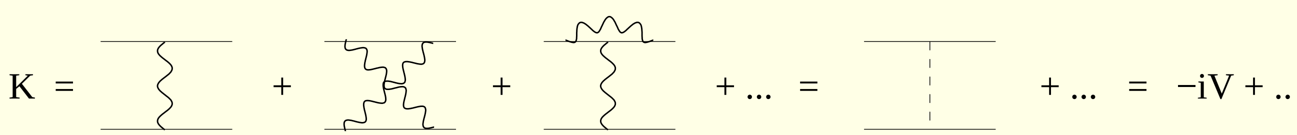


- 1951 Bethe–Salpeter equation:

$$\begin{cases} G = G_0 + G_0 K G \\ G_0 = g_0^D(p_1) \otimes g_0^D(p_2) \end{cases}$$



which reduces to the Schrödinger equation in the non-relativistic limit, $E^{(\mathrm{ext})} \sim mv^2$*,*$p^{(\mathrm{ext})} \sim mv$*:*



$$g_0^D(\textit{fermion/anti-fermion}) = \frac{i}{\pm p^0 + E/2 - \mathbf{p}^2/2m + i\epsilon}\,\frac{1 \pm \gamma^0}{2} + \dots$$

Non-relativistic expansions of the Bethe–Salpeter equation

The non-relativistic expansion may be implemented systematically at the level of the Bethe–Salpeter equation:

$K = K_V + \delta K$ where $K_V \approx -iV$ and $G_V = G_0 + G_0 K_V G_V$ can be solved
$G = G_V + G_V \delta K G$

- Lepage PRA 16(77)863, Barbieri Remiddi NPB 141(78)413

... and its problems

- cumbersome in perturbation theory;
- very poorly suited to achieve factorization (specially important in QCD).

<u>Ex.</u>

- It shows the difficulty of the approach the fact that going from the calculation of the $m\alpha^5$ correction in the hyperfine splitting of the positronium ground state to the $m\alpha^6 \ln \alpha$ term took twenty-five years!
  - `Karplus Klein PR 87(52)848, Caswell Lepage PRA (20)(79)36 Bodwin Yennie PR 43(78)267`

- With few exceptions no applications to QCD and quarkonium physics.
  - `Mödritsch Kummer ZPC 66(95)225`

... and its problems

- cumbersome in perturbation theory;
- very poorly suited to achieve factorization (specially important in QCD).

Why?

- All energy scales of the full dynamics contribute:
  each diagram has a complicated power counting and contributes to all orders in the coupling and velocity.

- Another way of saying is that the non-relativistic bound state dynamics, described by the Schrödinger equation at the soft scale $p \sim 1/r \sim mv$, gets entangled with the relativistic dynamics at the scale m (e.g. radiative corrections) and the low-energy dynamics at the ultrasoft scale mv^2 (e.g. the Lamb shift).

Effective Field Theories

Whenever a system H, described by a Lagrangian $\mathcal{L}$, is characterized by 2 scales $\Lambda \gg \lambda$, observables may be calculated by expanding one scale with respect to the other. An effective field theory makes the expansion in λ/Λ explicit at the Lagrangian level.

The EFT Lagrangian, $\mathcal{L}_{\mathrm{EFT}}$, suitable to describe H at scales lower than Λ is defined by
(1) a cut off $\Lambda \gg \mu \gg \lambda$;
(2) by some degrees of freedom that exist at scales lower than μ

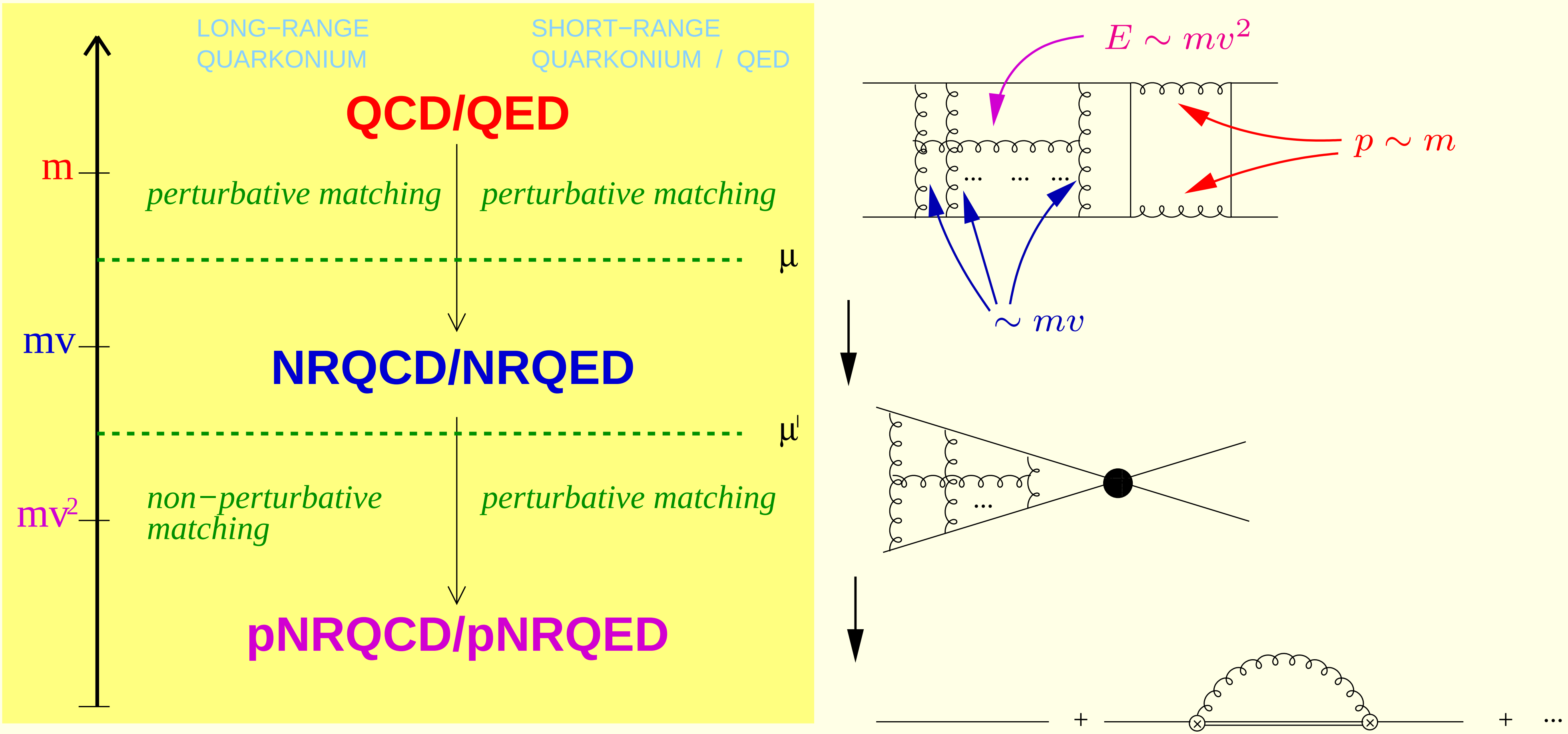
$\Rightarrow$ $\mathcal{L}_{\mathrm{EFT}}$ is made of all operators O_n that may be built from the effective degrees of freedom and are consistent with the symmetries of $\mathcal{L}$.

Effective Field Theories

$$\mathcal{L}_{\mathrm{EFT}} = \sum_n c_n(\Lambda/\mu) \frac{O_n(\mu, \lambda)}{\Lambda^n}$$

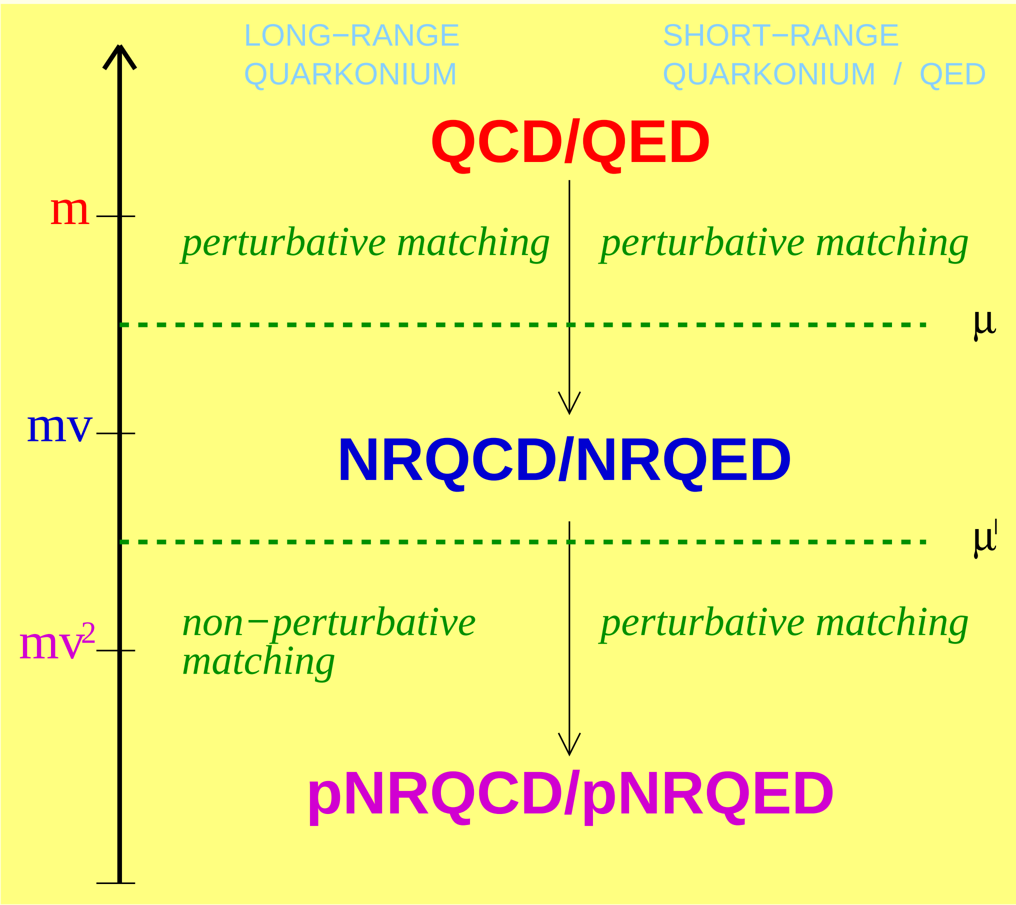
- Since at $\mu \sim \lambda$, $\langle O_n \rangle \sim \lambda^n$, the EFT is organized as an expansion in λ/Λ.
- The EFT is renormalizable order by order in λ/Λ.
- The matching coefficients $c_n(\Lambda/\mu)$ encode the non-analytic behaviour in Λ. They are calculated by imposing that $\mathcal{L}_{\mathrm{EFT}}$ and $\mathcal{L}$ describe the same physics at any finite order in the expansion: matching procedure.
- In QCD, if $\Lambda \gg \Lambda_{\mathrm{QCD}}$ then $c_n(\Lambda/\mu)$ may be calculated in perturbation theory.

