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Noether analysis for field theories in canonical noncommutative spacetimes

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Introduction

Quantum mechanics and general relativity, the two theories that profoundly changed our description of the fundamental laws of physics at the beginning of the last century, both continue to be very successful in their respective domains, which include the physics of atoms and other microscopic systems on one side and the motion of planets and other macroscopic bodies on the other side. However, these two theories turn out to be logically incompatible when applied to systems where they are both nonnegligible, such as particle-physics processes at energy scales of the order of the Planck energy, $E_P = \sqrt{\frac{\hbar c^5}{G}} \simeq 10^{28}\, eV$, or higher, which correspond to distance scales of length of the order of the Plank length, $L_P = \sqrt{\frac{\hbar G}{c^3}} \simeq 10^{-35}\, m$, and lower. The realization of this incompatibility has of course motivated a large research effort aimed at finding a new theory, the so-called "quantum gravity", that would unify quantum mechanics and general relativity, cure their incompatibility, but reproduce the two original theories in some appropriate limits. And it is rather clear that a profoundly new formalisation of the law of physics will be required for this sought new theory; in fact, the formalism of quantum mechanics (quantum field theory) does not allow the introduction of the gravitational field, which would cause unmanageable divergences, while the introduction of quantum matter fields within general relativity is not possible for many reasons, including the fact that quantum fields are defined in a given background spacetime geometry, whereas general relativity is a dynamical theory providing spacetime structure as the end result of the analysis.

While it is still unclear how exactly the correct quantum gravity should look like, through various indirect arguments the community is gaining confidence on some possible features of this new theory. In particular, numerous analyses ([4, 5]) suggest that there should be an absolute limitation, set by the Planck length, on the measurability of the position of a particle. For example, one arrives at this conclusion by introducing, however heuristically, gravitational effects within the famous thought experiment known as the "Heisenberg microscope"(which was originally used as an argument in favour of the Heisenberg uncertainty principle). The point is that in order to measure the position of a particle we need to let it interact with a probe, such as a photon. If we want a measure with precision δx, because of Heisenberg's indetermination principle we have to send a photon of energy of order $\frac{1}{\delta x}$, and therefore any increase in precision of the measurement requires an increase in the probe's energy. But, in taking into account gravitational effects, it must be considered that at some point the photon has energy high enough that the associated deformation of spacetime cannot be neglected, and introduces an additional source of uncertainty. In order to estimate this contribution to the uncertainty one can observe that the gravitational energy of the system when the probe is at a distance δx from the particle is of the order $\frac{L_P^2 ME}{\delta x}$, where M is the mass of the particle and E is the energy of the photon. This quantity has an uncertainty $\frac{L_P^2 M}{(\delta x)^2}$ due to the uncertainty on the energy of the photon (which of course is given by $\delta E \simeq \frac{1}{\delta x}$), and such an energy uncertainty translates into an uncertainty $\delta v \simeq \frac{L_P^2}{(\delta x)^2}$ on the velocity of the particle. Since the collision between the particle and the photon has a duration of order δx, from this uncertainty on the velocity one derives an uncertainty on the position of the particle of the order $\delta x \simeq L_P$.

The idea of spacetime noncommutativity, which is adopted in this thesis work, can be motivated on the basis of these arguments on Planck-scale limitations on the localisation of a particle. In ordinary quantum mechanics the uncertainty principle for simultaneous measurements of position and momentum is formalised describing coordinates and momenta as appropriate operators on a Hilbert space and assuming that the operators describing coordinates and the ones describing momenta satisfy the commutation relation $[x_i, p_j] = i\hbar\delta_{ij}$. One can attempt an analogous formalisation of the Planck-scale limit on the measurability of spacetime coordinates through some appropriate form of "space noncommutativity", *i.e.* treating the spacetime coordinates as noncommuting variables.

Besides its use in proposals in which it is assumed as a fundamental ingredient of the new theory, the idea of spacetime noncommutativity has also attracted interest from other perspectives. From the point of view of quantum field theory a spacetime noncommutativity at small length scales is expected to have implications somewhat similar to the ones of a discretisation of spacetime on a lattice, particularly for what concerns the presence of a natural ultraviolet cutoff that would avoid some divergences. As a regulator spacetime noncommutativity has the advantage, with respect to spacetime lattice discretisation, that it does not necessarily cause a loss of symmetries, as first pointed out by Snyder in 1947 in a renowned article [6].

Another reason of interest in this idea comes from the fact that a noncommutativity of spacetime coordinates sometimes emerges as an effective description of some limits of theories that are not originally formulated with spacetime noncommutativity. The simplest example of this mechanism is provided by a particle moving in the plain $x\,y$ subject to a constant magnetic field $\vec{B} \equiv B_z\hat{z}$ (the so-called "Landau problem" [1]). The Lagrangian of the system is $\mathcal{L} = \frac{mv^2}{2} + q\,x\,B_z v_y - q\,y\,B_z v_x$, so that the conjugate momenta are: $p_x = mv_x - q\,y\,B_z$ and

$p_y = mv_y + q\,x\,B_z$. Inserting these expressions in the commutators between coordinates and momenta one finds $[x, mv_x - q\,y\,B_z] = i\hbar$ and $[y, mv_y + q\,x\,B_z] = i\hbar$. In the large B field limit one can neglect the terms $m\,v_x$ and mv_y, obtaining the following commutation relation between coordinates:$[y, x] = i\frac{\hbar}{qB_z}$. Interestingly a similar mechanism is encountered in string theory, one of the most popular proposed approaches to the "quantum-gravity problem", which is not formulated in a noncommutative spacetime but is effectively described in terms of a noncommutative spacetime when studied in presence of a strong background of a certain tensor field [14].

In this thesis work we shall primarily adopt the perspective of spacetime noncommutativity introduced from the beginning as a fundamental description. From this perspective it is reasonable to contemplate a rather wide class of possible noncommutativity relations for the spacetime coordinates:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\Theta_{\mu\nu}(\hat{x}),$$

where $\Theta_{\mu\nu}(\hat{x})$ is a matrix for which a coordinate dependence is in general allowed. Different formulations of the matrix $\Theta_{\mu\nu}(\hat{x})$ provide different examples of noncommutative spacetimes. The two most studied possibilities are the so-called κ-Minkowski spacetime, characterised by the commutation relations:

$$[\hat{x}_j, \hat{x}_0] = i\lambda\hat{x}_j \qquad [\hat{x}_j, \hat{x}_k] = 0,$$

where λ is a length scale, usually assumed to be of the order of the Plank length, and the canonical (or θ-Minkowski) spacetime, in which the coordinates obey the relation:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu},$$

where $\theta_{\mu\nu}$ is a coordinate independent matrix.

In this thesis we shall investigate certain aspects of theories formulated in canonical noncommutative spacetime, specifically for the scenario in which $\theta_{\mu\nu}$ is an observer-independent matrix, taking the same form (and numerical values) for all observers. The primary objective of our work is a description of the symmetries of a theory describing Klein-Gordon-like scalar and massless fields in this type of spacetime. The first step of our work is the description of the algebra of functions over the canonical spacetime through a map from the commutative functions on Minkowski spacetime to the noncommutative functions. We describe this as a generalisation of the well-known Weyl map that was defined to map functions on the commutative phase space of classical mechanics to functions over the noncommutative phase space of quantum mechanics, in establishing a relationship between quantum observables and corresponding classical observables. This generalised Weyl map establishes an ordering convention that makes biunivocal the correspondence between commutative and noncommutative functions. We also observe that it is possible to define different such maps, adopting different ordering conventions. With the definition of these generalised Weyl maps we also define a deformed (noncommutative) product (the so-called "star product") between commutative functions that allows one to use commutative functions in analysing certain properties of the noncommutative functions of interest. [Section 1.1].

After this preliminary characterisation of our noncommutative spacetime we shall be ready for the symmetry analysis. The first observation is that the presence of the constant matrix $\theta_{\mu\nu}$ violates the classical Poincaré symmetries making the spacetime no longer invariant under the action of the generators of the Poincaré Lie algebra. [Section 1.2].

We shall then argue that this feature however does not amount to a complete loss of Poincaré invariance: actually in our theory one can uncover symmetries that could be described as an adaptation of Poincaré symmetry to the novel context of our noncommutative spacetime. These novel symmetries are described in terms of some generators, which can be introduced by a simple adaptation of the rules of action of the standard Poincaré generators: this is accomplished by stating that these generators act just like the classical Poincaré generators inside the Weyl map, *i.e.* their action on noncommutative functions is obtained by acting with the Weyl map on the action of the corresponding classical Poincaré generator on a commutative function. This leads to a set of generators that satisfies the same commutation relations of the classical Poincaré algebra, but has different properties for what concerns the law of action of the generators on products of functions, which in particular no longer satisfies the Leibniz rule. We shall denote this as the "θ-Poincaré" algebra, since the matrix $\theta_{\mu\nu}$ of course governs the deformation. [Section 1.3].

The deformed action on products of functions plays, as we shall show, a crucial role in verifying that the commutation relations that characterise the canonical spacetime are invariant under the action of the deformed algebra. This invariance of the commutation relations provides a first indication of the fact that indeed it is more correct to describe the symmetry situation as one of deformation of the Poincaré symmetry rather than a full break down of Poincaré symmetry. [Section 1.4].

While the constructive approach described above to find the algebra of symmetries of canonical spacetime is an original result of our work, one does find in the literature ([[11]]) a different argument to reach the same conclusion. This argument relies on the formal structure and properties of "Hopf algebras", which is a generalisation of the concept of Lie algebra that includes additional structures, the most important of which is the "coproduct map" (from the tensor product of two copies of the algebra on which the Hopf algebra acts to the algebra itself) that establishes the rule of action over products of element of the algebra representing the Hopf algebra . There is a standard procedure, called “twisting”, to obtain a nontrivial Hopf algebra from a standard Lie algebra, leaving the rules of commutation of generators unchanged, but introducing a “twist element” in the coproduct map. We show that there exists a twist element that transforms the Poincaré Lie algebra (viewed as a Hopf algebra with trivial coproduct) into the θ-Poincaré algebra. [Sections 1.5 and 1.6].

Another original part of our work is the definition of a new basis of generators for the θ-Poincaré algebra that is constructed using an unconventional (but perfectly acceptable) Weyl map to define the action of the deformed generators. We manage to establish that the new basis generators are obtainable from the basis adopted by most authors by use of a simple map of redefinition. We also observe that the form of the coproduct map, as written in the alternative basis of generators, also admit description in terms of a twist element. [Section 1.7].

Through the study of the properties of the generators of the deformed symmetry algebra one of course prepares the description of the symmetries of field theories written in canonical noncommutative spacetime, but in order to even formulate (for example by writing an action) such field theories one needs a notion of integral in the noncommutative spacetime, and of course for such a notion one cannot rely on the familiar commutative-space intuition. We argue in favour of certain rules of integration applicable for integration over the whole spacetime or just the spatial volume, and we show that these notion of integration is independent of the choice of ordering convention. [Section 1.8].

While these results reported in the first chapter were in large part already known (although previously derived following a more formal, rather than constructive, procedure), the second and the third chapters present a fully original analysis, whose main objective is a description of the symmetries of Klein-Gordon-like field theory in canonically noncommutative spacetime. We shall show that, upon a suitable adaptation of mathematical structures, the symmetries of such a theory can be analysed in complete analogy with the renowned Noether analysis of the symmetries of field theories written in commutative spacetime. In the second chapter we provide a description of the infinitesimal symmetry transformations of a field, which, as we shall show, requires an appropriate adaptation of the concept of transformation parameters. In the third chapter we perform a Noether-like analysis on the action of a scalar massless field and get some coserved quantities associated to the invariance of the action under the symmetry transformations.

The possibility of a Noether analysis of Hopf-algebra spacetime symmetries became realistic only very recently, when in [3] the first successful analysis of this type was reported, for the specific case of translation symmetries of k-Minkowski spacetime. We use the outcome of that recent study to argue that also in the case here of interest, the one of the θ-twisted Hopf algebra of symmetries of canonical spacetimes, the nontriviality of the coprouduct properties will result in nontrivial commutation relations involving the transformation parameters and the spacetime coordinates. [Section 2.1].

We first concentrate on the case of translation transformations: we make some conservative requirements on the differential rule of transformation for fields ϕ, assumed to be of the familiar form $\phi \to \phi + d\phi$, the most important of which is the validity of Leibniz rule for $d\phi$. We find that the translation parameters should satisfy trivial commutation relations with coordinates. [Section 2.2].

We subsequently try to apply the same procedure in the analysis of the Lorentz sector transformations (spatial rotations and boosts), but we discover that the imposition of the Leibniz rule cannot be achieved with acceptable choices of transformation parameters, for which (as done in [3]) we demand that, while possibly governed by a nontrivial product rule with the spacetime coordinates, they act on the spacetime coordinates by simple associative multiplication. [Section 2.3].

We find that however no obstruction is encountered upon considering simultaneously Lorentz-sector and translation transformations in a single differential. Within this setup one ends up concluding that the Lorentz-sector transformation parameters commute with the spacetime coordinates, but the commutators of translation parameters with spacetime coordinates should be proportional to the Lorentz-sector transformation parameters. This has the striking consequence that, while pure translations are admitted, it is not possible to perform a pure Lorentz-sector transformation, since if the corresponding parameter is different from zero then also at least one of the translation parameters should be different from zero. [Section 2.4].

In testing the dependence of these results on the above-mentioned choice of basis of generators, we find that, while qualitatively similar, in different bases the rules of commutation between transformation parameters and spacetime coordinates are not exactly the same. [Section 2.5].

Postponing the investigation of this dependence on the choice of basis at the level of the conserved charges associated to the symmetry transformations, we proceed with the introduction of a field theory, which is invariant under the transformations we constructed and could therefore be meaningfully analysed a la Noether for our purposes. We show that an appropriate choice for scalar fields is a Klein-Gordon-like equation constructed using the casimir of the deformed Poincaré algebra. [Section 2.6].

As the final step in our preparation for the Noether analysis we then construct an action that generates our chosen equation of motion and we show it to be invariant under the symmetry transformations we constructed. [Sections 3.1-3.2].

We then show that the variation of our action under a symmetry transformation takes a form that points to certain functionals of the fields as currents [Section 3.4], and by space integration of these currents we do obtain time-independent quantities, the conserved charges.[Sections 3.5]. Finally we return to the issue of the possible dependence on the choice of basis of generators, which one could fear in light of the mentioned basis dependence of the commutation relations between transformation parameters and spacetime coordinates. However, we show that, while at intermediate stages of the Noether analysis one does find terms that depend on the choice of basis, the final result for the conserved charges is basis independent. [Section 3.6].

Notations

The signature of the spacetime is $(-1, 1, 1, 1)$

Greek indices stand for the four spacetime coordinates (e.g. $\mu = 0, 1, 2, 3$)

Latin indices stand for just spatial coordinates (e.g. $j = 1, 2, 3$)

$\hat{x}_\mu$ are noncommuting coordinates, x_μ are the commuting ones

$\hat{G}$ indicates a generator of the deformed Poincaré algebra, G a generator of the classical one

$A_{[\mu}B_{\nu]}$ indicates that indices μ and ν are antisymmetrised: $A_{[\mu}B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu$

q, x stand for q^μ, x^μ respectively

qx stands for the scalar product $q^\mu x_\mu$

We write formulas involving delta functions and Fourier transforms up to 2π factors

Chapter 1

Canonical spacetime

In this thesis we shall study the so-called canonical (or θ-Minkowski) noncommutative spacetime, whose coordinates satisfy the commmutation relations:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \tag{1.1}$$

where $\theta_{\mu\nu}$ is an observer-independent and coordinate-independent antisymmetric matrix.

As we explained in the introduction, noncommutative spacetimes are in general interesting because they formalise a quantisation of spacetime due to the existence of a minimum uncertainty in measures of length. Canonical spacetime in particular is one of the most studied examples of noncommutative spacetimes (the other one that is most studied is known as κ-Minkowski, where the time coordinate does not commute with the spatial one, $[\hat{x}_j, \hat{x}_0] = i\lambda\hat{x}_j$) and moreover it emerges in the theory of strings in high external field limit.

There have already been some proposals for field theories over this kind of spacetime, but these studies have never provided a derivation of observable quantities associated to the spacetime symmetries. This is probably due to the fact that it has not yet been established a sort of Noether theorem valid for noncommutative spaces.

We are interested in the description of the algebra of symmetries of this spacetime in order to analyse a classical field theory invariant under the transformations associated to this algebra. The algebra of symmetries is clearly not trivial: the above equations break the classical Poincaré invariance of the spacetime, because of the presence of a constant matrix (see Sec. 1.2). It has been shown that this is not properly the case of a breaking of symmetry, but it is more correct to talk of a deformation, since (as we shall show later in the thesis) there actually exists an algebra of symmetries of canonical spacetime that results to be a deformation of the classical Poincaré algebra.

1.1 Weyl ordering and star product

When extending functions of (commutative) Minkowski spacetime coordinates to functions of (noncommutative) θ-Minkowski spacetime coordinates through the correspondence $x^\mu \rightarrow \hat{x}^\mu$ we make the only requirement that the commutative functions should be the $\theta \rightarrow 0$ limit of the noncommutative ones. This generates some ambiguities, since in general it is possible to associate more than one noncommutative function to a commutative one, all satisfying the above condition on the commutative limit. We want to avoid these ambiguities to make univocal the definition of the physical quantities (functions of coordinates) that have as their $\theta \rightarrow 0$ limit the classical ones. This problem is similar to that of " *translating* " the observables defined in classical mechanics (that are functions of the phase space coordinates: spacetime coordinates x and momenta p, that commute one with each other) to analogous observables defined in quantum mechanics, whose phase space coordinates $\hat{x}$ and $\hat{p}$ are no more commutative, but instead satisfy the commutation relation:

$$[\hat{x}, \hat{p}] = i\hbar. \tag{1.2}$$

A simple example of the ordering issues we have to deal with is found if we try to extend the classical function $xp \equiv px$ to its quantum equivalent. We find an entire class of candidates: $\alpha\hat{x}\hat{p} + \beta\hat{p}\hat{x}$, where $\alpha, \beta \in \mathbb{R}, \alpha + \beta = 1$.

It is well known that the ordering choice that agrees with experimental data is the one called "Weyl ordering", from the name of the scientist H. Weyl, that first defined it ([2]): the function of quantum coordinates that correctly extends a classical one is the one where $\hat{x}$ and $\hat{p}$ appear *full symmetrised*. In the example above it should be $\frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})$. To formalise this construction it is usual to define a map (the so-called *Weyl map*):

$$\Omega : f(x, p) \rightarrow \hat{f}(\hat{x}, \hat{p}),$$

that takes a function of the commutative coordinates of the phase space of classical mechanics and gives the corresponding function of the noncommutative coordinates of the phase space of quantum mechanics full-symmetrising the commutative function. Obviously the Weyl map is linear, so that it can be easily shown that its action on exponentials is:

$$\Omega\left(e^{\alpha x + \beta p}\right) = e^{\alpha\hat{x} + \beta\hat{p}}. \tag{1.3}$$

This formula allows us to write the action of the Weyl map on an arbitrary function of x and p using its Fourier expansion. In fact, given $f(x,p) = \int d\alpha\, d\beta\, \tilde{f}(\alpha, \beta) e^{i(\alpha x + \beta p)}$ its Weyl transformed can be written, using the linearity of the map Ω, in the form:

$$\Omega(f(x,p)) = \int d\alpha\, d\beta\, \tilde{f}(\alpha,\beta)\Omega\left(e^{i(\alpha x + \beta p)}\right) \equiv \int d\alpha\, d\beta\, \tilde{f}(\alpha,\beta) e^{i(\alpha\hat{x} + \beta\hat{p})} \equiv f_\Omega(\hat{x}, \hat{p}). \tag{1.4}$$

Now we want to extend the use of the Weyl map from the quantisation of phase space coordinates to the quantisation of spacetime coordinates. We define the generalised Weyl map $\Omega_w : f(x) \to \hat{f}(\hat{x})$ from classical Minkowski spacetime to noncommutative θ-Minkowski spacetime such that it transforms $f(x)$ in its full-symmetrised version in terms of noncommuting coordinates, for example $\Omega_w(x_1 x_2) = \frac{\hat{x}_1\hat{x}_2+\hat{x}_2\hat{x}_1}{2}$. In particular the action of the map on exponentials generates the so-called "*Weyl-ordered exponentials*", that is:

$$\Omega_w\left(e^{ikx}\right) = e^{ik\hat{x}}. \tag{1.5}$$

In this way, to a generic commutative function $f(x)$ whose Fourier transform reads $f(x) = \int d^4x\, \tilde{f}(k)e^{ikx}$ it corresponds, through the Weyl map Ω_w, the noncommutative function:

$$f_w(\hat{x}) \equiv \Omega_w(f(x)) = \int d^4k\, \tilde{f}(k)\Omega_w(e^{ikx}) \equiv \int d^4k\, \tilde{f}(k)e^{ik\hat{x}}. \tag{1.6}$$

Note that here k has not the meaning of a momentum, but it is just the conjugate variable to the $\hat{x}$'s. through the Fourier transform. The expression above shows us also the fact that the noncommutative exponentials of the form $e^{ik\hat{x}}$ are a basis for the space of functions on θ-Minkowski spacetime, like their commutative analogues.

Viceversa, one can define the inverse Weyl map in this way: if one has a generic noncommutative function $g(\hat{x}) = \sum a_{n_0,n_1,n_2,n_3} p_{n_0,n_1,n_2,n_3}(x_0,x_1,x_2,x_3)$, where p_{n_0,n_1,n_2,n_3} is a polynomial of grade n_i in the coordinate $\hat{x}_i$ $(i = 0,1,2,3)$[1] and a_{n_0,n_1,n_2,n_3} is the coefficient with which that polynomial appears, he has to put it in the canonical form corresponding to the full-symmetrised Weyl map (i.e. one has to full-symmetrise all the polynomials commuting the coordinates and taking into account the resulting commutators: $\hat{x}_1\hat{x}_2^2$ has to be put in the form $\frac{\hat{x}_1\hat{x}_2^2+\hat{x}_2\hat{x}_1\hat{x}_2+\hat{x}_2^2\hat{x}_1}{3} + i\theta_{12}$), then considers all the coordinates as commutative and writes down the resulting function (in the example above one would obtain $x_1x_2^2 + i\theta_{12} \equiv x_2^2x_1 + i\theta_{12} \equiv x_2x_1x_2 + i\theta_{12}$). In this way we have defined the map Ω_w^{-1} such that $\Omega_w\left(\Omega_w^{-1}(f(\hat{x}))\right) = f(\hat{x})$.

We have just defined a one-to-one correspondence between commutative functions and noncommutative ones, that makes the Weyl map so useful.

It is now important to notice that the map defined above is not the unique possible generalisation of the Weyl map used in quantum mechanics. All possible coordinates ordering choices are legitimate, if they reduce to the right commutative function in the commutative limit, i.e. in the limit in which $\theta_{\mu\nu} \to 0$ and so the coordinates return commutative. For example the ordering prescription $\Omega_\alpha(e^{ikx}) = e^{ik^1\hat{x}_1}e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}$ can not be used, since in the commutative limit $e^{ik^1\hat{x}_1}e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1} \xrightarrow{\theta_{\mu\nu}\to 0} e^{i(kx+k^1x_1)} \neq e^{ikx}$, while the similar ordering convention,

[1]Note that every polynomial can be put in the desired ordered form just commuting the coordinates, and this does not make vary the grade with which each coordinate appears, since the commutator is coordinate-independent.

$\Omega_\beta(e^{ikx}) = e^{\frac{i}{2}k^1\hat{x}_1}e^{ik^A\hat{x}_A}e^{\frac{i}{2}k^1\hat{x}_1}$, is legal. A map that will be useful later as a simple example of a different ordering convention is the one that, instead of full-symmetrising all the coordinates, takes the coordinate x_1 on the right and symmetrises the remaining ones:

$$\Omega_1\left(e^{ikx}\right) = e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}, \tag{1.7}$$

where $A = 0, 2, 3$. Obviously in general the noncommutative functions obtained from the same commutative one using different ordering conventions (i.e. different Weyl maps) result to be different, since for example:

$$\Omega_w\left(e^{ikx}\right) \equiv e^{ik\hat{x}} \neq e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1} \equiv \Omega_1\left(e^{ikx}\right), \tag{1.8}$$

but each choice of the generalised Weyl map defines a different one-to-one correspondence between commutative and noncommutative functions, in the same way that we explained for the full-symmetrising Weyl map. Moreover each map defines also a different basis of exponentials for the space of noncommutative functions [2] so we can just think at this ordering ambiguity as a choice of the basis for the space of noncommutative functions.

The utility of the generalised Weyl map is that we can choose not to work directly in the space of noncommutative functions, but to work with the commutative Minkowski spacetime coordinates, performing all the calculations inside the Weyl map and explicitating the map just at the end of the work. The only warning in this procedure is that one could ingenuously write $f_i(\hat{x})g_i(\hat{x}) = \Omega_i(f(x)g(x))$ (where Ω_i is a generic Weyl map and $f_i(\hat{x})$ indicates the noncommutative function corresponding to the commutative one $f(x)$ through the map Ω_i), but this is wrong, as one can convince himself observing that

$$e^{ik\hat{x}}e^{iq\hat{x}} \neq e^{i(k+q)\hat{x}} = \Omega_w(e^{i(k+q)x}) \equiv \Omega_w(e^{ikx}e^{iqx}). \tag{1.9}$$

It is so clear that if one wants to work inside the Weyl map, he can not use the ordinary product between commutative functions, but has to deform it ($f(x)g(x) \to f(x) \star g(x)$, and the new product is called "*star-product*") in a way that:

$$f_i(\hat{x})g_i(\hat{x}) = \Omega_i(f(x) \star_i g(x)). \tag{1.10}$$

Notice that we have posed the subscript i also to the product symbol because the product law depends on the Weyl map in consideration, as we shall see in a while. To find explicitly the star-product law let us first consider a product of exponentials, $e^{ik\hat{x}}e^{iq\hat{x}}$.

[2] For example, as one can see in (1.7), the map Ω_1, defines the basis $e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}$, in terms of which one can write each noncommutative function as:

$$f(\hat{x}) \equiv \Omega_1(f_1(x)) = \Omega_1\left(\int d^4k\tilde{f}_1(k)e^{ikx}\right) = \int d^4k\tilde{f}_1(k)e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}$$

It will be useful here and in the following to exploit the famous Baker-Campbell-Hausdorff (BCH) formula:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]+\dots}, \tag{1.11}$$

that has a closed expression if used for products exponentials of θ-Minkowski spacetime coordinates, since the commutator between θ-Minkowski coordinates is a constant that commutes with the coordinates themselves, making all the nested commutators to be null.

Turning back to the product of exponentials we can write it, using the BCH formula, in the following way:

$$e^{ik^\mu \hat{x}_\mu} e^{iq^\nu \hat{x}_\nu} = e^{i(k^\mu+q^\mu)\hat{x}_\mu+\frac{1}{2}[ik^\mu \hat{x}_\mu, iq^\nu \hat{x}_\nu]} \equiv e^{i(k^\mu+q^\mu)\hat{x}_\mu-\frac{i}{2}k^\mu q^\nu \theta_{\mu\nu}} \tag{1.12}$$

From this last expression, using (1.5), it follows:

$$e^{ik\hat{x}} e^{iq\hat{x}} = \Omega_w \left(e^{i(k+q)x} e^{-\frac{i}{2}k^\mu q^\nu \theta_{\mu\nu}} \right) \tag{1.13}$$

So, if we want to work inside the symmetric Weyl map, we have to use the following product rule for the exponentials (from the comparison of (1.13) and (1.10)):

$$\Omega_w \left(e^{ikx} \star_w e^{iqx} \right) \equiv \Omega_w \left(e^{i(k+q)x} e^{-\frac{i}{2}k^\mu q^\nu \theta_{\mu\nu}} \right) \tag{1.14}$$

It is now straightforward to extend the star-product rule (to be used inside the symmetric Weyl map) to generic functions using their Fourier transform:

$$f(x) \star g(x) = \int d^4k d^4q \; \tilde{f}(k) \tilde{g}(q) \; e^{ikx} \star e^{iqx} \equiv \int d^4k d^4q \; \tilde{f}(k) \tilde{g}(q) \; e^{i(k^\mu+q^\mu)x_\mu} e^{-\frac{i}{2}k^\mu q^\nu \theta_{\mu\nu}} \tag{1.15}$$

If we want to know the star product rule to be used inside a Weyl map different from the symmetric one, for example in the Ω_1 map, it suffices to write the product of $\hat{x}_1$-to-the-right ordered exponentials in the following way (we use twice the BCH formula (1.11)):

$$\begin{aligned}
\Omega_1(e^{ikx})\Omega_1(e^{iqx}) &\equiv e^{ik^A \hat{x}_A} e^{ik^1 \hat{x}_1} e^{iq^A \hat{x}_A} e^{iq^1 \hat{x}_1} = e^{ik^\mu \hat{x}_\mu} e^{iq^\nu \hat{x}_\nu} e^{-\frac{i}{2}k^A k^1 \theta_{A1}} e^{-\frac{i}{2}q^A q^1 \theta_{A1}} \\
&= e^{i(k^\mu+q^\mu)\hat{x}_\mu} e^{-\frac{i}{2}k^\mu q^\nu \theta_{\mu\nu}} e^{-\frac{i}{2}k^A k^1 \theta_{A1}} e^{-\frac{i}{2}q^A q^1 \theta_{A1}} \\
&= e^{i(k^A+q^A)\hat{x}_A} e^{i(k^1+q^1)\hat{x}_1} e^{\frac{i}{2}(k^A+q^A)(k^1+q^1)\theta_{A1}} e^{-\frac{i}{2}k^\mu q^\nu \theta_{\mu\nu}} e^{-\frac{i}{2}k^A k^1 \theta_{A1}} e^{-\frac{i}{2}q^A q^1 \theta_{A1}} \\
&= e^{i(k^A+q^A)\hat{x}_A} e^{i(k^1+q^1)\hat{x}_1} e^{\frac{i}{2}(k^A q^1+q^A k^1)\theta_{A1}} e^{-\frac{i}{2}k^\mu q^\nu \theta_{\mu\nu}}
\end{aligned} \tag{1.16}$$

to deduce, with steps analogous to the ones described above:

$$\begin{aligned}
\Omega_1 \left(e^{ikx} \star_1 e^{iqx} \right) &= \Omega_1 \left(e^{i(k+q)x} e^{\frac{i}{2}(k^A q^1+q^A k^1)\theta_{A1}} e^{-\frac{i}{2}k^\mu q^\nu \theta_{\mu\nu}} \right) \\
&\equiv \Omega_1 \left(e^{i(k+q)x} e^{-ik^1 q^A \theta_{1A}} e^{-\frac{i}{2}k^A q^B \theta_{AB}} \right)
\end{aligned} \tag{1.17}$$

1.2 Breaking of classical Poincaré algebra symmetry

We noted at the beginning of the chapter that the presence of a constant matrix in the commutation rules (1.1) makes the canonical spacetime non-Lorentz-invariant. To show this in more details let us briefly recall the main features of the classical Poincaré algebra.

