Introduction to Probabilistic Reasoning

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Uncertainty: some examples

Roll a die:

1, 2, 3, 4, 5, 6: ?

Toss a coin:

Head/Tail: ?

Having to perform a measurement:

Which numbers shall come out from our device ?

Having performed a measurement:

What have we learned about the value of the quantity of interest ?

Many other examples from real life:

Football, weather, tests/examinations, . . .

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Let us consider three outcomes:

$E_1 =$ ‘6’

$E_2 =$ ‘even number’

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- You will get a price if the event you chose will occur. On which event would you bet?

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- You will get a price if the event you chose will occur. On which event would you bet?
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- Imagine to repeat the experiment: which event do you expect to occur mostly? (More frequently)

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$\Rightarrow$ Many expressions to state our preference

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Can we use it for all other events of our interest?
($\longrightarrow$ two envelop 'paradox')

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Indeed, using David Hume's words,† *"this process of the thought or reasoning may seem trivial and obvious"*

A counting experiment

Imagine a small scintillation counter, with suitable threshold, placed

here

now

Fix the measuring time (e.g. 5 second each) and perform 20 measurements: 0, 0, 1, 0, 0, 0, 1, 2, 0, 0, 1, 1, 0.

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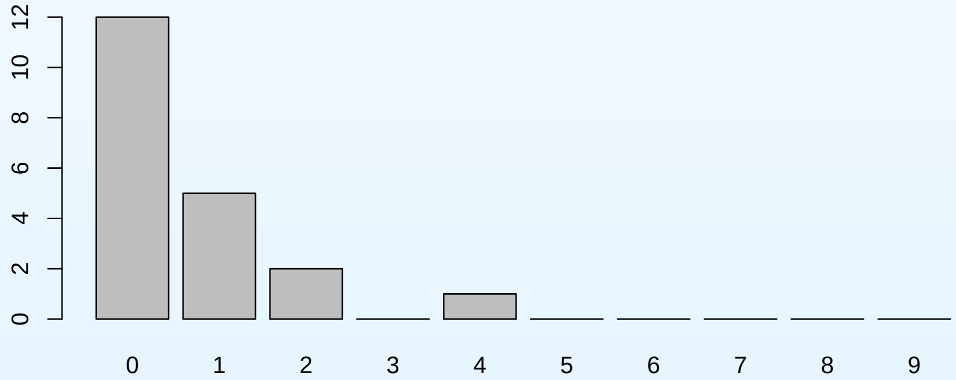

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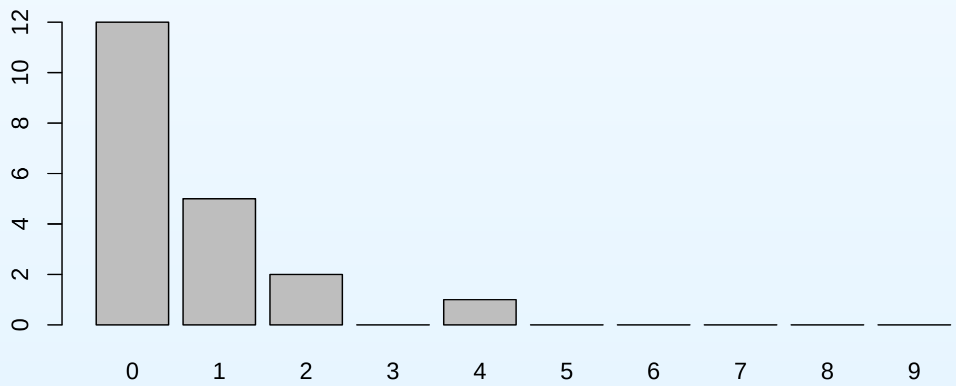
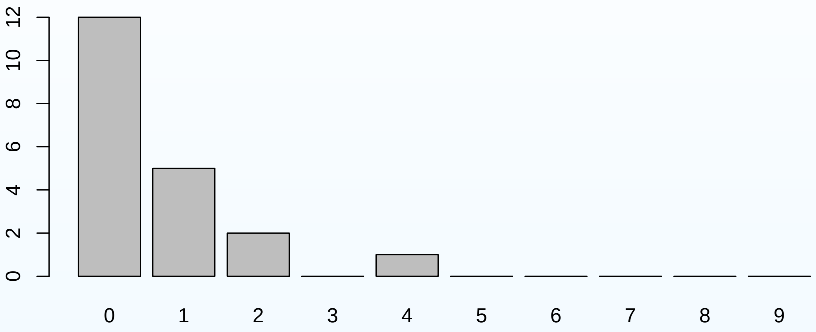


Think at the 21^{st} measurement:

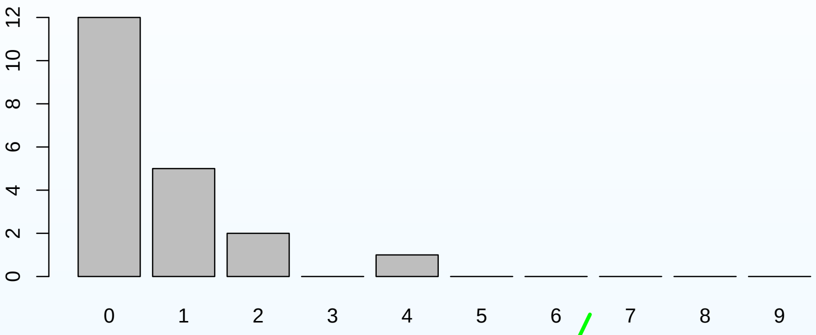
- Which outcome do you consider more likely? (0, 1, 2, 3, . . .)
- Why?

A counting experiment



$\Rightarrow$ Next ?

A counting experiment



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$P(0) > P(1) > P(2)$ $\surd$

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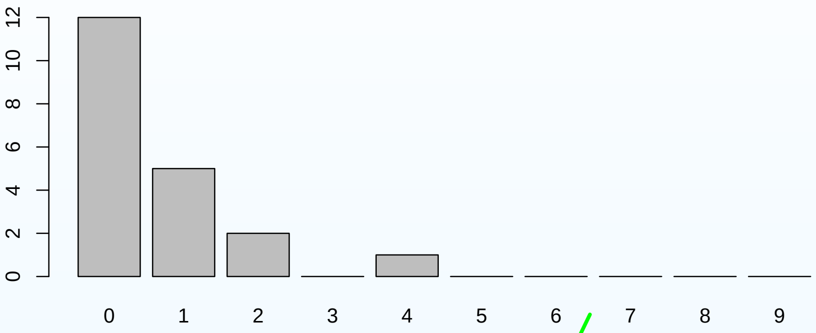

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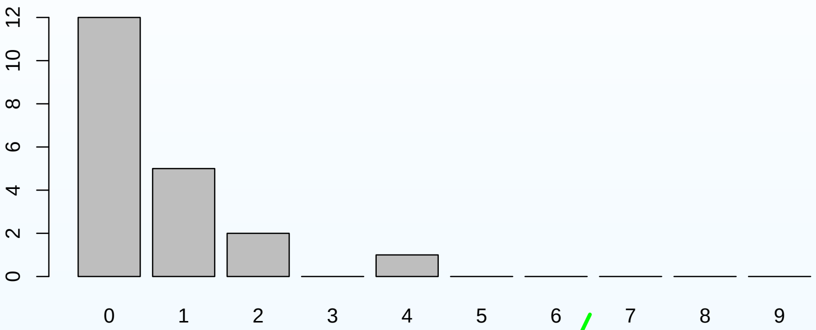

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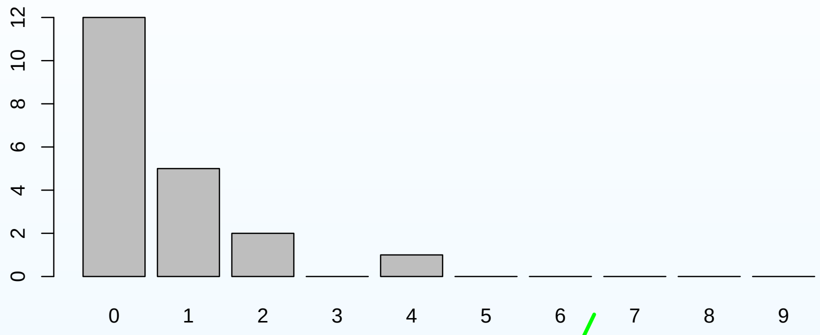

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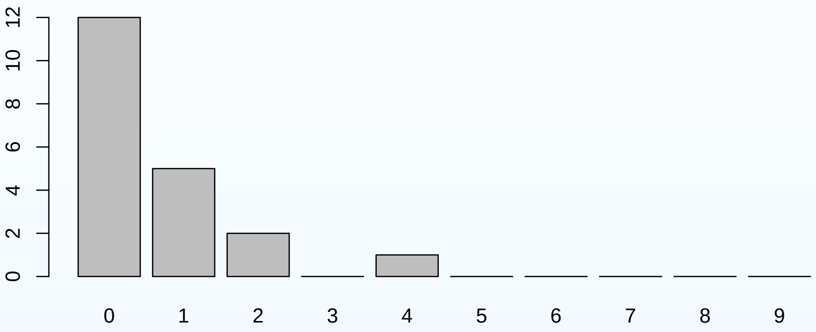
Not correct to say *"we cannot do it"*, or *"let us do other measurements and see"*:

> In real life we are asked to make assessments (and take decisions) with the information we have NOW. If, later, the information changes, we can (must!) use the update one (and perhaps update our opinion).

A counting experiment



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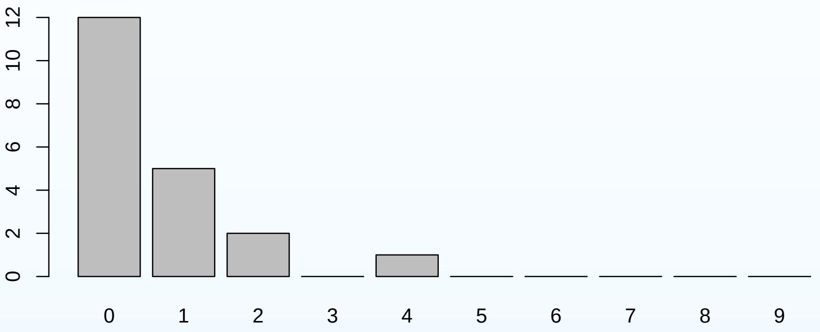

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We 'know' / 'assume' / 'hope' / 'guess' / 'postulate' regularity of nature

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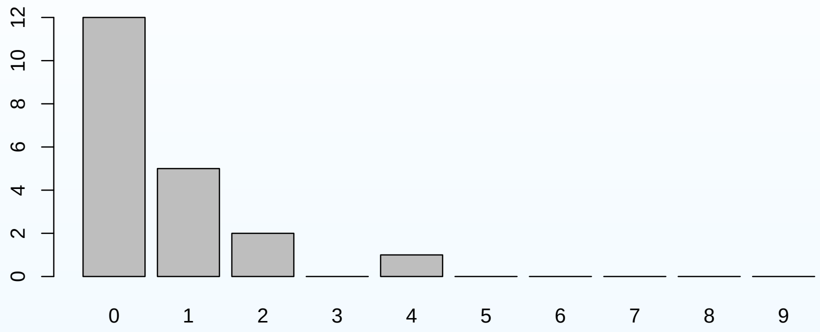

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A philosopher, physicist and mathematician joke

A philosopher, a physicist and a mathematician travel by train through Scotland.

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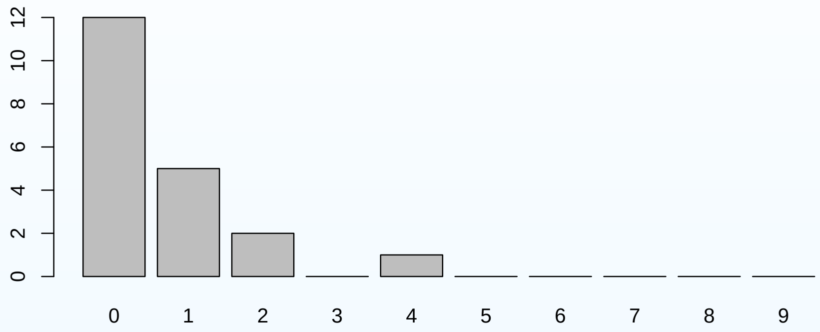
Physicists' statements about reality have plenty of tacit – mostly very reasonable! — assumptions that derive from experience and rationality.

$\Rightarrow$ We constantly use theory/models to link past and future!.

Transferring past to future



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Basic reasoning: assuming regularity of nature and a regular flow from the past to the future, we tend to believe that the effects that happened more frequently in the past will also occur more likely in the future.