EFTs for systems made of two heavy quarks/fermions



- They exploit the expansion in v/ factorization of low and high energy contributions.
- They are renormalizable order by order in v.
- In perturbation theory, RG techniques provide resummation of large logs.

EFTs for systems made of two heavy quarks/fermions

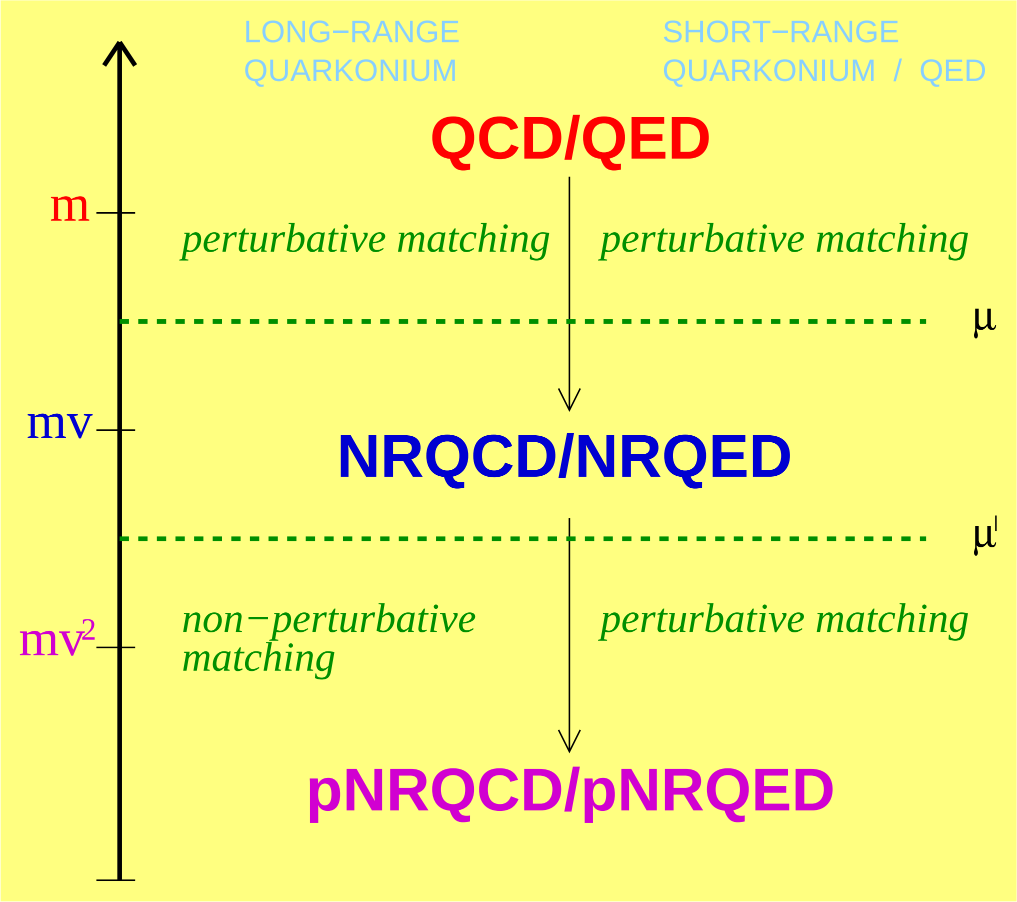


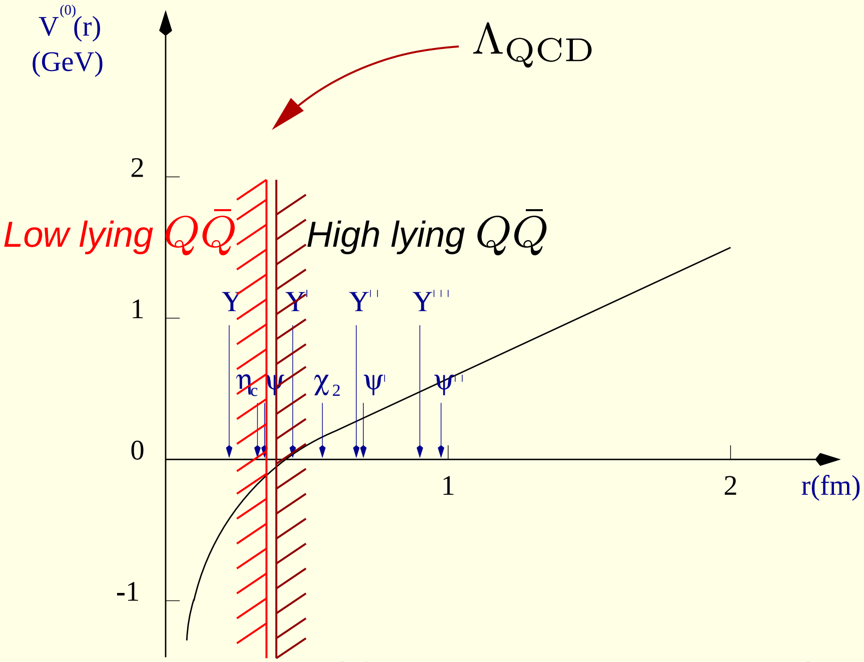
- Caswell Lepage PLB 167(86)437
- Lepage Thacker NP PS 4(88)199
- Bodwin et al PRD 51(95)1125, ...

- Pineda Soto NP PS 64(98)428
- Brambilla et al PRD 60(99)091502
- Brambilla et al NPB 566(00)275
- Kniehl et al NPB 563(99)200
- Luke Manohar PRD 55(97)4129
- Luke Savage PRD 57(98)413
- Grinstein Rothstein PRD 57(98)78
- Labelle PRD 58(98)093013
- Griesshammer NPB 579(00)313
- Luke et al PRD 61(00)074025
- Hoang Stewart PRD 67(03)114020, ...

- *for a review* Brambilla Pineda Soto Vairo RMP 77(04)1423

EFTs for systems made of two heavy quarks/fermions





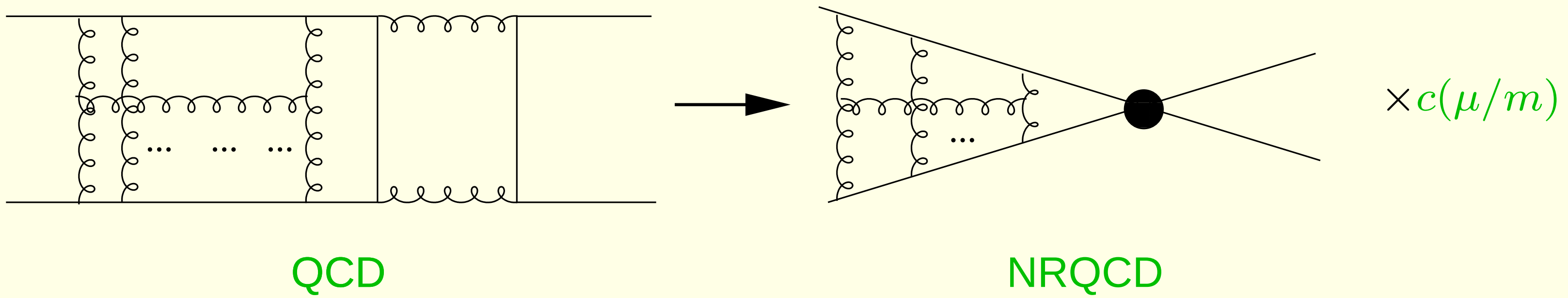
○ Godfrey Isgur PRD 32(85)189

A potential picture arises at the level of pNRQCD:

- the potential is perturbative if $mv \gg \Lambda_{\rm QCD}$
- the potential is non-perturbative if $mv \sim \Lambda_{\rm QCD}$