We shall call P_μ the four translation generators and $M_{\mu\nu}$ the antisymmetric matrix of the Lorentz sector generators, in terms of which the generators of spatial rotations, R_j, and boosts, N_j, read:

$$N_j = M_{0j} \qquad\qquad R_j = \frac{1}{2}\varepsilon_{jkl}M_{kl}. \tag{1.18}$$

The action of the algebra generators on a generic function $f(x)$ of the commutative coordinates x_μ is:

$$\begin{aligned} P_\mu f(x) &= i\partial_\mu f(x) \\ M_{\mu\nu}f(x) &= i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)f(x) \equiv ix_{[\mu}\partial_{\nu]}f(x), \end{aligned} \tag{1.19}$$

where ∂_μ stands for the derivative with respect to the coordinate x_μ. From these action rules it is easy to demonstrate that the Poincaré generators satisfy the Leibniz rule when acting on products of commutative functions $f(x)g(x)$:

$$\begin{aligned} P_\mu\left(f(x)g(x)\right) &\equiv i\partial_\mu\left(f(x)g(x)\right) = i\left(\partial_\mu f(x)\right)g(x) + if(x)\partial_\mu g(x) \\ &= \left(P_\mu f(x)\right)g(x) + f(x)P_\mu g(x) \\ M_{\mu\nu}\left(f(x)g(x)\right) &\equiv ix_{[\mu}\partial_{\nu]}\left(f(x)g(x)\right) = ix_{[\mu}\left(\partial_{\nu]}f(x)\right)g(x) + ix_{[\mu}f(x)\partial_{\nu]}g(x) \\ &= ix_{[\mu}\left(\partial_{\nu]}f(x)\right)g(x) + if(x)x_{[\mu}\partial_{\nu]}g(x) = \left(M_{\mu\nu}f(x)\right)g(x) + f(x)M_{\mu\nu}g(x). \end{aligned} \tag{1.20}$$

Notice that it has been of crucial importance for this demonstration the possibility, in the last line, of commuting the x_μ's with the function $f(x)$, in order to construct from the last expression of the third line the action of the generator $M_{\mu\nu}$ on the function $g(x)$.

From (1.19) also the commutation rules that define the Poincaré algebra easily follow:

$$\begin{aligned} [P_\mu, P_\nu] &= \;= 0 \\ [P_\alpha, M_{\mu\nu}] &= i\eta_{\alpha[\mu}P_{\nu]} \\ [M_{\mu\nu},\ M_{\alpha\beta}] &= i(\eta_{\beta[\mu}M_{\nu]\alpha} - \eta_{\alpha[\nu}M_{\mu]\beta}). \end{aligned} \tag{1.21}$$

The importance of the Poincaré algebra resides in the fact that it describes the symmetries of the Minkowski spacetime. This means that all the equations that we can write involving the

Minkowski coordinates have to be *covariant* under the action of the generators of the algebra in order to be well-written, i.e. if we apply the generators on both sides of the equations we have to still obtain equivalences. This is no more possible in canonical noncommutative spacetime, since even the most important equation that we can write using canonical coordinates, that is the equation that defines the commutation relations of the coordinates themselves (1.1), is not covariant under the action of the Lorentz sector generators:

$$\begin{aligned} M_{\mu\nu}[\hat{x}_\alpha,\hat{x}_\beta] &= [M_{\mu\nu}\hat{x}_\alpha,\hat{x}_\beta]+[\hat{x}_\alpha,M_{\mu\nu}\hat{x}_\beta] = [i\hat{x}_{[\mu}\eta_{\nu]\alpha},\hat{x}_\beta]+[\hat{x}_\alpha,i\hat{x}_{[\mu}\eta_{\nu]\beta}] = \theta_{\beta[\mu}\eta_{\nu]\alpha}-\theta_{\alpha[\mu}\eta_{\nu]\beta} \\ &\neq M_{\mu\nu}(i\theta_{\alpha\beta}) = 0. \end{aligned} \tag{1.22}$$

The problem resides in the fact that $\theta_{\mu\nu}$ has been defined to be a constant and coordinate independent matrix, while the left hand side of the equation (1.1) is a two-rank tensor. Now it is clear in what way the presence of a constant matrix breaks the invariance of canonical spacetime under the action of the classical Poincaré algebra, that thus can not be used to describe the symmetries of this kind of spacetime.

Since our purpose is to check if something similar to the Noether theorem holds for the symmetries of a field theory constructed over the canonical spacetime, we need first to find a set of "would-be-symmetries", i.e. we need to find a set of generators that could be a good candidate to describe the symmetries of this spacetime: first of all it is necessary for the equation (1.1) to be invariant under these new generators.

1.3 Translations and Lorentz sector transformations with classical action on Weyl-ordered fields

The simplest thing that one can try to do to define the symmetry algebra of the canonical noncommutative spacetime is to deform the classical Poincaré algebra, following the intuition that θ-Minkowski is only a deformation of the commutative Minkowski spacetime regulated by the parameters that appear in the matrix $\theta_{\mu\nu}$. Since in the section 1.1 we defined a set of maps that establishes a univocal correspondence between functions on the commutative spacetime and functions on the noncommutative one (generalised Weyl maps), we try to use this formalism to map not only commutative functions, but also the Poincaré generators acting on these functions. To be more precise, we shall assume the generators of the deformed algebra to have classical action through the (symmetric) Weyl map, i.e., calling $\hat{P}_\mu$, $\hat{M}_{\mu\nu}$ these generators, we shall set:

$$\begin{aligned} \hat{P}_\mu f(\hat{x}) &\equiv \hat{P}_\mu \Omega_w\left(f(x)\right) := \Omega_w\left(P_\mu f(x)\right) \\ \hat{M}_{\mu\nu} f(\hat{x}) &\equiv \hat{M}_{\mu\nu}\Omega_w\left(f(x)\right) := \Omega_w\left(M_{\mu\nu} f(x)\right), \end{aligned} \tag{1.23}$$

where the action of the generators P_μ, $M_{\mu\nu}$ on commutative functions has been defined in (1.19).

This is a quite conservative choice, since the generators defined in this way have still classical action on single coordinates:

$$\begin{aligned}\hat{P}_\mu \hat{x}_\alpha &= \Omega_w(P_\mu x_\alpha) = \Omega_w(i\eta_{\mu\alpha}) = i\eta_{\mu\alpha} \\ \hat{M}_{\mu\nu}\hat{x}_\alpha &= \Omega_w(M_{\mu\nu}x_\alpha) = \Omega_w(ix_{[\mu}\eta_{\nu]\alpha}) = i\hat{x}_{[\mu}\eta_{\nu]\alpha}\end{aligned} \tag{1.24}$$

and have the same commutation relations of the corresponding generators of the classical Poincaré algebra, as one can easily verify [3]:

$$\begin{aligned}\left[\hat{P}_\mu,\hat{P}_\nu\right]f(\hat{x}) &= \Omega_w([P_\mu,P_\nu]f(x))=0 \\ \left[\hat{P}_\alpha,\hat{M}_{\mu\nu}\right]f(\hat{x}) &= \Omega_w([P_\alpha,M_{\mu\nu}]f(x))=\Omega_w\left(i\eta_{\alpha[\mu}P_{\nu]}f(x)\right)=i\eta_{\alpha[\mu}\hat{P}_{\nu]}f(\hat{x}) \\ \left[\hat{M}_{\mu\nu},\hat{M}_{\alpha\beta}\right]f(\hat{x}) &= \Omega_w\left([M_{\mu\nu},M_{\alpha\beta}]f(x)\right)=\Omega_w\left(i(\eta_{\beta[\mu}M_{\nu]\alpha}-\eta_{\alpha[\nu}M_{\mu]\beta})f(x)\right)=i(\eta_{\beta[\mu}\hat{M}_{\nu]\alpha}-\eta_{\alpha[\nu}\hat{M}_{\mu]\beta})f(\hat{x}).\end{aligned} \tag{1.25}$$

Differences with the classical algebra arise obviously when considering the action on products of coordinates, i.e. on all functions different from a single coordinate, since ordering prescriptions become important. For example if we write the action on Weyl ordered (symmetric) exponentials, that, as we noted in Sec. 1.1, are a basis for the space of functions of noncommutative coordinates, we find:

$$\begin{aligned}\hat{P}_\mu e^{ik^\alpha \hat{x}_\alpha} &= \Omega_w\left(-k_\mu e^{ik^\alpha x_\alpha}\right) = -k_\mu e^{ik^\alpha \hat{x}_\alpha} \\ \hat{M}_{\mu\nu} e^{ik^\alpha \hat{x}_\alpha} &= \Omega_w\left(-x_{[\mu}k_{\nu]}e^{ik^\alpha x_\alpha}\right) = -k_{[\nu}\Omega_w\left(x_{\mu]}e^{ik^\alpha x_\alpha}\right)\end{aligned} \tag{1.26}$$

We see that we have to take the full-symmetrised version of what we would have obtained using the classical generators, that is for the generator $\hat{M}_{\mu\nu}$ we obtain $-k_{[\nu}\Omega_w\left(x_{\mu]}e^{ik^\alpha x_\alpha}\right)$ instead of $-k_{[\nu}x_{\mu]}e^{ik^\alpha x_\alpha}$.

It is more convenient to write the action in a differential form, noting that in the commutative case $x_\mu e^{ikx} = -i\frac{\partial}{\partial k^\mu}e^{ikx}$ and that, due to the linearity of the Weyl map[4], we can write $\Omega_w(\frac{\partial}{\partial k^\mu}e^{ikx}) = \frac{\partial}{\partial k^\mu}\Omega_w(e^{ikx}) = \frac{\partial}{\partial k^\mu}e^{ik\hat{x}}$. In this way the actions read:

$$\begin{aligned}\hat{P}_\mu e^{ik^\alpha \hat{x}_\alpha} &= -k_\mu e^{ik^\alpha \hat{x}_\alpha} \\ \hat{M}_{\mu\nu} e^{ik^\alpha \hat{x}_\alpha} &= i\frac{\partial}{\partial k^{[\mu}}k_{\nu]}e^{ik^\alpha \hat{x}_\alpha},\end{aligned} \tag{1.28}$$

[3]This is true in general for all the generators that have classical action through some Weyl map, as one can see generalising the demonstration used in this case.

[4]The linearity of the Weyl map lets us write, for a function $f(k,\hat{x}) \equiv \Omega_w(f_{(w)}(k,x))$, where for simplicity we here consider k to be a number and not a vector:

$$\begin{aligned}\Omega_w\left(\frac{\partial}{\partial k}f_{(w)}(k,x)\right) &\equiv \Omega_w\left(\lim_{\Delta k\to 0}\frac{f_{(w)}(k+\Delta k,x)-f_{(w)}(k,x)}{\Delta k}\right) \\ &= \lim_{\Delta k\to 0}\frac{\Omega_w(f_{(w)}(k+\Delta k,x))-\Omega(f_{(w)}(k))}{\Delta k} \equiv \frac{\partial}{\partial k^\mu}f(k,\hat{x})\end{aligned} \tag{1.27}$$

where the derivative with respect to k^μ does not act on k_ν, being the indices μ and ν anti-symmetrised. Obviously the use of the derivative with respect to k^μ does not removes ordering problems, but only hides them: for example, writing $\frac{\partial}{\partial k^\mu}e^{ik\hat{x}}$ instead of $\Omega_w\left(x_\mu e^{ikx}\right)$ is only a short notation, but does not modify the ordering convention of the final expression, because it is intended that the $\hat{x}^\mu$ taken down from the exponential $e^{ik\hat{x}}$ has to be put in a full-symmetrised form with respect to the exponential itself.

To formalise the difference between the classical Poincaré generators and the deformed ones as regards the action on products of functions let us calculate the action on products of Weyl ordered (symmetric) exponentials (that are good test functions as they constitute a basis for the space of noncommutative functions). Using the relations (1.28) and (1.12) we find some interesting issues. As regards the translation generators we find the action on the product of exponentials to be:

$$\begin{aligned} \hat{P}_\mu e^{ik^\alpha\hat{x}_\alpha}e^{iq^\beta\hat{x}_\beta} &= \hat{P}_\mu e^{i(k+q)^\alpha\hat{x}_\alpha}e^{-\frac{i}{2}k^\alpha\theta_{\alpha\beta}q^\beta} = -(k_\mu+q_\mu)e^{i(k+q)^\alpha\hat{x}_\alpha}e^{-\frac{i}{2}k^\alpha\theta_{\alpha\beta}q^\beta} \\ &= -(k_\mu+q_\mu)e^{ik^\alpha\hat{x}_\alpha}e^{iq^\beta\hat{x}_\beta} = \left(\hat{P}_\mu e^{ik^\alpha\hat{x}_\alpha}\right)e^{iq^\beta\hat{x}_\beta} + e^{ik^\alpha\hat{x}_\alpha}\left(\hat{P}_\mu e^{iq^\beta\hat{x}_\beta}\right), \end{aligned} \tag{1.29}$$

that is still classical in form (i.e. satisfies the Leibniz rule), while for the spatial rotation/boost generators we find:

$$\hat{M}_{\mu\nu}e^{ik^\alpha\hat{x}_\alpha}e^{iq^\beta\hat{x}_\beta} = \hat{M}_{\mu\nu}e^{i(k+q)^\alpha\hat{x}_\alpha}e^{-\frac{i}{2}k^\alpha\theta_{\alpha\beta}q^\beta} = i(k+q)_{[\nu}\left(\frac{\partial}{\partial(k+q)^{\mu]}}e^{i(k+q)^\alpha\hat{x}_\alpha}\right)e^{-\frac{i}{2}k^\alpha\theta_{\alpha\beta}q^\beta}. \tag{1.30}$$

Noting that:

$$\begin{aligned} \left(\frac{\partial}{\partial(k+q)^\mu}e^{i(k+q)^\alpha\hat{x}_\alpha}\right)e^{-\frac{i}{2}k^\alpha\theta_{\alpha\beta}q^\beta} &= \left(\frac{\partial}{\partial k^\mu}e^{i(k+q)^\alpha\hat{x}_\alpha}\right)e^{-\frac{i}{2}k^\alpha\theta_{\alpha\beta}q^\beta} \\ &= \left(\frac{\partial}{\partial k^\mu}+\frac{i}{2}\theta_{\mu\beta}q^\beta\right)\left(e^{i(k+q)^\alpha\hat{x}_\alpha}e^{-\frac{i}{2}k^\alpha\theta_{\alpha\beta}q^\beta}\right) \end{aligned} \tag{1.31}$$

and that a similar expression holds exchanging the rules of k and q, we can use the expression with the derivative with respect to k^μ for the part proportional to k_ν and the other (with k exchanged with q) for the part proportional to q_ν, to write:

$$\begin{aligned} \hat{M}_{\mu\nu}e^{ik^\alpha\hat{x}_\alpha}e^{iq^\beta\hat{x}_\beta} &= ik_{[\nu}\left[\left(\frac{\partial}{\partial k^{\mu]}}+\frac{i}{2}\theta_{\mu]\beta}q^\beta\right)e^{ik^\alpha\hat{x}_\alpha}e^{iq^\alpha\hat{x}_\alpha}\right]+ \\ &\quad +iq_{[\nu}\left[\left(\frac{\partial}{\partial q^{\mu]}}-\frac{i}{2}\theta_{\mu]\beta}k^\beta\right)e^{ik^\alpha\hat{x}_\alpha}e^{iq^\alpha\hat{x}_\alpha}\right] \\ &= \left(\hat{M}_{\mu\nu}e^{ik^\alpha\hat{x}_\alpha}\right)e^{iq^\alpha\hat{x}_\alpha}+e^{ik^\alpha\hat{x}_\alpha}\left(\hat{M}_{\mu\nu}e^{iq^\alpha\hat{x}_\alpha}\right)+ \\ &\quad +\frac{1}{2}\theta_{\beta[\mu}\left[\left(P_{\nu]}e^{ik^\alpha\hat{x}_\alpha}\right)\left(P^\beta e^{iq^\alpha\hat{x}_\alpha}\right)-\left(P^\beta e^{ik^\alpha\hat{x}_\alpha}\right)\left(P_{\nu]}e^{iq^\alpha\hat{x}_\alpha}\right)\right], \end{aligned} \tag{1.32}$$

that is a deformation of the Leibniz rule, since it contains a term that is the one coming from the Leibniz rule, plus a term that is proportional to the matrix $\theta_{\mu\nu}$ and involves the action of the translation parameters P_μ on both of the functions.

We thus have obtained an algebra whose structure (i.e. the commutators) is undeformed with respect to the classical Poincaré one, while the deformation appears evident in the action law on product of functions. Nevertheless this deformation is of a good kind, since the additional term contains only generators of the algebra itself, that thus remains closed also at this level. In a following section we will show that this kind of deformation can be formalised in more mathematical terms, since it results that we have just constructed a particular kind of *Hopf algebra* that comes out from the *twisting* of a Lie algebra. Now let us show that this algebra can be considered to be a good candidate to describe the symmetries of the θ-Minkowski spacetime.

1.4 Invariance under deformed Poincaré algebra

As we stressed at the end of section 1.2, for the algebra that we have constructed in the previous section to be a good candidate to describe the symmetries of the canonical spacetime the first requirement is that the commutation relations (1.1) have to be invariant under the action of the generators of the algebra. We find that this is indeed the case, in fact for the translation generators we have (using the action rule on products of functions (1.29) and the action on single coordinates (1.24)):

$$\hat{P}_\mu[\hat{x}_\alpha, \hat{x}_\beta] = [\hat{P}_\mu \hat{x}_\alpha, \hat{x}_\beta] + [\hat{x}_\alpha, \hat{P}_\mu \hat{x}_\beta] = [i\eta_{\mu\alpha}, \hat{x}_\beta] + [\hat{x}_\alpha, i\eta_{\mu\beta}] = 0 = \hat{P}_\mu\left(i\theta_{\alpha\beta}\right), \qquad (1.33)$$

where the last equation holds since the matrix $\theta_{\alpha\beta}$ is coordinate independent, as we stressed in the introduction of this chapter. For the Lorentz sector generators we have, using (1.32) and (1.24) :

$$\begin{aligned}
\hat{M}_{\mu\nu}[\hat{x}_\alpha, \hat{x}_\beta] &= [\hat{M}_{\mu\nu}\hat{x}_\alpha, \hat{x}_\beta] + [\hat{x}_\alpha, \hat{M}_{\mu\nu}\hat{x}_\beta] + \frac{1}{2}\theta_{\rho[\mu}\left(\{\hat{P}_{\nu]}\hat{x}_\alpha, \hat{P}^\rho \hat{x}_\beta\} - \{\hat{P}^\rho \hat{x}_\alpha, \hat{P}_{\nu]}\hat{x}_\beta\}\right) \\
&= [i\hat{x}_{[\mu}\eta_{\nu]\alpha}, \hat{x}_\beta] + [\hat{x}_\alpha, i\hat{x}_{[\mu}\eta_{\nu]\beta}] + \frac{1}{2}\theta_{\rho[\mu}\left(\{i\eta_{\nu]\alpha}, i\eta^\rho_\beta\} - \{i\eta^\rho_\alpha, i\eta_{\nu]\beta}\}\right) \\
&= \theta_{\beta[\mu}\eta_{\nu]\alpha} - \theta_{\alpha[\mu}\eta_{\nu]\beta} - [\theta_{\beta[\mu}\eta_{\nu]\alpha} - \theta_{\alpha[\mu}\eta_{\nu]\beta}] = 0 = \hat{M}_{\mu\nu}(i\theta_{\alpha\beta}) \qquad (1.34)
\end{aligned}$$

When we noticed that the Lorentz sector of the classical Poincaré algebra did not leave invariant the commutation rules (1.1) that define θ-Minkowski spacetime we talked of breaking of Lorentz symmetry. Now that we have discovered that it is possible to simply deform the Poincaré generators to obtain an algebra under which the space is invariant we are led to talk of a deformation of Lorentz symmetry instead of a breaking.

1.5 Hopf algebras and twist

In the previous sections we have found the properties of the algebra of symmetries of canonical spacetime in a constructive way, i.e. we have defined a deformation of the action of the classical Poincaré generators and we have deduced the properties of the new resulting algebra. Now we want to show that the findings of the previous sections can be codified in a more mathematical language. First of all we shall dedicate a section to a brief review about some algebraic tools, principally to define the notations, for a deeper treatise see [7, 8, 9, 10]. In the next section we shall show the connection between our findings and the ones coming from the use of these mathematical structures.

Algebras and coalgebras Given a field $\mathcal{K}$, an *algebra* $\mathcal{A}$ is a vector space over $\mathcal{K}$ provided with a map (*product*) $\mathtt{m} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ that satisfies the associative property:

$$\mathtt{m}(\mathtt{m}(a \otimes b) \otimes c) = \mathtt{m}(a \otimes \mathtt{m}(b \otimes c)), \tag{1.35}$$

and a map $\eta : \mathcal{K} \to \mathcal{A}$ such that:

$$\eta(1) = \mathcal{I}, \tag{1.36}$$

where $\mathcal{I}$ is the identity element of the algebra.
A *coalgebra* $\mathcal{C}$ is a vector space over $\mathcal{K}$ provided with a map (*coproduct*) $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ that satisfies the coassociative property:

$$c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}, \tag{1.37}$$

where we used the notation $\Delta c = c_{(1)} \otimes c_{(2)}$ (called Sweedler notation), standing for $\Delta c = \sum_i c^i_{(1)} \otimes c^i_{(2)}$, and a map (*counit*) $\epsilon : \mathcal{C} \to \mathcal{K}$ such that:

$$\epsilon(c_{(1)})c_{(2)} = c_{(1)}\epsilon(c_{(2)}) = c. \tag{1.38}$$

Bialgebras and Hopf algebras A *bialgebra* $\mathcal{B}$ is both an algebra and a coalgebra, with the compatibility conditions:

$$\Delta(\mathtt{m}(b \otimes c)) = \Delta(b)\Delta(c), \quad \Delta(\mathcal{I}) = \mathcal{I} \otimes \mathcal{I}, \tag{1.39}$$

$$\epsilon(\mathtt{m}(b \otimes c)) = \epsilon(b)\epsilon(c), \quad \epsilon(\mathcal{I}) = \mathcal{I}, \tag{1.40}$$

where in the right hand side of the first condition the product is defined in $\mathcal{B}\otimes\mathcal{B}$ in the following way: $(b \otimes c)(b' \otimes c') = \mathtt{m}(b \otimes b') \otimes \mathtt{m}(c \otimes c')$.
A *Hopf algebra* $\mathcal{H}$ is a bialgebra endowed with a map (*antipode*) $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ such that:

$$\mathtt{m}(\mathcal{S}(h_{(1)}) \otimes h_{(2)}) = \mathtt{m}(h_{(1)} \otimes \mathcal{S}(h_{(2)})) = \eta(\epsilon(h)). \tag{1.41}$$

Twist Given a Hopf algebra $\mathcal{H}$, a *twist* $\mathcal{F}$ is an invertible element of $\mathcal{H}\otimes\mathcal{H}$ such that, if $\mathcal{F}=\mathcal{F}^{(1)}\otimes\mathcal{F}^{(2)}$ (summation omitted), then:

$$\left(\mathcal{F}^{(1)}\otimes\mathcal{F}^{(2)}\otimes 1\right)(\Delta\otimes\mathcal{I})(\mathcal{F})=\left(1\otimes\mathcal{F}^{(1)}\otimes\mathcal{F}^{(2)}\right)(\mathcal{I}\otimes\Delta)(\mathcal{F}), \tag{1.42}$$

and

$$(\epsilon\otimes\mathcal{I})(\mathcal{F})=1=(\mathcal{I}\otimes\epsilon)(\mathcal{F}). \tag{1.43}$$

A *twisted Hopf algebra* $\mathcal{H}_{\mathcal{F}}$ can be obtained from a Hopf algebra $\mathcal{H}$ modifying the coproduct and the antipode of the original algebra in the following way:

$$\Delta_{\mathcal{F}}=\mathcal{F}\Delta\mathcal{F}^{-1} \qquad \text{and} \qquad \mathcal{S}_{\mathcal{F}}=\left[\mathtt{m}(\mathcal{I}\otimes\mathcal{S})(\mathcal{F})\right]\,\mathcal{S}\,\left[\mathtt{m}(\mathcal{S}\otimes\mathcal{I})(\mathcal{F}^{-1})\right]. \tag{1.44}$$

To specify the relation between the deformed Poincaré algebra defined in the previous section and the noncommutative algebra of coordinates of the θ-Minkowski spacetime it is also necessary to review briefly some notions about representations.

Representations Given an algebra $\mathcal{A}$ over the field $\mathcal{K}$ and a vector space $\mathcal{V}$, a *left representation* of $\mathcal{A}$ on $\mathcal{V}$ is a pair $(\mathcal{V},\rho)$, where $\rho:\mathcal{A}\otimes\mathcal{V}\to\mathcal{V}$ is a map such that:

$$\begin{aligned}\rho_{a\cdot b}(v) &= \rho_a(\rho_b(v))\\ \rho_{\mathcal{I}}(v) &= v,\end{aligned} \tag{1.45}$$

where $\rho_a(v)\equiv\rho(a\otimes v)$ and $a,b\in\mathcal{A}$, $v\in\mathcal{V}$ and finally $a\cdot b$ stands for the algebra multiplication $\mathtt{m}(a\otimes b)$.

Given a Hopf algebra $\mathcal{H}$ over a field $\mathcal{K}$ and an algebra A over $\mathcal{K}$, a *left algebra representation* of $\mathcal{H}$ is a *left action* of $\mathcal{H}$ on $\mathcal{A}$, i.e. is a pair $(\mathcal{A},\rho)$, where $\rho:\mathcal{H}\otimes\mathcal{A}\to\mathcal{A}$ is a linear map such that:

$$\begin{aligned}\rho_{h\cdot k}(a) &= \rho_h(\rho_k(a))\\ \rho_{\mathcal{I}}(a) &= a,\end{aligned} \tag{1.46}$$

that satisfies the additional conditions (if $\Delta h=h_{(1)}\otimes h_{(2)}$):

$$\begin{aligned}\rho_h(a\cdot b) &= \rho_{h_{(1)}}(a)\rho_{h_{(2)}}(b)\\ \rho_h(1) &= \epsilon(h)1.\end{aligned} \tag{1.47}$$

Since ρ defines the *left action* of the Hopf algebra over the algebra $\mathcal{A}$, we shall also write in the following $\rho_h(a)=ha$, intending the action of h on a. Note that the first of the above conditions regards the compatibility of the coproduct of the Hopf algebra $\mathcal{H}$ with the product of the algebra $\mathcal{A}$.

1.6 Twisting the classical Poincaré Lie algebra

In this section we shall take a Lie algebra [5] (in particular the Poincaré algebra $\mathcal{P}$) and the vector space on which its universal enveloping algebra[6] ($\mathcal{U}(\mathcal{P})$) is represented (the Minkowski spacetime $\mathcal{M}$), we shall make the algebra a Hopf algebra (in order to make the vector space into a commutative Lie algebra), then we shall deform the Hopf algebra by a twist and finally we shall see how the vector space has to be coherently modified in order to be the algebra on which the deformed Hopf algebra is represented. In this way we shall construct a twisted Poincaré algebra (in which the twist shall be regulated by the matrix $\theta_{\mu\nu}$) represented over a deformed (no more commutative) Minkowski spacetime (whose noncommutativity parameter shall result to be again the matrix $\theta_{\mu\nu}$), we shall see how the twisting is connected to the noncommutativity of the deformed Minkowski spacetime and we shall show that these objects are exactly the same of the ones constructed in the previous sections.

Let $\mathcal{P}$ be the Lie Poincaré algebra, defined by the commutation relations (1.21) and $\mathcal{U}(\mathcal{P})$ its universal enveloping algebra . Be $\mathcal{M}$ the Minkowski spacetime on which $\mathcal{U}(\mathcal{P})$ is represented. $\mathcal{U}(\mathcal{P})$ becomes a (trivial) Hopf algebra if we define the coproduct of a generic element $g \in \mathcal{U}(\mathcal{P})$ to be:

$$\Delta g \equiv g \otimes 1 + 1 \otimes g, \tag{1.48}$$

while the counit and antipode are defined in the following way:

$$\epsilon(g) \equiv 0 \qquad S(g) \equiv -g. \tag{1.49}$$

Now the vector space $\mathcal{M}$ can be made into an algebra, still remaining the space over which $\mathcal{U}(\mathcal{P})$ is represented; this thanks to the fact that we have defined a rule of action of $\mathcal{U}(\mathcal{P})$ over "products" of elements of $\mathcal{M}$, where the product has still to be defined.To do this last thing, first of all we make $\mathcal{M}$ into a Lie algebra, defining the brackets:

$$[x_i, x_j] \equiv 0, \tag{1.50}$$

where $x_i \in \mathcal{M}$ are elements of the basis of the Minkowski spacetime. Now we consider the universal enveloping algebra of $\mathcal{M}$, $\mathcal{U}(\mathcal{M})$, generated by the relations

$$x_i \otimes x_j - x_j \otimes x_i = 0. \tag{1.51}$$

[5] We remind that a Lie algebra $\mathcal{L}$ is a n-dimensional vector space on which is defined a bracket $[.,.] : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ that is antisymmetric and bilinear in its arguments and satisfies the Jacobi identities $[g,[h,k]] + [k,[g,h]] + [h,[k,g]] = 0$. Note also that, since the brakets are an element of the Lie algebra itself, they have to be of the form: $[g_i, g_j] = ic_{ijk}g_k$, where $g_i, i = \{1, ..., n\}$ are the basis element of the vecor space.

[6] The universal enveloping algebra (UEA) $\mathcal{U}(\mathcal{L})$ of a Lie algebra $\mathcal{L}$ is the quotient of the algebra of the tensor products of elements of $\mathcal{L}$ (called $\mathcal{T}(\mathcal{L})$) and the two-sided ideal in $\mathcal{T}(\mathcal{L})$ generated by the relations: $g_i \otimes g_j - g_j \otimes g_i - ic_{ijk}g_k = 0$.

This algebra is commutative and associative if we define its product to be the tensor product $x_i \otimes x_j$. Now we see that we can define the representation of the Hopf algebra $\mathcal{U}(\mathcal{P})$ on the algebra $\mathcal{U}(\mathcal{M})$ as stated above in the previous section; we have only to check that the generating relations of the algebra $\mathcal{U}(\mathcal{M})$ are left unchanged by the action of the map ρ_h, $h \in \mathcal{U}(\mathcal{P})$ (this corresponds to the first requirement of (1.47)):

$$\rho_h(x_i x_j - x_j x_i) = \rho_h(x_i)x_j + x_i\rho_h(x_j) - \rho_h(x_j)x_i - x_j\rho_h(x_i) = 0, \tag{1.52}$$

where the last equivalence holds since $\rho_h(x_i)$ is still an element of the commutative algebra $\mathcal{U}(\mathcal{M})$.

If we now twist the Hopf algebra $\mathcal{U}(\mathcal{P})$, the coproducts of its elements are deformed as stated in (1.44). It is now clear that the product of the algebra on which the Hopf algebra is represented has to be coherently deformed in order to leave the condition (1.47) still satisfied. In fact, after the twisting the equation (1.47) would become (a and b are elements of $\mathcal{U}(\mathcal{M})$):

$$\rho_h(a \cdot b) = \rho_{\mathcal{F}^{(1)}h_{(1)}\mathcal{F}^{-1(1)}}(a)\rho_{\mathcal{F}^{(2)}h_{(2)}\mathcal{F}^{-1(2)}}(b) \tag{1.53}$$

that in principle could no more be true. In order for the above equation to still be valid, the product $a \cdot b$ should also be modified, defining the new product $a \star_{\mathcal{F}} b$ such that:

$$a \star_{\mathcal{F}} b \equiv \mathcal{F}^{-1}(a \cdot b) = (\mathcal{F}^{-1^{(1)}}a) \cdot (\mathcal{F}^{-1^{(2)}}b). \tag{1.54}$$

It is easy to verify that this product satisfies the associative property (1.35), so that it is well-defined. The action of h an the deformed product of elements of $\mathcal{U}(\mathcal{M})$ is:

$$\begin{aligned}\rho_h(a \star_{\mathcal{F}} b) &= h\,\mathcal{F}^{-1}(a \cdot b) = \mathcal{F}^{-1}\mathcal{F}h\mathcal{F}^{-1}(a \cdot b) = \mathcal{F}^{-1}\Delta_T h(a \otimes b) = h^T_{(1)}(a) \star_{\mathcal{F}} h^T_{(2)}(b) \\ &\equiv \rho_{h^T_{(1)}}(a) \star_{\mathcal{F}} \rho_{h^T_{(2)}}(b),\end{aligned} \tag{1.55}$$

where in the second equivalence we have just inserted the unit element in the form $\mathcal{F}^{-1}\mathcal{F}$ and with Δ_T we indicate the twisted coproduct. This equation is the analogous of (1.47), and it is valid for construction.