Again, well expressed by Hume.†

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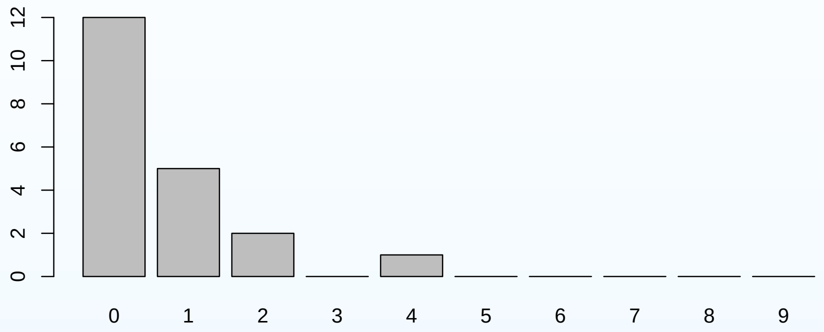

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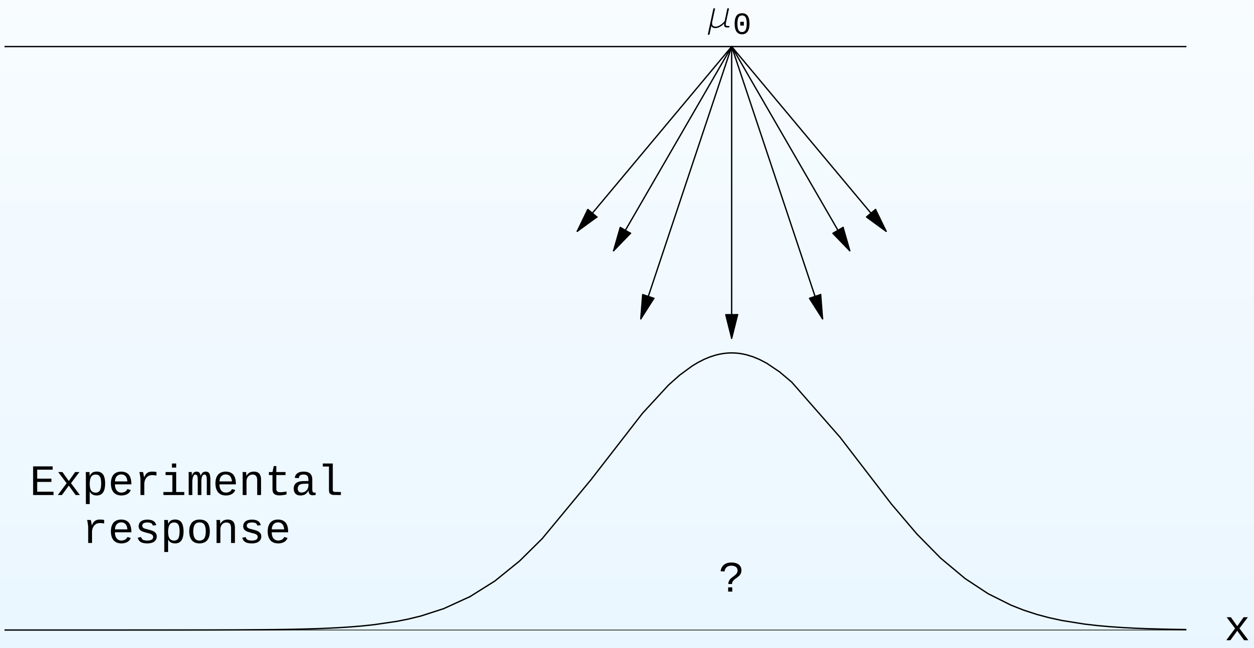
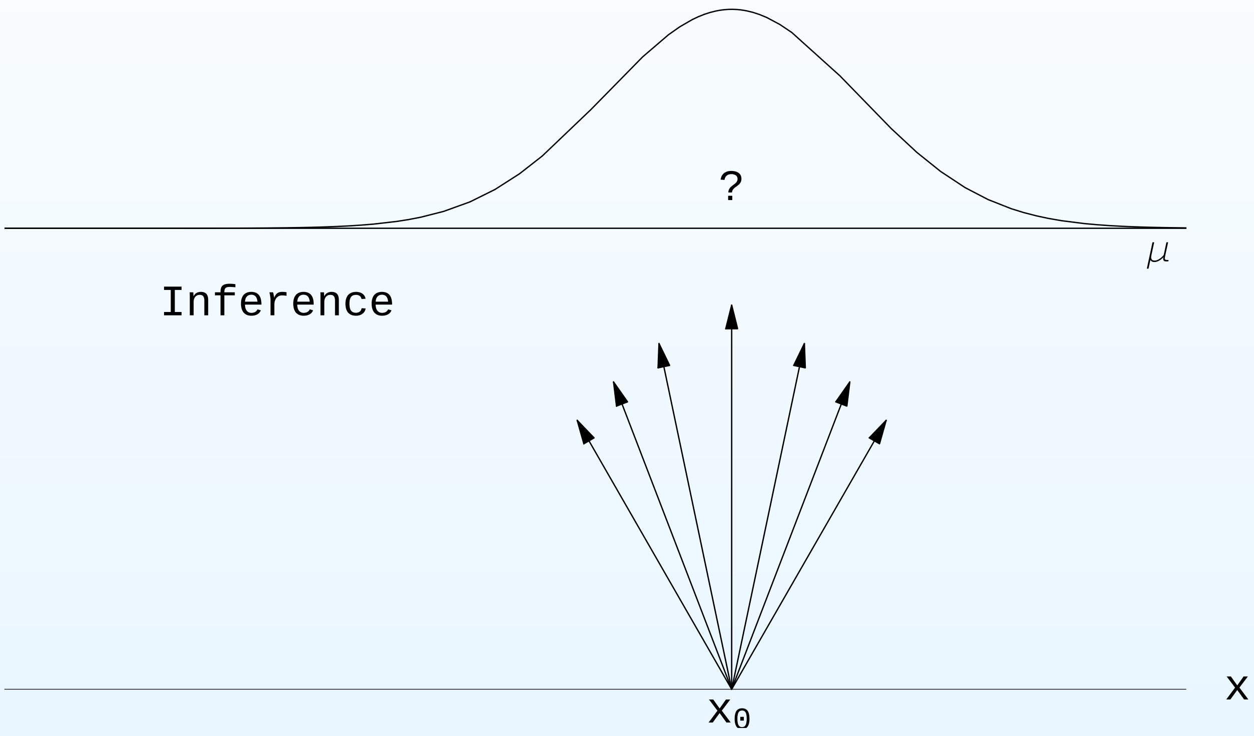
There is (in most cases) no way to get *directly* hints about $f(\mu\,|\,x)$.

Uncertainties in measurements



$f(x \mid \mu)$ experimentally accessible (though 'model filtered')

Uncertainties in measurements



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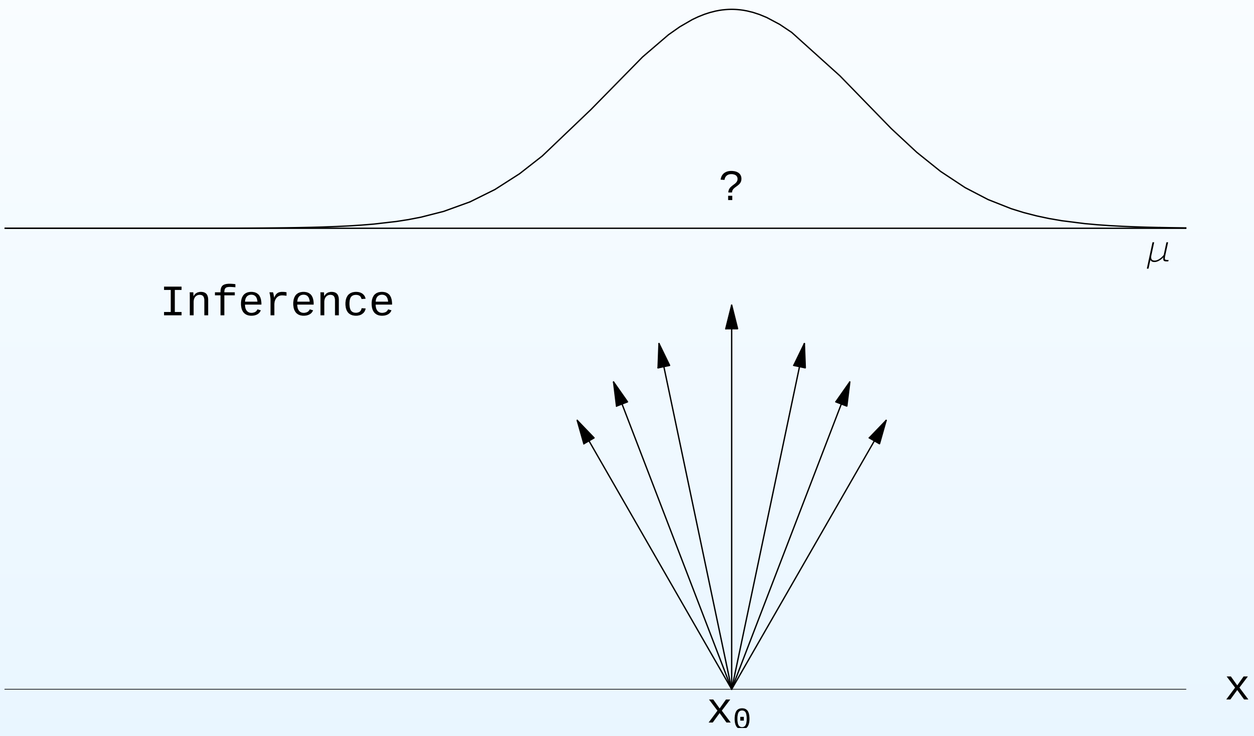

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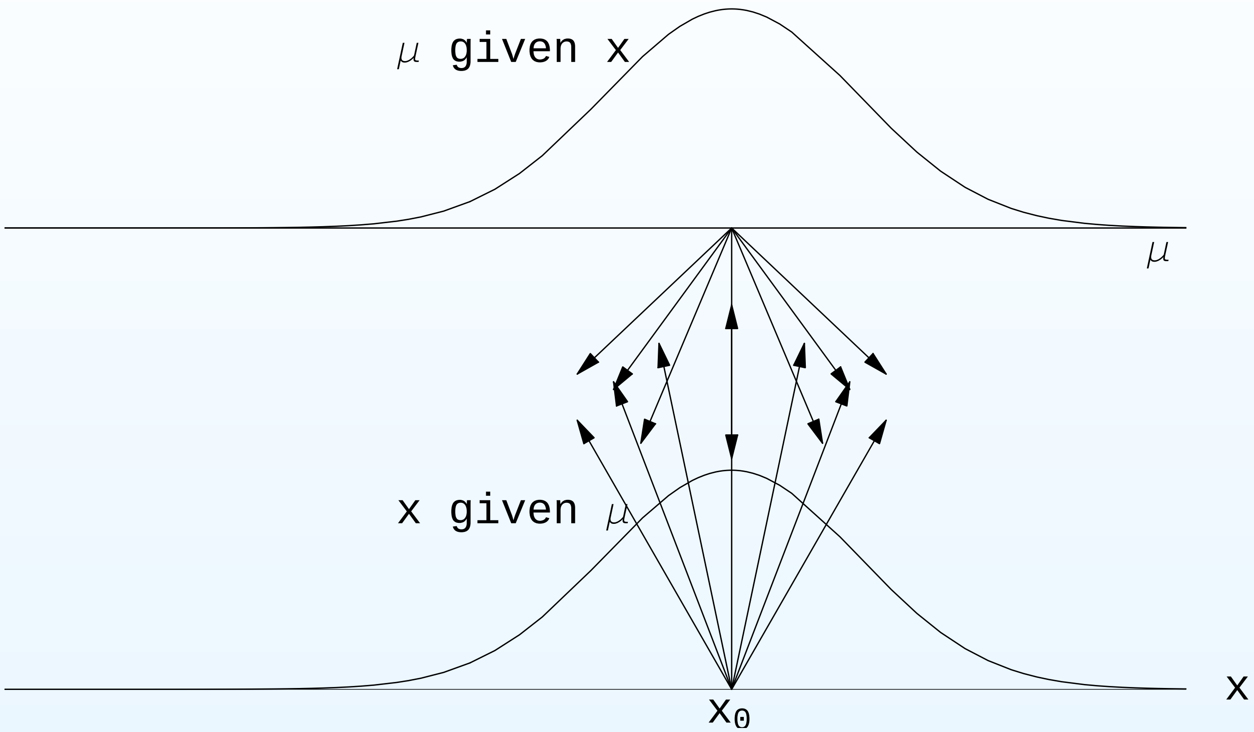
but logically accessible!

$\longrightarrow$ we need to learn how to do it

Uncertainties in measurements



- Review sources of uncertainties
- How measurement uncertainties are currently treated
- How to treat them logically using probability theory

  (But we also need to review what we mean by 'probability'!)