NRQCD

NRQCD is obtained by integrating out modes associated with the scale m



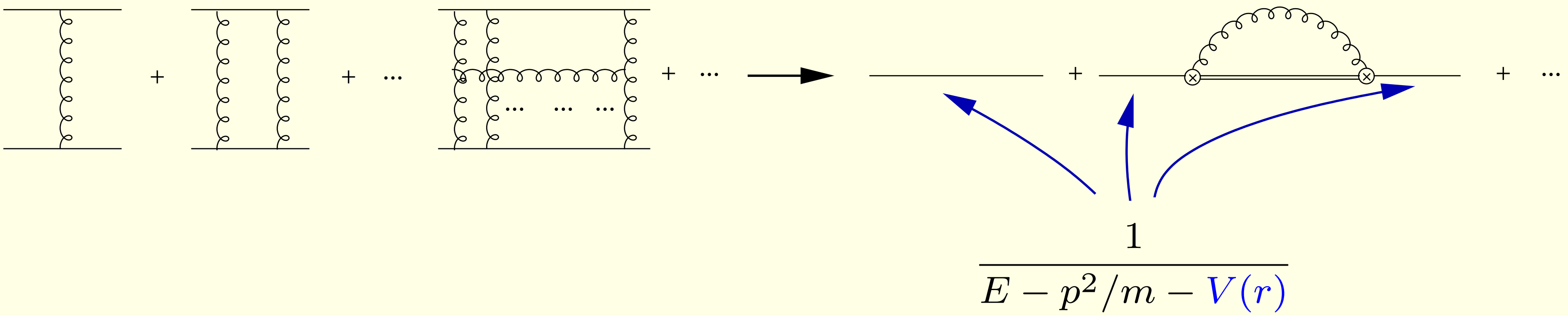
- The matching is perturbative.
- The Lagrangian is organized as an expansion in $1/m$ and $\alpha_s(m)$:

$$\mathcal{L}_{\rm NRQCD} = \sum_n c(\alpha_s(m/\mu)) \times O_n(\mu,\lambda)/m^n$$

Suitable to describe annihilation and production of quarkonium.

pNRQCD

pNRQCD is obtained by integrating out modes associated with the scale $\frac{1}{r} \sim mv$



- The Lagrangian is organized as an expansion in $1/m$, r, and $\alpha_s(m)$:

$$\mathcal{L}_{\mathrm{pNRQCD}} = \sum_k \sum_n \frac{1}{m^k} \times c_k(\alpha_s(m/\mu)) \times V(r\mu', r\mu) \times O_n(\mu', \lambda)\, r^n$$

Case 1: pNRQCD for $mv \gg \Lambda_{\rm QCD}$

- Degrees of freedom:

  Q-$\bar{Q}$ states, with energy $\sim \Lambda_{\rm QCD}, mv^2$ and momentum $\lesssim mv$
  $\Rightarrow$ (i) singlet $\rm S$ (ii) octet $\rm O$

  Gluons with energy and momentum $\sim \Lambda_{\rm QCD}, mv^2$

- Power counting:

  $p \sim \frac{1}{r} \sim mv$;
  all gauge fields are multipole expanded: $A(R,r,t) = A(R,t) + \mathbf{r}\cdot\boldsymbol{\nabla}A(R,t) + \ldots$
  and scale like $(\Lambda_{\rm QCD}$ or $mv^2)^{\rm dimension}$.

  Non-analytic behaviour in $r \rightarrow$ matching coefficients V

Case 1: pNRQCD for $mv \gg \Lambda_{\rm QCD}$

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} + \mathrm{Tr}\left\{ \mathrm{S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) \mathrm{S} + \mathrm{O}^\dagger \left(iD_0 - \frac{\mathbf{p}^2}{m} - V_o \right) \mathrm{O} \right\}$$

LO in r

$$\theta(T)\, e^{-iTH_s}$$

$$\theta(T)\, e^{-iTH_o} \left(e^{-i\int dt\, A^{\rm adj}} \right)$$

Case 1: pNRQCD for $mv \gg \Lambda_{\rm QCD}$

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} + {\rm Tr} \left\{ {\rm S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) {\rm S} + {\rm O}^\dagger \left(iD_0 - \frac{\mathbf{p}^2}{m} - V_o \right) {\rm O} \right\}$$

LO in r

The equation of motion of the singlet,

$$\left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) {\rm S} = 0,$$

is the Schrödinger equation!

Case 1: pNRQCD for $mv \gg \Lambda_{\rm QCD}$

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} + {\rm Tr}\left\{ {\rm S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) {\rm S} + {\rm O}^\dagger \left(iD_0 - \frac{\mathbf{p}^2}{m} - V_o \right) {\rm O} \right\}$$

LO in r

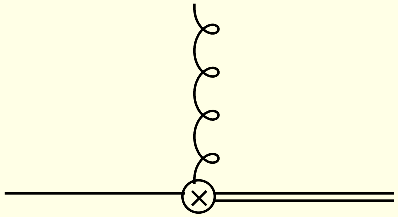
$$+ V_A {\rm Tr}\left\{ {\rm O}^\dagger \mathbf{r} \cdot g\mathbf{E}\, {\rm S} + {\rm S}^\dagger \mathbf{r} \cdot g\mathbf{E}\, {\rm O} \right\}$$

$$+ \frac{V_B}{2} {\rm Tr}\left\{ {\rm O}^\dagger \mathbf{r} \cdot g\mathbf{E}\, {\rm O} + {\rm O}^\dagger {\rm O} \mathbf{r} \cdot g\mathbf{E} \right\}$$

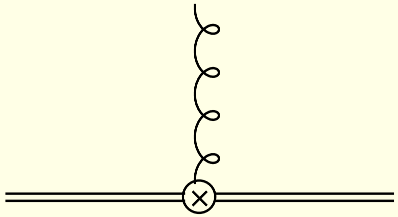
$$+ \cdots$$

NLO in r

Case 1: pNRQCD for $mv \gg \Lambda_{\rm QCD}$



$O^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{S}$



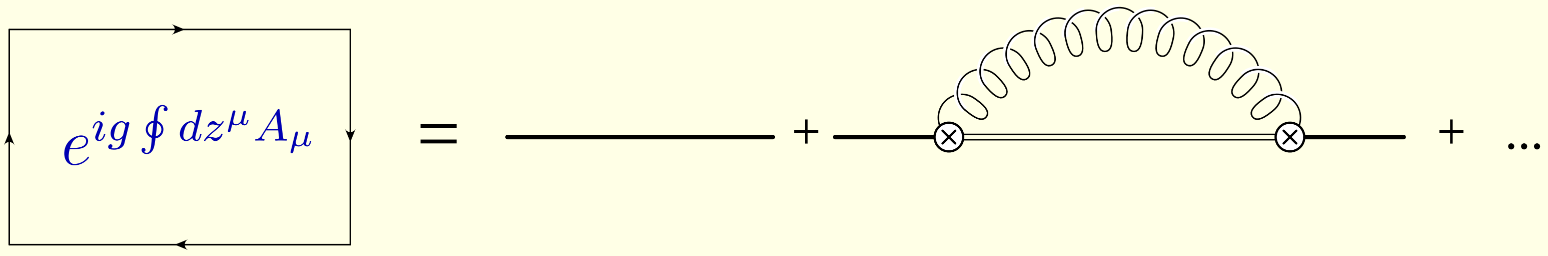
$O^\dagger \{\mathbf{r} \cdot g\mathbf{E},\, \mathrm{O}\}$

$$+V_A \mathrm{Tr}\left\{ \mathrm{O}^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{S} + \mathrm{S}^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{O} \right\}$$
$$+\frac{V_B}{2} \mathrm{Tr}\left\{ \mathrm{O}^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{O} + \mathrm{O}^\dagger \mathrm{O} \mathbf{r} \cdot g\mathbf{E} \right\}$$

NLO in r

- At leading order in the multipole expansion, the equation of motion of the EFT is the Schrödinger equation. Higher-order terms correct this picture (these higher order terms are responsible, for instance, for the Lamb shift).
- The Schrödinger potential, V_s, emerges as a Wilson coefficient of the EFT. As such, it undergoes renormalization, develops scale dependence and satisfies renormalization group equations, which allow to resum large logarithms.