Let us now see an example of twisting. Let us consider $\mathcal{U}(\mathcal{P})$ as a Hopf algebra (as was explained above) with the representation algebra $\mathcal{U}(\mathcal{M})$ (the algebra of functions on the Minkowski spacetime). We are going to twist $\mathcal{U}(\mathcal{P})$ through the twist element

$$\mathcal{F} \equiv e^{\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu}, \tag{1.56}$$

whose inverse is $\mathcal{F}^{(-1)} = e^{-\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu}$.

The coproducts of the generators of $\mathcal{U}(\mathcal{P})$ become:

$$\Delta_T P_\mu = e^{\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu}[P_\mu \otimes 1 + 1 \otimes P_\mu]e^{-\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu} = [P_\mu \otimes 1 + 1 \otimes P_\mu] \equiv \Delta P_\mu, \tag{1.57}$$

where the second equivalence holds since the translation generators P_μ commute one with each other. As regards the Lorentz sector generators:

$$\begin{aligned}
\Delta_T M_{\alpha\beta} &= e^{\frac{i}{2}\theta^{\mu\nu}P_\mu\otimes P_\nu}[M_{\alpha\beta}\otimes 1 + 1\otimes M_{\alpha\beta}]e^{-\frac{i}{2}\theta^{\mu\nu}P_\mu\otimes P_\nu}\\
&= \sum_{k=0}^{\infty}\left(\frac{i}{2}\right)^k\frac{(\theta^{\mu\nu})^k}{k!}P_\mu^{\ k}\otimes P_\nu^{\ k}[M_{\alpha\beta}\otimes 1 + 1\otimes M_{\alpha\beta}]\sum_{n=0}^{\infty}\left(-\frac{i}{2}\right)^n\frac{(\theta^{\rho\sigma})^n}{n!}P_\rho^{\ n}\otimes P_\sigma^{\ n}\\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n\left(\frac{i}{2}\right)^{k+n}\frac{(\theta^{\mu\nu})^k}{k!}\frac{(\theta^{\rho\sigma})^n}{n!}\left[P_\mu^{\ k}M_{\alpha\beta}P_\rho^{\ n}\otimes P_\nu^{\ k}P_\sigma^{\ n} + P_\mu^{\ k}P_\rho^{\ n}\otimes P_\nu^{\ k}M_{\alpha\beta}P_\sigma^{\ n}\right].
\end{aligned} \tag{1.58}$$

We have to commute $M_{\alpha\beta}$ with $P_\mu^{\,k}$. To evaluate the commutator we observe that:

$$\begin{aligned}
[P_\mu, M_{\alpha\beta}] &= i\eta_{\mu[\alpha}P_{\beta]}\\
\left[P_\mu^{\ 2}, M_{\alpha\beta}\right] &= 2i\eta_{\mu[\alpha}P_{\beta]}P_\mu\\
\left[P_\mu^{\ 3}, M_{\alpha\beta}\right] &= 3i\eta_{\mu[\alpha}P_{\beta]}P_\mu^{\ 2}\\
\Rightarrow \left[P_\mu^{\ k}, M_{\alpha\beta}\right] &= ki\eta_{\mu[\alpha}P_{\beta]}P_\mu^{\ k-1},
\end{aligned} \tag{1.59}$$

where there is no summation over the index μ. So the above expression becomes:

$$\begin{aligned}
\Delta_T M_{\alpha\beta} &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n\left(\frac{i}{2}\right)^{k+n}\frac{(\theta^{\mu\nu})^k}{k!}\frac{(\theta^{\rho\sigma})^n}{n!}\Big[M_{\alpha\beta}P_\mu^{\ k}P_\rho^{\ n}\otimes P_\nu^{\ k}P_\sigma^{\ n} +\\
&\quad + k\, i\,\eta_{\mu[\alpha}P_{\beta]}P_\mu^{\ k-1}P_\rho^{\ n}\otimes P_\nu^{\ k}P_\sigma^{\ n} + P_\mu^{\ k}P_\rho^{\ n}\otimes P_\nu^{\ k}P_\sigma^{\ n}M_{\alpha\beta} +\\
&\quad - n\, i\,\eta_{\sigma[\alpha}P_\mu^{\ k}P_\rho^{\ n}\otimes P_\nu^{\ k}P_\sigma^{\ n-1}P_{\beta]}\Big]\\
&= M_{\alpha\beta}\otimes 1 + 1\otimes M_{\alpha\beta} - \frac{1}{2}\theta^{\mu\nu}\eta_{\mu[\alpha}P_{\beta]}\otimes P_\nu - \frac{1}{2}\theta^{\rho\sigma}\eta_{\sigma[\alpha}P_\rho\otimes P_{\beta]}.
\end{aligned} \tag{1.60}$$

The coproducts just obtained are the same of those that characterise the generators defined in section 1.3, see equations (1.29) and (1.32). Since also the commutators are the same (and in particular remain classical in both cases) we deduce that these two sets of operators are the same: we have found an other way to construct the θ-Poincaré algebra, that so results to be a twisted algebra. Let us now find the deformed product corresponding to this twist element that has to be associated to the algebra $\mathcal{U}(\mathcal{M})$:

$$f(x)\star_{\mathcal{F}} g(x) = \mathcal{F}^{(-1)}(f(x)\otimes g(x)) = e^{-\frac{i}{2}\theta^{\mu\nu}P_\mu\otimes P_\nu}(f(x)\otimes g(x)). \tag{1.61}$$

This is the same of the star product that we used inside the symmetric Weyl map to map the product of two noncommutative functions, Cf. equations (1.14) and (1.15). We notice also that with this deformed product the commutator of elements of $\mathcal{U}(\mathcal{M})$ evaluated on the coordinates x_μ is:

$$[x_\alpha, x_\beta]_\mathcal{F} \equiv x_\alpha \star_\mathcal{F} x_\beta - x_\beta \star_\mathcal{F} x_\alpha = \frac{i}{2}\theta^{\mu\nu}\eta_{\mu\alpha}\eta_{\nu\beta} - \frac{i}{2}\theta^{\mu\nu}\eta_{\mu\beta}\eta_{\nu\alpha} = i\theta_{\alpha\beta}, \tag{1.62}$$

that are the same of θ-Minkowski.

We have thus found a correspondence between the construction of this section and the one of the previous section: the θ-Poincaré algebra results just from a twisting of the classical one through the twist element $\mathcal{F}$ defined above. The θ-Minkowski spacetime is the space on which the algebra of functions of Minkowski coordinates with a product deformed by the same twist is constructed.

1.7 Another possible ordering convention

In Sec. 1.3 we assumed the generators of the deformed Poincaré algebra to have classical action through the symmetric Weyl map . This is a quite arbitrary choice, so that one could wonder what happens if we define the generators with a different convention. For example we could define a new set of generators, that we shall call $\hat{P}^\mu_{(1)}$ and $\hat{M}^{(1)}_{\mu\nu}$, to have classical action through the map Ω_1, defined in (1.7):

$$\begin{aligned} \hat{P}^{(1)}_\mu f(\hat{x}) &\equiv \hat{P}^{(1)}_\mu \Omega_1\left(f_{(1)}(x)\right) := \Omega_1\left(P_\mu f_{(1)}(x)\right) \\ \hat{M}^{(1)}_{\mu\nu} f(\hat{x}) &\equiv \hat{M}^{(1)}_{\mu\nu} \Omega_1\left(f_{(1)}(x)\right) := \Omega_1\left(M_{\mu\nu} f_{(1)}(x)\right). \end{aligned} \tag{1.63}$$

It is easy to show that also this set of generators has still classical action on single coordinates and obeys the same commutation relations of the classical Poincaré algebra generators. Nevertheless these generators do not coincide with those defined through the symmetric Weyl map, since for example they have different action on symmetric exponentials. In fact, writing $e^{ik\hat{x}} = e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}e^{\frac{i}{2}k^Ak^1\theta_{A1}}$ thanks to the BCH formula

$$\begin{aligned} \hat{P}^{(1)}_\mu e^{ik^\alpha\hat{x}_\alpha} &= \Omega_1\left(-k_\mu e^{ik^\alpha x_\alpha}e^{\frac{i}{2}k^Ak^1\theta_{A1}}\right) = -k_\mu e^{ik^\alpha\hat{x}_\alpha}e^{\frac{i}{2}k^Ak^1\theta_{A1}} \\ \hat{M}^{(1)}_{\mu\nu} e^{ik^\alpha\hat{x}_\alpha} &= \Omega_1\left(-x_{[\mu}k_{\nu]}e^{ik^\alpha x_\alpha}e^{\frac{i}{2}k^Ak^1\theta_{A1}}\right) = -k_{[\nu}\Omega_1\left(x_{\mu]}e^{ik^\alpha x_\alpha}e^{\frac{i}{2}k^Ak^1\theta_{A1}}\right). \end{aligned} \tag{1.64}$$

Writing the last expression in a differential form, as it was done for the symmetric case:

$$\begin{aligned} \hat{M}^{(1)}_{\mu\nu} e^{ik^\alpha\hat{x}_\alpha} &= -k_{[\nu}\Omega_1\left([-i\frac{\partial}{\partial k^{\mu]}}e^{ik^\alpha x_\alpha}]e^{\frac{i}{2}k^Ak^1\theta_{A1}}\right) \\ &= -k_{[\nu}\Omega_1\left([-i\frac{\partial}{\partial k^{\mu]}} - \frac{1}{2}(\delta^A_{\mu]}k^1 + \delta^1_{\mu]}k^A)\theta_{A1}](e^{ik^\alpha x_\alpha}e^{\frac{i}{2}k^Ak^1\theta_{A1}})\right) \\ &= k_{[\nu}[i\frac{\partial}{\partial k^{\mu]}} + \frac{1}{2}(\delta^A_{\mu]}k^1 + \delta^1_{\mu]}k^A)\theta_{A1}]e^{ik^\alpha\hat{x}_\alpha}. \end{aligned} \tag{1.65}$$

Since the symmetric exponentials constitute a basis for the space of noncommutative functions, we can deduce from the comparison of the above equation and the first of (1.64) with (1.28) that:

$$\begin{aligned}\hat{P}^{(1)}_\mu &= \hat{P}_\mu \\ \hat{M}^{(1)}_{\mu\nu} &= \hat{M}_{\mu\nu} + \frac{1}{2}\hat{P}_{[\nu}(\delta^A_{\mu]}\hat{P}^1 + \delta^1_{\mu]}\hat{P}^A)\theta_{A1}.\end{aligned} \tag{1.66}$$

As regards coproducts, while the coproduct of $P^{(1)}_\mu$ remains obviously trivial, the coproduct of $M^{(1)}_{\mu\nu}$ becomes, using the preceding equation:

$$\begin{aligned}\Delta\hat{M}^{(1)}_{\mu\nu} &= \Delta\hat{M}_{\mu\nu} + \frac{1}{2}\Delta\hat{P}_{[\nu}\left(\delta^A_{\mu]}\Delta\hat{P}^1 + \delta^1_{\mu]}\hat{\Delta}P^A\right)\theta_{A1} \\ &= \hat{M}_{\mu\nu}\otimes 1 + 1\otimes\hat{M}_{\mu\nu} + \frac{1}{2}\theta_{\beta[\mu}\left(\hat{P}_{\nu]}\otimes\hat{P}^\beta - \hat{P}^\beta\otimes\hat{P}_{\nu]}\right) + \\ &\quad + \frac{1}{2}\theta_{A1}\left[\delta^A_{[\mu}\left(\hat{P}_{\nu]}\hat{P}^1\otimes 1 + 1\otimes\hat{P}_{\nu]}\hat{P}^1 + \hat{P}_{\nu]}\otimes\hat{P}^1 + \hat{P}^1\otimes\hat{P}_{\nu]}\right) + \right. \\ &\quad \left. + \delta^1_{[\mu}\left(\hat{P}_{\nu]}\hat{P}^A\otimes 1 + 1\otimes\hat{P}_{\nu]}\hat{P}^A + \hat{P}_{\nu]}\otimes\hat{P}^A + \hat{P}^A\otimes\hat{P}_{\nu]}\right)\right] \\ &= \hat{M}^{(1)}_{\mu\nu}\otimes 1 + 1\otimes\hat{M}^{(1)}_{\mu\nu} + \frac{1}{2}\theta_{\beta[\mu}\left(\hat{P}_{\nu]}\otimes\hat{P}^\beta - \hat{P}^\beta\otimes\hat{P}_{\nu]}\right) + \\ &\quad + \frac{1}{2}\theta_{A1}\left[\delta^A_{[\mu}\left(\hat{P}_{\nu]}\otimes\hat{P}^1 + \hat{P}^1\otimes\hat{P}_{\nu]}\right) + \delta^1_{[\mu}\left(\hat{P}_{\nu]}\otimes\hat{P}^A + \hat{P}^A\otimes\hat{P}_{\nu]}\right)\right],\end{aligned} \tag{1.67}$$

where in the first step we used the property of the coproduct (1.39) and in the last one we have expressed back $\hat{M}_{\mu\nu}$ in terms of $\hat{M}^{(1)}_{\mu\nu}$.

The fact that the new generators can be written in function of the symmetric generators without the participation of any other operator implies that the two sets of generators are actually two different basis for the same deformed Poincaré algebra. It is quite easy to show that any other set of generators of the kind $\hat{G}f(\hat{x}) = \Omega_i(Gf_i(x))$, where Ω_i is a generic Weyl map such that $\Omega_i(f_i(x)) = f(\hat{x})$ and G are the generators of the Poincaré algebra, is nothing else but a new basis of the same twisted Poincaré algebra. So in the following we shall talk of basis when referring to a certain set of generators that are the deformation of those of the classical Pincaré algebra through a classical action in some Weyl map. In particular we shall talk of symmetric and $\hat{x}_1$-to-the-right basis, intending the basis of generators of the twisted Poincaré algebra that have classical action, respectively, through the symmetric Weyl map and through the map that takes the coordinate x_1 on the right (i.e. Ω_1).

As a final remark we shall derive the twist element associated to this basis of generators from the star product to be used inside the Ω_1 Weyl map and shall show that it twists the generators of the classical Poincaré algebra leading to the x_1-to-the-right basis of generators.

As explained in the previous section, the twist element $\mathcal{F}_1$ can be derived studying the form of the star product. For the Weyl map Ω_1 we showed in section 1.1 that the star product of two exponentials is of the form (Cf. (1.16)):

$$e^{ikx} \star e^{iqx} = e^{i(k+q)x} e^{-\frac{i}{2}k^A q^B \theta_{AB}} e^{-ik^1 q^A \theta_{1A}}. \tag{1.68}$$

Since we know that this star product has to be of the form:

$$e^{ikx} \star e^{iqx} = \mathcal{F}_1^{-1}(e^{ikx} \otimes e^{iqx}), \tag{1.69}$$

we deduce that:

$$\mathcal{F}_1^{-1} = e^{-\frac{i}{2}\theta_{AB} P^A \otimes P^B} e^{-i\theta_{1A} P^1 \otimes P^A}, \tag{1.70}$$

and so the twist element is:

$$\mathcal{F}_1 = e^{\frac{i}{2}\theta_{AB} P^A \otimes P^B} e^{i\theta_{1A} P^1 \otimes P^A}. \tag{1.71}$$

Let us verify that with the use of this twist element we can obtain the right coproducts for the generators of the $\hat{x}_1$-to-the-right basis, i.e. a trivial coproduct for translation generators and the one written in (1.67) for the Lorentz sector generators.

As regards the $\hat{P}^{(1)}_\mu$'s, the triviality of the coproduct is immediately demonstrated. In fact from the twisting we obtain:

$$\begin{aligned} \Delta_{\mathcal{F}_1} P_\mu &= \mathcal{F}_1 \Delta P_\mu \mathcal{F}_1^{-1} = e^{\frac{i}{2}\theta_{AB} P^A \otimes P^B} e^{i\theta_{1A} P^1 \otimes P^A} [P_\mu \otimes 1 + 1 \otimes P_\mu] e^{-\frac{i}{2}\theta_{AB} P^A \otimes P^B} e^{-i\theta_{1A} P^1 \otimes P^A} \\ &= [P_\mu \otimes 1 + 1 \otimes P_\mu], \end{aligned} \tag{1.72}$$

where the last equivalence holds since the translation generators commute.

For the Lorentz sector generators $M_{\mu\nu}$ the calculation is less trivial:

$$\begin{aligned} \Delta_{\mathcal{F}_1} M_{\mu\nu} &= e^{\frac{i}{2}\theta_{AB} P^A \otimes P^B} e^{i\theta_{1A} P^1 \otimes P^A} [M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}] e^{-\frac{i}{2}\theta_{AB} P^A \otimes P^B} e^{-i\theta_{1A} P^1 \otimes P^A}. \\ &= e^{i\theta_{1A} P^1 \otimes P^A} e^{\frac{i}{2}\theta_{AB} P^A \otimes P^B} [M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}] e^{-\frac{i}{2}\theta_{AB} P^A \otimes P^B} e^{-i\theta_{1A} P^1 \otimes P^A}, \end{aligned} \tag{1.73}$$

where in the last step we used the commutativity of the translation parameters. Let us first calculate $\Delta'_{\mathcal{F}_1}(M_{\mu\nu}) \equiv e^{\frac{i}{2}\theta_{AB} P^A \otimes P^B} [M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}] e^{-\frac{i}{2}\theta_{AB} P^A \otimes P^B}$:

$$\begin{aligned} \Delta'_{\mathcal{F}_1}(M_{\mu\nu}) &= \sum_{k=0}^{\infty} \left(\frac{i}{2}\right)^k \frac{(\theta^{AB})^k}{k!} P_A^k \otimes P_B^k [M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}] \sum_{n=0}^{\infty} \left(-\frac{i}{2}\right)^n \frac{(\theta^{CD})^n}{n!} P_C^n \otimes P_D^n \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} (-1)^n \left(\frac{i}{2}\right)^{k+n} \frac{(\theta^{AB})^k}{k!} \frac{(\theta^{CD})^n}{n!} [P_A^k M_{\mu\nu} P_C^n \otimes P_B^k P_D^n + P_A^k P_C^n \otimes P_B^k M_{\mu\nu} P_D^n] \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} (-1)^n \left(\frac{i}{2}\right)^{k+n} \frac{(\theta^{AB})^k}{k!} \frac{(\theta^{CD})^n}{n!} [M_{\mu\nu} P_A^k P_C^n \otimes P_B^k P_D^n + \\ &\quad + i\, k\, \eta_{A[\mu} P_{\nu]} P_A^{k-1} \otimes P_B^k P_D^n + P_A^k P_C^n \otimes P_B^k P_D^n M_{\mu\nu} - i\, n\, \eta_{D[\mu} P_A^k P_C^n \otimes P_B^k P_{\nu]} P_D^{n-1}] \\ &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} - \frac{1}{2}\theta^{AB} \eta_{A[\mu} P_{\nu]} \otimes P_B - \frac{1}{2}\theta^{CD} \eta_{D[\mu} P_C \otimes P_{\nu]}. \end{aligned} \tag{1.74}$$

The final expression of the coproduct is then:

$$
\begin{aligned}
\Delta_{\mathcal{F}_1} M_{\mu\nu} &= e^{i\theta_{1A}P^1\otimes P^A}\Delta'_{\mathcal{F}_1}(M_{\mu\nu})e^{-i\theta_{1A}P^1\otimes P^A} = \sum_{k=0}^{\infty}\left(\frac{i}{2}\right)^k \frac{(\theta^{1A})^k}{k!}P_1^{\ k}\otimes P_A^{\ k}[M_{\mu\nu}\otimes 1+ \\
&\quad +1\otimes M_{\mu\nu} - \frac{1}{2}\theta^{AB}\eta_{A[\mu}P_{\nu]}\otimes P_B - \frac{1}{2}\theta^{CD}\eta_{D[\mu}P_C\otimes P_{\nu]}]\sum_{n=0}^{\infty}\left(-\frac{i}{2}\right)^n \frac{(\theta^{1B})^n}{n!}P_1^{\ n}\otimes P_B^{\ n} \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n\left(\frac{i}{2}\right)^{k+n}\frac{(\theta^{1A})^k}{k!}\frac{(\theta^{1B})^n}{n!}[P_1^{\ k}M_{\mu\nu}P_1^{\ n}\otimes P_A^{\ k}P_B^{\ n} + P_1^{\ k}P_1^{\ n}\otimes P_A^{\ k}M_{\mu\nu}P_B^{\ n}]+ \\
&\quad -\frac{1}{2}\theta^{AB}\eta_{A[\mu}P_{\nu]}\otimes P_B - \frac{1}{2}\theta^{CD}\eta_{D[\mu}P_C\otimes P_{\nu]} \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n\left(\frac{i}{2}\right)^{k+n}\frac{(\theta^{1A})^k}{k!}\frac{(\theta^{1B})^n}{n!}[M_{\mu\nu}P_1^{\ k}P_1^{\ n}\otimes P_A^{\ k}P_B^{\ n}+ \\
&\quad +i\,k\,\eta_{1[\mu}P_{\nu]}P_1^{\ k-1}\otimes P_A^{\ k}P_B^{\ n} + P_1^{\ k}P_1^{\ n}\otimes P_A^{\ k}P_B^{\ n}M_{\mu\nu} - i\,n\,\eta_{B[\mu}P_1^{\ k}P_1^{\ n}\otimes P_A^{\ k}P_{\nu]}P_B^{\ n-1}]+ \\
&\quad -\frac{1}{2}\theta^{AB}\eta_{A[\mu}P_{\nu]}\otimes P_B - \frac{1}{2}\theta^{CD}\eta_{D[\mu}P_C\otimes P_{\nu]} \\
&= M_{\mu\nu}\otimes 1 + 1\otimes M_{\mu\nu} - \frac{1}{2}\theta^{1A}\eta_{1[\mu}P_{\nu]}\otimes P_A - \frac{1}{2}\theta^{1B}\eta_{B[\mu}P_1\otimes P_{\nu]}]+ \\
&\quad -\frac{1}{2}\theta^{AB}\eta_{A[\mu}P_{\nu]}\otimes P_B - \frac{1}{2}\theta^{CD}\eta_{D[\mu}P_C\otimes P_{\nu]}.
\end{aligned}
\tag{1.75}
$$

This is easily seen to be coherent with (1.67).

A last important issue to point out is that, since in this section we saw that assuming a different ordering convention of the exponentials on which the generators of the deformed Poincaré algebra should have classical action corresponds only to a change in the basis of the algebra, we shall require in the following that all the physical quantities do not depend on the choice of this ordering convention.

1.8 Rules of integration and delta functions

Some clearly important tools that we shall need in studying a field theory in noncommutative spacetime are the integral over the whole spacetime and the spatial integral. In this section we are going to define these integrals and also the delta functions, whose definitions are closely related to those of integrals.

First of all we shall define the integral of a function $f(\hat{x})$ of the θ-Minkowski noncommutative coordinates over the whole spacetime. In analogy with the commutative case (in which this integral results to be just a real or complex number) we would like the integral in θ-Minkowski to be coordinate independent and commutative (since, like the commutative algebra of coordinates in Minkowski spacetime, the noncommutative algebra of coordinates of θ-Minkowski spacetime is built over the commutative field of complex numbers). Since we want to give a conservative definition, let us see how does the commutative case works. The integral of the function $g(x)$ can be written in the following way, expanding the function through its Fourier transform:

$$\int d^4x\, g(x) = \int d^4x d^4k \tilde{f}(k) e^{ikx} = \int d^4k \tilde{f}(k) \delta^{(4)}(k) = \tilde{f}(0). \tag{1.76}$$

We are thus tempted to define the integral of the noncommutative function $f(\hat{x})$ as its Fourier transform[7] calculated in zero, but it is easy to understand that to the same noncommutative function we can associate various "Fourier transforms", one for each basis of exponentials (see the next section). For example, we could extend the commutative definition both in this way:

$$\int d^4\hat{x} f(\hat{x}) = \tilde{f}_{(w)}(0), \tag{1.77}$$

or in this one:

$$\int d^4\hat{x} f(\hat{x}) = \tilde{f}_{(1)}(0). \tag{1.78}$$

It seems that this definition depends on the choice of the basis of exponentials that we use for the space of functions. But, as we shall see in the next section, two "Fourier transforms", $\tilde{f}_j(k)$ and $\tilde{f}_i(k)$, of the same function $f(\hat{x})$ differ only for a phase, whose exponent is proportional to k (for example $\tilde{f}_{(w)}(k) = \tilde{f}_{(1)}(k) e^{-\frac{i}{2}k^A k^1 \theta_{A1}}$), that thus gives no contribution in $k = 0$. So in zero all the "Fourier transforms" are equal and we can give the definition of integral:

$$\int d^4\hat{x} f(\hat{x}) = \tilde{f}(0), \tag{1.79}$$

without ambiguities and satisfying all our requirements.

From (1.79) various definitions of the $\delta^{(4)}(k)$ seem to follow if we require it to be an object such that $\int d^4k\, g(k)\delta^{(4)}(k - \bar{k}) = g(\bar{k})$. In fact, using the symmetric Weyl map one gets:

$$\tilde{f}(0) = \int d^4\hat{x} f(\hat{x}) = \int d^4\hat{x} d^4k\, \tilde{f}_{(w)}(k) e^{ik\hat{x}} \Rightarrow \delta^{(4)}(k) \equiv \int d^4\hat{x} e^{ik\hat{x}}, \tag{1.80}$$

while using the x_1-to-the-right Weyl map one gets:

$$\tilde{f}(0) = \int d^4\hat{x} f(\hat{x}) = \int d^4\hat{x} d^4k\, \tilde{f}_{(1)}(k) e^{ik^A \hat{x}_A} e^{ik^1 \hat{x}_1} \Rightarrow \delta^{(4)}(k) \equiv \int d^4\hat{x} e^{ik^A \hat{x}_A} e^{ik^1 \hat{x}_1}, \tag{1.81}$$

and $\int d^4\hat{x} e^{ik\hat{x}} \neq \int d^4\hat{x} e^{ik^A \hat{x}_A} e^{ik^1 \hat{x}_1}$. This would mean that the definition of the delta depends on the choice of the Weyl map; nevertheless, one can easily convince himself that these two

[7] We call "Fourier transform" of a noncommutative function $f(\hat{x})$, corresponding through some Weyl map to a commutative function $f(x)$ as in (1.6), the Fourier transform of the commutative function. More details on this will be given in the next section.

definitions (and all the others that one gets using different Weyl maps) are equivalent. In fact using the first one:

$$\int d^4\hat{x} e^{ik^A\hat{x}_A} e^{ik^1\hat{x}_1} \equiv \int d^4\hat{x} e^{ik\hat{x}} e^{-\frac{i}{2}k^A k^1 \theta_{A1}} = \delta^{(4)}(k) e^{-\frac{i}{2}k^A k^1 \theta_{A1}} = \delta^{(4)}(k), \tag{1.82}$$

where the last equivalence holds since the delta imposes $k = 0$, making the exponent to be null, while using the second one:

$$\int d^4\hat{x} e^{ik\hat{x}} \equiv \int d^4\hat{x} e^{ik^A\hat{x}_A} e^{ik^1\hat{x}_1} e^{\frac{i}{2}k^A k^1 \theta_{A1}} = \delta^{(4)}(k) e^{\frac{i}{2}k^A k^1 \theta_{A1}} = \delta^{(4)}(k). \tag{1.83}$$

So each of the definitions implies the other, thus making the delta a well-defined object:

$$\delta^{(4)}(k) \equiv \int d^4\hat{x} \Omega_i \left(e^{ikx} \right), \tag{1.84}$$

where with $\Omega_i \left(e^{ikx} \right)$ we denote a generic ordering convention for the exponential.

Now we want to define also the integral of a function $f(\hat{x})$ over the spatial coordinates. We start observing that in the commutative case the following equivalences hold:

$$\int d^3x \; g(x) = \int d^3x d^4k \; \tilde{g}(k) e^{ikx} = \int d^4k \; \tilde{g}(k) \delta^{(3)}(\vec{k}) e^{ik^0 x_0} = \int dk_0 \tilde{g}(0, k_0) e^{ik^0 x_0}, \tag{1.85}$$

so we propose the following definition for the noncommutative case:

$$\int d^3\hat{x} \; f(\hat{x}) \equiv \int dk_0 \; \tilde{f}(0, k_0) e^{ik^0 \hat{x}_0}. \tag{1.86}$$

Also in this case we want to check if the object $\int dk_0 \; \tilde{f}(0, k_0) e^{ik^0 \hat{x}_0}$ depends or not on which "Fourier transform" $\tilde{f}$ we choose to use. With a comparison between the symmetric Weyl map and the x_1-to-the-right map one sees that:

$$\int dk_0 \; \tilde{f}_{(w)}(0, k_0) e^{ik^0 \hat{x}_0} = \int dk_0 \; \tilde{f}_{(1)}(0, k_0) e^{-\frac{i}{2}k^A k^1 \theta_{A1}} |_{\vec{k}=0} e^{ik^0 \hat{x}_0} = \int dk_0 \; \tilde{f}_{(1)}(0, k_0) e^{ik^0 \hat{x}_0}, \tag{1.87}$$

so that there is no dependence on the choice between these two maps. One can wonder if there exists some particular map for which something could go wrong, but it is immediately clear that this is not the case, since, in the passage from the "Fourier transform" corresponding to the symmetric basis of exponentials to any other "Fourier transform" corresponding to any other basis of exponentials the difference will always be in a phase, in whose exponent the Fourier parameters k appear quadratically and saturated with a θ-matrix. Since $\theta_{\mu\nu}$ is antisymmetric it will never happen that we have only two k_0 multiplied (that is the only case that could make things to go wrong, since in all the other cases we have at least one spatial component of k, that has to be calculated in zero). In other words, all the "Fourier transforms" of a function are equivalent if calculated in $\vec{k} = 0$.