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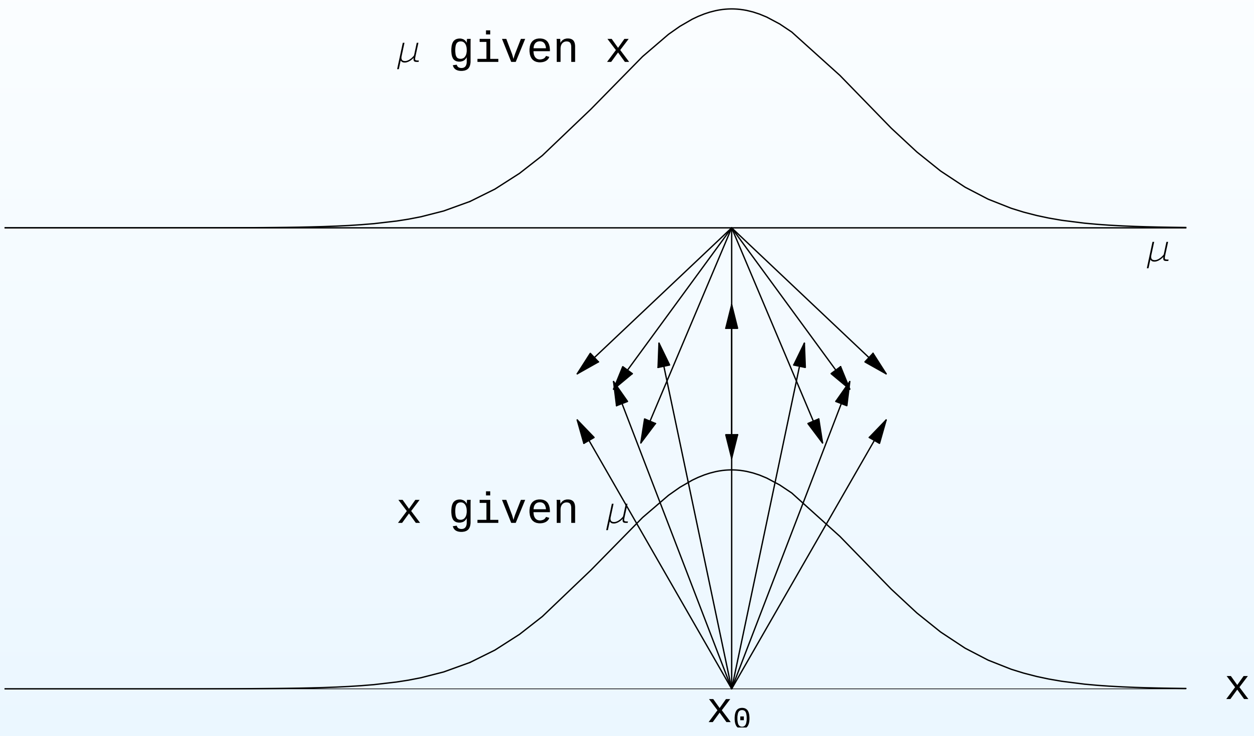

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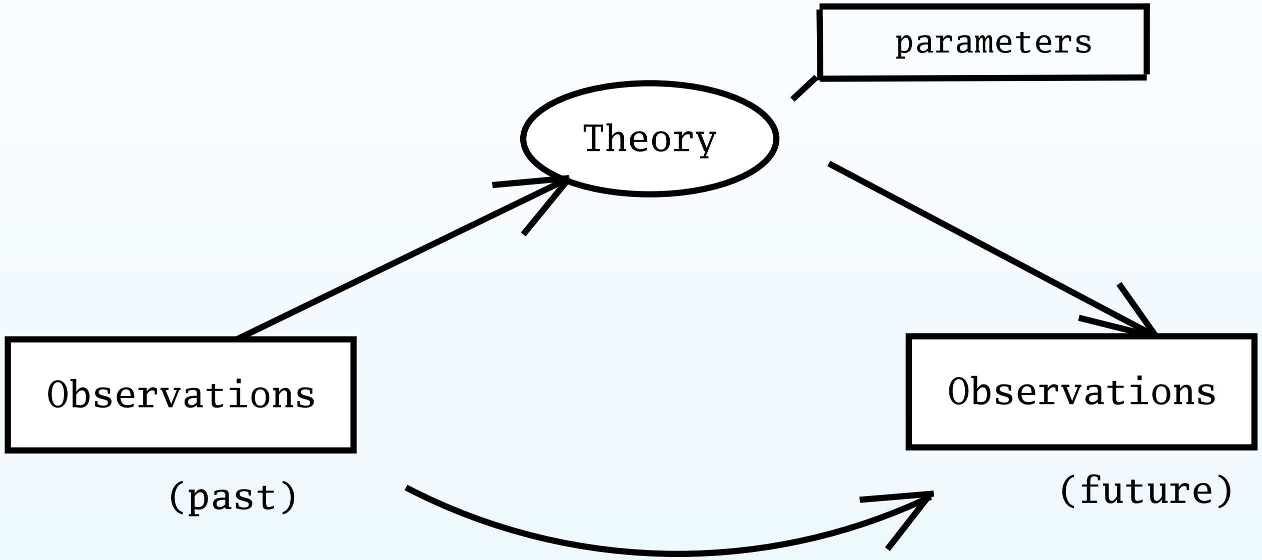
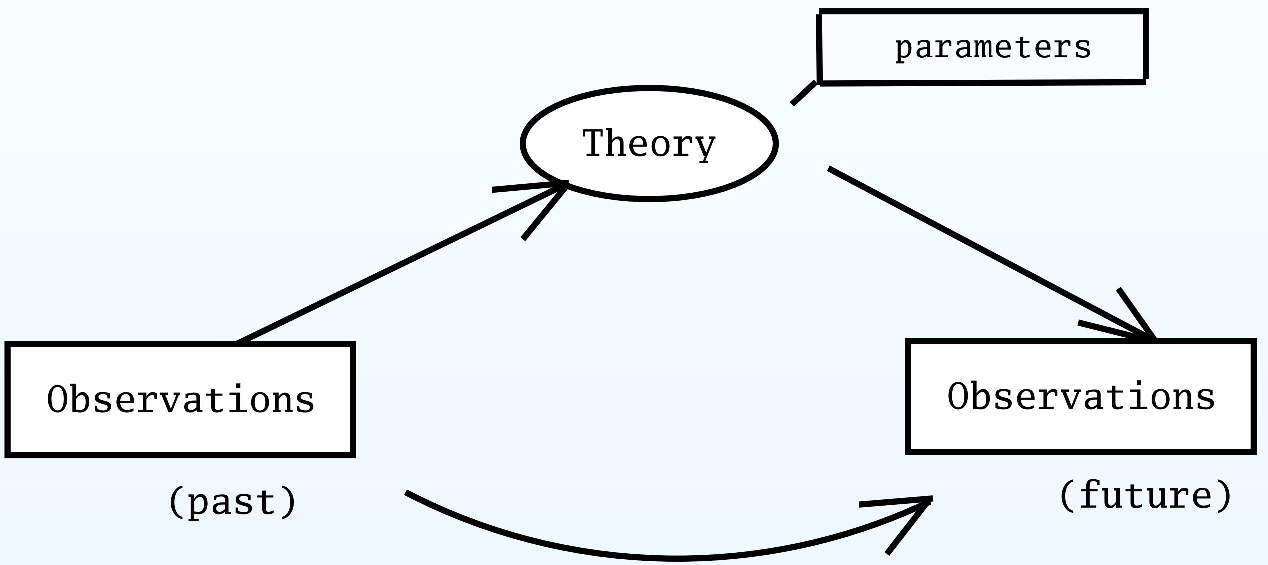
From past to future



Task of the ‘physicist’ (scientist, decision maker):

- Describe/understand the physical world
  $\Rightarrow$ inference of laws and their parameters
- Predict observations
  $\Rightarrow$ forecasting

From past to future



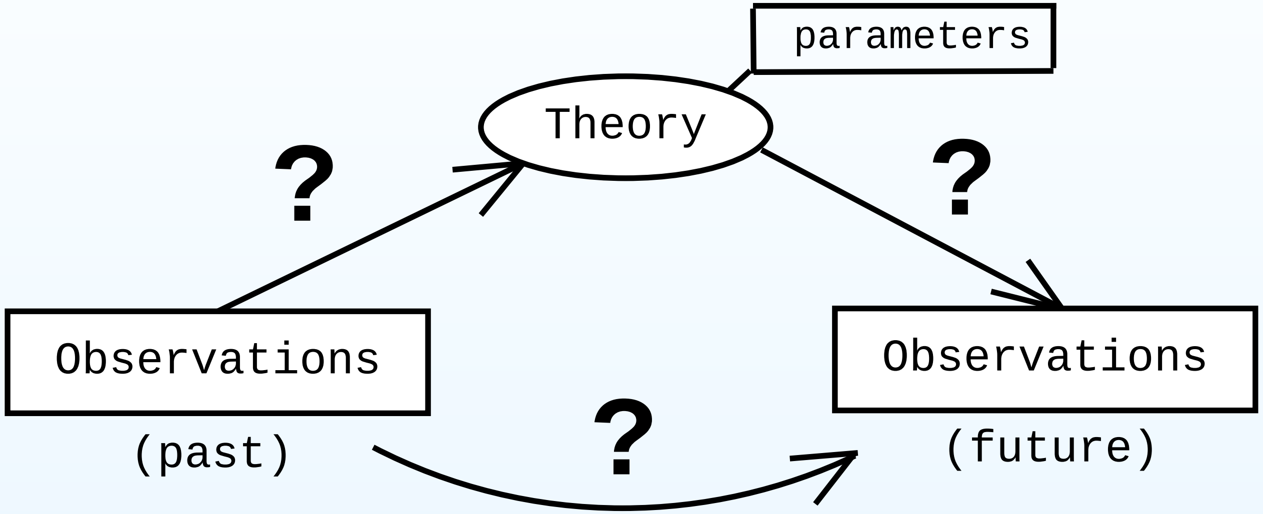
Process

- neither automatic
- nor purely contemplative

  $\longrightarrow$ 'scientific method'

  $\longrightarrow$ planned experiments ('actions') $\Rightarrow$ decision.

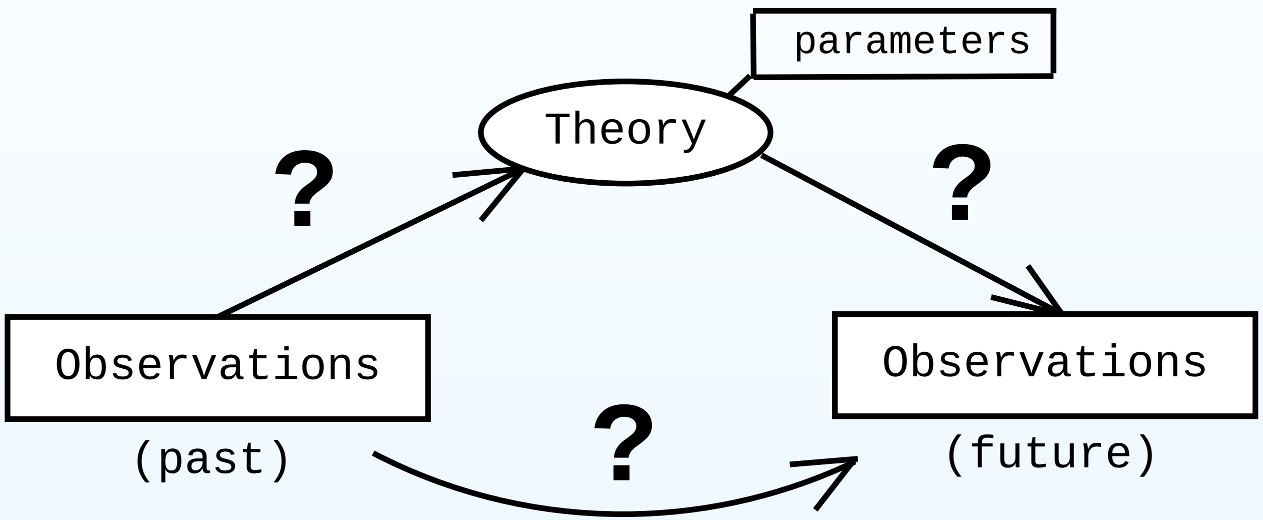
From past to future



⇒ Uncertainty:

1. Given the past observations, in general we are not sure about the theory parameter (and/or the theory itself)
2. Even if we were sure about theory and parameters, there could be internal (e.g. Q.M.) or external effects (initial/boundary conditions, ‘errors’, etc) that make the forecasting uncertain.

From past to future



⇒ Decision

- What is be best action ('experiment') to take in order 'to be confident' that what we would like will occur?
(Decision issues always assume uncertainty about future outcomes.)
- Before tackling problems of decision we need to learn to reason about uncertainty, possibly in a quantitative way.