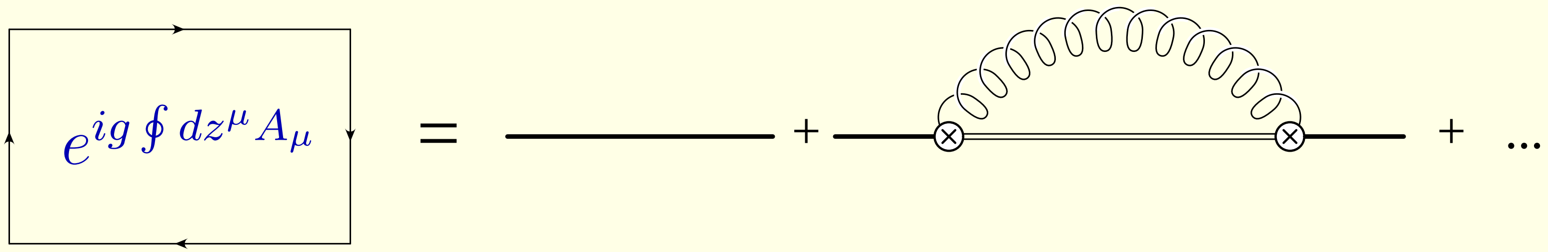
The static potential in perturbation theory



NRQCD

pNRQCD

The static potential in perturbation theory

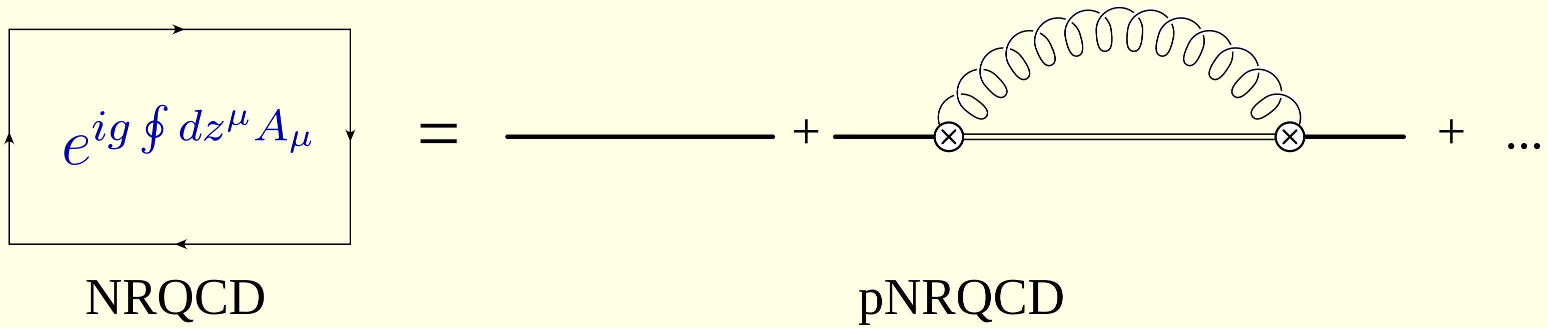


NRQCD pNRQCD

$$\lim_{T\to\infty} \frac{i}{T} \ln \langle \square \rangle = V_s(r,\mu) - i \frac{g^2}{N_c} V_A^2 \int_0^\infty dt\, e^{-it(V_o - V_s)} \langle \mathrm{Tr}(\mathbf{r}\cdot\mathbf{E}(t)\,\mathbf{r}\cdot\mathbf{E}(0))\rangle(\mu) + \dots$$

ultrasoft contribution

The static potential in perturbation theory



$$\lim_{T\to\infty} \frac{i}{T} \ln \langle \Box \rangle = V_s(r,\mu) - i\frac{g^2}{N_c} V_A^2 \int_0^\infty dt\, e^{-it(V_o - V_s)} \langle \mathrm{Tr}(r\cdot E(t)\, r\cdot E(0))\rangle(\mu) + \ldots$$

ultrasoft contribution

The μ dependence cancels between the two terms in the right-hand side:

- $V_s \sim \ln r\mu, \ln^2 r\mu$, ...
- *ultrasoft contribution* $\sim \ln(V_o - V_s)/\mu, \ln^2(V_o - V_s)/\mu, \ldots \ln r\mu, \ln^2 r\mu, \ldots$

- The static Wilson loop is known up to NNLO (N^{3}LO is under way).
  - `Schröder PLB 447(99)321, Smirnov et al PLB 668(08)293`
- The octet potential is known up to NNLO.
  - `Kniehl et al PLB 607(05)96`
- $V_A = 1 + \mathcal{O}(\alpha_\mathrm{s}^2)$.
  - `Brambilla et al PLB 647(07)185`
- The chromoelectric correlator $\langle \mathrm{Tr}(r \cdot E(t)\, r \cdot E(0)) \rangle$ is known up to NLO.
  - `Eidemüller Jamin PLB 416(98)415`

The static potential in perturbation theory

$$V_s(r,\mu) = -C_F \frac{\alpha_s(1/r)}{r}\Bigg[1 + a_1 \frac{\alpha_s(1/r)}{4\pi} + a_2 \left(\frac{\alpha_s(1/r)}{4\pi}\right)^2$$
$$+ \left(\frac{16\,\pi^2}{3} C_A^3 \ln r\mu + a_3\right)\left(\frac{\alpha_s(1/r)}{4\pi}\right)^3$$
$$+ \left(a_4^{L2} \ln^2 r\mu + \left(a_4^L + \frac{16}{9}\pi^2 C_A^3 \beta_0(-5 + 6\ln 2)\right)\ln r\mu + a_4\right)\left(\frac{\alpha_s(1/r)}{4\pi}\right)^4\Bigg]$$

$$a_4^{L2} = -\frac{16\pi^2}{3} C_A^3\,\beta_0$$

$$a_4^L = 16\pi^2 C_A^3\left[a_1 + 2\gamma_E\beta_0 + n_f\left(-\frac{20}{27} + \frac{4}{9}\ln 2\right) + C_A\left(\frac{149}{27} - \frac{22}{9}\ln 2 + \frac{4}{9}\pi^2\right)\right]$$

- Brambilla et al PRD 60(99)091502, PLB 647(07)185

The static potential in perturbation theory

$$
\begin{aligned}
V_s(r,\mu) \;&=\; -C_F\,\frac{\alpha_{\rm s}(1/r)}{r}\left[1 + a_1\,\frac{\alpha_{\rm s}(1/r)}{4\pi} + a_2\left(\frac{\alpha_{\rm s}(1/r)}{4\pi}\right)^2\right.\\
&+ \left(\frac{16\,\pi^2}{3}C_A^3\;\ln r\mu + a_3\right)\left(\frac{\alpha_{\rm s}(1/r)}{4\pi}\right)^3\\
&\left.+ \left(a_4^{L2}\ln^2 r\mu + \left(a_4^{L} + \frac{16}{9}\pi^2\,C_A^3\beta_0(-5+6\ln 2)\right)\ln r\mu + a_4\right)\left(\frac{\alpha_{\rm s}(1/r)}{4\pi}\right)^4\right]
\end{aligned}
$$

- The logarithmic contribution at N^{3}LO may be extracted from the one-loop calculation of the ultrasoft contribution;
- the single logarithmic contribution at N^{4}LO may be extracted from the two-loop calculation of the ultrasoft contribution.

The static potential in perturbation theory

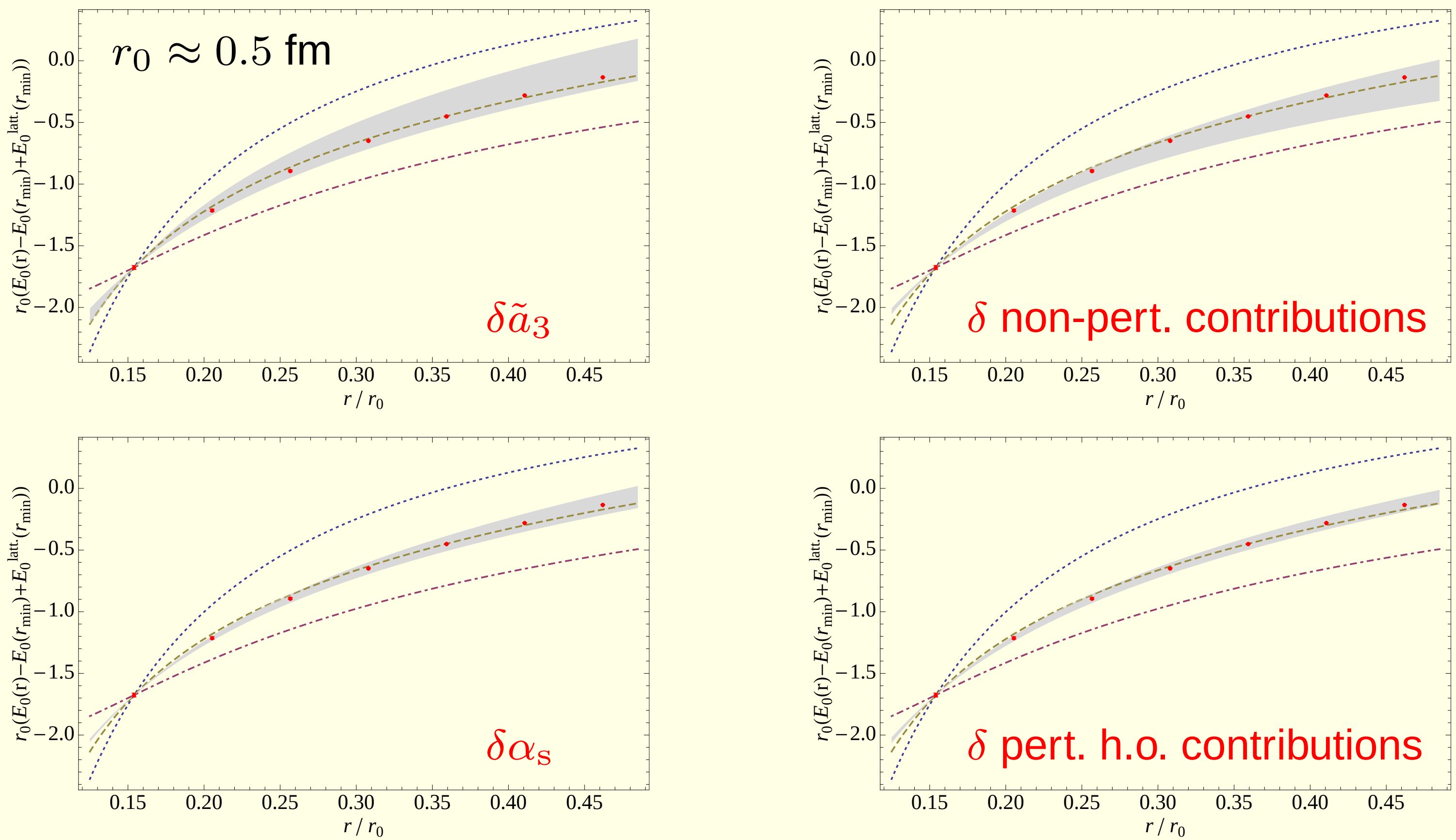
$$
\begin{aligned}
V_s(r,\mu) \;=\; & -C_F \frac{\alpha_s(1/r)}{r} \left[1 + a_1 \frac{\alpha_s(1/r)}{4\pi} + a_2 \left(\frac{\alpha_s(1/r)}{4\pi} \right)^2 \right. \\
& + \left(\frac{16\,\pi^2}{3} C_A^3 \; \ln r\mu + a_3 \right) \left(\frac{\alpha_s(1/r)}{4\pi} \right)^3 \\
& \left. + \left(a_4^{L2} \ln^2 r\mu + \left(a_4^L + \frac{16}{9}\pi^2 \, C_A^3 \beta_0 (-5 + 6\ln 2) \right) \ln r\mu + a_4 \right) \left(\frac{\alpha_s(1/r)}{4\pi} \right)^4 \right]
\end{aligned}
$$