The last tool we shall need in our work is the spatial delta $\delta^{(3)}(\vec{k})$. We are looking for an object such that:

$$\int d^3k\ g(\vec{k},k_0)\delta^{(3)}(\vec{k}) = g(0,k_0). \tag{1.88}$$

From (1.86), using the "time-to-the-right" basis of exponentials ($\Omega_R(e^{ik\hat{x}}) = e^{ik^j\hat{x}_j}e^{ik^0\hat{x}_0}$) to expand in "Fourier series" the function, we get:

$$\begin{aligned}\int dk_0\ \tilde{f}_{(R)}(0,k_0)e^{ik^0\hat{x}_0} &= \int d^3\hat{x}\ f(\hat{x}) = \int d^3\hat{x}d^4k\ \tilde{f}_{(R)}(\vec{k},k_0)e^{ik^j\hat{x}_j}e^{ik^0\hat{x}_0}\\ &= \int dk_0\ \left[\int d^3kd^3\hat{x}\ \tilde{f}_{(R)}(\vec{k},k_0)e^{ik^j\hat{x}_j}\right]e^{ik^0\hat{x}_0}.\end{aligned} \tag{1.89}$$

In this case we have that under the action of the operator $\int dk_0e^{ik^0\hat{x}_0}$ it holds the equivalence $\tilde{f}_{(R)}(0,k_0) = \int d^3kd^3\hat{x}\ \tilde{f}_{(R)}(\vec{k},k_0)e^{ik^j\hat{x}_j}$ and consequently it is coherent the definition $\delta^{(3)}(\vec{k}) \equiv \int d^3\hat{x}e^{ik^j\hat{x}_j}$. Generalising this result we define:

$$\delta^{(3)}(\vec{k}) \equiv \int d^3\hat{x}e^{ik^j\hat{x}_j}, \tag{1.90}$$

also outside the $\int dk_0e^{ik^0\hat{x}_0}$. Notice that in the first expression of (1.89) we could have used any "Fourier transform" of f, since they are all equivalent if calculated in zero. Instead, in the second equivalence the use of another basis of exponentials to expand the function $f(x)$ wouldn't have been equivalent, for example if we have used the symmetric basis $e^{ik\hat{x}}$ we would have got:

$$\begin{aligned}\int dk_0\ \tilde{f}(0,k_0)e^{ik^0\hat{x}_0} &= \int d^3\hat{x}d^4k\ \tilde{f}_{(w)}(\vec{k},k_0)e^{ik\hat{x}} = \int d^3\hat{x}d^4k\ \tilde{f}_{(w)}(\vec{k},k_0)e^{ik^j\hat{x}_j}e^{ik^0\hat{x}_0}e^{\frac{i}{2}k^ik^0\theta_{i0}}\\ &= \int dk_0\left[\int d^3kd^3\hat{x}\ \tilde{f}_{(w)}(\vec{k},k_0)e^{ik^j\hat{x}_j}e^{\frac{i}{2}k^ik^0\theta_{i0}}\right]e^{ik^0\hat{x}_0},\end{aligned} \tag{1.91}$$

but this is not a problem, since the definition (1.90) of the $\delta^{(3)}(\vec{k})$ is coherent with (1.88) also in this case.

1.8.1 Ciclicity

There is a useful property of the integral of two noncommutative functions $f(x)$, $g(x)$: it is symmetric under the exchange of two functions. In this section we shall show this. Writing the functions in terms of their Fourier transform with the Weyl-ordered exponentials (Cf. (1.6)) we get:

$$\int d^4\hat{x}\ f(\hat{x})g(\hat{x}) = \int d^4\hat{x}d^4kd^4q\ \tilde{f}_w(k)\tilde{g}_w(q)e^{ik\hat{x}}e^{iq\hat{x}}. \tag{1.92}$$

To apply the definition (1.84) of the $\delta^{(4)}(k)$, we have to reorder the exponentials using the BCH formula (1.11). In this way we get:

$$\begin{aligned}\int d^4\hat{x}\ f(\hat{x})g(\hat{x}) &= \int d^4\hat{x}d^4kd^4q\ \tilde{f}_w(k)\tilde{g}_w(q)e^{i(k+q)\hat{x}}e^{-ik^\mu q^\nu\theta_{\mu\nu}}\\ &= \int d^4kd^4q\ \tilde{f}_w(k)\tilde{g}_w(q)\delta^{(4)}(k+q)e^{-\frac{i}{2}k^\mu q^\nu\theta_{\mu\nu}}\end{aligned} \tag{1.93}$$

Now we integrate in d^4q:

$$\int d^4\hat{x}\ f(\hat{x})g(\hat{x}) = \int d^4k\ \tilde{f}_w(k)\tilde{g}_w(-k). \tag{1.94}$$

We see that the phase $e^{-\frac{i}{2}k^\mu q^\nu \theta_{\mu\nu}}$ has disappeared because of the antisymmetry of the matrix $\theta_{\mu\nu}$, and the remaining expression is symmetric for exchange of the two functions, since, changing the variable of integration defining $p \equiv -k$:

$$\int d^4k\ \tilde{f}_w(k)\tilde{g}_w(-k) = \int d^4p\ \tilde{f}_w(-p)\tilde{g}_w(p) \equiv \int d^4k \tilde{f}_w(-k)\tilde{g}_w(k). \tag{1.95}$$

Let us see what happens if we use a differently ordered exponentials basis to write the Fourier transforms of the functions $f(\hat{x})$ and $g(\hat{x})$, for example let us try to use the x_1-to-the-right-ordered basis of exponentials:

$$\int d^4\hat{x}\ f(\hat{x})g(\hat{x}) = \int d^4\hat{x} d^4k d^4q\ \tilde{f}_1(k)\tilde{g}_1(q) e^{ik^A\hat{x}_A} e^{ik^1\hat{x}_1} e^{iq^A\hat{x}_A} e^{iq^1\hat{x}_1}. \tag{1.96}$$

With similar steps to those described above we get:

$$\begin{aligned}
\int d^4\hat{x}\ f(\hat{x})g(\hat{x}) &= \int d^4\hat{x} d^4k d^4q\ \tilde{f}_1(k)\tilde{g}_1(q) e^{i(k+q)\hat{x}} e^{-\frac{i}{2}k^\mu q^\nu \theta_{\mu\nu}} e^{-\frac{i}{2}(k^A k^1 + q^A q^1)\theta_{A1}} \\
&= \int d^4k d^4q\ \tilde{f}_1(k)\tilde{g}_1(q) \delta^{(4)}(k+q) e^{-ik^\mu q^\nu \theta_{\mu\nu}} e^{-\frac{i}{2}(k^A k^1 + q^A q^1)\theta_{A1}} \\
&= \int d^4k\ \tilde{f}_1(k)\tilde{g}_1(-k) e^{-ik^A k^1 \theta_{A1}}
\end{aligned} \tag{1.97}$$

Also in this case we obtained that the integral of the product of two functions does not depend on the order of the functions.

1.8.2 Fourier transform

This section is dedicated to some remarks on the concept of "Fourier transform" of a noncommutative function. In section 1.1 we showed that it is possible to establish a one-to-one correspondence between noncommutative and commutative functions through the use of a Weyl map. From this correspondence it follows a generalisation of the concept of "Fourier transform". In fact, as we showed in (1.6), if a noncommutative function $f(\hat{x})$ corresponds through the symmetric Weyl map to the commutative function $f_w(x) = \int d^4k\ \tilde{f}_w(k)e^{ikx}$ (the index w means that the commutative function $f(x)$ corresponds to the noncommutative one $f(\hat{x})$ through the symmetric Weyl map Ω_w: $f(\hat{x}) = \Omega_w(f_w(x))$), then $f(\hat{x})$ can be written in the form:

$$f(\hat{x}) = \int d^4k\ \tilde{f}_w(k)e^{ik\hat{x}}. \tag{1.98}$$

It seems natural now to call $f_w(k)$ the "Fourier transform" of the noncommutative function $f(\hat{x})$.

It is clear that with definition the "Fourier transform" of a noncommutative function is not unique; in fact, if we use a different Weyl map to write $f(\hat{x})$ in terms of a commutative function the generalised Fourier transform of $f(\hat{x})$ results to be different. For example if we use the x_1-to-the-right map Ω_1, such that $f(\hat{x}) = \Omega_1(f_1(x))$, where $f_1(x) = \int d^4k\ \tilde{f}_1(k)e^{ikx}$, we get, in a way analogous to the one used to derive (1.6):

$$f(\hat{x}) = \int d^4k\ \tilde{f}_1(k)e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}. \tag{1.99}$$

In this case it would seem natural to define the "Fourier transform" of $f(\hat{x})$ to be $\tilde{f}_1(k)$.

There exists a simple relation between different "Fourier transforms" of a same noncommutative function. Let us see as an example the comparison between the "symmetric Fourier transform" and the "x_1-to-the-right Fourier transform": imposing the equivalence between (1.98) and (1.99) and exploiting the BCH formula (1.11) we get:

$$f(\hat{x}) = \begin{cases} \int d^4k\ \tilde{f}_w(k)e^{ik\hat{x}} = \int d^4k\ \tilde{f}_w(k)e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}e^{\frac{i}{2}k^Ak^1\theta_{A1}} \\ \int d^4k\ \tilde{f}_1(k)e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1} \end{cases} \quad \Rightarrow \quad \tilde{f}_w(k)e^{\frac{i}{2}k^Ak^1\theta_{A1}} = \tilde{f}_1(k) \tag{1.100}$$

It is also possible to show that in general two different "Fourier transforms" corresponding to different Weyl maps are always connected by a phase factor as in this case.

Chapter 2

Symmetry transformations

One aim of our work is to find the conserved charges associated to the invariance of a certain field theory under the action of the deformed Poincaré algebra that we defined in the previous chapter. To do this we shall follow as close as possible the steps of the standard Noether analysis that is usually performed in commutative spacetime. In preparation for that, the first step should be to define the transformation generated by the operator G of the deformed Poincaré algebra and regulated by the parameter ϵ. In particular, we want to define the transformation of the function $f(\hat{x})$ at the first order in the parameter ϵ. Classically this object is called differential and it is defined as $df(x) = i\epsilon G f(x)$, where ϵ is a real-valued parameter. The fundamental property characterising the differential (resulting to be of crucial importance in the development of the Noether analysis) is that it must satisfy the Leibniz rule:

$$d(f(x)g(x)) = (df(x))g(x) + f(x)dg(x). \tag{2.1}$$

Now we are looking for an object similar to the one defined in the commutative case (i.e. containing a transformation parameter and the generator acting on the function), that reduces to that one in the commutative limit and satisfies the Leibniz rule even for $\theta_{\mu\nu} \neq 0$.

2.1 Seeking guidance from a previous analysis of translations in another noncommutative spacetime

A problem similar to the one described above was previously (Cf. [3]) dealt with in the analysis of translation transformations in the other most-known noncommutative spacetime, the so-called κ-Minkowski [1], characterised by the commutation relations $[\bar{x}_j, \bar{x}_0] = i\lambda\bar{x}_j \quad [\bar{x}_i, \bar{x}_j] = 0$ (λ is a length scale). The symmetry algebra of κ-Minkowski is also a deformation of the

[1]In this section all the noncommutative object related to κ-Minkowski spacetime will have a bar upon them: the coordinates will be denoted with $\bar{x}^\mu$ and the κ-Poincaré generators with $\bar{G}$

Poincaré algebra (called κ-Poincaré). Its translation generators $\bar{P}_\mu$, expressed in the Majid-Ruegg basis (Cf. [12]), have the following action on time-to-the-right ordered exponentials: $P_\mu(e^{ik^j\bar{x}_j}e^{ik^0\bar{x}_0}) = k_\mu e^{ik^j\bar{x}_j}e^{ik^0\bar{x}_0}$ and, differently from the translation generators of θ-Poincaré, they have a non trivial coproduct: $\Delta(\bar{P}_\mu) = \bar{P}_\mu \otimes 1 + e^{-\lambda\bar{P}_0(1-\delta_{\mu 0})} \otimes \bar{P}_\mu$. In [3] it was shown that one can define the differential associated to translation transformations in the same way of the commutative case, setting $df(\bar{x}) \equiv i\epsilon^\mu \bar{P}_\mu f(\bar{x})$, but enforcing the Leibniz rule one obtains a nontrivial equation, due to the nontrivial coproduct of the $\bar{P}_\mu$. In fact from the Leibniz rule one gets:

$$d[f(\bar{x})g(\bar{x})] = [df(\bar{x})]g(\bar{x}) + f(\bar{x})dg(\bar{x}) \Rightarrow \epsilon^\mu \bar{P}_\mu[f(\bar{x})g(\bar{x})] = [\epsilon^\mu \bar{P}_\mu f(\bar{x})]g(\bar{x}) + f(\bar{x})\epsilon^\mu \bar{P}_\mu g(\bar{x}), \tag{2.2}$$

while using the coproduct of the translation generator one gets:

$$\epsilon^\mu \bar{P}_\mu[f(\bar{x})g(\bar{x})] = \epsilon^\mu[(\bar{P}_\mu f(\bar{x}))g(x) + (e^{-\lambda\bar{P}_0(1-\delta_{\mu 0})} f(\bar{x}))\bar{P}_\mu g(\bar{x})]. \tag{2.3}$$

Equating the last two expressions one obtains the condition[2]:

$$[f(\bar{x})\epsilon^\mu - \epsilon^\mu e^{-\lambda P_0(1-\delta_{\mu 0})} f(\bar{x})]\bar{P}_\mu g(\bar{x}) = 0. \tag{2.4}$$

It is evident that in order this equation to be satisfied the transformation parameter ϵ^μ can not behave as a commutative object with respect to the function $f(\bar{x})$, but has to obey the following commutation relations with coordinates:

$$[\epsilon_j, \bar{x}_0] = i\lambda\epsilon_j \qquad [\epsilon_j, \bar{x}_k] = [\epsilon_0, \bar{x}_\mu] = 0. \tag{2.5}$$

So we have found the most conservative generalisation of the differential of a function: it has the same form of the commutative case, but the transformation parameter, still acting by associative multiplication on the right, is no more a real-valued commutative number, but it is subject to nontrivial commutation rules with coordinates. In [3] it was shown that this definition is a good one, since it permits to perform the Noether analysis for the translational symmetries of a scalar field theory and find some time independent "charges".

Comforted by this success in κ-Minkowski spacetime, we are now going to try a similar procedure, that is general enough to make us hope in a success, to define the differentials of functions of θ-Minkowski spacetime.

2.2 Pure translations

Let us first concentrate on the translation sector of the twisted Poincaré algebra. As explained in the previous section, we shall try to describe the transformation of a function $f(\hat{x})$ of the

[2] It is important to stress that this equation can be obtained assuming that ϵ_μ acts by associative multiplication on the functions on his right.

canonical coordinates under the action of the translation generators $\hat{P}_\mu$. In this case we do not expect to find highly nontrivial features, since in the other space these came from the nontrivial coproduct properties of the translation generators, while in this case the translation generators have classical action on coordinates, obey classical commutation relations, and have trivial coproduct (Cf. Sec. (1.3)). Nevertheless, we shall follow the steps described in the previous section to define the differential $df(\hat{x})$, using this simple case as a check that the way of proceeding that resulted to be valid in a different noncommutative spacetime can be used also in the canonical one.

We define the transformation of the function $f(\hat{x})$ under the action of the translation generator $\hat{P}_\mu$ at the first order in the transformation parameter ϵ_μ to be:

$$d_P f(\hat{x}) = i\epsilon^\mu \hat{P}_\mu f(\hat{x}). \tag{2.6}$$

Also in this case we assume the parameter ϵ^μ to act by associative multiplication on the functions on his right, while we allow it to eventually satisfy nontrivial commutation rule with coordinates. To find these commutation rules we enforce the Leibniz rule (2.1):

$$\epsilon^\mu \hat{P}_\mu[f(\hat{x})g(\hat{x})] = \epsilon^\mu[\hat{P}_\mu f(\hat{x})]g(\hat{x}) + f(\hat{x})\hat{\epsilon}^\mu P_\mu g(\hat{x}); \tag{2.7}$$

considering instead the (trivial) coproduct of P_μ, (1.57) we have that:

$$\epsilon^\mu \hat{P}_\mu[f(\hat{x})g(\hat{x})] = \epsilon^\mu[(\hat{P}_\mu f(\hat{x}))g(\hat{x}) + f(\hat{x})\hat{P}_\mu g(\hat{x})]. \tag{2.8}$$

Since the two equations must be equivalent we get the condition:

$$[\epsilon^\mu f(\hat{x}) - f(\hat{x})\epsilon^\mu]\hat{P}_\mu g(\hat{x}) = 0. \tag{2.9}$$

This implies univocally that the transformation parameter ϵ^μ must satisfy trivial commutation relations with coordinates:

$$[\epsilon^\mu, \hat{x}^\nu] = 0, \tag{2.10}$$

that is what we have expected.

2.3 Space rotations and boosts

Now we have to deal with the nontrivial sector of the twisted Poincaré algebra. The space rotations and boosts generator $\hat{M}_{\mu\nu}$ is characterised by having classical action on single coordinates and classical commutation relations, but nontrivial coproduct. As happened for the translation sector of the κ-Poincaré algebra we expect that this nontriviality will provide a more complex structure for the differentials of functions of θ-Minkowski, such as a noncommutativity of the transformation parmeter.

We shall use the following parametrisation for the (first order) differential of a function $f(\hat{x})$ associated to a spatial rotation or boost transformation (the subscript of d stands for "Lorentz"):

$$d_L f(\hat{x}) = i\omega^{\mu\nu}\hat{M}_{\mu\nu}f(\hat{x}), \tag{2.11}$$

where the transformation parameter $\omega^{\mu\nu}$ is antisymmetric in the indices μ and ν. Also in this case we impose the Leibniz rule to evince the properties of this object. Equating the expression resulting from the imposition of the Leibniz rule (2.1) on $d_L(f(\hat{x})g(\hat{x}))$ and the expression obtained letting $\hat{M}_{\mu\nu}$ act through its coproduct law (1.60) on the product of the two functions $f(\hat{x})g(\hat{x})$ we get:

$$\begin{aligned} \left[f(\hat{x})\omega^{\mu\nu} - \omega^{\mu\nu}f(\hat{x})\right]\hat{M}_{\mu\nu}g(\hat{x}) &= \frac{1}{2}\omega^{\mu\nu}\theta_{\beta[\mu}\left[(\hat{P}_{\nu]}f(\hat{x}))\hat{P}^{\beta}g(\hat{x}) - (\hat{P}^{\beta}f(\hat{x}))\hat{P}_{\nu]}g(\hat{x})\right] \\ &= \frac{1}{2}\omega^{\mu\nu}\left[\theta_{\sigma[\mu}\eta_{\nu]\rho} - \theta_{\rho[\mu}\eta_{\nu]\sigma}\right](\hat{P}^{\rho}f(\hat{x}))\hat{P}^{\sigma}g(\hat{x}) \end{aligned} \tag{2.12}$$

This condition is not compatible with the previous assumption (that we made to obtain this equation) that $\omega^{\mu\nu}$ should act through simple associative multiplication on the functions on his right, since the commutator $[f(\hat{x}), \omega^{\mu\nu}]$ results to have an action on the function on his right $\hat{M}_{\mu\nu}g(\hat{x})$, that transforms it in an object proportional to $\hat{P}^{\sigma}g(\hat{x})$.

We have to conclude that it is not possible to consider a pure transformation of the Lorentz sector, regulated by a parameter $\omega^{\mu\nu}$ with trivial properties of action on the functions, that differs from its commutative analogous just for the fact that it does not commute with coordinates. Notice also that it makes not sense to use the equation (2.12) to deduce some different properties of action for $\omega^{\mu\nu}$, since that equation itself has been obtained assuming $\omega^{\mu\nu}$ to act only by associative multiplication.

2.4 No-pure-Lorentz-sector transformations

Let us study in more detail the equation (2.12): on the left hand side we have the commutator of the parameter with the function acting on $\hat{M}_{\mu\nu}g$, while on the right hand side we have something proportional to $\hat{P}_{\sigma}g$. The problem is clearly in the additional term that $\hat{M}_{\mu\nu}$ has in his coproduct with respect to the trivial form of the coproduct. To solve it we could think not to consider a pure Lorentz sector transformation, but to add a generator to the transformation of which we are defining the differential that in some way could compensate the presence of that additional term in the coproduct. We can try a definition of the differential of the kind (we shall call it $d_{\mathcal{L}}f$ since it is a "deformation" of the Lorentz sector differential, that was called $d_L f$):

$$d_{\mathcal{L}}f(\hat{x}) = i[\omega^{\mu\nu}M_{\mu\nu} + \sigma^A G_A]f(\hat{x}), \tag{2.13}$$

where A is a generic set of indices (like μ, or $\mu\nu$ etc.) and G_A is a generic operator of the Poincaré algebra. As regards the choice of the generator G_A, obviously it should be a certain combination of the generators of the algebra, that for the moment we shall assume to be only a linear combination, since this is the simplest choice. With this assumption, it is useless to include in the expression of G_A the generators of the Lorentz sector transformations $\hat{M}_{\mu\nu}$, since they appear yet in the differential, so it remains only to include the translation generators, that are the only generators of the algebra missing in the expression above. So the new differential results to be:

$$df(\hat{x}) = i\left[\epsilon^{\alpha}\hat{P}_{\alpha} + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right]f(\hat{x}). \tag{2.14}$$

Imposing this differential to satisfy the Leibniz rule we obtain:

$$[f(\hat{x}),\omega^{\mu\nu}]\,\hat{M}_{\mu\nu}g(\hat{x}) + [f(\hat{x}),\epsilon^{\alpha}]\,\hat{P}_{\alpha}g(\hat{x}) = \frac{1}{2}\omega^{\mu\nu}\left[\theta_{\sigma[\mu}\eta_{\nu]\rho} - \theta_{\rho[\mu}\eta_{\nu]\sigma}\right](\hat{P}^{\rho}f(\hat{x}))\hat{P}^{\sigma}g(\hat{x}), \tag{2.15}$$

that can be written in the form:

$$[f(\hat{x}),\omega^{\mu\nu}]\,\hat{M}_{\mu\nu}g(\hat{x}) + \left([f(\hat{x}),\epsilon^{\alpha}] - \frac{1}{2}\omega^{\mu\nu}\left[\theta^{\alpha}{}_{[\mu}\delta^{\rho}_{\nu]} - \theta^{\rho}{}_{[\mu}\delta^{\alpha}_{\nu]}\right](\hat{P}^{\rho}f(\hat{x}))\right)\hat{P}_{\alpha}g(\hat{x}) = 0. \tag{2.16}$$

In order this expression to be null, the two terms, the one proportional to $\hat{M}_{\mu\nu}g(\hat{x})$ and the one proportional to $\hat{P}_{\alpha}g(\hat{x})$, must be separately null, being the function $g(\hat{x})$ arbitrary. This amounts to the following requirements for the commutators of the transformation parameters with the coordinates:

$$\begin{aligned}
[f(\hat{x}),\omega^{\mu\nu}] &= 0, && (2.17)\\
[f(\hat{x}),\epsilon^{\alpha}] &= -\frac{1}{2}\omega^{\mu\nu}\Upsilon^{\alpha\beta}_{\mu\nu}\hat{P}_{\beta}f(\hat{x}), && (2.18)
\end{aligned}$$

where:

$$\Upsilon^{\alpha\beta}_{\mu\nu} = \theta_{[\mu}{}^{\alpha}\delta_{\nu]}{}^{\beta} - \theta_{[\mu}{}^{\beta}\delta_{\nu]}{}^{\alpha}. \tag{2.19}$$

These commutation relations are now coherent with the requirement that we made for the parameters, i.e. to have an associative multiplicative action on the functions on their right and eventually to obey nontrivial commutation relations with coordinates[3].

We have thus discovered that, while pure spatial rotations and boosts transformations are not allowed in canonical spacetime if we make some minimal and quite conservative requirements

[3]In terms of commutators with coordinates the above equations read:

$$\begin{aligned}
[\hat{x}^{\rho},\omega^{\mu\nu}] &= 0, && (2.20)\\
[x^{\rho},\epsilon^{\alpha}] &= -\frac{i}{2}\omega^{\mu\nu}\Upsilon^{\alpha\rho}_{\mu\nu}. && (2.21)
\end{aligned}$$

for the transformation parameters, they are instead permitted if performed contemporary to a translation, since the differential involving the whole Poincaré algebra generators exists and is compatible with our requirements. As a confirmation of this, we can observe that, according to (2.18), whenever $\omega^{\mu\nu}$ in non null, the commutator of ϵ^{α} with coordinates becomes non null too, causing the impossibility to fix the value of the translation parameter ϵ^{α} to zero. This means that if we are performing a spatial rotation or a boost, we can not know with arbitrary precision the amount of translation that we are contemporary subjected to, so that we can not fix the translation to be null and it is not possible to make a pure spatial rotation and/or a pure boost. Viceversa, we see from (2.17) and (2.18) that it is possible to perform a pure translation (i.e. $\epsilon^{\alpha} \neq 0$ and $\omega^{\mu\nu} = 0$), since making ϵ^{α} different from zero does not produce effects on the parameter $\omega^{\mu\nu}$. Moreover the commutation relation for ϵ^{α} that one obtains in this case are $[f(\hat{x}), \epsilon^{\alpha}] = 0$, that are reassuringly the same that we obtained when we performed the analysis of the differential associated to pure translation transformations in Sec. 2.2.

2.5 An ordering ambiguity?

The results in the previous section provide a satisfactory description of the twisted Poincaré algebra transformations in the symmetric basis of generators. In the first chapter we saw that it was possible to define other basis for the algebra of symmetries of canonical spacetime, and we argued that all the physical quantities of our theory should not depend on the choice of the basis of generators. To check this, in this section we are going to define the differential of a function $f(\hat{x})$ in terms of the $\hat{x}_1$-to-the-right basis of generators of the θ-Poincaré algebra.

Let us start with pure translations. Obviously, since $\hat{P}^{(1)}_{\mu} = \hat{P}_{\mu}$, the differential associated to pure translation transformations is the same of the one that we defined in Sec. 2.2. In fact:

$$d^{(1)}_P f(\hat{x}) \equiv i\epsilon^{\alpha}_{(1)} \hat{P}^{(1)}_{\alpha} f(\hat{x}) = i\epsilon^{\alpha}_{(1)} \hat{P}_{\alpha} f(\hat{x}), \tag{2.22}$$

and imposing the Leibniz rule on this differential we get the commutation relations:

$$\left[f(\hat{x}), \epsilon^{\alpha}_{(1)}\right] = 0, \tag{2.23}$$

that are the same relations holding for ϵ^{α} (Cf. (2.10)), so that we can say that also the transformation parameters associated to pure translations are the same, and so:

$$d^{(1)}_P f(\hat{x}) = d_P f(\hat{x}). \tag{2.24}$$

As regards spatial rotations and boosts, also in this case we find that it is not possible to define a pure Lorentz sector transformation, since, imposing the Leibniz rule on the differential:

$$d^{(1)}_L f(\hat{x}) \equiv i\omega^{\mu\nu}_{(1)} M^{(1)}_{\mu\nu} f(\hat{x}), \tag{2.25}$$

we get:

$$d_L^{(1)}\left(f(\hat{x})g(\hat{x})\right) = i\left(\omega^{\mu\nu}_{(1)}M^{(1)}_{\mu\nu}f(\hat{x})\right)g(\hat{x}) + if(\hat{x})\omega^{\mu\nu}_{(1)}M^{(1)}_{\mu\nu}g(\hat{x}), \tag{2.26}$$

while applying the coproduct rule (1.67):

$$d_L^{(1)}\left(f(\hat{x})g(\hat{x})\right) = i\omega^{\mu\nu}_{(1)}\left(M^{(1)}_{\mu\nu}f(\hat{x})\right)g(\hat{x}) + i\omega^{\mu\nu}_{(1)}f(\hat{x})M^{(1)}_{\mu\nu}g(\hat{x}) - \frac{i}{2}\omega^{\mu\nu}_{(1)}\chi^{\rho\sigma}_{\mu\nu}\left(\hat{P}_\rho f(\hat{x})\right)\left(\hat{P}_\sigma g(\hat{x})\right), \tag{2.27}$$

where we defined:

$$\chi^{\rho\sigma}_{\mu\nu} = \Upsilon^{\rho\sigma}_{\mu\nu} - \theta_A{}^1\left(\delta^A_{[\mu}\delta^\rho_{\nu]}\delta^\sigma_1 + \delta^A_{[\mu}\delta^\sigma_{\nu]}\delta^\rho_1 + \delta^1_{[\mu}\delta^\rho_{\nu]}\delta^\sigma_A + \delta^1_{[\mu}\delta^\sigma_{\nu]}\delta^\rho_A\right).$$

Equating the two above expressions we obtain the following condition for the commutator of the transformation parameters $\omega^{\mu\nu}_{(1)}$ with functions:

$$\left[f(\hat{x}),\omega^{\mu\nu}_{(1)}\right]M^{(1)}_{\mu\nu}g(\hat{x}) = -\frac{i}{2}\omega^{\mu\nu}_{(1)}\chi^{\rho\sigma}_{\mu\nu}\left(\hat{P}_\rho f(\hat{x})\right)\left(\hat{P}_\sigma g(\hat{x})\right). \tag{2.28}$$

This expression is of the same kind of the one obtained trying to define pure space rotations and boosts in the symmetric basis and, as that one, it is not compatible with the requirement that $\omega^{(1)}_{\mu\nu}$ should act by associative multiplication on the functions on his right. So, to get rid of the problem, also with this basis of generators we are led to define a differential associated to the whole Poincaré algebra :

$$d^{(1)}f(\hat{x}) \equiv i\left[\epsilon^\alpha_{(1)}\hat{P}^{(1)}_\alpha + \omega^{\mu\nu}_{(1)}\hat{M}^{(1)}_{\mu\nu}\right]f(\hat{x}) \equiv i\left[\epsilon^\alpha_{(1)}\hat{P}_\alpha + \omega^{\mu\nu}_{(1)}\hat{M}^{(1)}_{\mu\nu}\right]f(\hat{x}). \tag{2.29}$$

In the second expression we wanted to point out that in this case, though the translation generators are the same in both of the basis, the translation parameters could in principle be different (and actually they are), since their commutation relations are now determined not only by the translation parameters $\hat{P}^{(1)}_\mu \equiv \hat{P}_\mu$, but also by the coproduct law of $\hat{M}^{(1)}_{\mu\nu}$ that is different from $\hat{M}_{\mu\nu}$. Imposing the Leibniz rule to be satisfied by this differential we get the condition:

$$\left[f(\hat{x}),\omega^{\mu\nu}_{(1)}\right]\hat{M}^{(1)}_{\mu\nu}g(\hat{x}) + \left(\left[f(\hat{x}),\epsilon^\alpha_{(1)}\right] + \frac{1}{2}\omega^{\mu\nu}_{(1)}\chi^{\rho\alpha}_{\mu\nu}(\hat{P}_\rho f(\hat{x}))\right)\hat{P}_\alpha g(\hat{x}) = 0, \tag{2.30}$$

that leads to the commutation rules:

$$\left[f(\hat{x}),\omega^{\mu\nu}_{(1)}\right] = 0 \tag{2.31}$$

$$\left[f(\hat{x}),\epsilon^\alpha_{(1)}\right] = -\frac{1}{2}\omega^{\mu\nu}_{(1)}\chi^{\rho\alpha}_{\mu\nu}\left(\hat{P}_\rho f(\hat{x})\right). \tag{2.32}$$

These equations have the same physical meaning of the equations holding for the symmetric basis parameters, that is, they describe “no-pure” Lorentz sector transformations and make also possible to perform pure translations transformations, that are the same obtained when we defined the pure translation differential $d^{(1)}f(\hat{x})$. In spite of this good behaviour, it must be paid attention to the fact that here, if $\omega^{\mu\nu}_{(1)}$ is nonzero, the translation parameters are different from those appearing in $df(\hat{x})$, since they satisfy different commutation relations. So the translation transformations result to be the same when expressed in different basis if they are pure (i.e. if the Lorentz sector parameters are null) while they are different if performed together with a space rotation or a boost. The parameters associated to the generators of the Lorentz sector are instead the same, since they are both commutative, but we can not make a comparison between Lorentz sector transformations, since they can not be pure, so that it has no physical sense to make any assertion about them. The only thing that makes sense about Lorentz sector transformations is to compare the whole differentials $d^{(1)}f(\hat{x})$ and $df(\hat{x})$. We are now going to demonstrate that the two differentials are truly different: let us firstly write the differential $d^{(1)}f(\hat{x})$ in terms of the generators of the symmetric basis, using the relation (1.66) and remembering that $\omega^{\mu\nu}_{(1)} = \omega^{\mu\nu}$:

$$d^{(1)}f(\hat{x}) = i\left[\epsilon^{\alpha}_{(1)}\hat{P}_{\alpha} + \omega^{\mu\nu}\left(\hat{M}_{\mu\nu} + \frac{1}{2}\hat{P}_{[\nu}(\delta^{A}_{\mu]}\hat{P}^{1} + \delta^{1}_{\mu]}\hat{P}^{A})\theta_{A1}\right)\right]f(\hat{x}); \tag{2.33}$$

now, assuming by contradiction that the two differentials are equal, from a comparison of the above equation and (2.14) we get:

$$\epsilon^{\alpha} = \epsilon^{\alpha}_{(1)} + \frac{1}{2}\omega^{\mu\nu}\hat{\delta}^{\alpha}_{[\nu}\left(\delta^{A}_{\mu]}\hat{P}^{1} + \delta^{1}_{\mu]}\hat{P}^{A}\right)\theta_{A1}. \tag{2.34}$$

If we now calculate the commutator of ϵ^{α} with functions from this equation, if the two differentials are really equal we expect to find the same commutation relations of (2.18). Let us see if it is this the case:

$$\begin{aligned}
[f(\hat{x}), \epsilon^{\alpha}] &= \left[f(\hat{x}), \epsilon^{\alpha}_{(1)}\right] + \frac{1}{2}\omega^{\mu\nu}\delta^{\alpha}_{[\nu}\left(\delta^{A}_{\mu]}\left[f(\hat{x}), \hat{P}^{1}\right] + \delta^{1}_{\mu]}\left[f(\hat{x}), \hat{P}^{A}\right]\right)\theta_{A1} \\
&= -\frac{1}{2}\omega^{\mu\nu}\chi^{\rho\alpha}_{\mu\nu}\left(\hat{P}_{\rho}f(\hat{x})\right) - \frac{1}{2}\omega^{\mu\nu}\delta^{\alpha}_{[\nu}\left(\delta^{A}_{\mu]}\left[\hat{P}^{1}f(\hat{x})\right] + \delta^{1}_{\mu]}\left[\hat{P}^{A}f(\hat{x})\right]\right)\theta_{A1} \\
&= -\frac{1}{2}\omega^{\mu\nu}\left[\Upsilon^{\rho\alpha}_{\mu\nu} - \theta_{A}{}^{1}\left(\delta^{A}_{[\mu}\delta^{\alpha}_{1} + \delta^{1}_{[\mu}\delta^{\alpha}_{A}\right)\right]\left(\hat{P}_{\nu]}f(\hat{x})\right).
\end{aligned} \tag{2.35}$$

This is not the same of (2.18), so we came to a contradiction with our hypothesis and we can not state that the two differentials, $d^{(1)}f(\hat{x})$ and $df(\hat{x})$, are equal. So we have to face with an ordering ambiguity: what will be the "phisically observed" transformation? Since the choice of the basis of generators of the algebra is only a convention, it should not have any physical consequence, so it should not be possible to distinguish, through any physical measure, which of the bases of generators we are using to define the differential of the transformation. In the following chapter we shall show a way out of this ambiguity (Cf. Sec. 3.6).