About predictions

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*"Prediction is very difficult,
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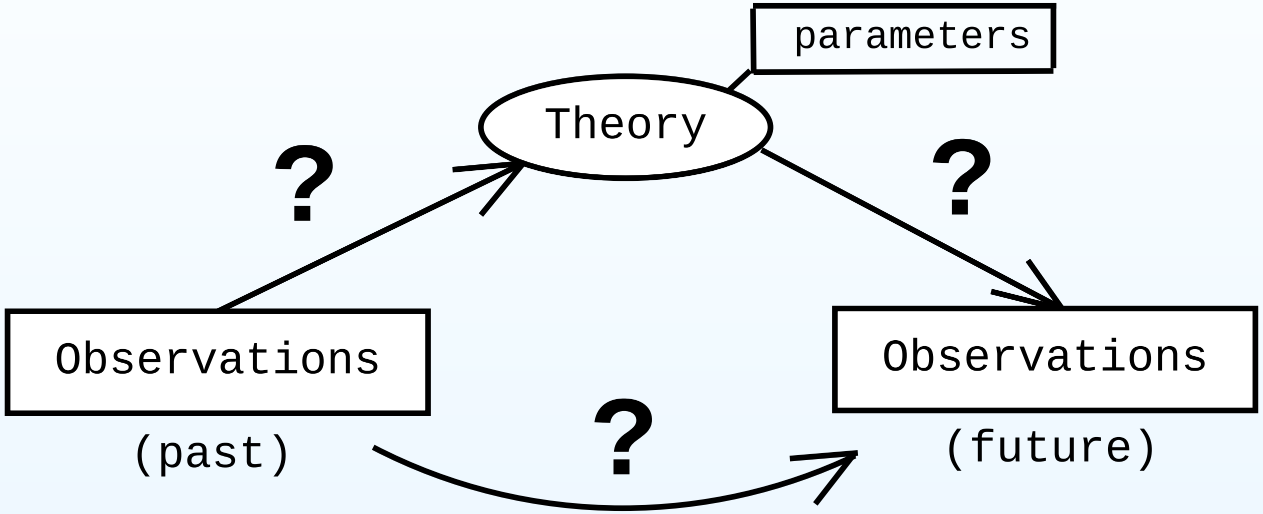
But, anyway:

“It is far better to foresee even without certainty than not to foresee at all” (Poincaré)

Deep source of uncertainty



Uncertainty:

Theory	— ? $\longrightarrow$		
Past observations	— ? $\longrightarrow$		
Theory	— ? $\longrightarrow$	Future observations	

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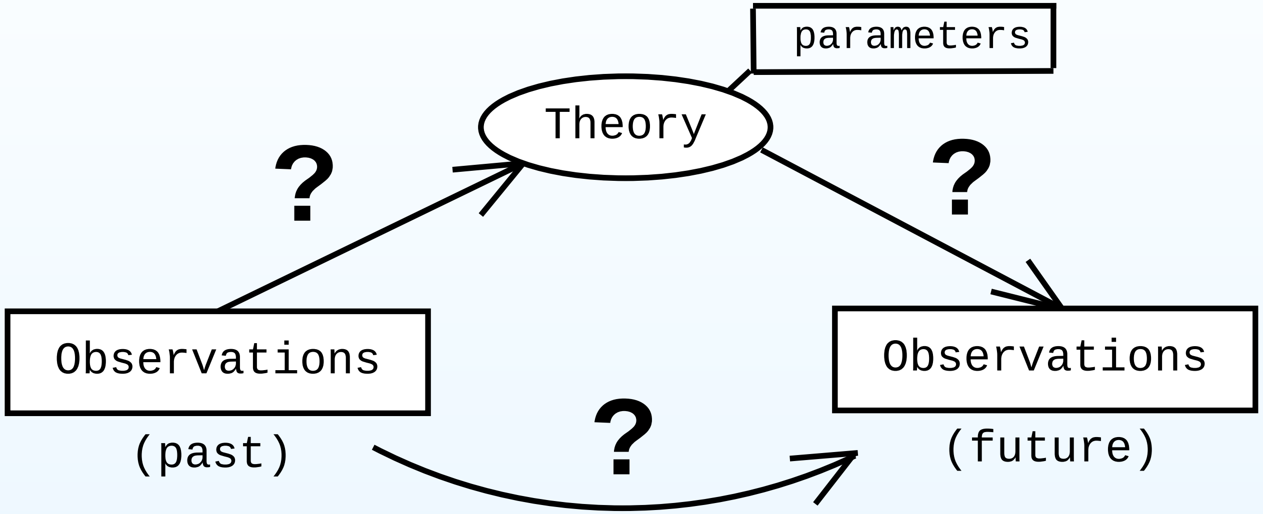


Uncertainty:

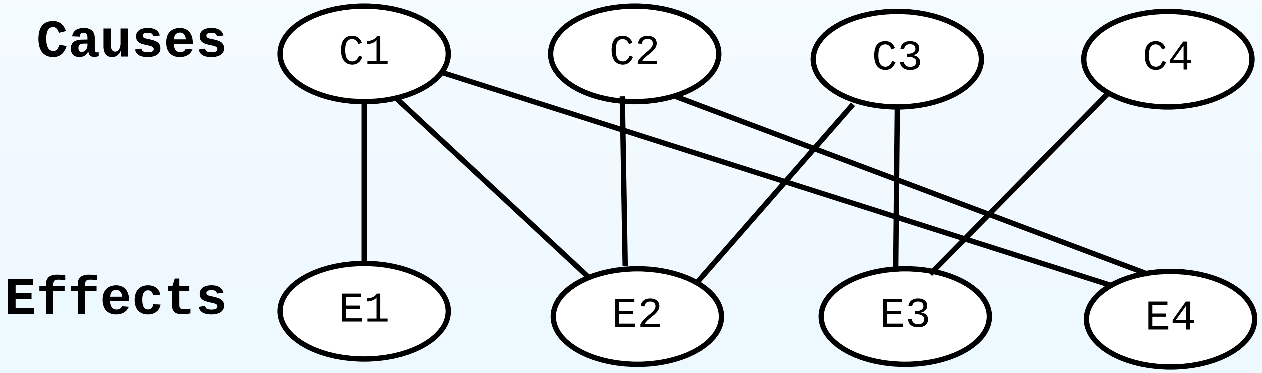
Theory	—?⟶	Future observations
Past observations	—?⟶	Theory
Theory	—?⟶	Future observations

$\Longrightarrow$ **Uncertainty about causal connections**

CAUSE $\Longleftrightarrow$ EFFECT

Causes $\longrightarrow$ effects

The same *apparent* cause might produce several,different effects



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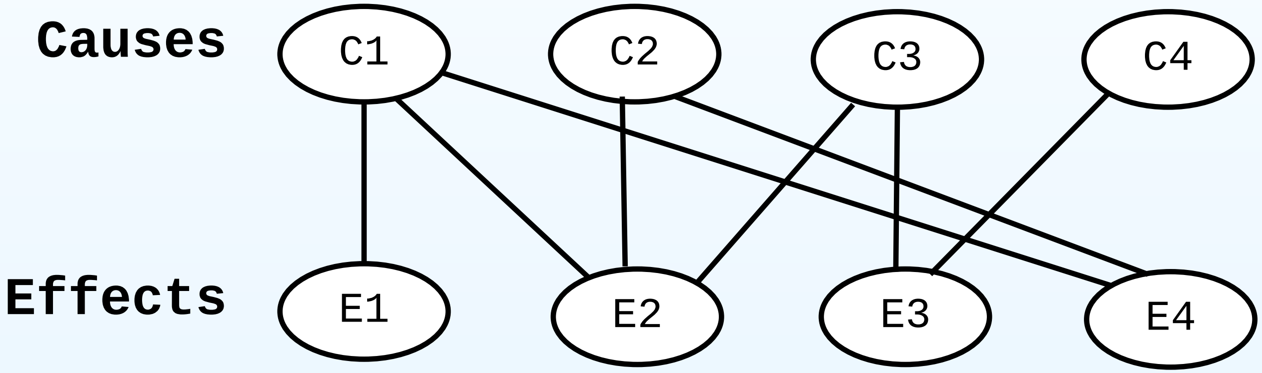

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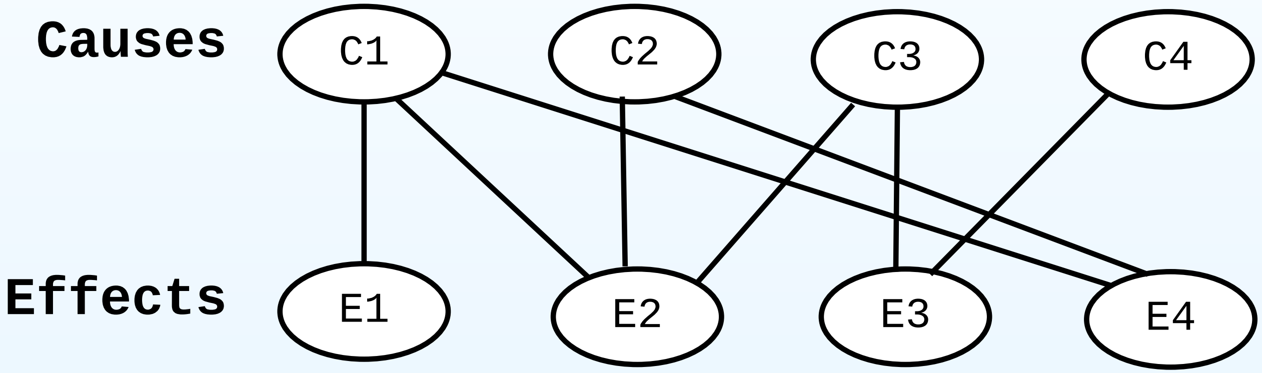

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$$\mathbf{E_2} \Rightarrow \{C_1,\, C_2,\, C_3\}?$$

The essential problem of the experimental method

“Now, these problems are classified as *probability of causes*, and are most interesting of all their scientific applications. I play at *écarté* with a gentleman whom I know to be perfectly honest. What is the chance that he turns up the king? It is 1/8. This is a problem of the *probability of effects*.

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I play with a gentleman whom I do not know. He has dealt ten times, and he has turned the king up six times. What is the chance that he is a sharper? This is a problem in the probability of causes. It may be said that it is the essential problem of the experimental method."

(H. Poincaré – *Science and Hypothesis*)

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The human mind is used to live — and survive — in conditions of uncertainty and has developed mental categories to handle it.

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As a matter of fact, although we are in a constant state of uncertainty about many events which might or might not occur,

- we can be "more or less *sure* — or *confident* — on something than on something else";
- "we consider something more or less *probable* (or *likely*)";
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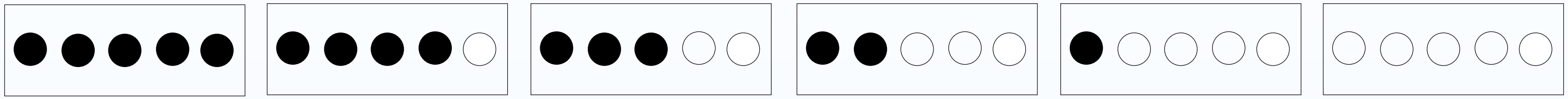
We can use similar expressions, all referring to the intuitive idea of probability.

The six box problem



H_0 H_1 H_2 H_3 H_4 H_5

Let us take randomly one of the boxes.

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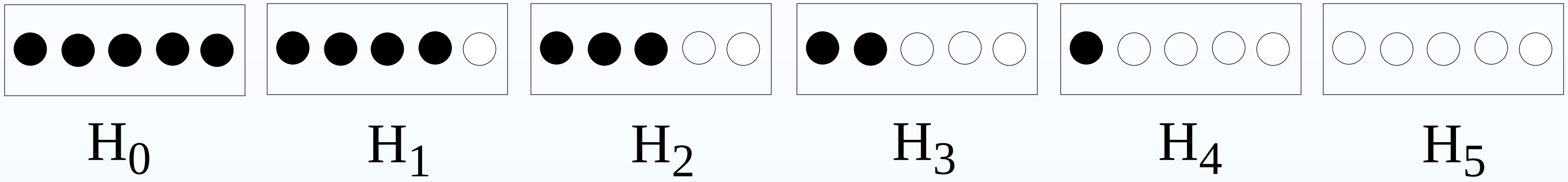

Let us take randomly one of the boxes.

We are in a state of uncertainty concerning several *events*, the most important of which correspond to the following questions:

(a) Which box have we chosen, $H_0, H_1, \ldots, H_5$?

(b) If we extract randomly a ball from the chosen box, will we observe a white ($E_W \equiv E_1$) or black ($E_B \equiv E_2$) ball?

Our certainty:

$$\begin{aligned} \cup_{j=0}^{5} \, H_j &= \Omega \\ \cup_{i=1}^{2} \, E_i &= \Omega \,. \end{aligned}$$

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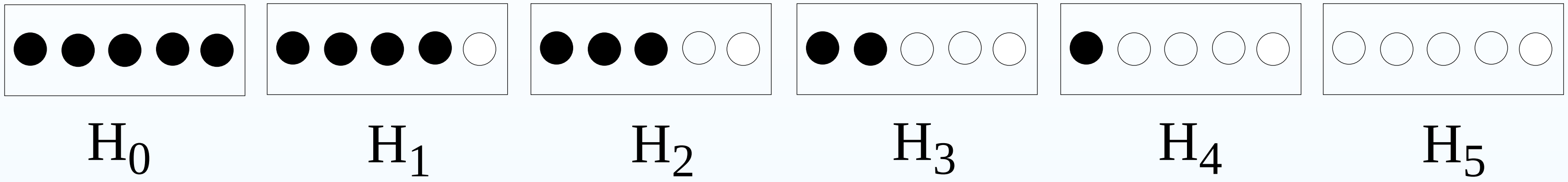

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  - Can we do it quantitatively, in an objective way?

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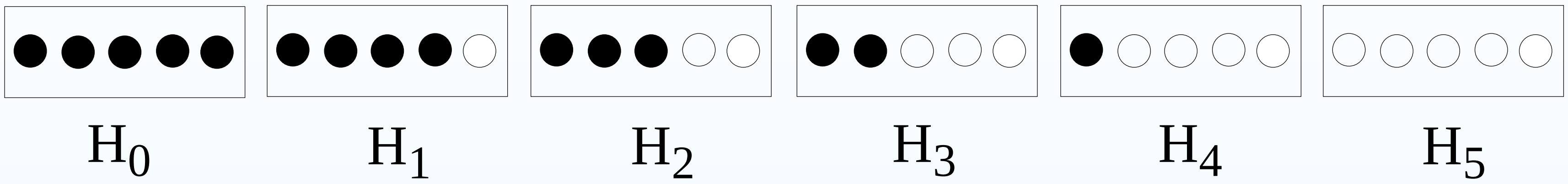

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- And after a sequence of extractions?