- The leading logarithmic contribution has been resummed using RG equations at LL accuracy.
  - Pineda Soto PLB 495(00)323
- The next-to-leading logarithmic contribution has been resummed using RG equations at NLL accuracy.
  - Brambilla Garcia Soto Vairo PRD 80(09)034016

Static quark-antiquark energy in perturbation theory

$$
\begin{aligned}
E_0(r) \quad = \quad & -\frac{C_F \alpha_s(1/r)}{r} \left\{ 1 + \frac{\alpha_s(1/r)}{4\pi} \left[a_1 + 2\gamma_E \beta_0 \right] \right. \\
& + \left(\frac{\alpha_s(1/r)}{4\pi} \right)^2 \left[a_2 + \left(\frac{\pi^2}{3} + 4\gamma_E^2 \right) \beta_0^2 + \gamma_E \left(4 a_1 \beta_0 + 2\beta_1 \right) \right] \\
& + \left(\frac{\alpha_s(1/r)}{4\pi} \right)^3 \left[\frac{16\pi^2}{3} C_A^3 \ln \frac{C_A \alpha_s(1/r)}{2} + \tilde{a}_3 \right] \\
& + \left(\frac{\alpha_s(1/r)}{4\pi} \right)^4 \left[a_4^{L2} \ln^2 \frac{C_A \alpha_s(1/r)}{2} + a_4^L \ln \frac{C_A \alpha_s(1/r)}{2} + \tilde{a}_4 \right] \\
& \left. + \qquad \cdots \qquad \right\}
\end{aligned}
$$

Static quark-antiquark energy at N^3LL vs lattice



- No evidence of violation of the OPE expansion up to 0.2 fm.
- By comparison, the 3 loop coefficient is $\tilde{a}_3 = 1.11^{+0.06}_{-0.03} \times 10^5$.

- Brambilla et al PRD 80(09)034016
- Necco Sommer NPB 622(02)328

Applications to quarkonium physics

- c and b masses at NNLO, N^{3}LO*, NNLL*;
- B_c mass at NNLO;
- B_c^*, η_c, η_b masses at NLL;
- Quarkonium 1P fine splittings at NLO;
- $\Upsilon(1S)$, η_b electromagnetic decays at NNLL;
- $\Upsilon(1S)$ and J/ψ radiative decays at NLO;
- $\Upsilon(1S) \to \gamma\eta_b$, $J/\psi \to \gamma\eta_c$ at NNLO;
- $t\bar{t}$ cross section at NNLL;
- ...

  - *for reviews* `Brambilla et al` *Heavy Quarkonium Physics* `CERN Yellow Report`
    `Vairo EPJA 31(07)728, IJMPA 22(07)5481`

Applications to QED bound states

Many QED calculations have remarkably benefitted from the EFT approach and corrections of very high order in perturbation theory have been calculated in the last years for many observables after decades of very slow or no progress ...

... just to mention that

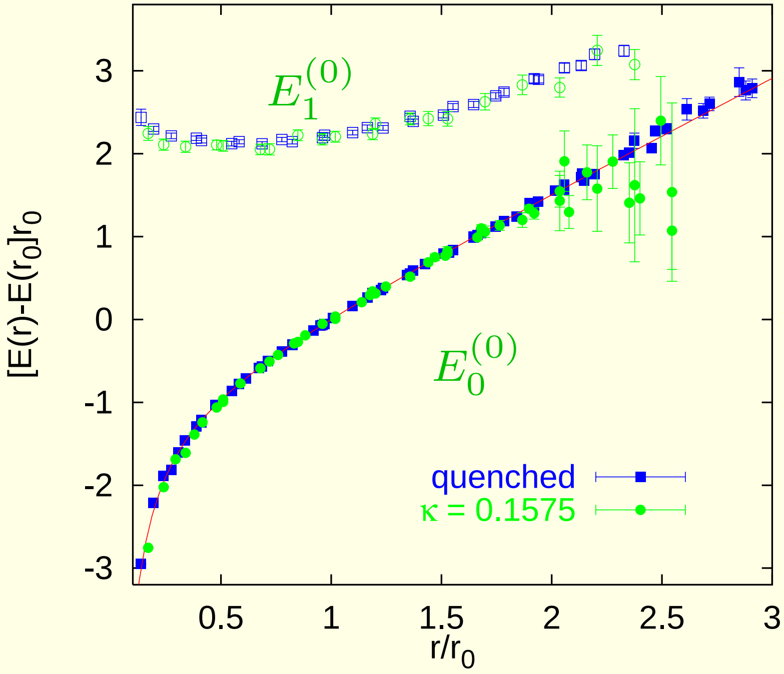
- for the hyperfine splitting of the positronium ground state the terms of order $m\alpha^6$, $m\alpha^7 \ln^2 \alpha$ and $m\alpha^7 \ln \alpha$ are now available!
  - *for reviews on positronium* `Karshenboim IJMPA 19(04)3879`
    `Penin IJMPA 19(04)3897`

Case 2: pNRQCD for $mv \sim \Lambda_{\rm QCD}$

- All scales above mv^2 are integrated out (including $\Lambda_{\rm QCD}$).

Case 2: pNRQCD for $mv \sim \Lambda_{\rm QCD}$

- All scales above mv^2 are integrated out (including $\Lambda_{\rm QCD}$).
- All gluonic excitations between heavy quarks are integrated out since they develop a gap of order $\Lambda_{\rm QCD}$ with the static $Q\bar{Q}$ energy.



○ Bali et al PRD 62(00)054503

($r_0 \simeq 0.5\,$fm)

Case 2: pNRQCD for $mv \sim \Lambda_{\rm QCD}$

- All scales above mv^2 are integrated out (including $\Lambda_{\rm QCD}$).
- All gluonic excitations between heavy quarks are integrated out since they develop a gap of order $\Lambda_{\rm QCD}$ with the static $Q\bar{Q}$ energy.

$\Rightarrow$ The singlet quarkonium field S of energy mv^2 is the only degree of freedom of pNRQCD (up to ultrasoft hadrons, e.g. pions).

Case 2: pNRQCD for $mv \sim \Lambda_{\rm QCD}$

$$\mathcal{L} = \mathrm{Tr}\left\{ \mathrm{S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) \mathrm{S} \right\}$$

- Brambilla Pineda Soto Vairo PRD 63(01)014023

Case 2: pNRQCD for $mv \sim \Lambda_{\rm QCD}$

$$\mathcal{L} = {\rm Tr}\left\{ {\rm S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) {\rm S} \right\}$$

∘ Brambilla Pineda Soto Vairo PRD 63(01)014023

- The potential V_s $({\rm Re}\, V_s + i\, {\rm Im}\, V_s)$ is non-perturbative:

  (a) to be determined from the lattice;
  - ∘ Bali PR 343(01)1

  (b) to be determined from QCD vacuum models.
  - ∘ Brambilla Vairo PRD 55(97)3974