2.6 Casimirs and equation of motion

An other important feature that is needed when studying the symmetries of a certain system, besides finding the expression of the transformations of the fields under the action of the generators of the symmetries, is to individuate the operators, called casimirs, invariant under the action of these generators, i.e. that commute with all the generators of the algebra:

$$[C, G] = 0, \tag{2.36}$$

where C is a casimir and G a generic generator of the algebra.

It is well known that the classical Poincaré algebra has the following invariant operators:

$$\Box \equiv P^\mu P_\mu \tag{2.37}$$
$$W^\lambda W_\lambda \equiv \varepsilon^{\lambda\kappa\rho\sigma} P_\kappa M_{\rho\sigma}\ \eta_{\tau\lambda} \varepsilon^{\tau\alpha\beta\gamma} P_\alpha M_{\beta\gamma}, \tag{2.38}$$

where $\Box$ is called the Klein-Gordon operator and W_λ is the Pauli-Lubanski vector. The importance of the first casimir resides in the fact that it establishes an invariant relation between energy and momentum carried by a field. For example a classical (i.e. commutative) and massless scalar field theory obeys the Klein-Gordon equation of motion, that is constructed with the first casimir of the Poincaré algebra in order to make it invariant under the action of the algebra itself.:

$$\Box\phi(x) = 0. \tag{2.39}$$

In this way, from this equation, expanding in Fourier transform the field, one gets:

$$0 = \Box\phi(x) \equiv \Box \int d^4k\ \tilde{\phi}(k) e^{ikx} = \int d^4k\ \tilde{\phi}(k) k^\mu k_\mu e^{ikx} \ \Rightarrow\ k^\mu k_\mu \equiv |\vec{k}|^2 - E^2 = 0. \tag{2.40}$$

This is the dispersion relation to which energy and momentum of a scalar field must obey, its invariance under Poincaré transformations being assured by the invariance of the first casimir.

Since in this work we want to concentrate on a scalar field theory, we shall be concerned only with an analogous of the first casimir of the Poincaré algebra, and shall not study the second casimir (for an in-depth discussion of the second casimir see [13]).

Defining the first casimir of the twisted Poincaré algebra is very simple, in fact it suffices to notice that the translation sector of this algebra remains classical at the level of commutators between the generators, so that the defining condition (2.36) gives the same result of the classical case and so the casimir remains invariate in form with respect to the one of the classical algebra:

$$\hat{\Box} \equiv \hat{P}^\mu \hat{P}_\mu, \tag{2.41}$$

and moreover it is map-independent, since the translation generators are the same assuming their classical action through any Weyl map. Also the equation of motion of a scalar field can be written in the Klein-Gordon commutative form:

$$\hat{P}^{\alpha}\hat{P}_{\alpha}\phi(\hat{x}) \equiv \hat{\Box}\phi(\hat{x}) = 0. \tag{2.42}$$

The importance of defining the equation of motion of the field using the casimir of the Poincaré algebra resides in the fact that in this way the algebra describes the symmetries also of the field theory, besides the ones of the spacetime. In fact the equation of motion (2.42) is invariant under the action of the transformation (2.14), in the sense that the transformed field $\phi(\hat{x})+d\phi(\hat{x})$ satisfies the same equation of the original one, as one can see in the following way[4]:

$$\Box(d\phi(\hat{x})) = i\hat{\Box}\left(\epsilon^{\alpha}\hat{P}_{\alpha} + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right)\phi(\hat{x}) = i\left(\epsilon^{\alpha}\hat{P}_{\alpha} + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right)(\hat{\Box}\phi(\hat{x})) = 0, \tag{2.43}$$

where in the second equivalence we used the fact that the casimir commutes with all the generators of the algebra and with the transformation parameters. It is important to emphasise that we could demonstrate the invariance of the equation of motion only thanks to the assumption (that will be treated in a deeper way in appendix 2.7) that the transformation parameters ϵ^{μ} and $\omega^{\mu\nu}$ commute with the translation generators $\hat{P}^{\mu}$. As we shall see later this assumption is of fundamental importance also in making the commutation rules (2.17) and (2.18) covariant under the action of the symmetry generators.

2.7 Covariance of transformation parameters and assumption $\left[\epsilon^{\alpha}, \hat{P}_{\beta}\right] = \left[\omega^{\mu\nu}, \hat{P}_{\beta}\right] = 0$

During our analysis it is of crucial importance the assumption that the transformation parameters should commute with the translation generators; this was necessary first of all to demonstrate the invariance of the equation of motion (2.42) and it will be used also in the following chapter to demonstrate the invariance of the actions (3.8) and (3.9) under the transformations (2.14) and (2.29), and to deduce the form of the currents associated to this invariance. In this section we want to explain why this assumption appears to be reasonable and deduce some interesting consequences about the covariance of the equations that define the transformation parameters, i.e. their commutation rules with coordinates.

When, in this chapter, we constructed the differential of functions under the transformations represented by the θ-Poincaré algebra generators, we tried to define the transformation parameters in the most conservative way possible, i.e. stating that they should be coordinate independent and should have an action on the functions on their right only by associative

[4]The demonstration of the invariance under the transformation (2.29) is very similar to this one

multiplication. The assumption that these parameters should commute with the translation generators $\hat{P}^\mu$ has been made to be coherent with the requirements just explained. In fact, in the commutative case the property of an object α of being coordinate independent can be expressed by the assertion that it satisfies the relation $P_\mu(\alpha f(x)) = \alpha P_\mu f(x)$, where P_μ is the (classical) translation generator and $f(x)$ is a generic commutative function. So in the noncommutative case we say that since the transformation parameters should be coordinate independent, they should satisfy the analogous relation:

$$\hat{P}_\mu\left(\alpha^A f(\hat{x})\right) = \alpha^A \hat{P}_\mu\left(f(\hat{x})\right),$$

where α^A is a generic transformation parameter with index A. In this analogy we are comforted by the observation that, looking at the action of the translation generators through the Weyl map, it is easy to see that they indeed satisfy this kind of relation with constant functions. Now, we do not know how to introduce inside the Weyl map the transformation parameters (because the Weyl map transforms commutative functions in noncommutative ones and the parameters are not functions), but since they are coordinate independent, we can think that the $\hat{P}_\mu$'s act on them as on constant functions. From the relation (2.7) written above it follows this commutation rule between parameters and translation generators:

$$\left[\hat{P}_\mu, \epsilon^\alpha\right] f(\hat{x}) \equiv \hat{P}_\mu\left(\epsilon^\alpha f(\hat{x})\right) - \epsilon^\alpha \hat{P}_\mu f(\hat{x}) \equiv \epsilon^\alpha \hat{P}_\mu f(\hat{x}) - \epsilon^\alpha \hat{P}_\mu f(\hat{x}) = 0,$$

so that the assumption (2.7) on the action of a transformation generator on the product of a parameter with a function results in an assumption on the commutation rules between parameters and translation generators.

We are now going to show that assuming a null commutator between of the translation generators and the parameters, the commutation relations (2.18) and (2.17) result covariant under translations, in the sense that if we apply to both sides of the equations the translation parameters $\hat{P}_\mu$ the equivalences still hold. Let us show this in detail. As regards the commutation rules of the translation parameters, for the left hand side of the equation it holds:

$$\begin{aligned} \hat{P}_\rho\left[f(\hat{x}), \epsilon^\alpha\right] &= \hat{P}_\rho(f(\hat{x})\epsilon^\alpha) - \hat{P}_\rho(\epsilon^\alpha f(\hat{x})) = (\hat{P}_\rho f(\hat{x}))\epsilon^\alpha - \epsilon^\alpha(\hat{P}_\rho f(\hat{x})) \\ &= \left[\hat{P}_\rho f(\hat{x}), \epsilon^\alpha\right] = -\frac{1}{2}\omega^{\mu\nu}\Upsilon^{\alpha\beta}_{\mu\nu}\hat{P}_\beta\hat{P}_\rho f(\hat{x}), \end{aligned} \tag{2.44}$$

while for the right hand side:

$$\hat{P}_\rho\left(-\frac{1}{2}\omega^{\mu\nu}\Upsilon^{\alpha\beta}_{\mu\nu}\hat{P}_\beta f(\hat{x})\right) = -\frac{1}{2}\omega^{\mu\nu}\Upsilon^{\alpha\beta}_{\mu\nu}\hat{P}_\rho\hat{P}_\beta f(\hat{x}), \tag{2.45}$$

and the two above expressions are equal. For the Lorentz sector parameter, $\hat{P}_\rho$ applied on the right hand side of the equation gives obviously zero, while applied on the left hand side:

$$\hat{P}_\rho[\omega_{\mu\nu}, f(\hat{x})] = \omega_{\mu\nu}\hat{P}_\rho f(\hat{x}) - \left(\hat{P}_\rho f(\hat{x})\right)\omega_{\mu\nu}, \tag{2.46}$$

that is zero since the parameter commutes with functions.

The covariance of the equations defining the transformation parameters assures us that the definition of the transformations is univocally determined and does not depend on the reference frame; so we would like to define also the action of the generators of the Lorentz sector on the transformation parameters in a way that the equations (2.18) and (2.17) result to be covariant also under the transformations defined by these generators.

As regards this last issue, in this work we shall be content to demonstrate that assuming a particular action of $\hat{M}_{\mu\nu}$ on the product of a function with a parameter, $\hat{M}_{\mu\nu}\left(f(\hat{x})\epsilon\right)$, the equations (2.18) and (2.17) can be made covariant. To do this, let us apply the Lorentz sector generators $\hat{M}_{\mu\nu}$ to the commutation rules (2.18) and (2.17). The action of $\hat{M}_{\mu\nu}$ on the right hand side of the first equation reads:

$$\begin{aligned} \hat{M}_{\mu\nu}\left[f(\hat{x}), \epsilon^{\alpha}\right] &= \hat{M}_{\mu\nu}\left(f(\hat{x})\epsilon^{\alpha}\right) - \hat{M}_{\mu\nu}\left(\epsilon^{\alpha} f(\hat{x})\right) \\ &= \hat{M}_{\mu\nu}\left(f(\hat{x})\epsilon^{\alpha}\right) - \epsilon^{\alpha}\hat{M}_{\mu\nu}\left(f(\hat{x})\right) - \left[\hat{M}_{\mu\nu}, \epsilon^{\alpha}\right] f(\hat{x}). \end{aligned} \tag{2.47}$$

Assuming that $\hat{M}_{\mu\nu}\left(f(\hat{x})\epsilon^{\alpha}\right) = \left(\hat{M}_{\mu\nu} f(\hat{x})\right)\epsilon^{\alpha}$ the above expression becomes:

$$\begin{aligned} \hat{M}_{\mu\nu}\left[f(\hat{x}), \epsilon^{\alpha}\right] &= \left[\left(\hat{M}_{\mu\nu} f(\hat{x})\right), \epsilon^{\alpha}\right] - \left[\hat{M}_{\mu\nu}, \epsilon^{\alpha}\right] f(\hat{x}) \\ &= -\frac{1}{2}\omega^{\rho\sigma}\Upsilon^{\alpha\beta}_{\rho\sigma}\hat{P}_{\beta}\hat{M}_{\mu\nu} f(\hat{x}) - \left[\hat{M}_{\mu\nu}, \epsilon^{\alpha}\right] f(\hat{x}), \end{aligned} \tag{2.48}$$

while applying $M_{\mu\nu}$ on the left hand side of (2.18) we get:

$$\begin{aligned} \hat{M}_{\mu\nu}\left(-\frac{1}{2}\omega^{\rho\sigma}\Upsilon^{\alpha\beta}_{\rho\sigma}\hat{P}_{\beta} f(\hat{x})\right) &= -\frac{1}{2}\omega^{\rho\sigma}\Upsilon^{\alpha\beta}_{\rho\sigma}\hat{M}_{\mu\nu}\hat{P}_{\beta} f(\hat{x}) - \frac{1}{2}\left[\omega^{\rho\sigma}, \hat{M}_{\mu\nu}\right]\Upsilon^{\alpha\beta}_{\rho\sigma}\hat{P}_{\beta} f(\hat{x}) \\ &= -\frac{1}{2}\omega^{\rho\sigma}\Upsilon^{\alpha\beta}_{\rho\sigma}\hat{P}_{\beta}\hat{M}_{\mu\nu} f(\hat{x}) - \frac{1}{2}\omega^{\rho\sigma}\Upsilon^{\alpha\beta}_{\rho\sigma}\left[\hat{M}_{\mu\nu}, \hat{P}_{\beta}\right] f(\hat{x}) + \\ &\quad -\frac{1}{2}\left[\omega^{\rho\sigma}, \hat{M}_{\mu\nu}\right]\Upsilon^{\alpha\beta}_{\rho\sigma}\hat{P}_{\beta} f(\hat{x}) \\ &= -\frac{1}{2}\omega^{\rho\sigma}\Upsilon^{\alpha\beta}_{\rho\sigma}\hat{P}_{\beta}\hat{M}_{\mu\nu} f(\hat{x}) - \frac{i}{2}\omega^{\rho\sigma}\Upsilon^{\alpha\beta}_{\rho\sigma}\eta_{\beta[\nu}P_{\mu]} f(\hat{x}) + \\ &\quad -\frac{1}{2}\left[\omega^{\rho\sigma}, \hat{M}_{\mu\nu}\right]\Upsilon^{\alpha\beta}_{\rho\sigma}\hat{P}_{\beta} f(\hat{x}). \end{aligned} \tag{2.49}$$

From a comparison of the two equations above we get a first condition on the commutators of the Lorentz sector generators with parameters:

$$-\left[\hat{M}_{\mu\nu}, \epsilon^{\alpha}\right] f(\hat{x}) = -\frac{i}{2}\omega^{\rho\sigma}\Upsilon^{\alpha\beta}_{\rho\sigma}\eta_{\beta[\nu}P_{\mu]} f(\hat{x}) - \frac{1}{2}\left[\omega^{\rho\sigma}, \hat{M}_{\mu\nu}\right]\Upsilon^{\alpha\beta}_{\rho\sigma}\hat{P}_{\beta} f(\hat{x}). \tag{2.50}$$

To gain a second condition we apply $\hat{M}_{\mu\nu}$ to both sides of equation (2.17). As regards the left hand side:

$$\begin{aligned} \hat{M}_{\mu\nu}\left[f(\hat{x}),\omega^{\alpha\beta}\right] &= \left(\hat{M}_{\mu\nu}f(\hat{x})\right)\omega^{\alpha\beta} - \hat{M}_{\mu\nu}\left(\omega^{\alpha\beta}f(\hat{x})\right) \\ &= \left[\hat{M}_{\mu\nu}f(\hat{x}),\omega^{\alpha\beta}\right] - \left[\hat{M}_{\mu\nu},\omega^{\alpha\beta}\right]f(\hat{x}) = -\left[\hat{M}_{\mu\nu},\omega^{\alpha\beta}\right]f(\hat{x}), \end{aligned} \tag{2.51}$$

where analogously to what it has been done for the case of the translation parameter we assumed $\hat{M}_{\mu\nu}\left(f(\hat{x})\omega^{\alpha\beta}\right) = \left(\hat{M}_{\mu\nu}f(\hat{x})\right)\omega^{\alpha\beta}$. Since the right hand side is null, we get the condition:

$$\left[\hat{M}_{\mu\nu},\omega^{\alpha\beta}\right]f(\hat{x}) = 0, \tag{2.52}$$

so that from the previous condition (2.50) we deduce:

$$\left[\hat{M}_{\mu\nu},\epsilon^{\alpha}\right]f(\hat{x}) = \frac{i}{2}\omega^{\rho\sigma}\Upsilon^{\alpha\beta}_{\rho\sigma}\eta_{\beta[\nu}P_{\mu]}f(\hat{x}) \tag{2.53}$$

So the system that defines the commutation rules that $\hat{M}_{\mu\nu}$ has to obey with the transformation parameters in order to make the defining relations of the parameters themselves covariant under the action of spatial rotations and boosts has a solution that moreover results to be unique.

Chapter 3

Noether analysis

We are looking for a physical theory invariant under the transformation that we defined in the previous chapter (Cf. Sec. 2.4), in order to verify if results analogous to those stated in the commutative Minkowski spacetime by the Noether theorem hold, and thus be reassured that this tansformation is a real symmetry. In particular we want to study a scalar, massless field theory defined by the equation of motion (2.42), that, as we have shown in the last section, is invariant under the action of the transformation (2.14) and makes us hope to find conserved charges associated to this invariance.

The first step of our search for conserved quantities will be to find, guided by the analogy with the classical case, an action that generates the desired equation of motion (2.42) and moreover results to be invariant itself under the action of the transformation we are considering. In this way we shall be allowed to follow step-by-step the (commutative) Noether analysis, that starts just from the study of the variation of the action that generates the theory in consideration.

3.1 Invariant actions in commutative spacetimes

Let us briefly recall the standard procedure used in classical field theory to evaluate the variation of a generic action under a certain transformation generated by a unitary operator A.

The generic action $S[\phi] = \int d^4x\mathcal{L}[\phi(x)]$ varies in the following way under a generic coordinate transformation $x \to x'(x)$ that produces a field variation $\phi(x) \to \phi'(x')$[1]:

$$\delta S = \int d^4x\mathcal{L}[\phi'(x')] - \mathcal{L}[\phi(x)] \equiv \int d^4x\mathcal{L}[\phi'(x')] - \mathcal{L}[\phi(x')] + \mathcal{L}[\phi(x')] - \mathcal{L}[\phi(x)] \qquad (3.1)$$

If the transformation is infinitesimal (regulated by the infinitesimal parameter ϵ) and is generated by the operator A, we shall write $dx \equiv x'(x) - x = i\epsilon Ax$.

[1] Note that we do not consider the variation of d^4x, since the transformation is assumed to be unitary.

The total variation of the field is then:

$$\delta_T\phi = \phi'(x') - \phi(x) = \phi'(x') - \phi(x') + \phi(x') - \phi(x) = \delta\phi(x') + d\phi = \delta\phi(x) + i\epsilon A\phi$$

where $\delta\phi$ is the variation of the functional form of the field (at the first order one has $\delta\phi(x') = \delta\phi(x)$), while $d\phi(x)$ is the differential.

In this case the action variation (3.1) can be written as:

$$\delta S = \int d^4x\mathcal{L}[\phi(x') + \delta\phi(x')] - \mathcal{L}[\phi(x')] + d\mathcal{L} = \int d^4x\, \delta\mathcal{L}[\phi(x')] + i\epsilon A\mathcal{L}[\phi(x)],$$

where $\delta\mathcal{L}$ is the functional variation of the Lagrangian.

Let us remember here a fact that will be important later, i.e. that if the field is invariant under the action of A its total variation is null, then:

$$\delta_T\phi = 0 \quad\Rightarrow\quad \delta\phi(x) = -d\phi(x) = -i\epsilon A\phi(x). \tag{3.2}$$

Note that at the moment we are not making assumptions on the invariance of the Lagrangian under the action of A, so we can not write $\delta\mathcal{L} = -d\mathcal{L}$.

Let us now specialise to the case of the actions

$$S_1 = \int d^4x\mathcal{L}_1 = \frac{1}{2}\int d^4x\; \phi(x)P^\alpha P_\alpha\phi(x) \tag{3.3}$$

and

$$S_2 = \int d^4x\mathcal{L}_2 = \frac{1}{2}\int d^4x\; (P^\alpha\phi)P_\alpha\phi, \tag{3.4}$$

that generate variationally the equation of motion $\Box\phi(x) = 0$, i.e. the commutative version of (2.42).

Neglecting terms of the second order in the variation of the field $\delta\phi$ the functional variations of the Lagrangians are:

$$\begin{aligned}
\delta\mathcal{L}_1[\phi(x')] &\equiv \mathcal{L}_1[\phi(x') + \delta\phi(x')] - \mathcal{L}_1[\phi(x')] = \frac{1}{2}[\delta\phi(x')P^\alpha P_\alpha\phi(x') + \phi(x')P^\alpha P_\alpha\delta\phi(x')] \\
\delta\mathcal{L}_2[\phi(x')] &= \frac{1}{2}[P_\alpha(\delta\phi)P^\alpha\phi + (P_\alpha\phi)P^\alpha(\delta\phi)]
\end{aligned} \tag{3.5}$$

and so the action variation is, for S_1:

$$\begin{aligned}
\delta S_1 &= \frac{1}{2}\int d^4x\; \delta\phi(x')P^\alpha P_\alpha\phi(x') + \phi(x')P^\alpha P_\alpha\delta\phi(x') + i\int d^4x\; \epsilon A\mathcal{L}_1[\phi(x)] \\
&= \frac{1}{2}\int d^4x\; \delta\phi(x)P^\alpha P_\alpha\phi(x) + \phi(x)P^\alpha P_\alpha\delta\phi(x) + 2i\epsilon A\mathcal{L}_1[\phi(x)] \\
&= -\frac{i}{2}\int d^4x\; (\epsilon A\phi(x))P^\alpha P_\alpha\phi(x) + \phi(x)P^\alpha P_\alpha(\epsilon A\phi(x)) - 2\epsilon A\mathcal{L}_1[\phi(x)],
\end{aligned} \tag{3.6}$$

where in the first equivalence we changed the variable of integration assuming the transformation generated by A to be unitary (and so with unitary Jacobian), while in the second equivalence we exploited the relation (3.2). With analogous steps we find the variation of S_2 to be:

$$\delta S_2 \quad = \quad -\frac{i}{2}\int d^4x\ P_\alpha(\epsilon A\phi(x))P^\alpha\phi(x) + (P_\alpha\phi(x))P^\alpha(\epsilon A\phi(x)) - 2\epsilon A\mathcal{L}_2[\phi(x)] \tag{3.7}$$

Now we have found the expression of the variation of the actions (3.3) and (3.4) under a generic unitary transformation; a similar procedure can be applied for all the actions that have quadratic combinations of the fields. To check if the actions (3.3) and (3.4) are invariant under the transformation generated by A it is just necessary to verify (3.6) and (3.7) to be null, without using the equation of motion of the field, since the action must be invariant if calculated on every field and not only along the field that minimises it. We do not perform the explicit calculation in this classical case, since it is straightforward.

3.2 Invariance of tha actions in canonical spacetime

Let us now return to θ-Minkowski spacetime. We shall demonstrate that the following two actions are invariant under the twisted Poincaré algebra:

$$S = \frac{1}{2}\int d^4\hat{x}\ \phi(\hat{x})\hat{\Box}\phi(\hat{x}) \equiv \frac{1}{2}\int d^4\hat{x}\ \phi(\hat{x})\hat{P}^\alpha\hat{P}_\alpha\phi(\hat{x}) \tag{3.8}$$

and:

$$S' = \frac{1}{2}\int d^4\hat{x}\ \left(\hat{P}^\alpha\phi(\hat{x})\right)\hat{P}_\alpha\phi(\hat{x}). \tag{3.9}$$

These are the noncommuting analogous of (3.3) and (3.4) and are map-independent, since the translation generators are the same in all the maps. The equation of motion (2.42) holds for fields that minimise both of the actions, as it is shown in the next section, and moreover we shall find that these actions generate the same currents associated to invariance under the transformations of the Poincaré algebra, so that we can say that they are phisicall equivalent. Nevertheless it is instructive to perform anyway the explicit calculation with both of the actions; in fact, classically the two actions are equivalent up to border terms of the kind $\int d^4x\ P^\alpha\left[\phi(x)P_\alpha\phi(x)\right]$, that we set to zero thanks to the divergence theorem; but in the noncommutative case we do not have at our disposal a similar theorem, thus we can not state *a priori* the equivalence of the two actions, and the following equivalence of the currents. So it is interesting the fact that we still find, through an explicit calculation, that the two actions result to be equivalent also as reguards conserved quantities, thanks to a "good" behaviour of the transformation parameters.

In the following we shall demonstrate the invariance of the two actions under the coordinate transformation $\hat{x}^\rho \to \hat{x}^\rho + d\hat{x}^\rho \equiv \hat{x}^\rho - (\epsilon^\alpha\delta^\rho_\alpha + \omega^{\mu\nu}\hat{x}_{[\mu}\delta^\rho_{\nu]})$ that is of the kind of (2.14). In the next section we shall calculate the currents associated to this invariance. We start from the formulas (3.6) and (3.7) respectively, that have been obtained only through steps that are allowed also in the noncommutative case. Substituting $\epsilon A \to \left[\epsilon^\mu \hat{P}_\mu + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right]$ we get [2]:

$$\begin{aligned}\delta S &= -\frac{i}{2}\int d^4\hat{x}\, \left(\left[\epsilon^\mu \hat{P}_\mu + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x})\right)\hat{P}^\alpha\hat{P}_\alpha\phi(\hat{x}) + \phi(\hat{x})\hat{P}^\alpha\hat{P}_\alpha\left[\epsilon^\mu \hat{P}_\mu + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x}) + \\ &\quad - \left[\epsilon^\mu \hat{P}_\mu + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\left[\phi(\hat{x})\hat{P}^\alpha\hat{P}_\alpha\phi(\hat{x})\right]. \end{aligned} \tag{3.10}$$

Since $\epsilon^\mu \hat{P}_\mu + \omega^{\mu\nu}\hat{M}_{\mu\nu}$ satisfies the Leibniz rule for construction:

$$\delta S = -\frac{i}{2}\int d^4\hat{x}\ -\phi(\hat{x})(\left[\epsilon^\mu \hat{P}_\mu + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\hat{P}^\alpha\hat{P}_\alpha\phi(\hat{x})) + \phi(\hat{x})\hat{P}^\alpha\hat{P}_\alpha(\left[\epsilon^\mu \hat{P}_\mu + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x})), \tag{3.11}$$

and this expression is zero since $\hat{P}^\alpha\hat{P}_\alpha$ commutes both with the generators of the algebra and with the parameters ϵ^μ and $\omega^{\mu\nu}$, since we assumed the transformation parameters to commute with the translation generators (Cf. appendix 2.7).

So the action S is invariant under the action of the θ-Poincaré algebra generators. Let us see what happens for the action S':

$$\begin{aligned}\delta S' &= -\frac{i}{2}\int d^4\hat{x}\ \hat{P}_\alpha(\left[\epsilon^\mu \hat{P}_\mu + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x}))\hat{P}^\alpha\phi(\hat{x}) + (\hat{P}_\alpha\phi(\hat{x}))\hat{P}^\alpha(\left[\epsilon^\mu \hat{P}_\mu + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x})) + \\ &\quad - \left[\epsilon^\mu \hat{P}_\mu + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\left[\left(\hat{P}^\alpha\phi(\hat{x})\right)\hat{P}_\alpha\phi(\hat{x})\right] \qquad (3.12)\\ &= \frac{1}{2}\int d^4\hat{x}\ \omega^{\mu\nu}(\hat{P}_{[\nu}\phi(\hat{x}))\hat{P}_{\mu]}\phi(\hat{x}) + (\hat{P}_{[\mu}\phi(\hat{x}))\omega^{\mu\nu}\hat{P}_{\nu]}\phi(\hat{x}), \qquad (3.13)\end{aligned}$$

where we used:

$$[P_\alpha, \epsilon^\mu P_\mu + \omega^{\mu\nu}M_{\mu\nu}] = \omega^{\mu\nu}[P_\alpha, M_{\mu\nu}] = i\omega^{\mu\nu}\eta_{\alpha[\mu}P_{\nu]}. \tag{3.14}$$

Commuting $\omega^{\mu\nu}$ with $P_\mu\phi(x)$, using (2.17), we get:

$$\delta S' = \frac{1}{2}\int d^4x\ \omega^{\mu\nu}(P_{[\nu}\phi(x))P_{\mu]}\phi(x) - \omega^{\mu\nu}(P_{[\nu}\phi(x))P_{\mu]}\phi(x) = 0 \tag{3.15}$$

So the two actions are both invariant, so that it makes sense to perform the Noether analysis in order to find currents and then conserved charges associated to this invariance.

3.3 Deriving the equation of motion from an action

We want to demonstrate that the equation of motion (2.42), $\Box\phi(\hat{x}) = 0$, is generated by both of the actions (3.8) and (3.9):

$$S = \frac{1}{2}\int d^4\hat{x}\ \phi(\hat{x})\hat{\Box}\phi(\hat{x}) \tag{3.16}$$

$$S' = \frac{1}{2}\int d^4\hat{x}\ \left(\hat{P}^\alpha\phi(\hat{x})\right)\hat{P}_\alpha\phi(\hat{x}). \tag{3.17}$$

[2] In this case, and in the following, it is not possible to use the ciclicity of the integral of a product of two functions because of the presence of the transformation parameters inside the integral itself.