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This toy experiment is conceptually very close to what we do in Physics

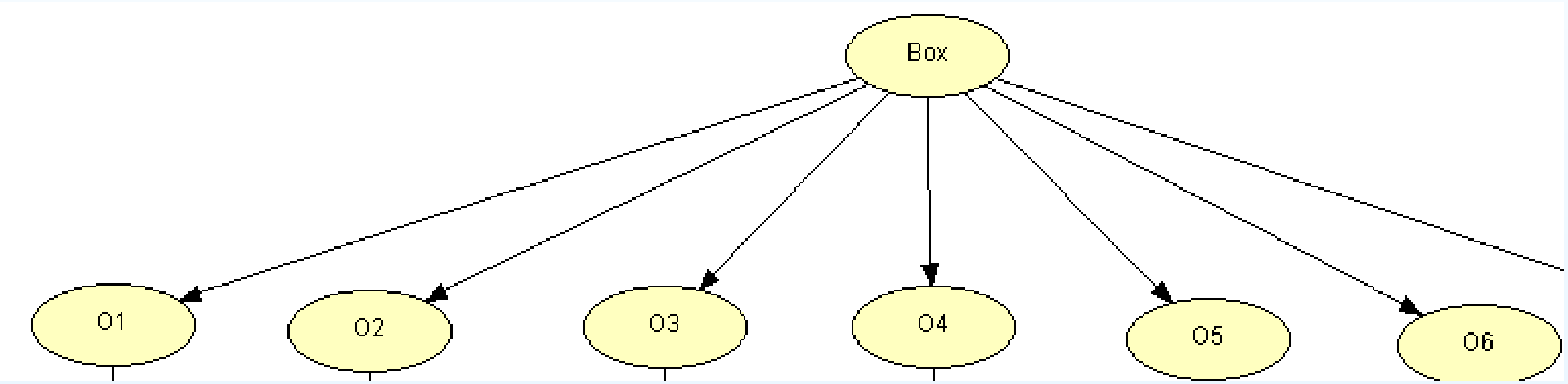
- try to guess what we cannot see (the electron mass, a branching ratio, etc)

  . . . from what we can see (somehow) with our senses.

The rule of the game is that we are not allowed to watch inside the box! (As we cannot open and electron and read its properties, like we read the MAC address of a PC interface)

Cause-effect representation

box content $\longrightarrow$ observed color



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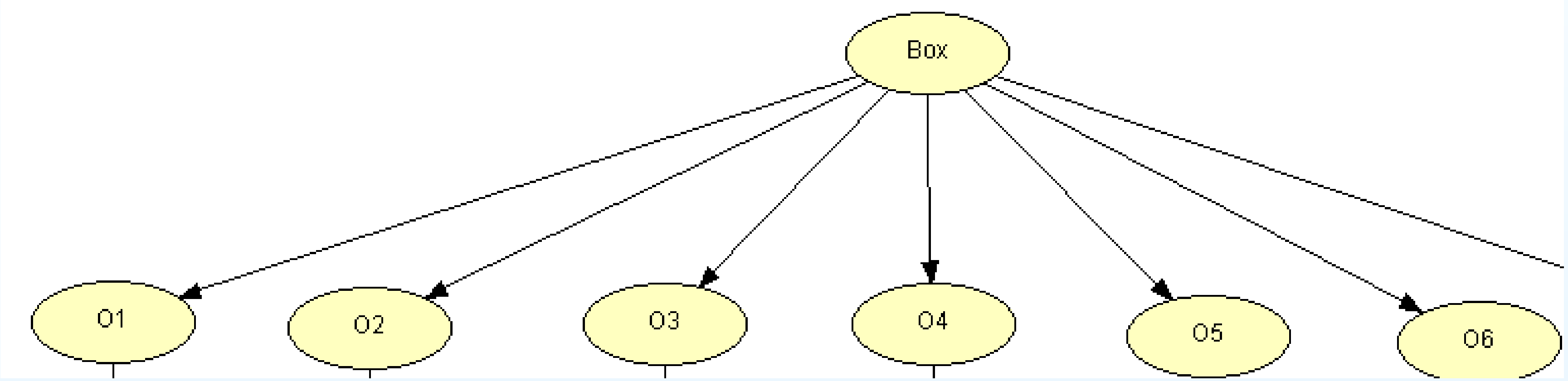
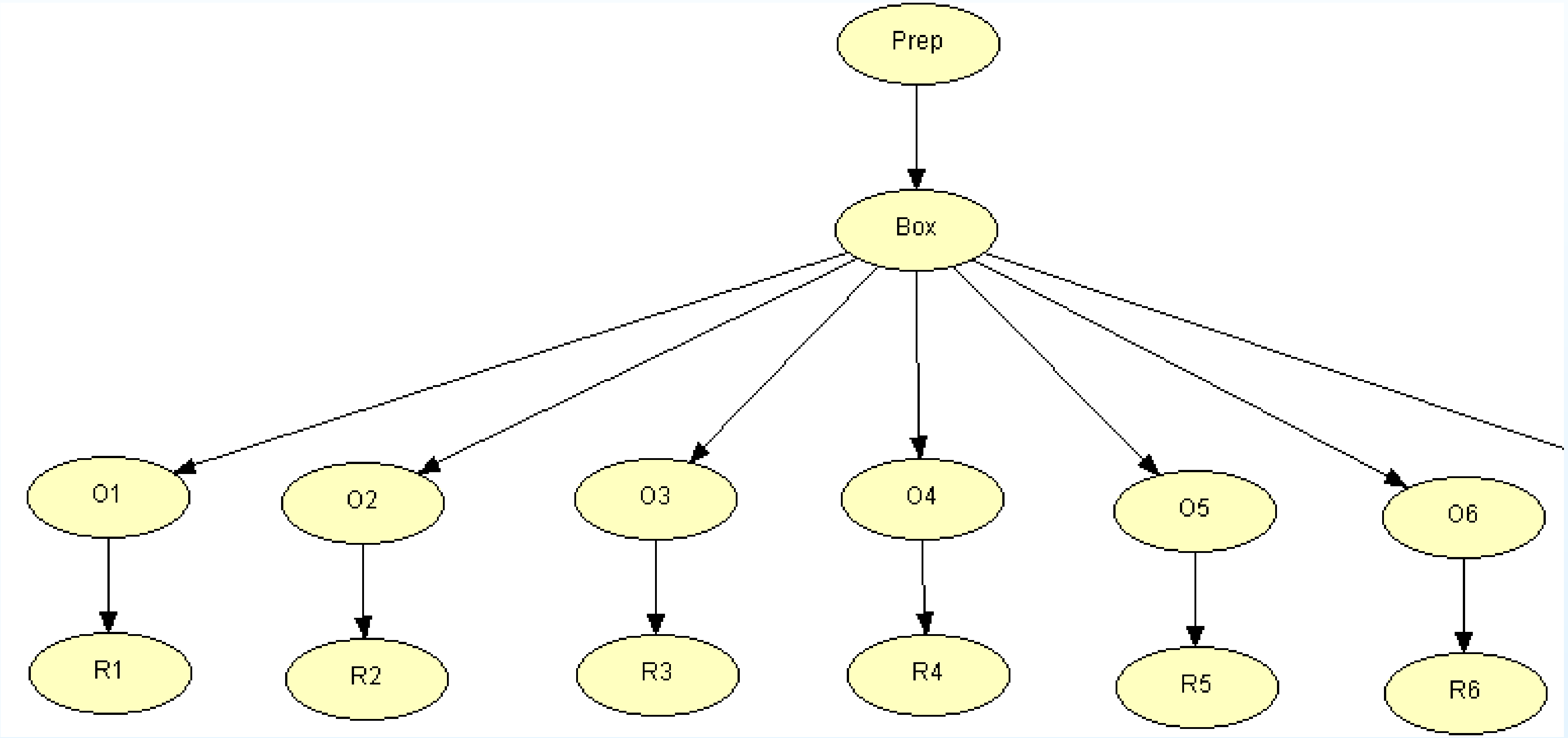


An effect might be the cause of another effect $\longrightarrow$

A network of causes and effects



Doing Science in conditions of uncertainty

The constant status of uncertainty does not prevent us from doing Science (in the sense of Natural Science and not just Mathematics)

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Indeed

"It is scientific only to say what is more likely and what is less likely" (Feynman)

Una rete di credenze ('belief network')

"Le teorie sul mondo si riferiscono alle credenze che nutriamo sul funzionamento del nostro mondo, sulla natura della rete causale in cui viviamo e sui possibili influssi delle nostre decisioni sull'ambiente esterno.

Importanti lati di queste teorie sul mondo riguardano le credenze sull'intreccio probabilistico (o deterministico) del mondo e le percezioni del rapporto di causalità.

(Max H. Bazerman, *Quanto sei (a)morale?*, Il Sole 24 Ore)

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Importanti lati di queste teorie sul mondo riguardano le credenze sull’intreccio probabilistico (o deterministico) del mondo e le percezioni del rapporto di causalità.

. . . .

I manager per avere successo devono possedere un’accurata conoscenza del loro mondo o, se non ce l’hanno, devono sapere come procurarsela.”

(Max H. Bazerman, *Quanto sei (a)morale?*, Il Sole 24 Ore)

How to quantify all that?

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- **Falsificationist approach**

  [and statistical variations over the theme].

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- **Falsificationist approach**

  [and statistical variations over the theme].

- **Probabilistic approach**

  [In the sense that probability theory is used throughly]

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and considered to be the *key to scientific progress*.

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$\Longrightarrow$ Causes that cannot produce observed effects are ruled out ('falsified').

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It seems OK, but it is naive for several aspects.

Let start realizing that the method is analogous with method of the proof by contradiction of classical, deductive logic.

- Assume that a hypothesis is true
- Derive 'all' logical consequence
- If (at least) one of the consequences is known to be false, then the hypothesis is declared false.

Falsificationism? OK, but...

- What to do of all hypotheses that are not falsified? (Limbus? Get stuck?)

Falsificationism? OK, but. . .

- What to do of all hypotheses that are not falsified? (Limbus? Get stuck?)
- What to do is nothing of what can be observed is incompatible with the hypothesis (or with many hypotheses)?

  E.g. H_i being a Gaussian $f(x\,|\,\mu_i,\sigma_i)$

  $\Rightarrow$ Given any pair or parameters $\{\mu_i,\sigma_i\}$, all values of x between $-\infty$ and $+\infty$ are possible.

  $\Rightarrow$ Having observed any value of x, none of H_i can be, strictly speaking, falsified.

Falsificationism and statistics

. . . then, statisticians have invented the “hypothesis tests”

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in which the impossible is replaced by the improbable!

Falsificationism and statistics

…then, statisticians have invented the "hypothesis tests"

in which the impossible is replaced by the improbable!

But from the impossible to the improbable there is not just a question of quantity, but a question of quality.

This mechanism, logically flawed, is particularly perverse, because deeply rooted in most people, due to education, but is not supported by logic.

$\Rightarrow$ Basically responsible of all fake claims of discoveries in the past decades.

[I am particularly worried about claims concerning our health, or the status of the planet, of which I have no control of the experimental data.]

In summary

A) **if** $C_i \not\longrightarrow E$, and **we observe** E
$\Rightarrow C_i$ is impossible ('false')

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~~"most likely false"~~

Example 1

Playing lotto

H: “I play honestly at lotto, betting on a rare combination”

E: “I win”

$$H \xrightarrow[\text{“practically impossible”}]{\hspace{4cm}} E$$

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$$\begin{tikzcd}[column sep=huge] H \arrow[r, "\text{“practically impossible”}"'] & E \arrow[l, bend left=60, "\text{“practically to exclude”}"] \end{tikzcd}$$

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$$H \xrightarrow[\text{“practically impossible”}]{} E$$

“practically to exclude”

$\Rightarrow$ almost certainly I have cheated. . .

(or it is false that I won. . .)

Example 2

An Italian citizen is selected at random to undergo an AIDS test. Performance of clinical trial is not perfect, as customary.

Toy model:

$$
\begin{aligned}
P(\text{Pos}\,|\,\text{HIV}) &= 100\% \\
P(\text{Pos}\,|\,\overline{\text{HIV}}) &= 0.2\% \\
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H_1='HIV' (Infected) E_1 = Positive

H_2='$\overline{\text{HIV}}$' (Healthy) E_2 = Negative

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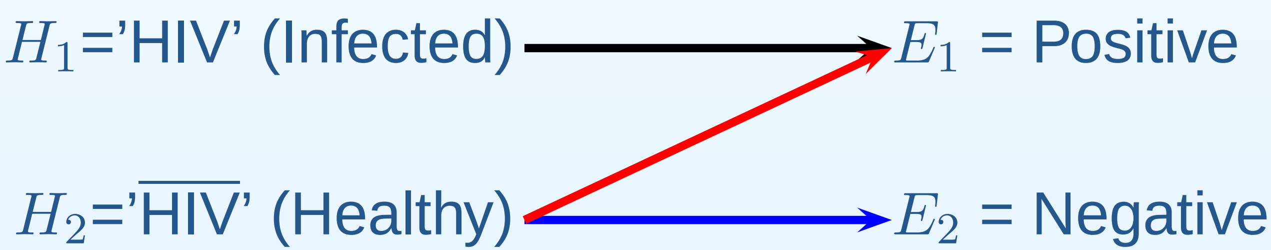

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Result: $\Rightarrow$ <u>Positive</u>

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Infected or healthy?