Case 2: pNRQCD for $mv \sim \Lambda_{\rm QCD}$

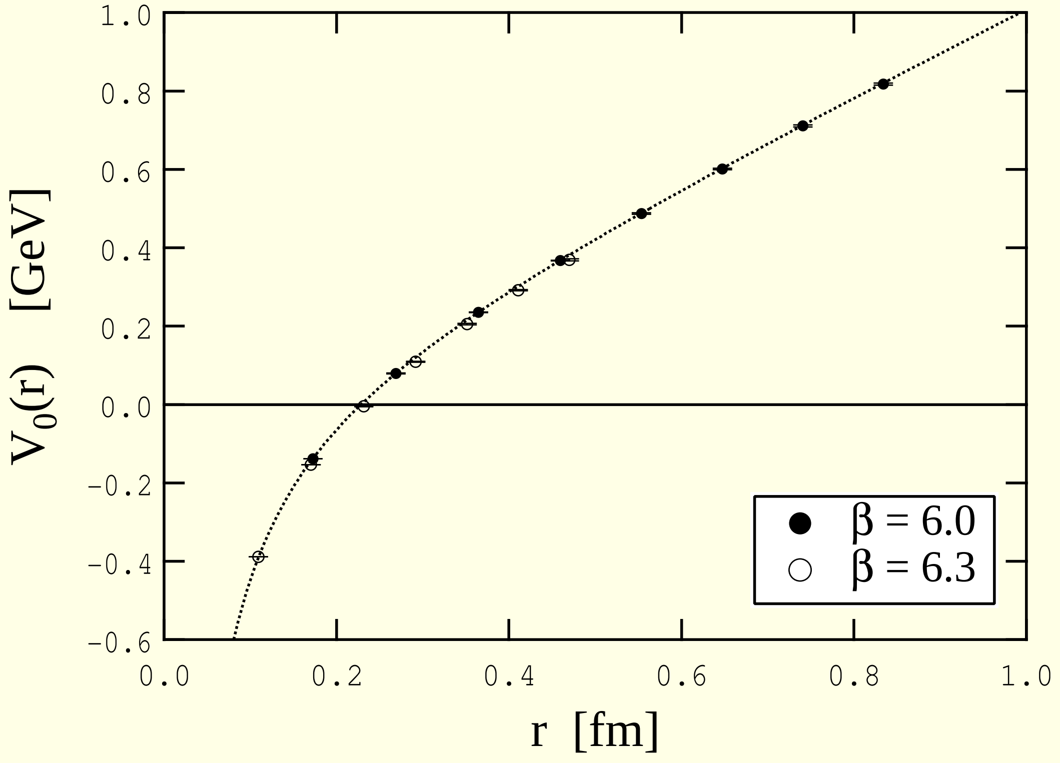
$$\mathcal{L} = \mathrm{Tr}\left\{ \mathrm{S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) \mathrm{S} \right\}$$

◦ Brambilla Pineda Soto Vairo PRD 63(01)014023

- (Without light hadrons) the Schrödinger equation is exact!
  ... which confirms the physical picture underlying potential models for heavy quarks.

The non-perturbative static potential

$$V_s^{(0)} = \lim_{T\to\infty} \frac{i}{T} \ln \langle \; \Box \; \rangle$$



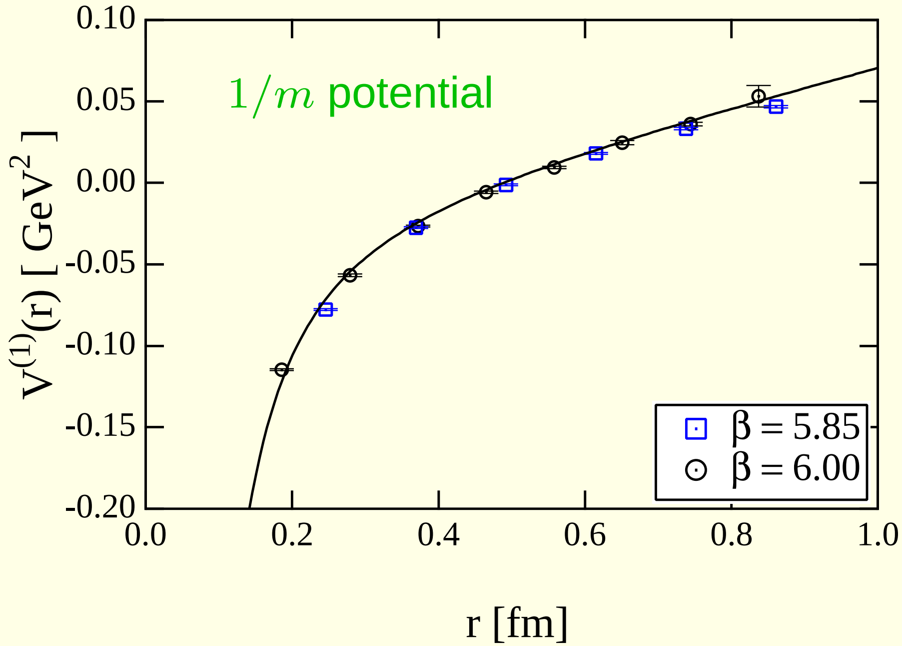
○ Koma Koma NPB 769(07)79

The non-perturbative $1/m$ potential

$1/m$ and $1/m^2$ potentials may be expressed in terms of expectation values of field insertions in a static Wilson loop.

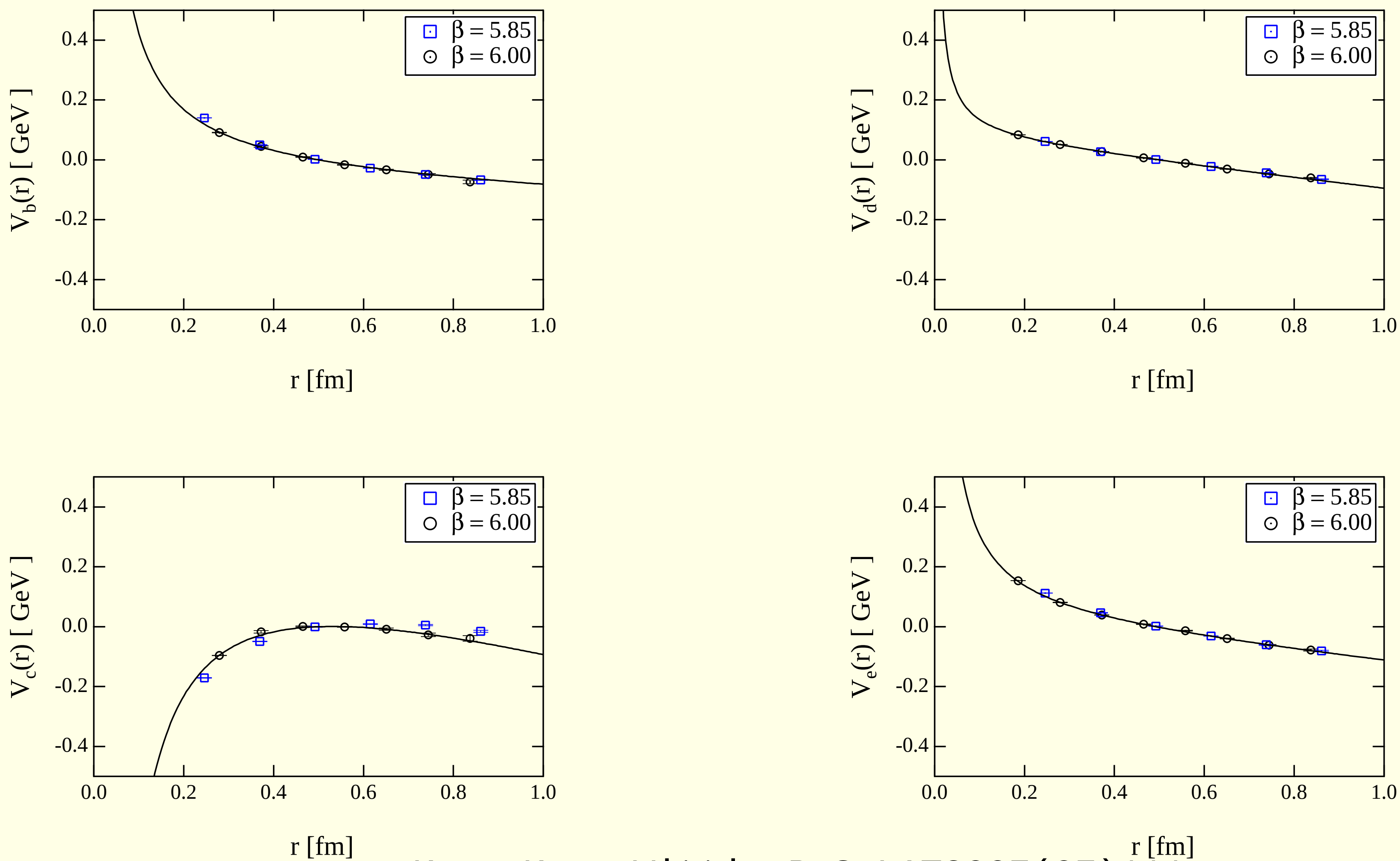
- Brambilla Pineda Soto Vairo PRD 63(01)014023
- Pineda Vairo PRD 63(01)054007

Lattice provides a non-perturbative determination of the potentials.