To do this, following the variational method, we write a generic variation of the field $\delta\phi$ and calculate the corresponding variation of the action, imposing it to be null for every variation of the field, that thus results to be the one that minimises the action.

Let us consider first the action S:[3]

$$S[\phi+\delta\phi]-S[\phi]=\frac{1}{2}\int d^4\hat{x}\ \left[(\phi+\delta\phi)\hat{\Box}(\phi+\delta\phi)-\phi\hat{\Box}\phi\right]. \tag{3.18}$$

Taking only the first order in the variation of the field:

$$S[\phi+\delta\phi]-S[\phi]=\frac{1}{2}\int d^4\hat{x}\ \left[(\delta\phi)\hat{\Box}\phi+\phi\hat{\Box}\delta\phi\right]. \tag{3.19}$$

Now we use a little trick: we write the expression above in terms of the Fourier transform of the field ϕ and of the variation of the field $\delta\phi$, considered as an independent function (we use the symmetric basis of exponentials $e^{ik\hat{x}}$ for the expansion, so when writing, in the following, $\tilde{\phi}(k)$ we intend $\tilde{\phi}_{(w)}(k)$):

$$S[\phi+\delta\phi]-S[\phi]=\frac{1}{2}\int d^4\hat{x}d^4kd^4q\ \left[e^{ik\hat{x}}\hat{\Box}e^{iq\hat{x}}+e^{iq\hat{x}}\hat{\Box}e^{ik\hat{x}}\right]\tilde{\delta\phi}(k)\tilde{\phi}(q). \tag{3.20}$$

Now we can apply the operator $\hat{\Box}$ to the exponentials:

$$S[\phi+\delta\phi]-S[\phi]=\frac{1}{2}\int d^4\hat{x}d^4kd^4q\ \left[q^\mu q_\mu e^{ik\hat{x}}e^{iq\hat{x}}+k^\mu k_\mu e^{iq\hat{x}}e^{ik\hat{x}}\right]\tilde{\delta\phi}(k)\tilde{\phi}(q). \tag{3.21}$$

To apply the definition (1.84) of the $\delta^{(4)}(k)$ we need to rewrite the exponentials through the BCH formula (1.11):

$$\begin{aligned} S[\phi+\delta\phi]-S[\phi] &= \frac{1}{2}\int d^4\hat{x}d^4kd^4q\ \left[q^\mu q_\mu e^{-\frac{i}{2}k^\mu q^\nu\theta_{\mu\nu}}+k^\mu k_\mu e^{-\frac{i}{2}q^\mu k^\nu\theta_{\mu\nu}}\right]\tilde{\delta\phi}(k)\tilde{\phi}(q)e^{i(q+k)\hat{x}} \\ &= \frac{1}{2}\int d^4kd^4q\ \left[q^\mu q_\mu e^{-\frac{i}{2}k^\mu q^\nu\theta_{\mu\nu}}+k^\mu k_\mu e^{-\frac{i}{2}q^\mu k^\nu\theta_{\mu\nu}}\right]\tilde{\delta\phi}(k)\tilde{\phi}(q)\delta^{(4)}(k+q). \end{aligned} \tag{3.22}$$

Integrating in one of the Fourier parameters:

$$S[\phi+\delta\phi]-S[\phi]=\int d^4k\ k^\mu k_\mu\tilde{\delta\phi}(k)\tilde{\phi}(-k), \tag{3.23}$$

where the exponentials containing $\theta_{\mu\nu}$ have disappeared because of the antisymmetry of this matrix. Now, since we want the variation of the action to be null for an arbitrary variation of the field $\delta\phi$, in order the above integral to be null it is necessary that $k^\mu k_\mu\tilde{\phi}(k)=0$. From this it follows:

$$k^\mu k_\mu\tilde{\phi}(k)=0\Rightarrow\int d^4k\ k^\mu k_\mu\tilde{\phi}(k)e^{ik\hat{x}}\equiv\int d^4k\ \tilde{\phi}(k)\Box e^{ik\hat{x}}\equiv\Box\phi(\hat{x})=0, \tag{3.24}$$

[3]To enlighten the notation we write just ϕ instead of $\phi(\hat{x})$

that is the equation of motion we were looking for.

Let us follow the same steps to find the equation of motion generated by the action S':

$$\begin{aligned}
S'[\phi+\delta\phi]-S'[\phi] &= \frac{1}{2}\int d^4\hat{x}\ \left[\left(\hat{P}^\mu(\phi+\delta\phi)\right)\hat{P}_\mu(\phi+\delta\phi)-\left(\hat{P}^\mu\phi\right)\hat{P}_\mu\phi\right] \\
&= \frac{1}{2}\int d^4\hat{x}\ \left[\left(\hat{P}^\mu(\delta\phi)\right)\hat{P}_\mu\phi+\left(\hat{P}^\mu\phi\right)\hat{P}_\mu(\delta\phi)\right] \\
&= \frac{1}{2}\int d^4\hat{x}d^4kd^4q\ \left[\left(\hat{P}^\mu e^{ik\hat{x}}\right)\hat{P}_\mu e^{iq\hat{x}}+\left(\hat{P}^\mu e^{iq\hat{x}}\right)\hat{P}_\mu(e^{ik\hat{x}})\right]\tilde{\delta\phi}(k)\tilde{\phi}(q) \\
&= \frac{1}{2}\int d^4\hat{x}d^4kd^4q\ \left[k^\mu q_\mu e^{ik\hat{x}}e^{iq\hat{x}}+q^\mu k_\mu e^{iq\hat{x}}(e^{ik\hat{x}})\right]\tilde{\delta\phi}(k)\tilde{\phi}(q) \\
&= \frac{1}{2}\int d^4kd^4q\ \left[k^\mu q_\mu e^{ik^\mu q^\nu\theta_{\mu\nu}}+q^\mu k_\mu(e^{iq^\mu k^\nu\theta_{\mu\nu}})\right]\tilde{\delta\phi}(k)\tilde{\phi}(q)\delta^{(4)}(k+q) \\
&= \int d^4k\ [-k^\mu k_\mu]\,\tilde{\delta\phi}(k)\tilde{\phi}(-k).
\end{aligned} \tag{3.25}$$

Imposing this last expression to be null for any variation $\delta\phi$ it follows:

$$k^\mu k_\mu\tilde{\phi}(k)=0\ \Rightarrow\ \Box\phi(\hat{x})=0. \tag{3.26}$$

3.4 Currents

Let us now go on with the Noether analysis. Classically, this kind of analysis for an action that is known to be invariant under a certain transformation generated by the operators G_A (A is a generic set of indices) is accomplished writing the action variation in the form of the integral of a four-divergence (using the equation of motion).

$$\delta S=\int d^4x\ \epsilon^A P_\mu J^\mu_A=\epsilon^A\int d^4x\ P_\mu J^\mu_A. \tag{3.27}$$

Since for a symmetry transformation the action variation must be null for an arbitrary value of the transformation parameter, the integral must be null, so that we have:

$$\int d^4xP_iJ^i_A+P_0J^0_A=0. \tag{3.28}$$

Applying the divergence theorem we get:

$$\int dx_0\ J^i(\vec{x},x_0)|_\Sigma=\int d^3x[J^0(\vec{x},T_2)-J^0(\vec{x},T_1)], \tag{3.29}$$

where Σ is the surface around the (infinite) spatial volume in which we integrate with the $\int d^3x$, while T_1 and T_2 are the extreme of integration (placed at plus and minus infinite time) of the temporal integral. J^i results to be null when calculated on the surface Σ, that is the boundary of an infinite space volume, so we have:

$$0 = \int d^3x [J^0(T_2) - J^0(T_1)]. \tag{3.30}$$

This equation states the conservation of the quantity $\int d^3x J^0$, that so is the time independent charge associated to the invariance of the theory under the action of the transformation G [4].

In the noncommutative case we shall see that we are not able to follow the same steps of the classical analysis (since we have not a divergence theorem), but, since we are able to put the variation in the same four-divergence form of the classical case, we proceed by analogy and we obtain conserved charges (Cf. also Sec. 3.5).

Let us first perform the analysis for the first action, S, starting from the expression (3.10). Using the equation of motion (2.42) it becomes:

$$\delta S \;=\; -\frac{i}{2}\int d^4\hat{x}\; \phi(\hat{x})\hat{P}^\alpha \hat{P}_\alpha(\left[\epsilon^\mu \hat{P}_\mu + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x})). \tag{3.31}$$

Using the trivial coproduct of $\hat{P}_\alpha$:

$$\begin{aligned}\delta S \;&=\; -\frac{i}{2}\int d^4\hat{x}\; \hat{P}^\alpha \left[\phi(\hat{x})\hat{P}_\alpha(\left[\epsilon^\mu \hat{P}_\mu + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x}))\right] - \left[\hat{P}^\alpha\phi(\hat{x})\right]\hat{P}_\alpha(\left[\epsilon^\mu \hat{P}_\mu + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x}))\\ &=\; -\frac{i}{2}\int d^4\hat{x}\; \hat{P}^\alpha \left[\phi(\hat{x})\hat{P}_\alpha(\left[\epsilon^\mu \hat{P}_\mu + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x})) - \left[\hat{P}_\alpha\phi(\hat{x})\right](\left[\epsilon^\mu \hat{P}_\mu + \omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x}))\right].\end{aligned} \tag{3.32}$$

Now we take all the transformation parameters on the same side of the integrand, e.g. on the left. We can do it just exploiting the commutation relations with functions (2.18) and (2.17) and assuming that the parameters commute with the translation generators, as we did in demonstrating the invariance of the action:

$$\begin{aligned}\delta S \;=\; -\frac{i}{2}\int d^4\hat{x}\; \hat{P}^\alpha \Big[&\epsilon^\mu \phi(\hat{x})\hat{P}_\alpha\hat{P}_\mu\phi(\hat{x}) + \omega^{\mu\nu}\phi(\hat{x})\hat{P}_\alpha\hat{M}_{\mu\nu}\phi(\hat{x}) - \epsilon^\mu\left(\hat{P}_\alpha\phi(\hat{x})\right)\hat{P}_\mu\phi(\hat{x}) +\\ &-\omega^{\mu\nu}\left(\hat{P}_\alpha\phi(\hat{x})\right)\hat{M}_{\mu\nu}\phi(\hat{x}) - \frac{1}{2}\omega^{\mu\nu}\Upsilon^{\sigma\rho}_{\mu\nu}\left[\left(\hat{P}_\rho\phi(\hat{x})\right)\hat{P}_\alpha\hat{P}_\sigma\phi(\hat{x}) - \left(\hat{P}_\rho\hat{P}_\alpha\phi(\hat{x})\right)\hat{P}_\sigma\phi(\hat{x})\right]\Big],\end{aligned} \tag{3.33}$$

where $\Upsilon^{\sigma\rho}_{\mu\nu}$ was defined in (2.19). We have thus obtained an expression of the kind:

$$\delta S = -\frac{i}{2}\int d^4\hat{x}\; \epsilon^\mu \hat{P}_\alpha T^\alpha_\mu + \omega^{\mu\nu}\hat{P}_\alpha J^\alpha_{\mu\nu}, \tag{3.34}$$

[4]More precisely, it is possible to demonstrate the conservation of the charge $\int d^3x\; J^0$ in an arbitrary time interval performing the same kind of analysis at the level of Lagrangian, i.e. in a differential form. The technical difficulties that one would encounter trying to follow this analysis in a noncommutative spacetime are of the same kind of the ones encountered when carrying out the analysis at the level of the action.

where:

$$
\begin{aligned}
T^{\alpha}_{\mu} &= \phi(\hat{x})\hat{P}^{\alpha}\hat{P}_{\mu}\phi(\hat{x}) - \left(\hat{P}^{\alpha}\phi(\hat{x})\right)\hat{P}_{\mu}\phi(\hat{x}) \\
J^{\alpha}_{\mu\nu} &= \phi(\hat{x})\hat{P}^{\alpha}\hat{M}_{\mu\nu}\phi(\hat{x}) - \left(\hat{P}^{\alpha}\phi(\hat{x})\right)\hat{M}_{\mu\nu}\phi(\hat{x}) + \\
&\quad -\frac{1}{2}\Upsilon^{\sigma\rho}_{\mu\nu}\left[\left(\hat{P}_{\rho}\phi(\hat{x})\right)\hat{P}^{\alpha}\hat{P}_{\sigma}\phi(\hat{x}) - \left(\hat{P}_{\rho}\hat{P}^{\alpha}\phi(\hat{x})\right)\hat{P}_{\sigma}\phi(\hat{x})\right]
\end{aligned}
\tag{3.35}
$$

Now, if we were dealing with commutative objects, as explained at the beginning of this section, we would have stated that, since the variation of the action is known to be null for every value of the transformation parameters ϵ^{μ} and $\omega^{\mu\nu}$ (the action is invariant) , then the two integrals (the one proportional to ϵ^{μ} and the one proportional to $\omega^{\mu\nu}$) must be separately null themselves. At this point we would have applied the divergence theorem to the two integrals $\int d^4x\ P_{\alpha}T^{\alpha}_{\mu}$ and $\int d^4x\ P_{\alpha}J^{\alpha}_{\mu\nu}$ to deduce the time independence of the quantities: $\int d^3x\ T^0_{\mu}$ and $\int d^3x\ J^0_{\mu\nu}$, that are thus the conserved charges associated to the invariance of the action under the translation and Lorentz sector transformations, respectively. In noncommutative spacetime we cannot follow these steps, because we do not know if a sort of divergence theorem holds. So we have to be content with proceeding in analogy with the commutative case and defining the "would-be-conserved-currents" as the terms in the integral (3.34) on which the translation generators P^{α} act (i.e. the quantities in (3.35)) and the "would-be-conserved charges" as the spatial integral (defined in Sec. 1.8) of the zeroth component of these currents:

$$
Q_{\mu} \equiv \int d^3\hat{x}\ T^0_{\mu} = \int d^3\hat{x}\ \phi(x)\hat{P}^0\hat{P}_{\mu}\phi(\hat{x}) - \left(\hat{P}^0\phi(\hat{x})\right)\hat{P}_{\mu}\phi(\hat{x}) \tag{3.36}
$$

$$
\begin{aligned}
K_{\mu\nu} \equiv \int d^3\hat{x}\ J^0_{\mu\nu} &= \int d^3\hat{x}\ \phi(\hat{x})\hat{P}^0\hat{M}_{\mu\nu}\phi(\hat{x}) - \left(\hat{P}^0\phi(\hat{x})\right)\hat{M}_{\mu\nu}\phi(\hat{x}) + \\
&\quad -\frac{1}{2}\Upsilon^{\alpha\beta}_{\mu\nu}\left[\left(\hat{P}_{\beta}\phi(\hat{x})\right)\hat{P}^0\hat{P}_{\alpha}\phi(\hat{x}) - \left(\hat{P}_{\beta}\hat{P}^0\phi(\hat{x})\right)\hat{P}_{\alpha}\phi(\hat{x})\right]
\end{aligned}
\tag{3.37}
$$

(3.38)

Obviously we can not in principle state anything about the time independence of the charges just defined, so in the next section we shall explicitate them to verify that they are indeed conserved. As regards the physical interpretation of the "would-be-conserved charges" (and currents) associated to the transformation parameter $\omega^{\mu\nu}$, we encounter some ambiguities. In fact, in Sec. 2.3 we showed that it is not possible to execute pure Lorentz sector transformations, so that it is not clear the meaning of saying that (3.37) is the charge associated to invariance under spatial rotations and boosts. Instead it makes sense to associate the charge (3.36) to invariance under translations, since pure translation transformations are allowed.

Let us now follow the steps just described to calculate the currents associated to the invariance of the action S' under the same Poincaré transformations. We do this to show another interesting case (besides the one that permits us to obtain conserved charges just proceeding by analogy with the classical case, even if we do not have at our disposal the divergence theorem) in which, even without all the tools of the commutative analysis, we find, after explicit calculations, that the same results hold, that we would have expected in analogy with the commutative case. In fact, as we have yet explained in Sec. 3.2, the commutative counterparts of S and S', i.e. S_1 and S_2, are equivalent up to border terms (that are null for the divergence theorem), so we know "*a priori*" that they will lead to the same conserved currents when we perform Noether analysis (and so are physically equivalent). Also the actions S and S' differ just for a four-divergence, but we can not say, without an explicit verification, that they are equivalent since, without the divergence theorem, we can not state that the integral of the four-divergence is null. Here we show that, besides all, the two actions do lead to the same currents associated to invariance under the twisted Poincaré algebra, and so are physically equivalent. We start from equation (3.12) and use the (trivial) coproduct of the translation generator $\hat{P}_\alpha$:

$$\begin{aligned}\delta S' &= -\frac{i}{2}\int d^4\hat{x}\hat{P}_\alpha\left\{\left(\left[\epsilon^\mu\hat{P}_\mu+\omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x})\right)\hat{P}^\alpha\phi(\hat{x})\right\}+\\&\quad-\left(\left[\epsilon^\mu\hat{P}_\mu+\omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x})\right)\hat{P}_\alpha\hat{P}^\alpha\phi(\hat{x})+\\&\quad+\hat{P}^\alpha\left\{\left(\hat{P}_\alpha\phi(\hat{x})\right)\left[\epsilon^\mu\hat{P}_\mu+\omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x})\right\}-\left(\hat{P}_\alpha\hat{P}^\alpha\phi(\hat{x})\right)\left[\epsilon^\mu\hat{P}_\mu+\omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x})+\\&\quad-\left[\epsilon^\mu\hat{P}_\mu+\omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\hat{P}^\alpha\left[\phi(\hat{x})\hat{P}_\alpha\phi(\hat{x})\right]+\left[\epsilon^\mu\hat{P}_\mu+\omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\left[\phi(\hat{x})\hat{P}^\alpha\hat{P}_\alpha\phi(\hat{x})\right].\end{aligned}\tag{3.39}$$

Using the equation of motion and taking the generator $\hat{P}_\alpha$ that is in the last line of the equation above on the left of $\left[\epsilon^\mu\hat{P}_\mu+\omega^{\mu\nu}\hat{M}_{\mu\nu}\right]$:

$$\begin{aligned}\delta S' &= -\frac{i}{2}\int d^4\hat{x}\;\hat{P}_\alpha\left\{\left(\left[\epsilon^\mu\hat{P}_\mu+\omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x})\right)\hat{P}^\alpha\phi(\hat{x})+\left(\hat{P}^\alpha\phi(\hat{x})\right)\left[\epsilon^\mu\hat{P}_\mu+\omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x})+\right.\\&\quad\left.-\left[\epsilon^\mu\hat{P}_\mu+\omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\left[\phi(\hat{x})\hat{P}_\alpha\phi(\hat{x})\right]\right\}+i\omega^{\mu\nu}\eta_{\alpha[\mu}\hat{P}_{\nu]}\left[\phi(\hat{x})\hat{P}^\alpha\phi(\hat{x})\right]\\&= -\frac{i}{2}\int d^4\hat{x}\;\hat{P}_\alpha\left\{\left(\hat{P}^\alpha\phi(\hat{x})\right)\left[\epsilon^\mu\hat{P}_\mu+\omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\phi(\hat{x})-\phi(\hat{x})\left[\epsilon^\mu\hat{P}_\mu+\omega^{\mu\nu}\hat{M}_{\mu\nu}\right]\hat{P}_\alpha\phi(\hat{x})\right\}+\\&\quad+i\omega^{\mu\nu}\eta_{\alpha[\mu}\hat{P}_{\nu]}\left[\phi(\hat{x})\hat{P}^\alpha\phi(\hat{x})\right],\end{aligned}\tag{3.40}$$

where in the second passage we just applied the Leibniz rule for $\epsilon^\mu\hat{P}_\mu+\omega^{\mu\nu}\hat{M}_{\mu\nu}$. Taking on the left the transformation parameters:

$$\begin{aligned}
\delta S' &= -\frac{i}{2}\int d^4\hat{x}\ \hat{P}_\alpha\Big[\epsilon^\mu\left(\hat{P}^\alpha\phi(\hat{x})\right)\hat{P}_\mu\phi(\hat{x})+\omega^{\mu\nu}\left(\hat{P}^\alpha\phi(\hat{x})\right)\hat{M}_{\mu\nu}\phi(\hat{x})-\epsilon^\mu\phi(\hat{x})\hat{P}_\mu\hat{P}_\alpha\phi(\hat{x})+\\
&\quad -\omega^{\mu\nu}\phi(\hat{x})\hat{M}_{\mu\nu}\hat{P}_\alpha\phi(\hat{x})+\frac{1}{2}\omega^{\mu\nu}\Upsilon^{\sigma\rho}_{\mu\nu}\left[\left(\hat{P}_\rho\phi(\hat{x})\right)\hat{P}_\alpha\hat{P}_\sigma\phi(\hat{x})-\left(\hat{P}_\rho\hat{P}_\alpha\phi(\hat{x})\right)\hat{P}_\sigma\phi(\hat{x})\right]\Big]+\\
&\quad +i\omega^{\mu\nu}\eta_{\alpha[\mu}\hat{P}_{\nu]}\left[\phi(\hat{x})\hat{P}^\alpha\phi(\hat{x})\right]\\
&= -\frac{i}{2}\int d^4\hat{x}\ \hat{P}_\alpha\Big[\epsilon^\mu\left(\hat{P}^\alpha\phi(\hat{x})\right)\hat{P}_\mu\phi(\hat{x})+\omega^{\mu\nu}\left(\hat{P}^\alpha\phi(\hat{x})\right)\hat{M}_{\mu\nu}\phi(\hat{x})-\epsilon^\mu\phi(\hat{x})\hat{P}_\mu\hat{P}_\alpha\phi(\hat{x})+\\
&\quad -\omega^{\mu\nu}\phi(\hat{x})\hat{P}_\alpha\hat{M}_{\mu\nu}\phi(\hat{x})+\frac{1}{2}\omega^{\mu\nu}\Upsilon^{\sigma\rho}_{\mu\nu}\left[\left(\hat{P}_\rho\phi(\hat{x})\right)\hat{P}_\alpha\hat{P}_\sigma\phi(\hat{x})-\left(\hat{P}_\rho\hat{P}_\alpha\phi(\hat{x})\right)\hat{P}_\sigma\phi(\hat{x})\right]\Big]+\\
&\quad +i\omega^{\mu\nu}\hat{P}_{[\mu}\left[\phi(\hat{x})\hat{P}_{\nu]}\phi(\hat{x})\right]+i\omega^{\mu\nu}\hat{P}_{[\nu}\left[\phi(\hat{x})\hat{P}_{\mu]}\phi(\hat{x})\right], \qquad (3.41)
\end{aligned}$$

where evidently the last line is null. So also the variation of this action can be put in the reassuring form:

$$\delta S' = -\frac{i}{2}\int d^4\hat{x}\ \epsilon^\mu\hat{P}_\alpha T'^{\alpha}_{\mu}+\omega^{\mu\nu}\hat{P}_\alpha J'^{\alpha}_{\mu\nu}, \qquad (3.42)$$

where:

$$\begin{aligned}
T'^{\alpha}_{\mu} &= -\Big\{\phi(\hat{x})\hat{P}^\alpha\hat{P}_\mu\phi(\hat{x})-\left(\hat{P}^\alpha\phi(\hat{x})\right)\hat{P}_\mu\phi(\hat{x})\Big\}\\
J'^{\alpha}_{\mu\nu} &= -\Big\{\phi(\hat{x})\hat{P}^\alpha\hat{M}_{\mu\nu}\phi(\hat{x})-\left(\hat{P}^\alpha\phi(\hat{x})\right)\hat{M}_{\mu\nu}\phi(\hat{x})+\\
&\quad -\frac{1}{2}\Upsilon^{\sigma\rho}_{\mu\nu}\left[\left(\hat{P}_\rho\phi(\hat{x})\right)\hat{P}^\alpha\hat{P}_\sigma\phi(\hat{x})-\left(\hat{P}_\rho\hat{P}^\alpha\phi(\hat{x})\right)\hat{P}_\sigma\phi(\hat{x})\right]\Big\}.
\end{aligned} \qquad (3.43)$$

We see that the currents associated to the invariance of the action S' are the same, up to a sign that has no physical importance, of those associated to the action S.

3.4.1 Currents generated by non symmetric basis of generators

In the previous section we derived the currents associated to the transformations of the twisted Poincaré algebra using for the variation of the fields the expression of the differential (2.14), obtained using the symmetric basis of the algebra. But in Sec. 2.5 we defined an other differential, (2.29), associated to the x_1-to-the-right basis of the algebra, that resulted to be different from the previous one. In order to verify is this difference has some physical consequences, we shall perform the Noether analysis also with this other differential, to check if it leads to charges different from those associated to the transformation (2.14). Obviously, it is not necessary to change the action (3.8) in order to express it in term of the new basis, since it is basis independent; in fact it contains only the operator $\Box$, that is the same in all the bases (Cf. Sec. 2.6). Moreover we shall not report here the demonstration that the action is invariant also under the action of this transformation, since it is very similar to the one performed for the symmetric basis of generators.

We are now going to derive the currents associated to this transformation, starting from the equation (3.32), in which we have only to substitute $\epsilon^\mu \hat{P}_\mu + \omega^{\mu\nu}\hat{M}_{\mu\nu}$ with $\epsilon^\beta_{(1)}\hat{P}_\beta + \omega^{\mu\nu}\hat{M}^{(1)}_{\mu\nu}$:

$$\delta S = -\frac{i}{2}\int d^4\hat{x}\ \hat{P}^\alpha \left[\phi(\hat{x})\hat{P}_\alpha(\left[\epsilon^\beta_{(1)}\hat{P}_\beta + \omega^{\mu\nu}\hat{M}^{(1)}_{\mu\nu}\right]\phi(\hat{x})) - \left[\hat{P}_\alpha\phi(\hat{x})\right](\left[\epsilon^\beta_{(1)}\hat{P}_\beta + \omega^{\mu\nu}\hat{M}^{(1)}_{\mu\nu}\right]\phi(\hat{x}))\right]. \tag{3.44}$$

Using the commutation rules (2.32) we can take all the transformation parameters on the left of the integrand (note that also in this case we assume the parameters to commute with the translation generators):

$$\begin{aligned}\delta S \ = \ & -\frac{i}{2}\int d^4\hat{x}\ \hat{P}^\alpha \Big[\epsilon^\beta_{(1)}\phi(\hat{x})\hat{P}_\alpha\hat{P}_\beta\phi(\hat{x}) + \omega^{\mu\nu}\phi(\hat{x})\hat{P}_\alpha\hat{M}^{(1)}_{\mu\nu}\phi(\hat{x}) - \epsilon^\beta_{(1)}\left(\hat{P}_\alpha\phi(\hat{x})\right)\hat{P}_\beta\phi(\hat{x}) + \\ & -\omega^{\mu\nu}\left(\hat{P}_\alpha\phi(\hat{x})\right)\hat{M}^{(1)}_{\mu\nu}\phi(\hat{x}) - \frac{1}{2}\omega^{\mu\nu}\chi^{\sigma\rho}_{\mu\nu}\left[\left(\hat{P}_\rho\phi(\hat{x})\right)\hat{P}_\alpha\hat{P}_\sigma\phi(\hat{x}) - \left(\hat{P}_\rho\hat{P}_\alpha\phi(\hat{x})\right)\hat{P}_\sigma\phi(\hat{x})\right]\Big],\end{aligned} \tag{3.45}$$

where $\chi^{\sigma\rho}_{\mu\nu}$ was defined in (2.5). This is the expression of the variation of the action we were looking for, since it is of the form:

$$\delta S = -\frac{i}{2}\int d^4\hat{x}\ \epsilon^\beta_{(1)}\hat{P}_\alpha T^\alpha_{\beta\,(1)} + \omega^{\mu\nu}\hat{P}_\alpha J^{\alpha\,(1)}_{\mu\nu}, \tag{3.46}$$

where in this case the would-be-conserved currents are defined in the following way:

$$\begin{aligned} T^\alpha_{\beta\,(1)} &= \phi(\hat{x})\hat{P}^\alpha\hat{P}_\beta\phi(\hat{x}) - \left(\hat{P}^\alpha\phi(\hat{x})\right)\hat{P}_\beta\phi(\hat{x}) \\ J^{\alpha\,(1)}_{\mu\nu} &= \phi(\hat{x})\hat{P}^\alpha\hat{M}^{(1)}_{\mu\nu}\phi(\hat{x}) - \left(\hat{P}^\alpha\phi(\hat{x})\right)\hat{M}^{(1)}_{\mu\nu}\phi(\hat{x}) + \\ & -\frac{1}{2}\chi^{\sigma\rho}_{\mu\nu}\left[\left(\hat{P}_\rho\phi(\hat{x})\right)\hat{P}^\alpha\hat{P}_\sigma\phi(\hat{x}) - \left(\hat{P}_\rho\hat{P}^\alpha\phi(\hat{x})\right)\hat{P}_\sigma\phi(\hat{x})\right]. \end{aligned} \tag{3.47}$$

We see that, while the current associated to the translation parameter are the same of those that we obtained using the other basis of generators, the current associated to the transformation parameter $\omega^{\mu\nu}$ is different from the corresponding one of the symmetric basis (the second of (3.35)). So we have not yet solved the ordering ambiguity that arises from the use of different bases of generators, and we have to go further in the analysis to compare the charges.

3.5 Conserved charges

Now we want to demonstrate that the charges (3.36) and (3.37) are effectively time-independent. We consider only the currents associated to the action S, since the ones associated to S' are equivalent to these ones only up to a minus sign: this of course ensures us that they would lead to equivalent charges. The demonstration will be done writing an explicit expression of the charges expanding the fields in terms of their Fourier transform (Cf. Sec. 1.8.2), with the Fourier parameters kept on-shell by a delta function:

$$\phi(\hat{x}) = \int d^4k\ \tilde{\phi}_{(w)}(k)\delta(k^2)e^{ik^\beta\hat{x}_\beta}, \tag{3.48}$$

where $\delta(k^2) \equiv \delta(k^\rho k_\rho) \equiv \delta(|\vec{k}|^2 - k_0^2)$. We have used the (symmetric) Weyl-ordered basis of exponentials for simplicity, because in the expressions (3.36) and (3.37) there appear the generators of the symmetric basis of the twisted Poincaré algebra, that have a simpler action law on symmetric exponentials than on the others.