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Being $P(\text{Pos} \,|\, \overline{\text{HIV}}) = 0.2\%$ and having observed 'Positive', can we say

- "It is practically impossible that the person is healthy, since it was practically impossible that an healthy person would result positive"?

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Instead, $P(\text{HIV}\,|\,\text{Pos, random Italian}) \approx 45\%$

(We will see in the sequel how to evaluate it correctly)

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... which might result into very bad decisions!

‘Standard’ statistical tests, p-values, etc

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  But
  - as far as logic is concerned, the situation is worsened (. . . although p-values ‘often, by chance work’).
- Mistrust statistical tests, unless you know the details of what it has been done.

$\longrightarrow$ You might take <u>bad decisions</u>!

Conflict: natural thinking $\Leftrightarrow$ cultural superstructure

Why? 'Who' is responsible?

- Since beginning of '900 it is dominant an unnatural approach to probability, in contrast to that of the founding fathers (Poisson, Bernoulli, Bayes, Laplace, Gauss, . . .).

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$\Rightarrow$ BUT people think naturally in terms of probability of causes, and use p-values as if they were probabilities of null hypotheses. $\Rightarrow$ Terrible mistakes!

What to do?

$\Rightarrow$ Back to the past

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But benefitting of

- Theoretical progresses in probability theory
- Advance in computation (both symbolic and numeric)

  $\longrightarrow$ many frequentistic ideas had their *raison d'être* in the computational barrier (and many simplified – often simplistic – methods were ingeniously worked out)

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⇒ Use consistently probability theory

- ◦ "It's easy if you try"
- ◦ But first you have to recover the intuitive idea of probability.

What is probability?

Standard textbook definitions

$$p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible equiprobable cases}}$$

$$p = \frac{\#\,\text{times the event has occurred}}{\#\,\text{independent trials under same conditions}}$$

Standard textbook definitions

It is easy to check that ‘scientific’ definitions suffer of circularity

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$$p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible equally possible cases}}$$

$$p = \frac{\#\,\text{times the event has occurred}}{\#\,\text{independent trials under same conditions}}$$

Laplace: *"lorsque rien ne porte à croire que l'un de ces cas doit arriver plutot que les autres"*

Pretending that replacing 'equi-probable' by 'equi-possible' is just cheating students (as I did in my first lecture on the subject. . .).

Standard textbook definitions

It is easy to check that 'scientific' definitions suffer of circularity, plus other problems

$$p \quad = \quad \frac{\#\,\text{favorable cases}}{\#\,\text{possible equiprobable cases}}$$

$$p \quad = \quad \lim_{n\rightarrow\infty} \quad \frac{\#\,\text{times the event has occurred}}{\#\,\text{independent trials under same condition}}$$

Future $\Leftrightarrow$ Past (believed so)

$n \longrightarrow \infty$: $\longrightarrow$ *"usque tandem?"*

$\longrightarrow$ *"in the long run we are all dead"*

$\longrightarrow$ It limits the range of applications

Definitions $\longrightarrow$ evaluation rules

Very useful evaluation rules

$$A) \quad p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible equiprobable cases}}$$

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If the implicit beliefs are well suited for each case of application.

Definitions $\longrightarrow$ evaluation rules

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BUT they cannot define the concept of probability!

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If the implicit beliefs are well suited for each case of application.

In the probabilistic approach we are going to see

- Rule A will be recovered immediately (under the assumption of equiprobability, when it applies).
- Rule B will result from a theorem (under well defined assumptions).

What is probability?

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It is what everybody knows what it is before going at school

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→ how much we are confident that something is true

What <u>is</u> probability?

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$\rightarrow$ how much we are confident that something is true

$\rightarrow$ how much we believe something

What <u>is</u> probability?

It is what everybody knows what it is before going at school

→ how much we are confident that something is true

→ how much we believe something

→ "A measure of the degree of belief that an event *will* occur"

[Remark: 'will' does not imply future, but only uncertainty.]

Or perhaps you prefer this way. . .

"Given the state of our knowledge about everything that could possible have any bearing on the coming true[1]*. . . ,*

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"Given the state of our knowledge about everything that could possible have any bearing on the coming true[1]*. . . , the numerical probability p of this event is to be a real number by the indication of which we try* in some cases *to setup a quantitative measure of the strength of our conjecture or anticipation, founded on the said knowledge, that the event comes true"*

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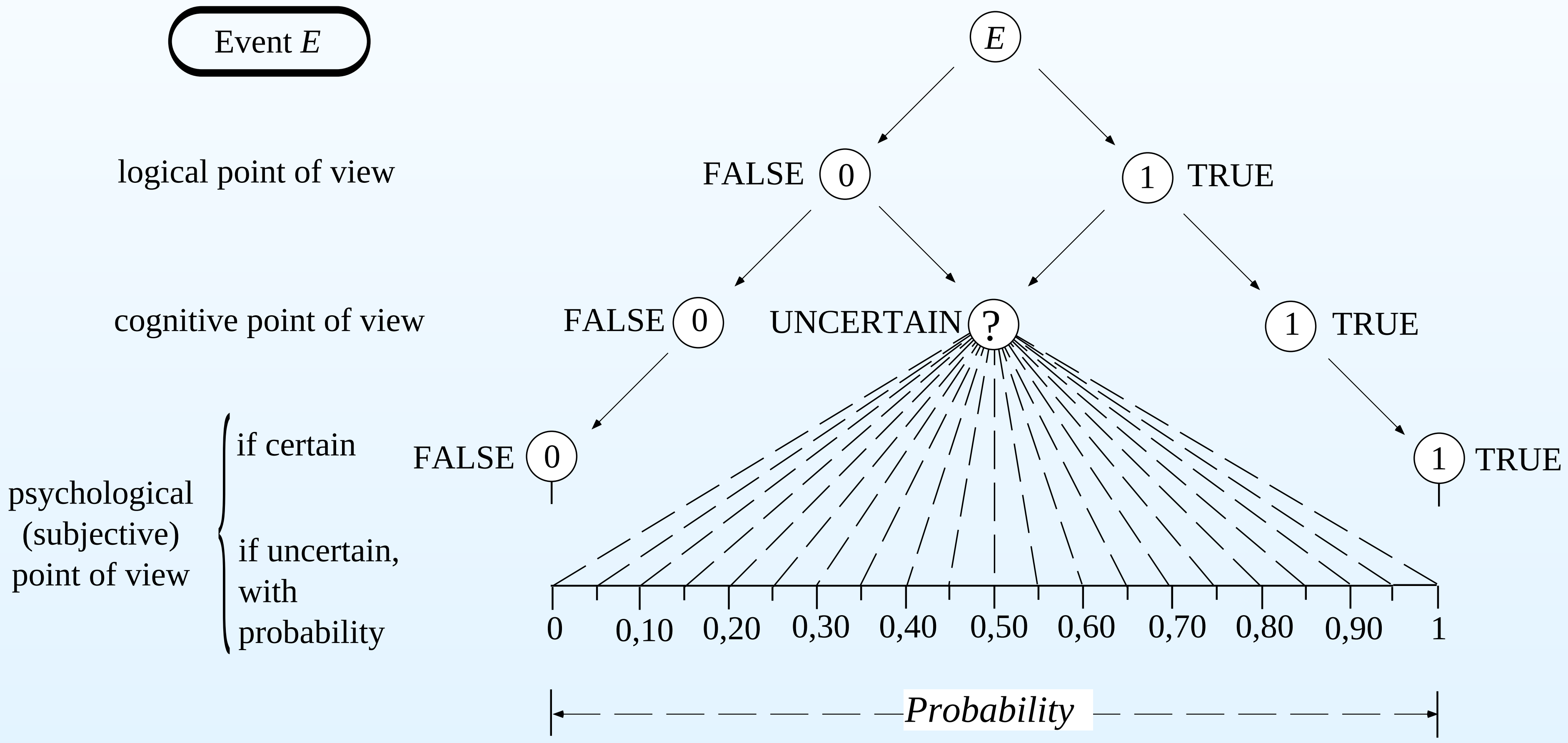
(E. Schrödinger, *The foundation of the theory of probability - I*, Proc. R. Irish Acad. 51A (1947) 51)

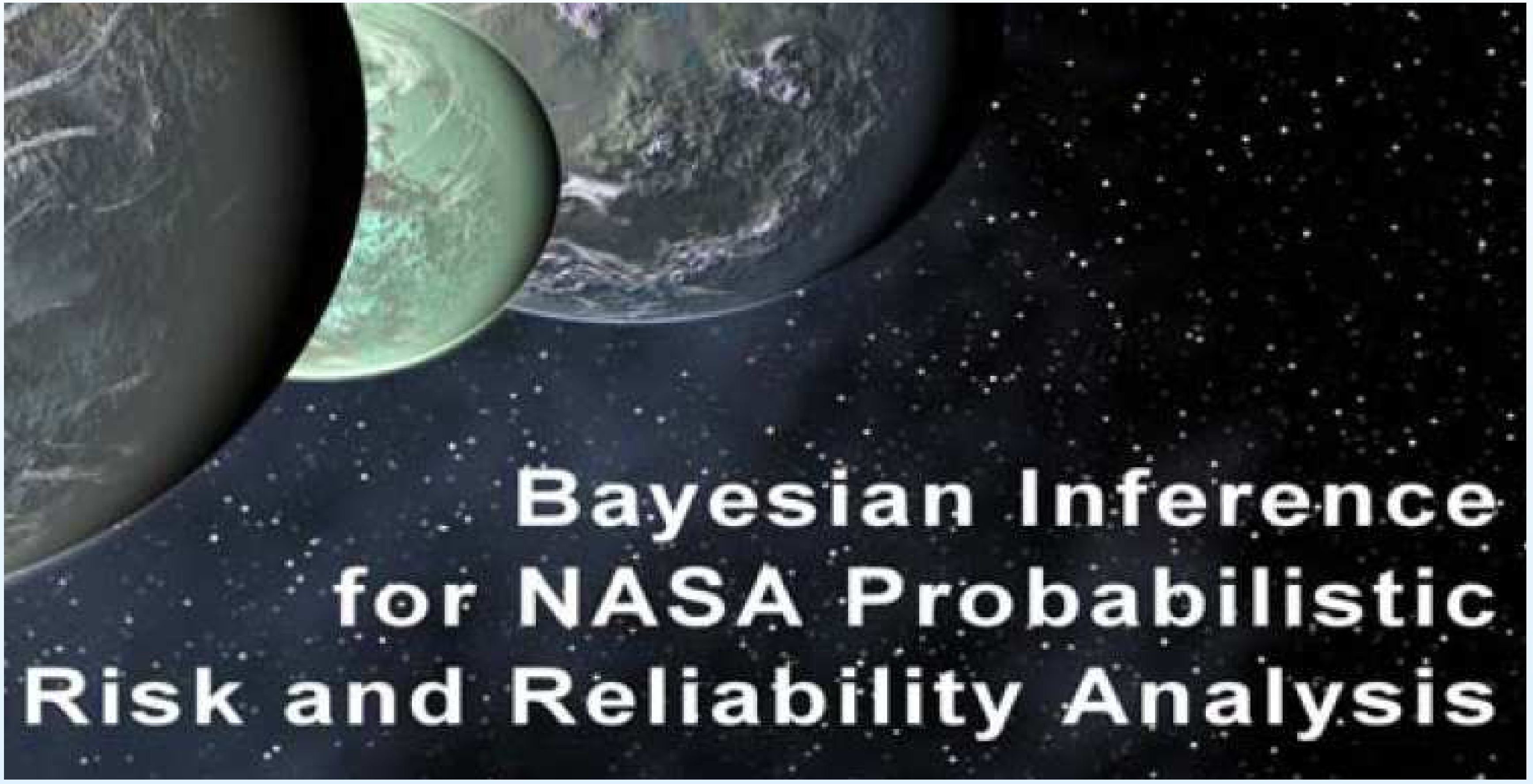
[1]*While in ordinary speech "to come true" usually refers to an event that is envisaged before it has happened, we use it here in the general sense, that the verbal description turns out to agree with actual facts.*

False, True and probable



An helpful diagram

The previous diagram seems to help the understanding of the concept of probability