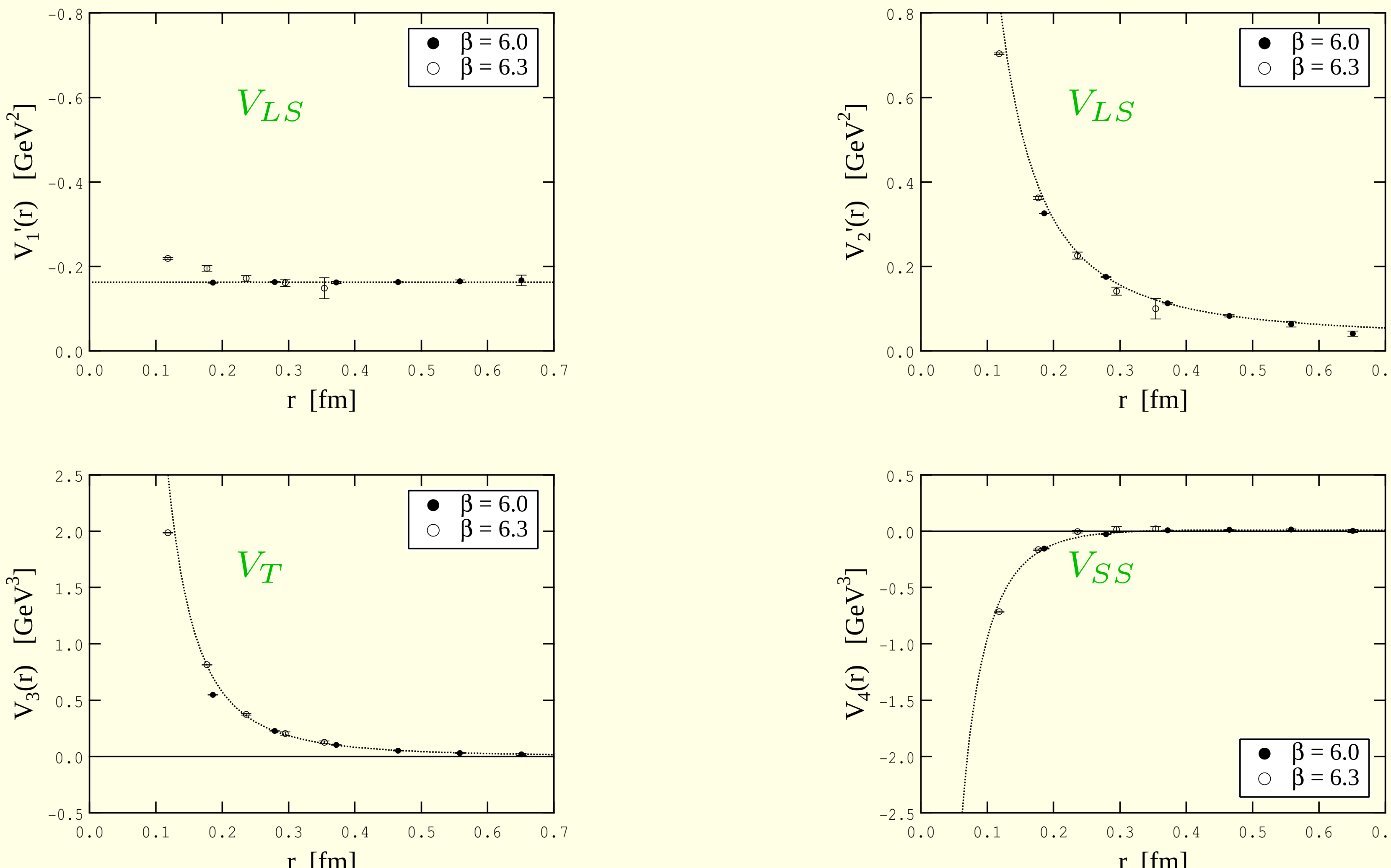
- Koma Koma Wittig PoS LAT2007(07)111

The non-perturbative spin-independent p^2/m^2 potentials



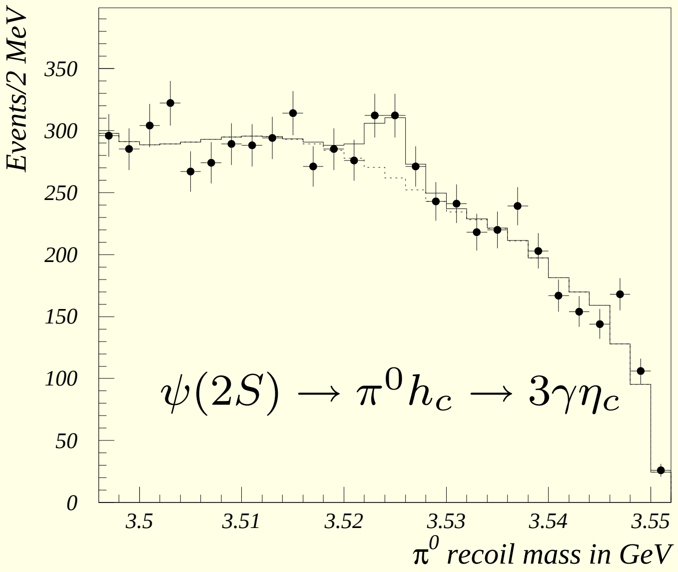
○ Koma Koma Wittig PoS LAT2007(07)111

The non-perturbative spin-dependent $1/m^2$ potentials



○ Koma Koma NPB 769(07)79

h_c



$$M_{h_c} = 3524.4 \pm 0.6 \pm 0.4 \text{ MeV}$$

○ CLEO PRL 95(05)102003

Also

$$M_{h_c} = 3525.8 \pm 0.2 \pm 0.2 \text{ MeV}, \qquad \Gamma < 1 \text{ MeV}$$

○ E835 PRD 72(05)032001

- To be compared with $M_{\text{c.o.g.}}(1P) = 3525.36 \pm 0.2 \pm 0.2 \text{ MeV}$.

Non-relativistic EFTs are equivalent, order by order in v, to the original relativistic quantum field theory. In particular, this applies also to Poincaré invariance, which is apparently badly broken, but actually is not. Poincaré invariance manifests itself in the EFT by constraining the form of the potentials.

- Dirac RMP 21(49)392, Foldy PR 122(61)275

Poincaré invariance

For any Poincaré invariant theory the generators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ of time translations, space translations, rotations, and Lorentz boosts satisfy the Poincaré algebra:

$$
\begin{aligned}
[\mathbf{P}^i, \mathbf{P}^j] &= 0 \\
[\mathbf{P}^i, H] &= 0 \\
[\mathbf{J}^i, \mathbf{P}^j] &= i\epsilon_{ijk}\mathbf{P}^k \\
[\mathbf{J}^i, H] &= 0 \\
[\mathbf{J}^i, \mathbf{J}^j] &= i\epsilon_{ijk}\mathbf{J}^k \\
[\mathbf{P}^i, \mathbf{K}^j] &= -i\delta_{ij}H \\
[H, \mathbf{K}^i] &= -i\mathbf{P}^i \\
[\mathbf{J}^i, \mathbf{K}^j] &= i\epsilon_{ijk}\mathbf{K}^k \\
[\mathbf{K}^i, \mathbf{K}^j] &= -i\epsilon_{ijk}\mathbf{J}^k
\end{aligned}
$$

$$
\begin{aligned}
h = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V^{(0)}(r) + \frac{V^{(1)}(r)}{m_1} + \frac{V^{(1)}(r)}{m_2} \\
+ \frac{V^{(2,0)}}{m_1^2} + \frac{V^{(0,2)}}{m_2^2} + \frac{V^{(1,1)}}{m_1 m_2} + \dots
\end{aligned}
$$

$$H = m_1 + m_2 + h$$

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$$

$$\mathbf{J} = \mathbf{x}_1 \times \mathbf{p}_1 + \mathbf{x}_2 \times \mathbf{p}_2 + \mathbf{S}_1 + \mathbf{S}_2$$

$$
\begin{aligned}
\mathbf{K} = -t\mathbf{P} + \frac{1}{2}\sum_{i=1}^{2}\left(\left\{\mathbf{x}_i, m_i + \frac{\mathbf{p_i}^2}{2m_i} + \frac{V^{(0)}}{2} + \frac{V^{(1)}}{m_i} + \dots\right\}\right. \\
\left. - \frac{\mathbf{S}_i \times \mathbf{p}_i}{m_i}(1 + \dots)\right)
\end{aligned}
$$

The algebra constraints the potentials:

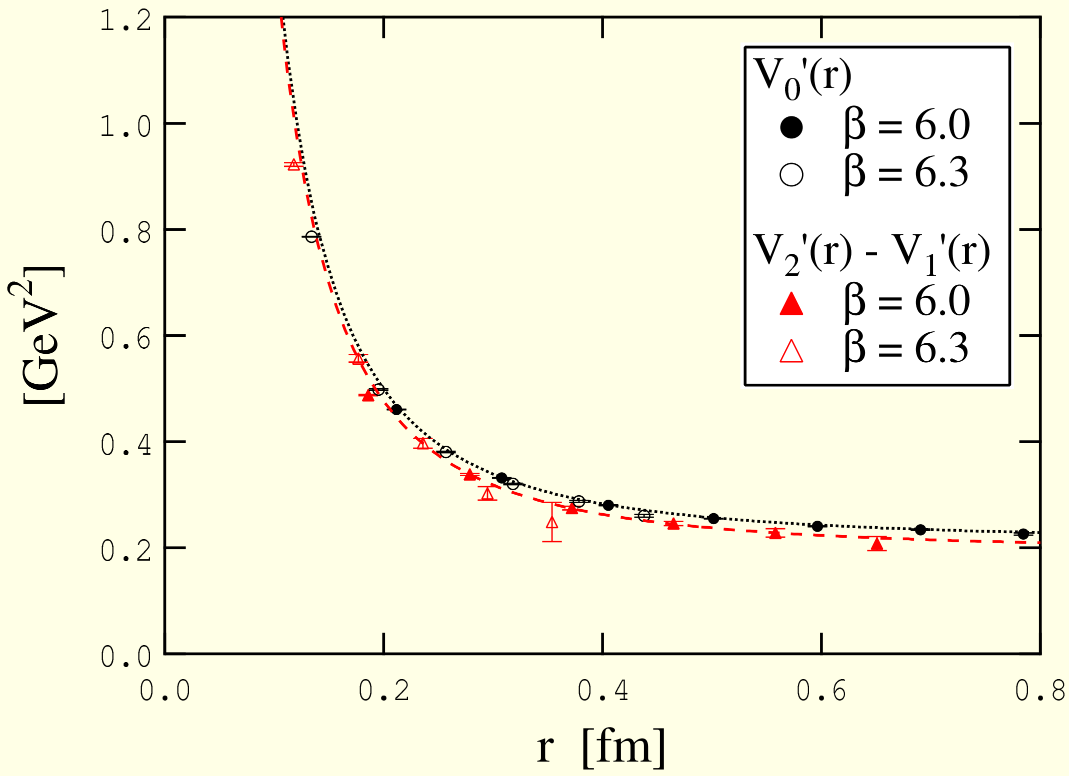
- $V_{LS}^{(2,0)} - V_{L_2 S_1}^{(1,1)} + \frac{V^{(0)\prime}}{2r} = 0$
- $V_{\mathbf{L}^2}^{(2,0)}(r) + V_{\mathbf{L}^2}^{(0,2)}(r) - V_{\mathbf{L}^2}^{(1,1)}(r) + \frac{r}{2} V^{(0)\prime}(r) = 0$
- $-2(V_{\mathbf{p}^2}^{(2,0)}(r) + V_{\mathbf{p}^2}^{(0,2)}(r)) + 2V_{\mathbf{p}^2}^{(1,1)}(r) - V^{(0)}(r) + rV^{(0)\prime}(r) = 0$
- ... ...