We first perform the calculation for the charge associated to invariance under translation, (3.36), that is quite straightforward, but gives an outline of the much more elaborated procedure to be followed for the other charge. Writing the fields as in (3.48) the charge takes the form:

$$Q_\mu = \int d^3\hat{x}\ d^4k\ d^4q\ \tilde{\phi}_{(w)}(k)\tilde{\phi}_{(w)}(q)\delta(k^2)\delta(q^2)\left[e^{ik\hat{x}}\hat{P}^0\hat{P}_\mu e^{iq\hat{x}} - \left(\hat{P}^0 e^{ik\hat{x}}\right)\hat{P}_\mu e^{iq\hat{x}}\right]. \tag{3.49}$$

Since the action of the translation generators is classical on the exponentials (Cf. (1.28)):

$$Q_\mu = \int d^3\hat{x}\ d^4k\ d^4q\ \tilde{\phi}_{(w)}(k)\tilde{\phi}_{(w)}(q)\delta(k^2)\delta(q^2)\left(q^0 - k^0\right)q_\mu\, e^{ik\hat{x}}e^{iq\hat{x}}. \tag{3.50}$$

To integrate in the spatial coordinates we first need to apply twice the Baker-Campbell-Hausdorff formula (1.11) to write the product of exponentials in the form:

$$e^{ik^\beta\hat{x}_\beta}e^{iq^\alpha\hat{x}_\alpha} = e^{i(k+q)^i\hat{x}_i}e^{i(k+q)^0\hat{x}_0}e^{\frac{i}{2}(k+q)^i(k+q)^0\theta_{i0}}e^{-\frac{i}{2}k^\beta q^\alpha\theta_{\beta\alpha}}; \tag{3.51}$$

then, remembering that we defined (Cf. Sec. (1.8)) $\int d^3\hat{x}e^{ik^i\hat{x}_i} \equiv \delta^{(3)}(\vec{k})$, we can integrate the charge to obtain:

$$\begin{aligned} Q_\mu \;&=\; \int d^4k\ d^4q\ \tilde{\phi}_{(w)}(k)\tilde{\phi}_{(w)}(q)\delta(k^2)\delta(q^2)\left(q^0 - k^0\right)q_\mu\,\delta^{(3)}(\vec{k}+\vec{q})e^{i(k^0+q^0)\hat{x}_0}\cdot \\ &\quad \cdot e^{\frac{i}{2}(k+q)^i(k+q)^0\theta_{i0}}e^{-\frac{i}{2}k^\beta q^\alpha\theta_{\beta\alpha}}. \end{aligned} \tag{3.52}$$

Integrating in d^3k we get:

$$\begin{aligned} Q_\mu \;&=\; \int dk_0 d^4q\ \tilde{\phi}_{(w)}(-\vec{q},k_0)\tilde{\phi}_{(w)}(q)\delta(|\vec{q}^2| - k_0^2)\delta(q^2)\left(q^0 - k^0\right)q_\mu\, e^{i(k^0+q^0)\hat{x}_0}\cdot \\ &\quad \cdot e^{-\frac{i}{2}(-q^i\delta_i^\beta + k^0\delta_0^\beta)q^\alpha\theta_{\beta\alpha}}. \end{aligned} \tag{3.53}$$

Now let us observe that the following property holds for the delta function[5]:

$$\delta(f(k)) = \sum_{i;f(k_i)=0}\frac{\delta(k-k_i)}{\left|\frac{\partial f(k)}{\partial k}|_{k=k_i}\right|} \Rightarrow \delta(|\vec{k}|^2 - k_0^2) = \frac{1}{2|\vec{k}|}\Big[\delta(k_0 + |\vec{k}|) + \delta(k_0 - |\vec{k}|)\Big], \tag{3.54}$$

[5]The delta function with argument a Fourier parameter is a commutative object, so it has all the properties holding for the usual delta function.

so, performing the integral in dk_0, we get:

$$\begin{aligned} Q_\mu &= \int \frac{d^4q}{2|\vec{q}|}\, \tilde{\phi}_{(w)}(q)\delta(q^2)\Big\{\tilde{\phi}_{(w)}(-\vec{q},|\vec{q}|)\left(q^0+|\vec{q}|\right) q_\mu\, e^{i(-|\vec{q}|+q^0)\hat{x}_0} e^{\frac{i}{2}q^i(q^0-|\vec{q}|)\theta_{i0}} + \\ &\quad +\tilde{\phi}_{(w)}(-\vec{q},-|\vec{q}|)\left(q^0-|\vec{q}|\right) q_\mu\, e^{i(|\vec{q}|+q^0)\hat{x}_0} e^{\frac{i}{2}q^i(q^0+|\vec{q}|)\theta_{i0}}\Big\}. \end{aligned} \quad (3.55)$$

Now the charge is easily seen to be conserved. In fact the presence of the $\delta(q^2)$ imposes $q_0 = \pm|\vec{q}|$, so, when the exponents $i(q^0 \mp |\vec{q}|)\hat{x}_0$ and $\frac{i}{2}q^i(q^0 \mp |\vec{q}|)\theta_{i0}$ are non vanishing (thus leading to a temporary dependence of the charge), the terms $q^0 \pm |\vec{q}|$, to which the exponentials are proportional, are null (thus nullifying the effect of the time dependence of the exponential); viceversa, when the terms $q^0 \pm |\vec{q}|$ are non-null, the exponentials are equal to one. Taking this in mind, one can easily convince himself that the above expression of the charge is equivalent to:

$$Q_\mu = \int \frac{d^4q}{2|\vec{q}|}\, \tilde{\phi}_{(w)}(q)\delta(q^2)q_\mu\Big\{\tilde{\phi}_{(w)}(-\vec{q},|\vec{q}|)\left(q^0+|\vec{q}|\right) + \tilde{\phi}_{(w)}(-\vec{q},-|\vec{q}|)\left(q^0-|\vec{q}|\right)\Big\}. \quad (3.56)$$

Finally, we got four charges, each associated to the translation in one direction. Their expression is not only evidently time-independent, but is also of the same form of the corresponding commutative charges, with the only difference that hare we have the Fourier transform $\tilde{\phi}_{(w)}(q)$ associated to the expansion of the field in terms of exponentials with a particular ordering convention, and this function could in principle vary if we change the basis for the expansion (Cf. comments at the end of this section next subsection), while in the commutative case we do not have to make distinctions between different ordering convention for the exponentials to use in the Fourier expansion. In the next section we shall see what form takes the translational charge if expressed in terms of Fourier transforms associated to another basis of exponentials. Beside this, this similarity in form of the charges could have been expected, since the translation sector of the θ-Poicaré algebra is classical, i.e. has not only classical action and algebra, but has also trivial coproduct. Nevertheless, this two facts, the one that we got conserved charges and the one that the form of this charges is not so different from the commutative ones (and, most important, has the correct commutative limit), reassures us, almost for the translation sector, for that now it seems effectively to make sense the description of the symmetries of the theory through a transformation of the kind (2.14), and to follow Noether analysis to find the associated conserved charges .

Now let us turn to the "Lorentz sector charges". Also in this case it is useful to write the charges (3.37) in terms of the field expansion (3.48):

$$K_{\mu\nu} = \int d^3\hat{x}d^4kd^4q\ \tilde{\phi}_{(w)}(k)\tilde{\phi}_{(w)}(q)\delta(k^\mu k_\mu)\delta(q^\mu q_\mu)\left\{e^{ik\hat{x}}\hat{P}^0\hat{M}_{\mu\nu}e^{iq\hat{x}} - \left(\hat{P}^0e^{ik\hat{x}}\right)\cdot\right.$$
$$\left.\cdot\hat{M}_{\mu\nu}(e^{iq\hat{x}}) - \frac{1}{2}\Upsilon^{\alpha\beta}_{\mu\nu}\left[\left(\hat{P}_\beta e^{ik\hat{x}}\right)\hat{P}^0\hat{P}_\alpha e^{iq\hat{x}} - \left(\hat{P}_\beta\hat{P}^0e^{ik\hat{x}}\right)\hat{P}_\alpha e^{iq\hat{x}}\right]\right\} \tag{3.57}$$

Let us first study separately the term proportional to $\Upsilon^{\alpha\beta}_{\mu\nu}$, that we call (A) and has an expression very similar to the translational charges.

$$(A) = \frac{1}{2}\int d^3\hat{x}d^4kd^4q\ \tilde{\phi}_{(w)}(k)\tilde{\phi}_{(w)}(q)\delta(k^\mu k_\mu)\delta(q^\mu q_\mu)\Upsilon^{\alpha\beta}_{\mu\nu}(k^0-q^0)k_\beta q_\alpha(e^{ik^\beta\hat{x}_\beta})(e^{iq^\beta\hat{x}_\beta}) \tag{3.58}$$

Writing the product of exponentials in the form (3.51) and performing the spatial integration:

$$(A) = \frac{1}{2}\int d^4kd^4q\ \tilde{\phi}_{(w)}(k)\tilde{\phi}_{(w)}(q)\delta(k^\mu k_\mu)\delta(q^\mu q_\mu)\Upsilon^{\alpha\beta}_{\mu\nu}(k^0-q^0)k_\beta q_\alpha\delta^{(3)}(\vec{k}+\vec{q})e^{i(k+q)^0\hat{x}_0}\cdot$$
$$\cdot e^{\frac{i}{2}(k+q)^i(k+q)^0\theta_{i0}}e^{-\frac{i}{2}k^\beta q^\alpha\theta_{\beta\alpha}}. \tag{3.59}$$

Now we integrate in d^4k, using for the $\delta(k^2)$ the expression (3.54):

$$(A) = \frac{1}{2}\int d^4q\ \frac{1}{2|\vec{q}|}\tilde{\phi}_{(w)}(q)\delta(q^\mu q_\mu)\Upsilon^{\alpha\beta}_{\mu\nu}\Big\{\tilde{\phi}_{(w)}(-\vec{q},|\vec{q}|)(-|\vec{q}|-q^0)(-q_i\delta^i_\beta+|\vec{q}|\delta^0_\beta)q_\alpha\cdot$$
$$\cdot e^{i(-|\vec{q}|+q^0)\hat{x}_0}e^{\frac{i}{2}q^i(q^0-|\vec{q}|)\theta_{i0}} + \tilde{\phi}_{(w)}(-\vec{q},-|\vec{q}|)(|\vec{q}|-q^0)(-q_i\delta^i_\beta-|\vec{q}|\delta^0_\beta)q_\alpha\cdot$$
$$e^{i(|\vec{q}|+q^0)\hat{x}_0}e^{\frac{i}{2}q^i(q^0+|\vec{q}|)\theta_{\beta\alpha}}\Big\}. \tag{3.60}$$

Integrating in dq_0, being careful while explicitating the delta because the presence of the terms $\pm|\vec{q}|-q^0$ imposes respectively $q^0=\mp|\vec{q}|$ in order the integral not to be null, one gets:

$$(A) = -\frac{1}{2}\int d^3q\ \frac{1}{4|\vec{q}|^2}\Upsilon^{\alpha\beta}_{\mu\nu}\Big\{\tilde{\phi}_{(w)}(\vec{q},-|\vec{q}|)\tilde{\phi}_{(w)}(-\vec{q},|\vec{q}|)2|\vec{q}|(-q_i\delta^i_\beta+|\vec{q}|\delta^0_\beta)(q_i\delta^i_\alpha-|\vec{q}|\delta^0_\alpha)+$$
$$-\tilde{\phi}_{(w)}(\vec{q},|\vec{q}|)\tilde{\phi}_{(w)}(-\vec{q},-|\vec{q}|)2|\vec{q}|(-q_i\delta^i_\beta-|\vec{q}|\delta^0_\beta)(q_i\delta^i_\alpha+|\vec{q}|\delta^0_\alpha)\Big\}. \tag{3.61}$$

This expression is null since is of the form $\int d^3q\ \left(f(\vec{q})-f(-\vec{q})\right)\equiv 0$. So we have demonstrated that the term proportional to $\Upsilon^{\alpha\beta}_{\mu\nu}$ does not give contribution to the charge $K_{\mu\nu}$, that thus reads:

$$K_{\mu\nu} = \int d^3\hat{x}d^4kd^4q\ \tilde{\phi}_{(w)}(k)\tilde{\phi}_{(w)}(q)\delta(k^2)\delta(q^2)\left\{e^{ik\hat{x}}\hat{P}^0\hat{M}_{\mu\nu}e^{iq\hat{x}} - \left(\hat{P}^0e^{ik\hat{x}}\right)\hat{M}_{\mu\nu}(e^{iq\hat{x}})\right\}. \tag{3.62}$$

For this charge the calculation will be a little bit more laborious, since we have to threat the derivative with respect to q^μ, that was not present in the previous cases. This causes some complications, since this derivative, as it was explained in Sec. 1.3, implies some ordering issues, that we shall threat below. To make the generators act on the exponentials in the correct way it is useful to observe that the following equivalences hold:

$$\begin{aligned}\hat{P}_0\hat{M}_{\mu\nu}\left(e^{ik\hat{x}}\right) &= \Omega_w\left(P_0 M_{\mu\nu}\, e^{ikx}\right) = \Omega_w\left(-\partial_0\left(x_{[\mu}\partial_{\nu]}\, e^{ikx}\right)\right)\\ &= \Omega_w\left(-\left[\eta_{0[\mu}+x_{[\mu}\partial_0\right]\left(\partial_{\nu]}\, e^{ikx}\right)\right) = -i\left[\eta_{0[\mu}+k_0\frac{\partial}{\partial k^{[\mu}}\right]\left(k_{\nu]}\, e^{ik\hat{x}}\right)\\ &= -i\frac{\partial}{\partial k^{[\mu}}\left[k_{\nu]}k_0\, e^{ik\hat{x}}\right].\end{aligned} \tag{3.63}$$

So:

$$K_{\mu\nu} = i\int d^3\hat{x}d^4kd^4q\;\tilde{\phi}_{(w)}(k)\tilde{\phi}_{(w)}(q)\delta(k^2)\delta(q^2)q_{[\nu}\frac{\partial}{\partial q^{\mu]}}\left[(k^0-q^0)e^{ik\hat{x}}e^{iq\hat{x}}\right]. \tag{3.64}$$

Using formula (3.51) to reorder the exponentials and integrating in the spatial variables:

$$\begin{aligned}K_{\mu\nu} &= i\int d^4kd^4q\;\tilde{\phi}_{(w)}(k)\tilde{\phi}_{(w)}(q)\delta(k^2)\delta(q^2)q_{[\nu}\frac{\partial}{\partial q^{\mu]}}\left[(k^0-q^0)\delta^{(3)}(\vec{k}+\vec{q})e^{i(k+q)^0\hat{x}_0}\,\cdot\right.\\ &\qquad\left.\cdot\, e^{\frac{i}{2}(k+q)^i(k+q)^0\theta_{i0}}e^{-\frac{i}{2}k^\mu q^\nu\theta_{\mu\nu}}\right].\end{aligned} \tag{3.65}$$

Now we see that the integral is of the form $\int d^4kd^4q\;f(q)g(k)\frac{\partial}{\partial q^\mu}[h(q,k)]$, where f, g, h are generic functions of q, k, and k and q together respectively so we can take $\int d^4k\;g(k)$ into the derivative with respect to q^μ:

$$\begin{aligned}K_{\mu\nu} &= i\int d^4q\;\tilde{\phi}_{(w)}(q)\delta(q^2)q_{[\nu}\frac{\partial}{\partial q^{\mu]}}\Big[\int d^4k\;\tilde{\phi}_{(w)}(k)\delta(k^2)(k^0-q^0)\,\cdot\\ &\qquad\cdot\delta^{(3)}(\vec{k}+\vec{q})e^{i(k+q)^0\hat{x}_0}e^{\frac{i}{2}(k+q)^i(k+q)^0\theta_{i0}}e^{-\frac{i}{2}k^\mu q^\nu\theta_{\mu\nu}}\Big]\end{aligned} \tag{3.66}$$

Now we can perform the integration in d^4k directly inside the derivative with respect to q^μ:

$$\begin{aligned}K_{\mu\nu} &= i\int d^4q\;\tilde{\phi}_{(w)}(q)\delta(q^2)q_{[\nu}\frac{\partial}{\partial q^{\mu]}}\Big[\frac{\tilde{\phi}_{(w)}(-\vec{q},|\vec{q}|)}{2|\vec{q}|}(-|\vec{q}|-q^0)e^{i(-|\vec{q}|+q^0)\hat{x}_0}e^{\frac{i}{2}(q^i\delta_i^\mu+|\vec{q}|\delta_0^\mu)q^\nu\theta_{\mu\nu}}+\\ &\quad+\frac{\tilde{\phi}_{(w)}(-\vec{q},-|\vec{q}|)}{2|\vec{q}|}(|\vec{q}|-q^0)e^{i(|\vec{q}|+q^0)\hat{x}_0}e^{\frac{i}{2}(q^i\delta_i^\mu-|\vec{q}|\delta_0^\mu)q^\nu\theta_{\mu\nu}}\Big]\end{aligned} \tag{3.67}$$

The next step should be to explicitate the derivative, but before doing this we have to notice that, because of the definition of the derivative with respect to q^μ that we gave when we introduced it (Cf. (1.26), (1.28) and the discussion below them), when we apply the derivative to the exponential $e^{i(\pm|\vec{q}|+q^0)\hat{x}_0}$, we have to take in consideration ordering issues, since the derivative would take down from the exponential an $\hat{x}_0$, that has to be put in the full symmetrised form together with the exponential itself:

$$\begin{aligned}\frac{\partial}{\partial q^\mu}\left(e^{i(-a|\vec{q}|+q^0)\hat{x}_0}\right) &\equiv \Omega_w\left[\frac{\partial}{\partial q^\mu}\left(e^{i(-a|\vec{q}|+q^0)\hat{x}_0}\right)\right] = \Omega_w\left[i(-a\frac{\delta^i_\mu q^i}{|\vec{q}|}+\delta^0_\mu)x_0\left(e^{i(-a|\vec{q}|+q^0)x_0}\right)\right] \\ &= i(-a\frac{\delta^i_\mu q^i}{|\vec{q}|}+\delta^0_\mu)\Omega_w\left[x_0\left(e^{i(-a|\vec{q}|+q^0)x_0}\right)\right] \end{aligned} \tag{3.68}$$

In the following we shall use this expression where we leave indicated the Weyl map. Explicitating the derivative the charge reads:

$$\begin{aligned}K_{\mu\nu} &= -i\sum_{a=\{1,-1\}}\int d^4q\ \tilde{\phi}_{(w)}(q)\delta(q^2)q_{[\nu}\Bigg\{\Bigg[\frac{\delta^j_{\mu]}}{2|\vec{q}|}\frac{\partial\tilde{\phi}_{(w)}(-\vec{q},a|\vec{q}|)}{\partial q^j}(-a|\vec{q}|-q^0)+\frac{\tilde{\phi}_{(w)}(-\vec{q},a|\vec{q}|)}{2|\vec{q}|}\cdot \\ &\cdot(-a\frac{q_i\delta^i_{\mu]}}{|\vec{q}|}-\eta^0_{\mu]})-\frac{\tilde{\phi}_{(w)}(-\vec{q},a|\vec{q}|)}{2|\vec{q}|^3}\delta^i_{\mu]}q_i(-a|\vec{q}|-q^0)+\frac{i}{2|\vec{q}|}\tilde{\phi}_{(w)}(-\vec{q},a|\vec{q}|)(-a|\vec{q}|-q^0)\Bigg(\frac{1}{2}(\delta^i_{\mu]}\delta^\beta_i+ \\ &+a\frac{q_i\delta^i_{\mu]}}{|\vec{q}|}\delta^\beta_0)q^\alpha\theta_{\beta\alpha}+\frac{1}{2}(q^i\delta^\beta_i+a|\vec{q}|\delta^\beta_0)\delta^\alpha_{\mu]}\theta_{\beta\alpha}\Bigg)\Bigg]e^{i(-a|\vec{q}|+q^0)\hat{x}_0}+\frac{i}{2|\vec{q}|}\tilde{\phi}_{(w)}(-\vec{q},a|\vec{q}|)(-a|\vec{q}|-q^0)\cdot \\ &\cdot(-a\frac{q_i\delta^i_{\mu]}}{|\vec{q}|}+\delta^0_{\mu]})\Omega_w\left[x_0e^{i(-a|\vec{q}|+q^0)x_0}\right]\Bigg\}e^{\frac{i}{2}(q^i\delta^\beta_i+a|\vec{q}|\delta^\beta_0)q^\alpha\theta_{\beta\alpha}}. \end{aligned} \tag{3.69}$$

We can now integrate in dq_0:

$$\begin{aligned}K_{\mu\nu} &= -i\sum_{a,b=\{1,-1\}}\int\frac{d^3q}{4|\vec{q}|}\ \tilde{\phi}_{(w)}(\vec{q},b|\vec{q}|)(q_i\delta^i_{[\nu}+b|\vec{q}|\delta^0_{[\nu})\Bigg\{(b-a)\Bigg[\delta^j_{\mu]}\frac{\partial\tilde{\phi}_{(w)}(-\vec{q},a|\vec{q}|)}{\partial q^j}+ \\ &-\frac{\tilde{\phi}_{(w)}(-\vec{q},a|\vec{q}|)}{|\vec{q}|^2}\delta^i_{\mu]}q_i+i\tilde{\phi}_{(w)}(-\vec{q},a|\vec{q}|)\Bigg(\frac{1}{2}(\delta^i_{\mu]}\delta^\beta_i+a\frac{q_i\delta^i_{\mu]}}{|\vec{q}|}\delta^\beta_0)(q^i\delta^\alpha_i-b|\vec{q}|\delta^\alpha_0)\theta_{\beta\alpha}+ \\ &+\frac{1}{2}(q^i\delta^\beta_i+a|\vec{q}|\delta^\beta_0)\delta^\alpha_{\mu]}\theta_{\beta\alpha}\Bigg)\Bigg]e^{-i(a+b)|\vec{q}|\hat{x}_0}-\frac{\tilde{\phi}_{(w)}(-\vec{q},a|\vec{q}|)}{|\vec{q}|}(a\frac{q_i\delta^i_{\mu]}}{|\vec{q}|}+\eta^0_{\mu]})e^{-i(a+b)|\vec{q}|\hat{x}_0}+ \\ &+i\tilde{\phi}_{(w)}(-\vec{q},a|\vec{q}|)(b-a)(-a\frac{q_i\delta^i_{\mu]}}{|\vec{q}|}+\delta^0_{\mu]})\Omega_w\left[x_0e^{-i(a+b)|\vec{q}|x_0}\right]\Bigg\}e^{\frac{i}{2}(a+b)q^i|\vec{q}|\theta_{0i}}. \end{aligned} \tag{3.70}$$

In this case the simple discussion on the time independence that we made for the translational charge is no more valid, since we do not have only terms of the type $f(\vec{q})(b-a)e^{i(b+a)\hat{x}_0}$, but also a term proportional to $\hat{x}_0$ and one of the type $f(\vec{q})e^{i(b+a)\hat{x}_0}$, so we have to check by an explicit calculation whether this charge is conserved or not. It is convenient to separate the calculation: first we consider the case in which $a=b$, then the one in which $a=-b$.

1)Case $a=b$:

$$K_{\mu\nu}=i\sum_{a=\{1,-1\}}\int\frac{d^3q}{4|\vec{q}|^2}\,\tilde{\phi}_{(w)}(\vec{q},a|\vec{q}|)(q_i\delta^i_{[\nu}+a|\vec{q}|\delta^0_{[\nu})\tilde{\phi}_{(w)}(-\vec{q},a|\vec{q}|)(a\frac{q_i\delta^i_{\mu]}}{|\vec{q}|}+\eta^0_{\mu]})e^{ia|\vec{q}|(q^i\theta_{0i}-2\hat{x}_0)},\tag{3.71}$$

and this expression is null since it contains $(q_i\delta^i_{[\nu}+a|\vec{q}|\delta^0_{[\nu})(a\frac{q_i\delta^i_{\mu]}}{|\vec{q}|}+\eta^0_{\mu]})$, that is symmetric in the indices μ and ν, but for they have to be antisymmetrised, giving a null result.

2)Case $a=-b$

$$\begin{aligned}K_{\mu\nu} \;=\; & -i\sum_{a=\{1,-1\}}\int\frac{d^3q}{4|\vec{q}|}\,\tilde{\phi}_{(w)}(\vec{q},-a|\vec{q}|)(q_i\delta^i_{[\nu}-a|\vec{q}|\delta^0_{[\nu})\Bigg\{(-2a)\Bigg[\delta^j_{\mu]}\frac{\partial\tilde{\phi}_{(w)}(-\vec{q},a|\vec{q}|)}{\partial q^j}+\\ & +i\tilde{\phi}_{(w)}(-\vec{q},a|\vec{q}|)\Bigg(\frac{1}{2}(\delta^i_{\mu]}\delta^\beta_i+a\frac{q_i\delta^i_{\mu]}}{|\vec{q}|}\delta^\beta_0)(q^i\delta^\alpha_i+a|\vec{q}|\delta^\alpha_0)\theta_{\beta\alpha}+\\ & +\frac{1}{2}(q^i\delta^\beta_i+a|\vec{q}|\delta^\beta_0)\delta^\alpha_{\mu]}\theta_{\beta\alpha}\Bigg)\Bigg]+\Bigg(a\frac{q_i\delta^i_{\mu]}}{|\vec{q}|}-\eta^0_{\mu]}\Bigg)\tilde{\phi}_{(w)}(-\vec{q},a|\vec{q}|)\Bigg(\frac{1}{|\vec{q}|}+2ia\hat{x}_0\Bigg)\Bigg\}.\end{aligned}\tag{3.72}$$

The term proportional to $a\frac{q_i\delta^i_{\mu]}}{|\vec{q}|}-\eta^0_{\mu]}$ (that contains the time coordinate $\hat{x}_0$) is of the kind of the one discussed above in the case $a=b$, so gives null contribution, ensuring us that the charge is effectively time independent. The term proportional to $\theta_{\alpha\beta}$, multiplied by $(q_i\delta^i_{[\nu}-a|\vec{q}|\delta^0_{[\nu})$, it is easily seen to be symmetric in the indices α and β, but these indices are saturated with the corresponding ones of $\theta_{\alpha\beta}$, that is an antisymmetric matrix. So also this term gives null contribution to the charge.

The final expression of the charge is then, explicitating the sum on a:

$$\begin{aligned}K_{\mu\nu} \;=\; & i\int\frac{d^3q}{2|\vec{q}|}\,\Bigg\{q_i\delta^j_{[\mu}\delta^i_{\nu]}\Bigg[\tilde{\phi}_{(w)}(\vec{q},-|\vec{q}|)\,\frac{\partial\tilde{\phi}_{(w)}(-\vec{q},|\vec{q}|)}{\partial q^j}-\tilde{\phi}_{(w)}(\vec{q},|\vec{q}|)\,\frac{\partial\tilde{\phi}_{(w)}(-\vec{q},-|\vec{q}|)}{\partial q^j}\Bigg]+\\ & -|\vec{q}|\delta^j_{[\mu}\delta^0_{\nu]}\Bigg[\tilde{\phi}_{(w)}(\vec{q},-|\vec{q}|)\,\frac{\partial\tilde{\phi}_{(w)}(-\vec{q},|\vec{q}|)}{\partial q^j}+\tilde{\phi}_{(w)}(\vec{q},|\vec{q}|)\,\frac{\partial\tilde{\phi}_{(w)}(-\vec{q},-|\vec{q}|)}{\partial q^j}\Bigg]\Bigg\}.\end{aligned}\tag{3.73}$$

So we got six independent charges, that we would like to say being three associated to the invariance of the action under space rotations and three associated to invariance under boosts. This is not possible since, as explained in the previous section, pure spatial rotations or pure boosts are not allowed.

Nevertheless, as we found studying the translational charges, these ones are equal in form to those that one would say to be the correspondent ones in the commutative spacetime, but for the dependence on the ordering convention of the Fourier transform of the fields $\tilde{\phi}_{(w)}$. Also in this case we shall show in the next section the form that these charges take when expressed in terms of the Fourier transform of the field associated to the x_1-to-the-right-ordered exponentials, and this will lead to a way out from the ordering ambiguity that arises from the fact of having different differentials associated with different bases of the θ-Poincaré algebra.

3.6 Weyl-map independence

In this section we shall treat in detail the problem of the map dependence. We have two points to tackle.

The first one, that is more a problem of internal coherence of the formalism constructed in this work than a problem of a possible map dependence of the physics of the system under study, regards the fact that the expressions of the charges (3.56) and (3.73) could in principle depend on the choice of ordering convention used to expand the fields in (3.36) and (3.37). This means that we may obtain a result different from (3.56) if we use, for example, the x_1-to-the-right-ordered exponentials for the expansion, writing:

$$\phi(\hat{x}) = \int d^4k\ \tilde{\phi}_{(1)}(k)\delta(k^2)e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}, \tag{3.74}$$

and we make the symmetric generators act on this non-symmetric ordered exponentials, finally writing back the expression obtained in terms of $\tilde{\phi}_{(w)}(k)$, using the relation (1.100). This should not happen, because the following series of equivalences holds:

$$\begin{aligned}
\int d^3\hat{x}\ \left[\hat{F}(\phi(\hat{x}))\hat{G}(\phi(\hat{x}))\right] &= \int d^3\hat{x}d^4kd^4q\ \tilde{\phi}_{(w)}(k)\tilde{\phi}_{(w)}(q)\left[\hat{F}(e^{ik\hat{x}})\hat{G}(e^{iq\hat{x}})\right] \\
&= \int d^3\hat{x}d^4kd^4q\ \tilde{\phi}_{(w)}(k)\tilde{\phi}_{(w)}(q)\left[\hat{F}(e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1})\hat{G}(e^{iq^A\hat{x}_A}e^{iq^1x_1})\right]e^{\frac{i}{2}k^Ak^1\theta_{A1}}e^{\frac{i}{2}q^Aq^1\theta_{A1}} \\
&= \int d^3\hat{x}d^4kd^4q\ \tilde{\phi}_{(1)}(k)\tilde{\phi}_{(1)}(q)\left[\hat{F}(e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1})\hat{G}(e^{iq^A\hat{x}_A}e^{iq^1x_1})\right],
\end{aligned} \tag{3.75}$$

where $\hat{F}$ and $\hat{G}$ stand for generic operators (or products of operators) of the Poincaré algebra, so that the first of the above expressions stands for a generic term appearing in the expressions of the charges (3.36) and (3.37). The second equivalence holds since the operators $\hat{F}$ and $\hat{G}$ act only on coordinates and not on the exponentials containing only Fourier parameters. To check this point, we shall explicitate the charges (3.36) and (3.37) through (3.74) and shall compare the expression obtained with the one resulting from writing (3.56) and (3.73) in terms of $\tilde{\phi}_{(1)}$.

The second (but with more physical consequences) issue regards the map dependence of the transformation to which the fields are subjected (Cf. Sec. 2.5). We have to check if the charges associated to a transformation expressed in a basis different from the symmetric one (for example (2.29)) result to be the same of the charges corresponding to the symmetric basis transformation. This correspond to the requirement that the physics of the system should not depend on the convention used for the choice of the basis of the Poincaré algebra. We found in Sec. 3.4.1 that the translation currents resulting from the use of the x_1-to-the-right basis of generators were the same of those associated to the symmetric basis, so that it does not need any check on the charges, while the currents "associated" to the Lorentz sector transformations are different depending on the basis used to describe the transformation, thus leading to this map ambiguity.