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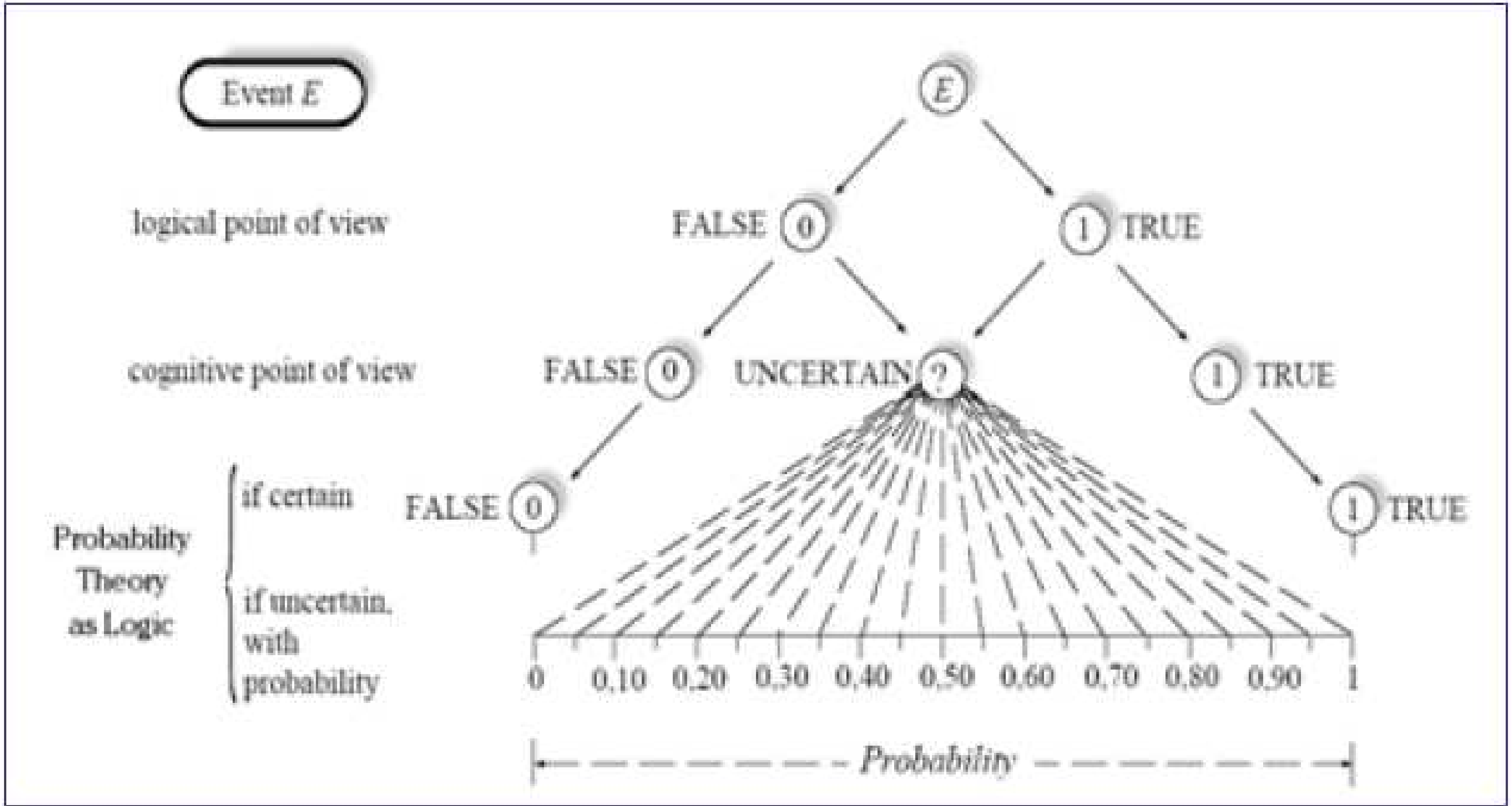

- Figure 2-1. Graphical abstraction of probability as a measure of information (adapted from "Probability and Measurement Uncertainty in Physics" by D'Agostini, [1995]).

(…but NASA guys are afraid of 'subjective', or 'psycological')

Uncertainty $\longrightarrow$ probability

Probability is related to uncertainty and not (only) to the results of repeated experiments

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> *"If we were not ignorant there would be no probability, there could only be certainty. But our ignorance cannot be absolute, for then there would be no longer any probability at all. Thus the problems of probability may be classed according to the greater or less depth of our ignorance."* (Poincaré)

Uncertainty $\longrightarrow$ probability

Probability is related to uncertainty and not (only) to the results of repeated experiments

The state of information can be different from subject to subject

$\Rightarrow$ intrinsic subjective nature.

- No negative meaning: only an acknowledgment that several persons might have different information and, therefore, necessarily different opinions.

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- No negative meaning: only an acknowledgment that several persons might have different information and, therefore, necessarily different opinions.
- *"Since the knowledge may be different with different persons or with the same person at different times, they may anticipate the same event with more or less confidence, and thus different numerical probabilities may be attached to the same event"* (Schrödinger)

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Probability is related to uncertainty and not (only) to the results of repeated experiments

Probability is always conditional probability

$$\text{`}P(E)\text{'} \;\longrightarrow\; P(E\,|\,I) \;\longrightarrow\; P(E\,|\,I(t))$$

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$$`P(E)' \quad\longrightarrow\quad P(E\,|\,I) \quad\longrightarrow\quad P(E\,|\,I(t))$$

- *"Thus whenever we speak loosely of 'the probability of an event,' it is always to be understood: probability with regard to a certain given state of knowledge*" (Schrödinger)

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- *“Thus whenever we speak loosely of ‘the probability of an event,’ it is always to be understood: probability with regard to a certain given state of knowledge*” (Schrödinger)
- Some examples:
  - tossing a die;
  - ’three box problems’;
  - two envelops’ paradox.

Unifying role of subjective probability

- Wide range of applicability

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- Probability statements all have the same meaning no matter to what they refer and how the number has been evaluated.
  - $P(\text{rain next Saturday}) = 68\%$
  - $P(\text{Inter will win Italian champion league}) = 68\%$
  - $P(\text{free neutron decays before 17 s}) = 68\%$
  - $P(\text{White ball from a box with 68W+32B}) = 68\%$

Unifying role of subjective probability

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- Probability statements all have the same meaning no matter to what they refer and how the number has been evaluated.
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  They all convey unambiguously the <u>same confidence</u> on something.

Unifying role of subjective probability

- Wide range of applicability
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  They all convey unambiguously the same confidence on something.
- You might agree or disagree, but at least You know what this person has in his mind. (NOT TRUE with “C.L.’s”!)
- If a person has these beliefs and he/she has the chance to win a rich prize bound to one of these events, he/she has no rational reason to chose an event instead than the others.

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- Probability not bound to a single evaluation rule.
- In particular, combinatorial and frequency based 'definitions' are easily recovered as evaluation rules under well defined hypotheses.
- Keep separate concept from evaluation rule.

From the concept of probability to the probability theory

Ok, it looks nice, . . . but "how do we deal with 'numbers'?"

From the concept of probability to the probability theory

- Formal structure: we need a mathematical structure in order to ‘propagate’ probability values to other, logically connected events:
  - basic rules
  - logic (mathematics)

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Coherent bet (de Finetti, Ramsey - 'Dutch book argument')

It is well understood that bet odds can express confidence†

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Coherent bet $\longrightarrow$ A bet acceptable in both directions:
- You state your confidence fixing the bet odds
- ...but somebody else chooses the direction of the bet
- best way to honestly assess beliefs.

$\longrightarrow$ see later for details, examples, objections, etc

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Consistency arguments (Cox, + Good, Lucas)

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Lindley’s ‘calibration’ against ‘standards’

$\longrightarrow$ analogy to measures (we need to measure ’befiefs’)

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Lindley’s ‘calibration’ against ‘standards’

$\longrightarrow$ analogy to measures (we need to measure ’befiefs’)

$\Rightarrow$ reference probabilities provided by simple cases in which equiprobability applies (coins, dice, turning wheels,. . .).

- Example: You are offered to options to receive a price: a) if E happens, b) if a coin will show head. Etc. . . .

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Lindley's 'calibration' against 'standards'

$\longrightarrow$ Rational under everedays expressions like "there are 90 possibilities in 100" to state beliefs in situations in which the real possibilities are indeed only 2 (e.g. dead or alive)

- Example: a question to a student that has to pass an exam: a) normal test; b) pass it is a uniform random x will be ≤ 0.8.

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  Lindley's 'calibration' against 'standards'
- Also based on coherence, but it avoids the 'repulsion' of several person when they are asked to think directly in terms of bet (it is proved that many persons have reluctance to bet money).

Basic rules of probability

They all lead to

$$
\begin{array}{ll}
1. & 0 \le P(A) \le 1 \\
2. & P(\Omega) = 1 \\
3. & P(A \cup B) = P(A) + P(B) \quad [\text{if } P(A \cap B) = \emptyset\,] \\
 & \\
4. & P(A \cap B) = P(A \,|\, B) \cdot P(B) = P(B \,|\, A) \cdot P(A)\,,
\end{array}
$$

where

- Ω stands for 'tautology' (a proposition that is certainly true $\longrightarrow$ referring to an event that is certainly true) and $\emptyset = \overline{\Omega}$.
- $A \cap B$ is true only when both A and B are true (logical AND)

  (shorthands 'A, B' or $A\,B$ often used $\longrightarrow$ logical product)
- $A \cup B$ is true when at least one of the two propositions is true (logical OR)

Basic rules of probability

Remember that probability is always conditional probability!

1. $0 \leq P(A \,|\, I) \leq 1$
2. $P(\Omega \,|\, I) = 1$
3. $P(A \cup B \,|\, I) = P(A \,|\, I) + P(B \,|\, I) \quad [\text{if } P(A \cap B \,|\, I) = \emptyset]$
4. $P(A \cap B \,|\, I) = P(A \,|\, B, I) \cdot P(B \,|\, I) = P(B \,|\, A, I) \cdot P(A \,|\, I)$

I is the background condition (related to information I)

$\longrightarrow$ usually implicit (we only care on ‘re-conditioning’)

Subjective $\neq$ arbitrary

Crucial role of the coherent bet

- You claim that <u>this</u> coin has 70% to give head?
  No problem with me: you place 70€ on head, I 30€ on tail and who wins take 100€.
  $\Rightarrow$ If OK with you, let's start.

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  $\Rightarrow$ Just reverse the bet

  (Like sharing goods, e.g. a cake with a child)

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(Like sharing goods, e.g. a cake with a child)

$\Rightarrow$ Take into account all available information *in the most 'objective way'*
(Even that someone has a different opinion!)

$\Rightarrow$ It might seem paradoxically, but the 'subjectivist' is much more 'objective' than those who blindly use so-called objective methods.

Summary on probabilistic approach

- Probability means how much we believe something
- Probability values obey the following basic rules

1. $0 \le P(A) \le 1$
2. $P(\Omega) = 1$
3. $P(A \cup B) = P(A) + P(B) \quad [\text{if } P(A \cap B) = \emptyset]$
4. $P(A \cap B) = P(A \,|\, B) \cdot P(B) = P(B \,|\, A) \cdot P(A)\,,$

- All the rest by logic

$\longrightarrow$ And, please, be coherent!

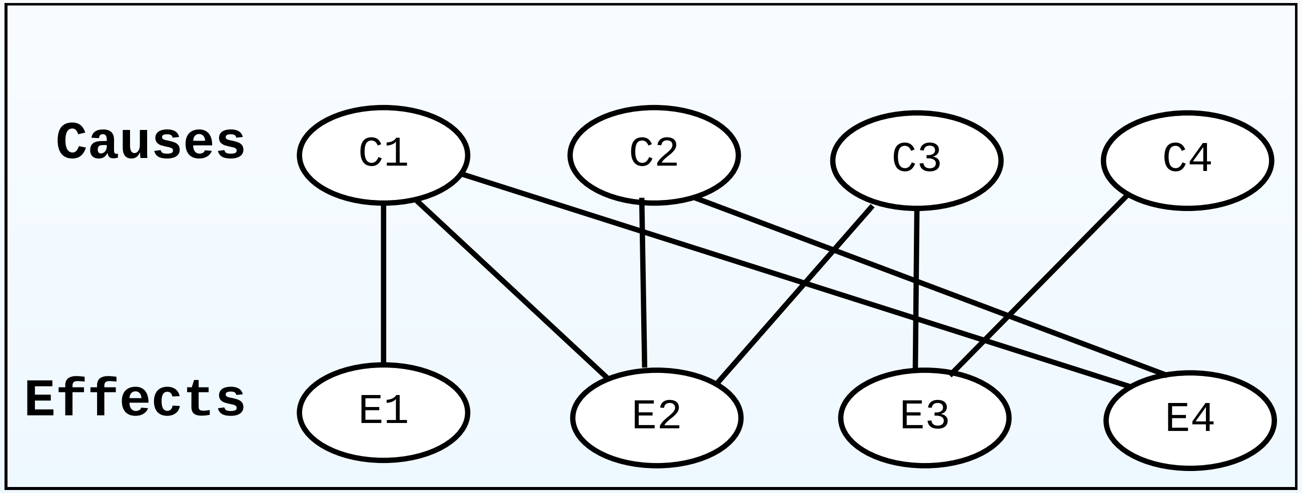
Inference

Inference

$\Rightarrow$ How do we learn from data in a probabilistic framework?