◦ Gromes ZPC 26(84)401, Barchielli Brambilla Prosperi NCA 103(90)59

◦ Brambilla Gromes Vairo PRD 64(01)076010, PLB 576(03)314

Constraint on the spin-dependent potentials

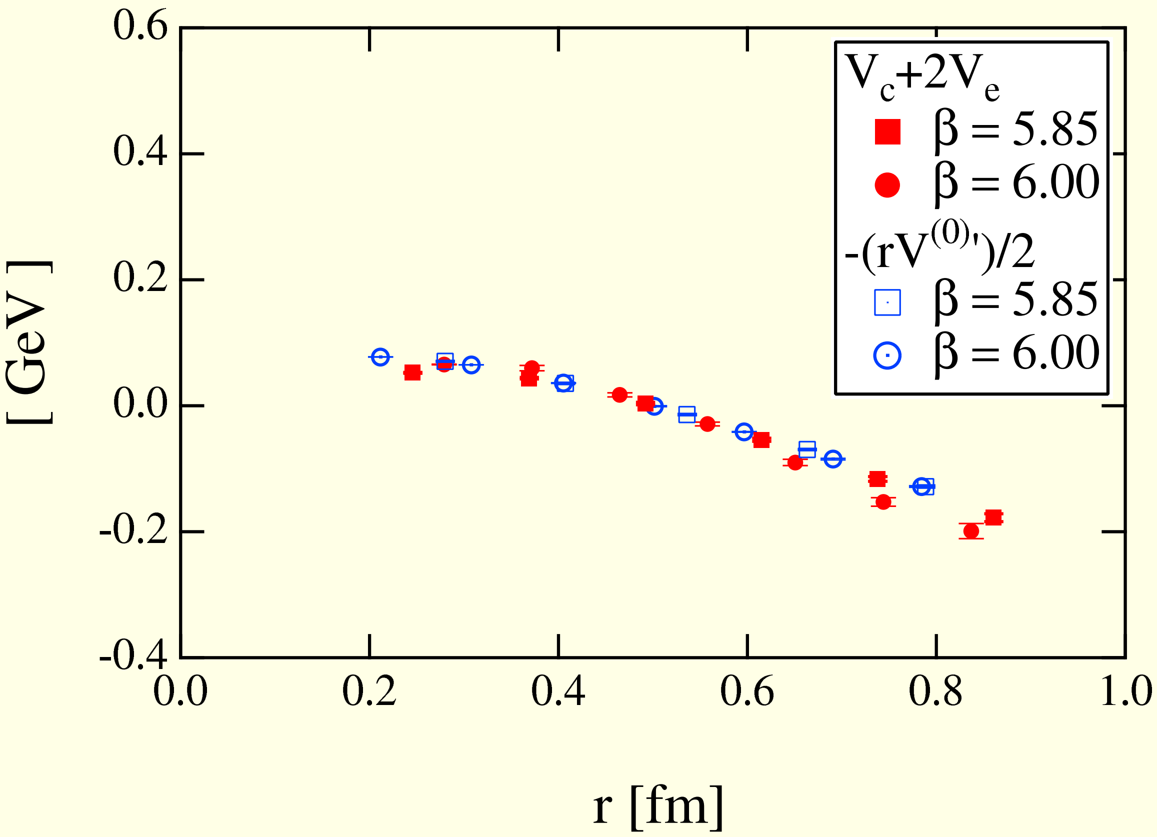
A lattice determination of $V_{LS}^{(2,0)} - V_{L_2 S_1}^{(1,1)} + \frac{V^{(0)\prime}}{2r} = 0$



○ M.Koma Y.Koma NPB 769(07)79

Constraint on the spin-independent potentials I

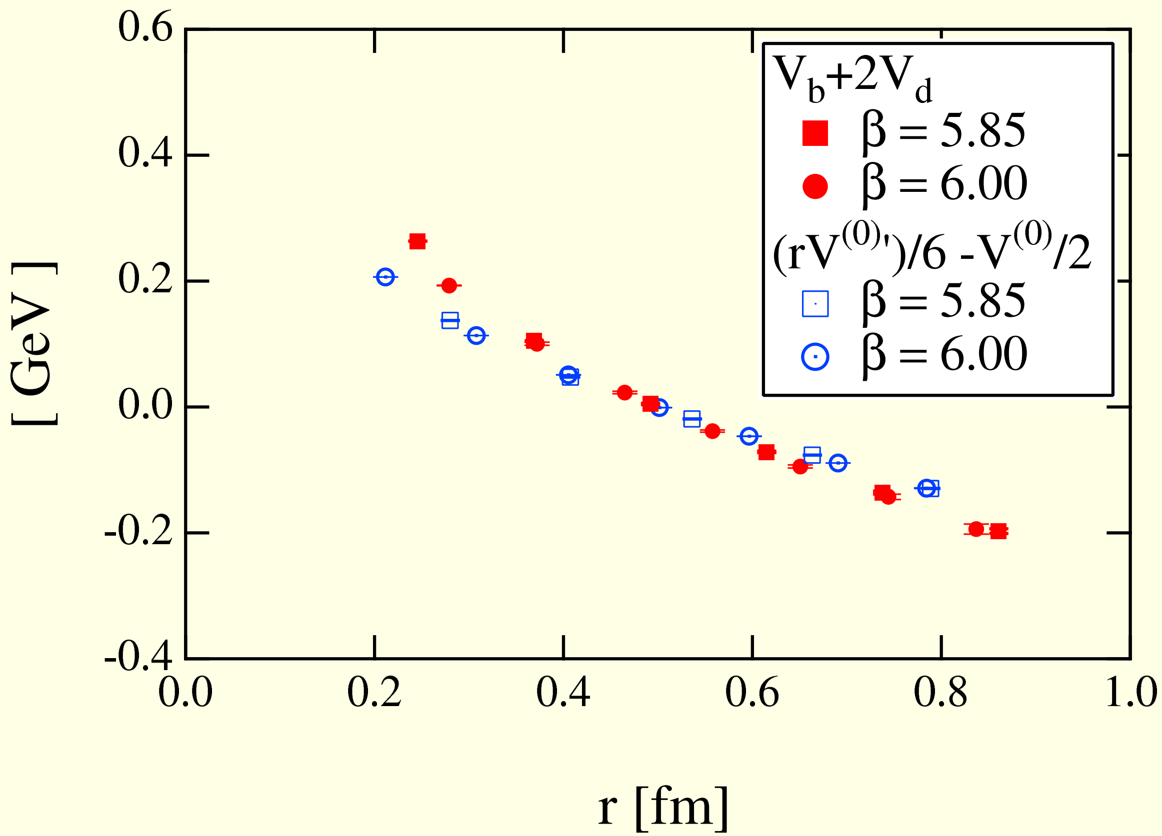
A lattice determination of $V_{\mathbf{L}^2}^{(2,0)}(r) + V_{\mathbf{L}^2}^{(0,2)}(r) - V_{\mathbf{L}^2}^{(1,1)}(r) + \frac{r}{2}V^{(0)\prime}(r) = 0$



○ M.Koma Y.Koma Wittig PoS LAT2007(07)111

Constraint on the spin-independent potentials II

A lattice determination of $-2(V_{\mathbf{p}^2}^{(2,0)}(r) + V_{\mathbf{p}^2}^{(0,2)}(r)) + 2V_{\mathbf{p}^2}^{(1,1)}(r) - V^{(0)}(r) + rV^{(0)\prime}(r) = 0$



○ M.Koma Y.Koma Wittig PoS LAT2007(07)111

Conclusions

The history of non-relativistic bound states in the quantum theory had in the last century a peculiar spiral behaviour. It started with the Schrödinger equation of the hydrogen atom and seemed to have written its ultimate chapter with the Bethe–Salpeter equation in the fifties. However, in face of the enormous difficulties in treating bound states in field theory by means of the Bethe–Salpeter equation, a long journey started in the seventies that took us back to the Schrödinger equation.

In a sense, the Schrödinger equation we have come back to is not like the one we have started with. It encompasses all the complexity of the Bethe–Salpeter equation, all the richness of field theory, in the systematic setting of non-relativistic effective field theories. The counting rules and structure of the EFTs have allowed to perform calculation with unprecedented precision, where higher-order perturbative calculations were possible, and to systematically factorize short from long range contributions where observables were sensitive to the non-perturbative, infrared dynamics of QCD.