First of all we we shall give an answer to the first question. So let us rewrite the expression of the translational charge (3.56) in terms of the Fourier transform of the field associated to the x_1-to-the-right basis of exponentials $\tilde{\phi}_{(1)}(k)$ using the relation (1.100):

$$\begin{aligned} Q_\mu &= \int \frac{d^4q}{2|\vec{q}|}\tilde{\phi}_{(1)}(q)e^{-\frac{i}{2}q^A q^1\theta_{A1}}\delta(q^2)q_\mu\Big\{\tilde{\phi}_{(1)}(-\vec{q},|\vec{q}|)e^{-\frac{i}{2}(q^i\delta^i_A+|\vec{q}|\delta^A_0)q^1\theta_{A1}}(q^0+|\vec{q}|)+ \\ &+\tilde{\phi}_{(1)}(-\vec{q},-|\vec{q}|)e^{-\frac{i}{2}(q^i\delta^i_A-|\vec{q}|\delta^A_0)q^1\theta_{A1}}(q^0-|\vec{q}|)\Big\} \\ &= \int \frac{d^4q}{2|\vec{q}|}\tilde{\phi}_{(1)}(q)\delta(q^2)q_\mu e^{-iq^i q^1\delta^i_A\theta_{A1}}\Big\{\tilde{\phi}_{(1)}(-\vec{q},|\vec{q}|)e^{-\frac{i}{2}(q^0+|\vec{q}|)q^1\theta_{01}}(q^0+|\vec{q}|)+ \\ &+\tilde{\phi}_{(1)}(-\vec{q},-|\vec{q}|)e^{-\frac{i}{2}(q^0-|\vec{q}|)q^1\theta_{01}}(q^0-|\vec{q}|)\Big\}. \end{aligned} \tag{3.76}$$

This expression has to be compared with the one obtained expanding the fields in (3.36) in terms of the x_1-to-the-right exponentials (Cf. 3.74). In this case the charge reads:

$$\begin{aligned} Q_\mu &= \int d^3x d^4k d^4q\ \tilde{\phi}_{(1)}(k)\tilde{\phi}_{(1)}(q)\delta(k^2)\delta(q^2)\Big[e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}\hat{P}^0\hat{P}_\mu(e^{iq^A\hat{x}_A}e^{iq^1\hat{x}_1})+ \\ &-\Big(\hat{P}^0(e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1})\Big)\hat{P}_\mu(e^{iq^A\hat{x}_A}e^{ik^1\hat{x}_1})\Big]. \end{aligned} \tag{3.77}$$

The action of the translation generators is always classical on the exponentials written with any ordering convention, so:

$$Q_\mu = \int d^3x d^4k d^4q\ \tilde{\phi}_{(1)}(k)\tilde{\phi}_{(1)}(q)\delta(k^2)\delta(q^2)q_\mu(q^0-k^0)e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}e^{iq^A\hat{x}_A}e^{iq^1\hat{x}_1}. \tag{3.78}$$

Now the steps will be exactly the same of the calculation performed in Sec. 3.5, so here we shall not give all the details again. We only remind that the definition (1.90) of the $\delta^{(3)}(\vec{k})$ is map independent. So we get:

$$\begin{aligned} Q_\mu &= \int \frac{d^4q}{2|\vec{q}|}\ \tilde{\phi}_{(1)}(q)\delta(q^2)q_\mu e^{-iq^i\delta^A_i q^1\theta_{A1}}\Big\{\tilde{\phi}_{(1)}(-\vec{q},|\vec{q}|)(q^0+|\vec{q}|)e^{i(-|\vec{q}|+q^0)\hat{x}_0}e^{-\frac{i}{2}(|\vec{q}|+q^0)q^1\theta_{01}}\ . \\ &\cdot e^{-\frac{i}{2}q^i(|\vec{q}|-q^0)\theta_{i0}}+\tilde{\phi}_{(1)}(-\vec{q},-|\vec{q}|)(q^0-|\vec{q}|)e^{i(|\vec{q}|+q^0)\hat{x}_0}e^{-\frac{i}{2}(q^0-|\vec{q}|)q^1\theta_{01}}e^{\frac{i}{2}q^i(|\vec{q}|+q^0)\theta_{i0}}\Big\} \\ &= \int \frac{d^4q}{2|\vec{q}|}\ \tilde{\phi}_{(1)}(q)\delta(q^2)q_\mu e^{-iq^i\delta^A_i q^1\theta_{A1}}\Big\{\tilde{\phi}_{(1)}(-\vec{q},|\vec{q}|)(q^0+|\vec{q}|)e^{-\frac{i}{2}(|\vec{q}|+q^0)q^1\theta_{01}}+ \\ &+\tilde{\phi}_{(1)}(-\vec{q},-|\vec{q}|)(q^0-|\vec{q}|)e^{-\frac{i}{2}(q^0-|\vec{q}|)q^1\theta_{01}}\Big\}, \end{aligned} \tag{3.79}$$

that is the same expression of (3.76), so that for translational charge we have not the firs kind of map ambiguity, nor the second, since we saw that the current derived using the differential $d_{(1)}f(\hat{x})$ is the same of the one derived using $df(\hat{x})$. It is interesting to note that with the charge written in terms of $\tilde{\phi}_{(1)}$ it is more evident its θ-dependence, that is no more only hidden inside the Fourier transform as happened in (3.56), and its form is no more classical, but differs from that one for some phases (that are equal to one for $\theta_{\mu\nu} \to 0$, so that the classical limit is preserved) multiplying the various terms.

Now we are going to threat the "Lorentz sector charges". To answer the first question about map dependence we shall follow the same steps described above for translational charges. The expression (3.73), written in terms of $\tilde{\phi}_{(1)}$, reads:

$$\begin{aligned} K_{\mu\nu} \;=\; & i\int \frac{d^3q}{2|\vec{q}|}\ \Big\{ q_i\delta^j_{[\mu}\delta^i_{\nu]} \Big[\tilde{\phi}_{(1)}(\vec{q},-|\vec{q}|)e^{-\frac{i}{2}|\vec{q}|q^1\theta_{01}} \frac{\partial}{\partial q^j}\Big[\tilde{\phi}_{(1)}(-\vec{q},|\vec{q}|)\, e^{-\frac{i}{2}(q^i\delta^A_i+|\vec{q}|\delta^A_0)q^1\theta_{A1}}\Big] + \\ & -\tilde{\phi}_{(1)}(\vec{q},|\vec{q}|)e^{\frac{i}{2}|\vec{q}|q^1\theta_{01}}\ \frac{\partial}{\partial q^j}\Big[\tilde{\phi}_{(1)}(-\vec{q},-|\vec{q}|)\, e^{-\frac{i}{2}(q^i\delta^A_i-|\vec{q}|\delta^A_0)q^1\theta_{A1}}\Big]\Big] + \\ & -|\vec{q}|\delta^j_{[\mu}\delta^0_{\nu]}\Big[\tilde{\phi}_{(1)}(\vec{q},-|\vec{q}|)e^{-\frac{i}{2}|\vec{q}|q^1\theta_{01}}\ \frac{\partial}{\partial q^j}\Big[\tilde{\phi}_{(1)}(-\vec{q},|\vec{q}|)\, e^{-\frac{i}{2}(q^i\delta^A_i+|\vec{q}|\delta^A_0)q^1\theta_{A1}}\Big] + \\ & +\tilde{\phi}_{(1)}(\vec{q},|\vec{q}|)e^{\frac{i}{2}|\vec{q}|q^1\theta_{01}}\ \frac{\partial}{\partial q^j}\Big[\tilde{\phi}_{(1)}(-\vec{q},-|\vec{q}|)\, e^{-\frac{i}{2}(q^i\delta^A_i-|\vec{q}|\delta^A_0)q^1\theta_{A1}}\Big]\Big]\Big\}\ e^{-\frac{i}{2}q^i\delta^A_iq^1\theta_{A1}}. \end{aligned} \tag{3.80}$$

Let us calculate separately the derivatives:

$$\begin{aligned} & \tfrac{\partial}{\partial q^j}\Big[\tilde{\phi}_{(1)}(-\vec{q},|\vec{q}|)\, e^{-\frac{i}{2}(q^i\delta^A_i+|\vec{q}|\delta^A_0)q^1\theta_{A1}}\Big] = \\ & = \Big[\tfrac{\partial\tilde{\phi}_{(1)}(-\vec{q},|\vec{q}|)}{\partial q^j} - \tfrac{i}{2}\tilde{\phi}_{(1)}(-\vec{q},|\vec{q}|)(\delta^A_j + \tfrac{q_j}{|\vec{q}|}\delta^A_0)q^1\theta_{A1}\Big]\ e^{-\frac{i}{2}(q^i\delta^A_i+|\vec{q}|\delta^A_0)q^1\theta_{A1}}; \\ & \tfrac{\partial}{\partial q^j}\Big[\tilde{\phi}_{(1)}(-\vec{q},-|\vec{q}|)\, e^{-\frac{i}{2}(q^i\delta^A_i-|\vec{q}|\delta^A_0)q^1\theta_{A1}}\Big] = \\ & = \Big[\tfrac{\partial\tilde{\phi}_{(1)}(-\vec{q},-|\vec{q}|)}{\partial q^j} - \tfrac{i}{2}\tilde{\phi}_{(1)}(-\vec{q},-|\vec{q}|)(\delta^A_j - \tfrac{q_j}{|\vec{q}|}\delta^A_0)q^1\theta_{A1}\Big]\ e^{-\frac{i}{2}(q^i\delta^A_i-|\vec{q}|\delta^A_0)q^1\theta_{A1}}. \end{aligned} \tag{3.81}$$

Then:

$$\begin{aligned} K_{\mu\nu} \;=\; & i\int \frac{d^3q}{2|\vec{q}|}\ \Big\{(q_i\delta^j_{[\mu}\delta^i_{\nu]} - |\vec{q}|\delta^j_{[\mu}\delta^0_{\nu]})\ \tilde{\phi}_{(1)}(\vec{q},-|\vec{q}|)e^{-i|\vec{q}|q^1\theta_{01}}\ \Big[\frac{\partial\tilde{\phi}_{(1)}(-\vec{q},|\vec{q}|)}{\partial q^j} + \\ & -\frac{i}{2}\tilde{\phi}_{(1)}(-\vec{q},|\vec{q}|)(\delta^A_j + \frac{q_j}{|\vec{q}|}\delta^A_0)q^1\theta_{A1}\Big] - (q_i\delta^j_{[\mu}\delta^i_{\nu]} + |\vec{q}|\delta^j_{[\mu}\delta^0_{\nu]})\tilde{\phi}_{(1)}(\vec{q},|\vec{q}|)e^{i|\vec{q}|q^1\theta_{01}}\cdot \\ & \cdot\Big[\frac{\partial\tilde{\phi}_{(1)}(-\vec{q},-|\vec{q}|)}{\partial q^j} - \frac{i}{2}\tilde{\phi}_{(1)}(-\vec{q},-|\vec{q}|)(\delta^A_j - \frac{q_j}{|\vec{q}|}\delta^A_0)q^1\theta_{A1}\Big]\Big\}\ e^{-iq^i\delta^A_iq^1\theta_{A1}}. \end{aligned} \tag{3.82}$$

The terms proportional to θ_{A1} give null contribution, since if considered together they are of the form $\int d^3q\,[f(\vec{q})-f(-\vec{q})]$, that is zero. So the final form of $K_{\mu\nu}$ in terms of $\tilde{\phi}_{(1)}$ is:

$$\begin{aligned} K_{\mu\nu} &= i\int\frac{d^3q}{2|\vec{q}|}\left\{(q_i\delta^j_{[\mu}\delta^i_{\nu]}-|\vec{q}|\delta^j_{[\mu}\delta^0_{\nu]})\,\tilde{\phi}_{(1)}(\vec{q},-|\vec{q}|)\frac{\partial\tilde{\phi}_{(1)}(-\vec{q},|\vec{q}|)}{\partial q^j}e^{-i|\vec{q}|q^1\theta_{01}}+\right.\\ &\left.-(q_i\delta^j_{[\mu}\delta^i_{\nu]}+|\vec{q}|\delta^j_{[\mu}\delta^0_{\nu]})\tilde{\phi}_{(1)}(\vec{q},|\vec{q}|)\frac{\partial\tilde{\phi}_{(1)}(-\vec{q},-|\vec{q}|)}{\partial q^j}e^{i|\vec{q}|q^1\theta_{01}}\right\}e^{-iq^i\delta^A_i q^1\theta_{A1}}. \end{aligned} \tag{3.83}$$

We have to compare this expression of the charge with the one obtained using the field expansion (3.74) in terms of x_1-to-the-right ordered exponentials in (3.37). In this last case one gets:

$$\begin{aligned} K_{\mu\nu} &= \int d^3\hat{x}d^4kd^4q\;\tilde{\phi}_{(1)}(k)\tilde{\phi}_{(1)}(q)\delta(k^\mu k_\mu)\delta(q^\mu q_\mu)\left\{e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}\hat{P}^0\hat{M}_{\mu\nu}(e^{iq^A\hat{x}_A}e^{iq^1\hat{x}_1})+\right.\\ &-\left(\hat{P}^0(e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1})\right)\hat{M}_{\mu\nu}(e^{iqA\hat{x}_A}e^{iq^1\hat{x}_1})-\frac{1}{2}\Upsilon^{\alpha\beta}_{\mu\nu}\left[\left(\hat{P}_\beta(e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1})\right)\cdot\right.\\ &\left.\left.\cdot\;\hat{P}^0\hat{P}_\alpha(e^{iq^A\hat{x}_A}e^{iq^1\hat{x}_1})-\left(\hat{P}_\beta\hat{P}^0(e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1})\right)\hat{P}_\alpha(e^{iq^A\hat{x}_A}e^{iq^1\hat{x}_1})\right]\right\}. \end{aligned} \tag{3.84}$$

Let us first consider the term proportional to $\Upsilon^{\alpha\beta}_{\mu\nu}$, that we shall call (B). Since the translation generators have still classical action on the exponentials it holds:

$$\begin{aligned} (B) &= \frac{1}{2}\int d^3\hat{x}d^4kd^4q\;\tilde{\phi}_{(1)}(k)\tilde{\phi}_{(1)}(q)\delta(k^2)\delta(q^2)\Upsilon^{\alpha\beta}_{\mu\nu}(k^0-q^0)k_\beta q_\alpha e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}e^{iq^A\hat{x}_A}e^{iq^1\hat{x}_1}\\ &= \frac{1}{2}\int d^3\hat{x}d^4kd^4q\;\tilde{\phi}_{(1)}(k)\tilde{\phi}_{(1)}(q)\delta(k^2)\delta(q^2)\Upsilon^{\alpha\beta}_{\mu\nu}(k^0-q^0)k_\beta q_\alpha e^{i(k+q)^i\hat{x}_i}e^{i(k+q)^0\hat{x}_0}\cdot\\ &\quad\cdot e^{-\frac{i}{2}(k^Ak^1+q^Aq^1)\theta_{A1}}e^{\frac{i}{2}(k+q)^i(k+q)^0\theta_{i0}}e^{-\frac{i}{2}k^\beta q^\alpha\theta_{\beta\alpha}}. \end{aligned} \tag{3.85}$$

From this point the steps to follow are very similar to the ones of 3.5, so we shall give only the final result:

$$\begin{aligned} (B) &= \frac{1}{2}\int d^3q\frac{1}{2|\vec{q}|}\Upsilon^{\alpha\beta}_{\mu\nu}\left\{\tilde{\phi}_{(1)}(\vec{q},-|\vec{q}|)\tilde{\phi}_{(1)}(-\vec{q},|\vec{q}|)(q_i\delta^i_\beta-|\vec{q}|\delta^0_\beta)(q_i\delta^i_\alpha-|\vec{q}|\delta^0_\alpha)e^{-i|\vec{q}|\delta^A_0 q^1\theta_{A1}}+\right.\\ &\left.-\tilde{\phi}_{(1)}(\vec{q},|\vec{q}|)\tilde{\phi}_{(1)}(-\vec{q},-|\vec{q}|)(q_i\delta^i_\beta+|\vec{q}|\delta^0_\beta)(q_i\delta^i_\alpha+|\vec{q}|\delta^0_\alpha)e^{i|\vec{q}|\delta^A_0 q^1\theta_{A1}}\right\}e^{-iq^j\delta^A_j q^1\theta_{A1}}. \end{aligned} \tag{3.86}$$

This term is null since it is of the form $\int d^3q[f(\vec{q})-f(-\vec{q})]=0$. As regards the remaining term of the charge, we see that the action of $\hat{M}_{\mu\nu}$ on non-symmetric ordered exponentials is not classical; in particular, on the x_1-to-the-right ordered ones it is:

$$\begin{aligned} \hat{M}_{\mu\nu}(e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}) &= \hat{M}_{\mu\nu}e^{ik\hat{x}}e^{-\frac{i}{2}k^Ak^1\theta_{A1}}=i\left(\frac{\partial}{\partial k^{[\mu}}k_{\nu]}e^{ik\hat{x}}\right)e^{-\frac{i}{2}k^Ak^1\theta_{A1}}\\ &= i\left(\frac{\partial}{\partial k^{[\mu}}+\frac{i}{2}(\delta^A_{[\mu}k^1+\delta^1_{[\mu}k^A)\theta_{A1}\right)k_{\nu]}\left[e^{ik\hat{x}}e^{-\frac{i}{2}k^Ak^1\theta_{A1}}\right]; \end{aligned} \tag{3.87}$$

moreover it holds, similarly to what explained in Sec. 3.5:

$$\hat{P}^0 \hat{M}_{\mu\nu}(e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}) = -i\left(\frac{\partial}{\partial k^{[\mu}} + \frac{i}{2}(\delta^A_{[\mu}k^1 + \delta^1_{[\mu}k^A)\theta_{A1}\right)k^0 k_{\nu]}\left[e^{ik\hat{x}}e^{-\frac{i}{2}k^A k^1\theta_{A1}}\right], \tag{3.88}$$

so:

$$\begin{aligned} K_{\mu\nu} = {} & i\int d^3\hat{x}d^4kd^4q\ \tilde{\phi}_{(1)}(k)\tilde{\phi}_{(1)}(q)\delta(k^\mu k_\mu)\delta(q^\mu q_\mu)q_{[\nu}\left(\frac{\partial}{\partial q^{\mu]}} + \frac{i}{2}(\delta^A_{\mu]}q^1 + \delta^1_{\mu]}q^A)\theta_{A1}\right)\cdot \\ & \cdot\left[(k^0-q^0)e^{i(k+q)^i\hat{x}_i}e^{i(k+q)^0\hat{x}_0}e^{-\frac{i}{2}(k^Ak^1+q^Aq^1)\theta_{A1}}e^{\frac{i}{2}(k+q)^i(k+q)^0\theta_{i0}}e^{-\frac{i}{2}k^\beta q^\alpha\theta_{\beta\alpha}}\right]. \end{aligned} \tag{3.89}$$

Noe we shall threat separately the term that contains the derivative with respect to q^μ (that we shall call (C)) and the term proportional to θ_{A1}, that we shall call (D). This last one results to be, with very simple passages that we shall omit:

$$\begin{aligned} (D) = {} & \int\frac{d^3q}{4|\vec{q}|}\theta_{A1}\left\{\tilde{\phi}_{(1)}(-\vec{q},|\vec{q}|)\tilde{\phi}_{(1)}(\vec{q},-|\vec{q}|)(q_j\delta^j_{[\nu} - |\vec{q}|\delta^0_{[\nu})\left[\delta^A_{\mu]}q^1 + \delta^1_{\mu]}(q^j\delta^A_j + |\vec{q}|\delta^A_0)\right]\cdot\right. \\ & \cdot e^{-i|\vec{q}|\delta^A_0 q^1\theta_{A1}} - \tilde{\phi}_{(1)}(-\vec{q},-|\vec{q}|)\tilde{\phi}_{(1)}(\vec{q},|\vec{q}|)(q_j\delta^j_{[\nu} + |\vec{q}|\delta^0_{[\nu})\left[\delta^A_{\mu]}q^1 + \delta^1_{\mu]}(q^j\delta^A_j - |\vec{q}|\delta^A_0)\right]\cdot \\ & \left.\cdot\ e^{i|\vec{q}|\delta^A_0 q^1\theta_{A1}}\right\}e^{-iq^j\delta^A_j q^1\theta_{A1}}, \end{aligned} \tag{3.90}$$

and this expression results to be null, since it is again of the form $\int d^3q[f(\vec{q}) - f(-\vec{q})]$. The term containing the derivative $\frac{\partial}{\partial q^\mu}$ becomes, after integration in the spatial coordinates and in d^4k, that, as explained in Sec. 3.5, can be performed inside the derivative:

$$\begin{aligned} (C) = K_{\mu\nu} = {} & i\sum_{a=\{1,-1\}}\int d^4q\ \tilde{\phi}_{(1)}(q)\delta(q^\mu q_\mu)q_{[\nu}\frac{\partial}{\partial q^{\mu]}}\left[\frac{\tilde{\phi}_{(1)}(-\vec{q},a|\vec{q}|)}{2|\vec{q}|}(-a|\vec{q}| - q^0)\cdot\right. \\ & \left.\cdot e^{i(-a|\vec{q}|+q^0)\hat{x}_0}e^{-\frac{i}{2}((q^j\delta^A_j + a|\vec{q}|\delta^A_0)q^1 + q^Aq^1)\theta_{A1}}e^{\frac{i}{2}(q^j\delta^\beta_j + a|\vec{q}|\delta^\beta_0)q^\alpha\theta_{\beta\alpha}}\right]. \end{aligned} \tag{3.91}$$

Since the argument of the derivative contains only one more exponential with respect to the expression (3.67), it is easy to show that:

$$
\begin{aligned}
K_{\mu\nu} \;=\; & -i \sum_{a,b=\{1,-1\}} \int \frac{d^3q}{4|\vec{q}|}\, \tilde{\phi}_{(1)}(\vec{q}, b|\vec{q}|)(q_j\delta^j_{[\nu} + b|\vec{q}|\delta^0_{[\nu})\Bigg\{\Bigg[\delta^j_{\mu]}\frac{\partial\tilde{\phi}_{(1)}(-\vec{q}, a|\vec{q}|)}{\partial q^j}(b-a) + \\
& -\frac{\tilde{\phi}_{(1)}(-\vec{q}, a|\vec{q}|)}{|\vec{q}|}(a\frac{q_i\delta^i_{\mu]}}{|\vec{q}|} + \delta^0_{\mu]}) - \frac{\tilde{\phi}_{(1)}(-\vec{q}, a|\vec{q}|)}{|\vec{q}|^2}\delta^i_{\mu]}q_i(b-a) + \frac{i}{2}\tilde{\phi}_{(1)}(-\vec{q}, a|\vec{q}|)(b-a)\,\cdot \\
& \cdot\Bigg((\delta^i_{\mu]}\delta^\beta_i + a\frac{q_i\delta^i_{\mu]}}{|\vec{q}|}\delta^\beta_0)(q^j\delta^\alpha_j - b|\vec{q}|\delta^\alpha_0)\theta_{\beta\alpha} + (q^i\delta^\beta_i + a|\vec{q}|\delta^\beta_0)\delta^\alpha_{\mu]}\theta_{\beta\alpha} - ((q^j\delta^A_j + a|\vec{q}|\delta^A_0 + \\
& + q^j\delta^A_j - b|\vec{q}|\delta^A_0)\delta^1_{\mu]} + (\delta^j_{\mu]}\delta^A_j + a\frac{q_j\delta^j_{\mu]}}{|\vec{q}|}\delta^A_0 + \delta^A_{\mu]})q^1)\theta_{A1}\Bigg)\Bigg]e^{-i(a+b)|\vec{q}|\hat{x}_0} + i\tilde{\phi}_{(1)}(-\vec{q}, a|\vec{q}|)\,\cdot \\
& \cdot(b-a)(-a\frac{q_i\delta^i_{\mu]}}{|\vec{q}|} + \delta^0_{\mu]})\Omega_w\Big[x_0 e^{-i(a+b)|\vec{q}|x_0}\Big]\Bigg\}e^{\frac{i}{2}(q^i\delta^\beta_i + a|\vec{q}|\delta^\beta_0)(q^j\delta^\alpha_j - b|\vec{q}|\delta^\alpha_0)\theta_{\beta\alpha}}\,\cdot \\
& \cdot e^{-\frac{i}{2}((q^j\delta^A_j + a|\vec{q}|\delta^A_0)q^1 + (q^j\delta^A_j - b|\vec{q}|\delta^A_0)q^1)\theta_{A1}},
\end{aligned}
\tag{3.92}
$$

where the term with $a = b$ is null since it is symmetric in the (antisymmetrised) indices μ and ν. So it remains only the term with $a = -b$, that is (the calculations omitted are very similar to those of Sec. 3.5):

$$
\begin{aligned}
K_{\mu\nu} \;=\; & -i\int\frac{d^3q}{4|\vec{q}|}\Bigg\{-\tilde{\phi}_{(1)}(\vec{q}, -|\vec{q}|)(q_j\delta^j_{[\nu} - |\vec{q}|\delta^0_{[\nu})\Big\{2\delta^j_{\mu]}\frac{\partial\tilde{\phi}_{(1)}(-\vec{q}, |\vec{q}|)}{\partial q^j} - i\tilde{\phi}_{(1)}(-\vec{q}, |\vec{q}|)\,\cdot \\
& \cdot\Big(2(q^j\delta^A_j + |\vec{q}|\delta^A_0)\delta^1_{\mu]} + (\delta^j_{\mu]}\delta^A_j + \frac{q_j\delta^j_{\mu]}}{|\vec{q}|}\delta^A_0 + \delta^A_{\mu]})q^1\Big)\theta_{A1}\Big\}e^{-i|\vec{q}|\delta^A_0 q^1\theta_{A1}} + \tilde{\phi}_{(1)}(\vec{q}, |\vec{q}|)\,\cdot \\
& \cdot(q_j\delta^j_{[\nu} + |\vec{q}|\delta^0_{[\nu})\Big\{2\delta^j_{\mu]}\frac{\partial\tilde{\phi}_{(1)}(-\vec{q}, -|\vec{q}|)}{\partial q^j} - i\tilde{\phi}_{(1)}(-\vec{q}, -|\vec{q}|)\Big(2(q^j\delta^A_j - |\vec{q}|\delta^A_0)\delta^1_{\mu]} + \\
& + (\delta^j_{\mu]}\delta^A_j - \frac{q_j\delta^j_{\mu]}}{|\vec{q}|}\delta^A_0 + \delta^A_{\mu]})q^1\Big)\theta_{A1}\Big\}e^{i|\vec{q}|\delta^A_0 q^1\theta_{A1}}\Bigg\}e^{-iq^j\delta^A_j q^1\theta_{A1}},
\end{aligned}
\tag{3.93}
$$

The terms proportional to θ_{A1} are again of the form $\int d^3q\,[f(\vec{q}) - f(-\vec{q})] \equiv 0$, so this expression of the charge results to be the same of (3.83).

So, also for the "Lorentz sector charges" we have found that the formalism is coherent, and, as we saw for translational case, written with the "Fourier transforms" of the fields associated to non symmetric exponentials the charges result to be no more classical in form, since there appear some phases.

Now we shall concentrate on the second issue regarding map independence. We shall derive, in the usual way, the "would-be-conserved-charges"

$$K^{(1)}_{\mu\nu} \equiv \int d^3\hat{x} J^{0\,(1)}_{\mu\nu}, \tag{3.94}$$

associated to the second of the currents (3.47), that were obtained varying the action with the use of the differential (2.29) associated to the $\hat{x}_1$-to-the-right basis of generators of the θ-Poincaré algebra.

Since in these currents the $\hat{x}_1$-to-the-right map generators appear, that have classical action on $\hat{x}_1$-to-the-right-ordered exponentials, it it convenient to use this basis to expand the fields. So, using (3.74), the charges become:

$$\begin{aligned} K^{(1)}_{\mu\nu} &= \int d^3\hat{x} d^4k d^4q\; \tilde{\phi}_{(1)}(k)\tilde{\phi}_{(1)}(q)\delta(k^2)\delta(q^2)\Big[e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}\hat{P}^0\hat{M}^{(1)}_{\mu\nu}(e^{iq^A\hat{x}_A}e^{iq^1\hat{x}_1}) + \\ &\quad - \left(\hat{P}^0(e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1})\right)\hat{M}^{(1)}_{\mu\nu}(e^{iq^A\hat{x}_A}e^{iq^1\hat{x}_1}) - \frac{1}{2}\chi^{\sigma\rho}_{\mu\nu}\Big[\left(\hat{P}_\rho(e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1})\right)\cdot \\ &\quad \cdot\hat{P}^0\hat{P}_\sigma(e^{iq^A\hat{x}_A}e^{iq^1\hat{x}_1}) - \left(\hat{P}_\rho\hat{P}^0(e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1})\right)\hat{P}_\sigma(e^{iq^A\hat{x}_A}e^{iq^1\hat{x}_1})\Big]. \end{aligned} \tag{3.95}$$

The term proportional to $\chi^{\sigma\rho}_{\mu\nu}$ is of the same kind of the one proportional to $\Upsilon^{\alpha\beta}_{\mu\nu}$ that appears in (3.84), except for that it has the coefficient χ instead of Υ. That term was shown to be null independently to the detailed expression of the coefficient Υ, so that demonstration is valid also in this case.

The action of $\hat{P}^0\hat{M}_{\mu\nu}$ on the $\hat{x}_1$-to-the-right-ordered exponentials is:

$$\begin{aligned} \hat{P}^0\hat{M}^{(1)}_{\mu\nu}(e^{iq^A\hat{x}_A}e^{iq^1\hat{x}_1}) &= \Omega_1\left(-i\partial^0(x_{[\mu}k_{\nu]}e^{ikx})\right) = \Omega_1\left(-i(\delta^0_{[\mu} + ix_{[\mu}k^0)k_{\nu]}e^{ikx}\right) \\ &= -i(\delta^0_{[\mu} + k^0\frac{\partial}{\partial k^{[\mu}})k_{\nu]}\left(e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}\right) = -i\frac{\partial}{\partial k^{[\mu}}\left(k^0k_{\nu]}e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}\right). \end{aligned} \tag{3.96}$$

so the resulting charge is:

$$K^{(1)}_{\mu\nu} = i\int d^3\hat{x} d^4k d^4q\; \tilde{\phi}_{(1)}(k)\tilde{\phi}_{(1)}(q)\delta(k^2)\delta(q^2)\frac{\partial}{\partial q^{[\mu}}\Big[q_{\nu]}(k^0 - q^0)e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}e^{iq^A\hat{x}_A}e^{iq^1\hat{x}_1}\Big]. \tag{3.97}$$

If we compare this expression with (3.89), we see that that one is the same of this one, except for that it contains an additional term proportional to θ_{A1}, that instead is shown to be null (see immediately after eq. (3.89)). So the final expression of $K^{(1)}_{\mu\nu}$ is just the same of (3.89), that is (3.84). This means that the charge associated to the transformations generated by the $\hat{x}_1$-to-the-right basis of the agebra of θ-Poincaré are the same of those associated to the symmetric basis, since (3.89) is nothing else but the expression of the "symmetric charges" in terms of the Fourier transform $\tilde{\phi}_{(1)}$ instead of $\tilde{\phi}_{(w)}$. So there isn't actually any map dependece of the charges:

the possibility of writing various differentials, that are different in dependence of the basis of generators used to write the transformation does not result in the presence of different charges, so that physically there are no ordering ambiguities.

Conclusions

In this thesis work we have proposed a generalisation of the Noether scheme of symmetry analysis suitable for the study of a scalar massless field theory described by a Klein-Gordon-like equation of motion in canonical noncommutative spacetime. We have shown that, when the noncommutativity parameters are observer independent, this kind of noncommutative spacetime is characterised by a deformation (rather than a breaking) of the Poincaré symmetries, and we provided an explicit construction, within a Weyl-map setup, of the generators of the deformed algebra of symmetries. This symmetry algebra (that we called θ-Poincaré algebra) was shown to be a Hopf-algebra twist of the classical Poincaré algebra.

On the basis of the properties of the deformed generators of the algebra we have proposed a definition of the action of symmetry transformations on fields, whose main characteristic is the necessity of noncommutative transformation parameters. A striking consequence of the noncommutativity of transformation parameters is the emergence of an obstruction for the realization of a pure spatial rotation transformation or a pure boost transformation: whenever at least one of the parameters of the Lorentz sector is non-null, also at least one of the translation parameters must be non-null.

Adopting our proposed description of symmetry transformation it turned out to be possible to perform a complete Noether analysis, including explicit formulas for the conserved charges associated to the invariance of the theory under the action of the θ-Poincaré algebra.

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