From causes to effects and back

Our original problem:



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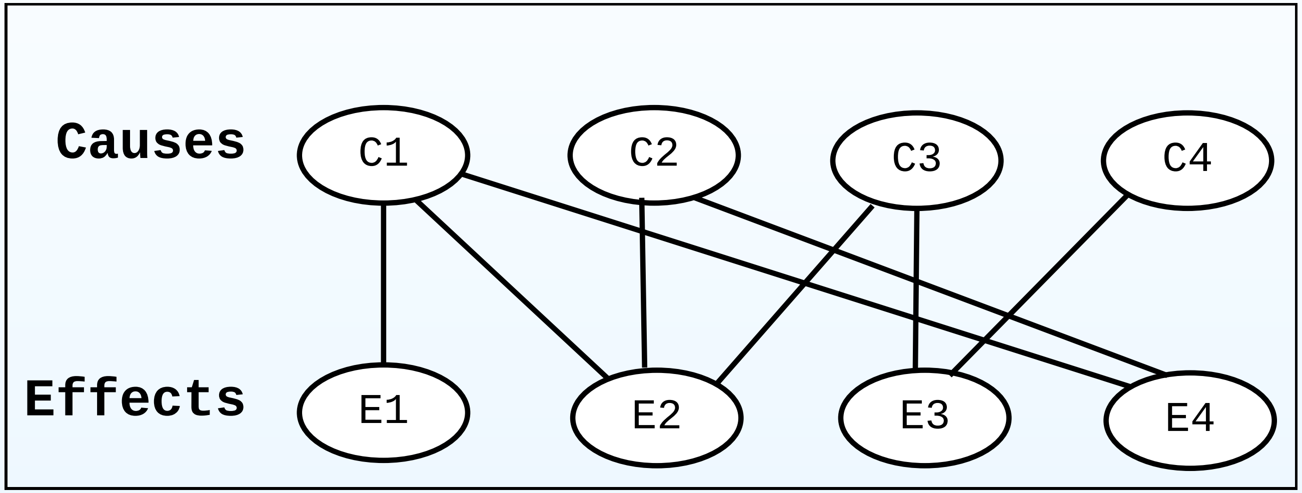

Our conditional view of probabilistic causation

$$\boxed{P(E_i \,|\, C_j)}$$

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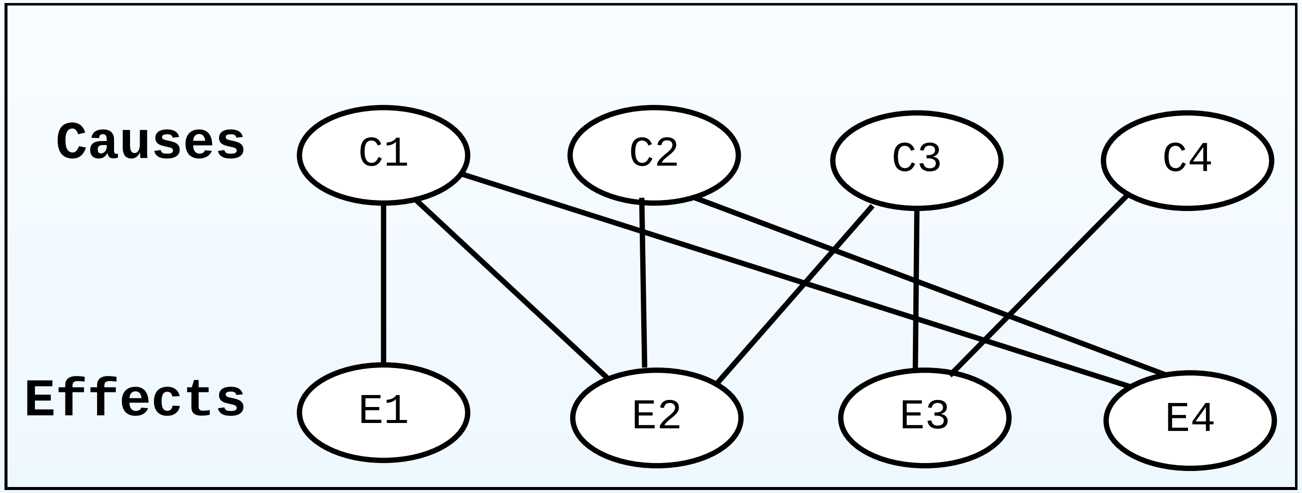

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Our conditional view of probabilistic inference

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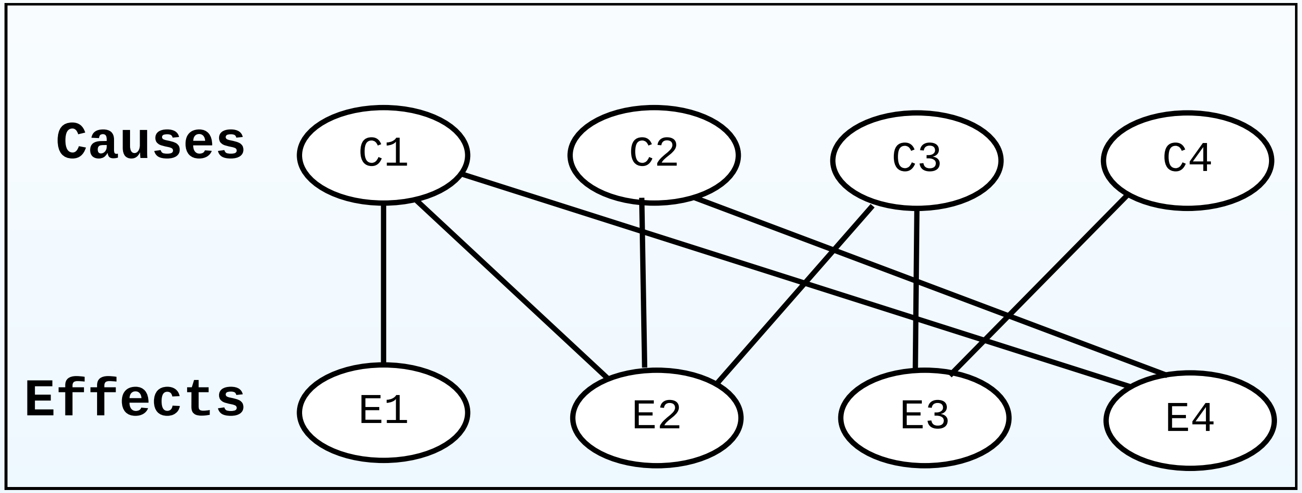

Our conditional view of probabilistic causation

$$\boxed{P(E_i \,|\, C_j)}$$

Our conditional view of probabilistic inference

$$\boxed{P(C_j \,|\, E_i)}$$

The fourth basic rule of probability:

$$\boxed{P(C_j, E_i) = P(E_i \,|\, C_j)\, P(C_j) = P(C_j \,|\, E_i)\, P(E_i)}$$

Symmetric conditioning

Let us take basic rule 4, written in terms of hypotheses H_j and effects E_i, and rewrite it this way:

$$\frac{P(H_j \,|\, E_i)}{P(H_j)} = \frac{P(E_i \,|\, H_j)}{P(E_i)}$$

"The condition on E_i changes in percentage the probability of H_j as the probability of E_i is changed in percentage by the condition H_j."

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Got 'after' Calculated 'before'

(where 'before' and 'after' refer to the knowledge that E_i is true.)

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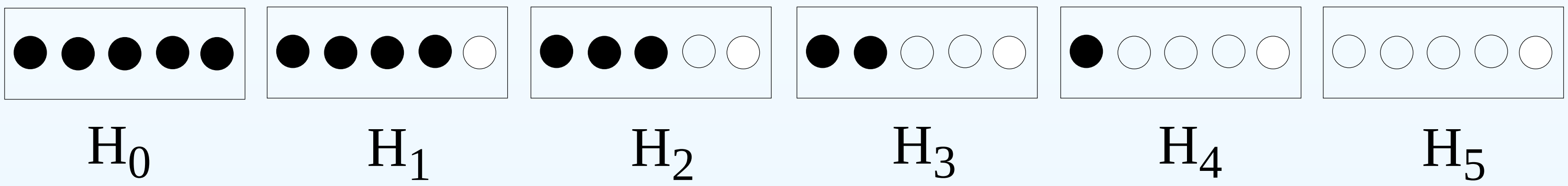
"post illa observationes" *"ante illa observationes"*

(Gauss)

Application to the six box problem



Remind:

- $E_1 =$ White
- $E_2 =$ Black

Collecting the pieces of information we need

Our tool:

$$\boxed{P(H_j \,|\, E_i, I) = \frac{P(E_i \,|\, H_j, I)}{P(E_i \,|\, I)}\, P(H_j \,|\, I)}$$

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Our prior belief about H_j

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Probability of E_i under a well defined hypothesis H_j
It corresponds to the 'response of the apparatus in measurements.
$\rightarrow$ likelihood (traditional, rather confusing name!)

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Probability of E_i taking account all possible H_j

$\rightarrow$ How much we are confident that E_i will occur.

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Probability of E_i taking account all possible H_j
$\rightarrow$ How much we are confident that E_i will occur.
Easy in this case, because of the symmetry of the problem.
But already after the first extraction of a ball our opinion about the box content will change, and symmetry will break.

Collecting the pieces of information we need

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'decomposition law': $P(E_i \,|\, I) = \sum_j P(E_i \,|\, H_j, I) \cdot P(H_j \,|\, I)$

($\rightarrow$ Easy to check that it gives $P(E_i \,|\, I) = 1/2$ in our case).

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We are ready!

$\longrightarrow$ R program

First extraction

After first extraction (and reintroduction) of the ball:

- $P(H_j)$ changes
- $P(E_j)$ for next extraction changes

Note: The box is exactly in the same status as before

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Where is probability?

$\longrightarrow$ Certainly not in the box!

Bayes theorem

The formulae used to *infer* H_i and
to *predict* $E_j^{(2)}$ are related to the name of Bayes

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Neglecting the background state of information I:

$$\frac{P(H_j \,|\, E_i)}{P(H_j)} = \frac{P(E_i \,|\, H_j)}{P(E_i)}$$

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$$P(H_j \,|\, E_i) \propto P(E_i \,|\, H_j) \cdot P(H_j)$$

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Different ways to write the

Bayes' Theorem

Updating the knowledge by new observations

Let us repeat the experiment:

$\boxed{\text{Sequential use of Bayes theorem}}$

Old posterior becomes new prior, and so on

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$$P(H_j\,|\,E^{(1)},E^{(2)}) \;\propto\; P(E^{(2)}\,|\,H_j,E^{(1)})\cdot P(H_j\,|\,E^{(1)})$$

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Let us repeat the experiment:

$\boxed{\text{Sequential use of Bayes theorem}}$

Old posterior becomes new prior, and so on

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\begin{aligned}
P(H_j \,|\, E^{(1)}, E^{(2)}) &\propto P(E^{(2)} \,|\, H_j, E^{(1)}) \cdot P(H_j \,|\, E^{(1)}) \\
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Updating the knowledge by new observations

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P(H_j \,|\, \text{data}) &\propto P(\text{data} \,|\, H_j) \cdot P_0(H_j)
\end{aligned}$$

$\boxed{\text{Bayesian inference}}$

Updating the knowledge by new observations

Let us repeat the experiment:

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Learning from data using probability theory

Solution of the AIDS test problem

$$\begin{aligned} P(\text{Pos}\,|\,\text{HIV}) &= 100\% \\ P(\text{Pos}\,|\,\overline{\text{HIV}}) &= 0.2\% \\ P(\text{Neg}\,|\,\overline{\text{HIV}}) &= 99.8\% \end{aligned}$$

We miss something: $P_\circ(\text{HIV})$ and $P_\circ(\overline{\text{HIV}})$: Yes! We need some input from our best knowledge of the problem. Let us take $P_\circ(\text{HIV}) = 1/600$ and $P_\circ(\overline{\text{HIV}}) \approx 1$ (the result is rather stable against *reasonable* variations of the inputs!)

$$\begin{aligned} \frac{P(\text{HIV}\,|\,\text{Pos})}{P(\overline{\text{HIV}}\,|\,\text{Pos})} &= \frac{P(\text{Pos}\,|\,\text{HIV})}{P(\text{Pos}\,|\,\overline{\text{HIV}})} \cdot \frac{P_\circ(\text{HIV})}{P_\circ(\overline{\text{HIV}})} \\ &= \frac{\approx 1}{0.002} \times \frac{0.1/60}{\approx 1} = 500 \times \frac{1}{600} = \frac{1}{1.2} \end{aligned}$$

Odd ratios and Bayes factor

$$
\begin{aligned}
\frac{P(\mathsf{HIV}\,|\,\mathsf{Pos})}{P(\overline{\mathsf{HIV}}\,|\,\mathsf{Pos})} &= \frac{P(\mathsf{Pos}\,|\,\mathsf{HIV})}{P(\mathsf{Pos}\,|\,\overline{\mathsf{HIV}})}\cdot\frac{P_\circ(\mathsf{HIV})}{P(\overline{\mathsf{HIV}})}\\
&= \frac{\approx 1}{0.002}\times\frac{0.1/60}{\approx 1} = 500\times\frac{1}{600} = \frac{1}{1.2}\\
\Rightarrow P(\mathsf{HIV}\,|\,\mathsf{Pos}) &= 45.5\%\,.
\end{aligned}
$$

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There are some advantages in expressing Bayes theorem in terms of odd ratios:

- There is no need to consider all possible hypotheses (how can we be sure?)
  We just make a comparison of any couple of hypotheses!

Odd ratios and Bayes factor

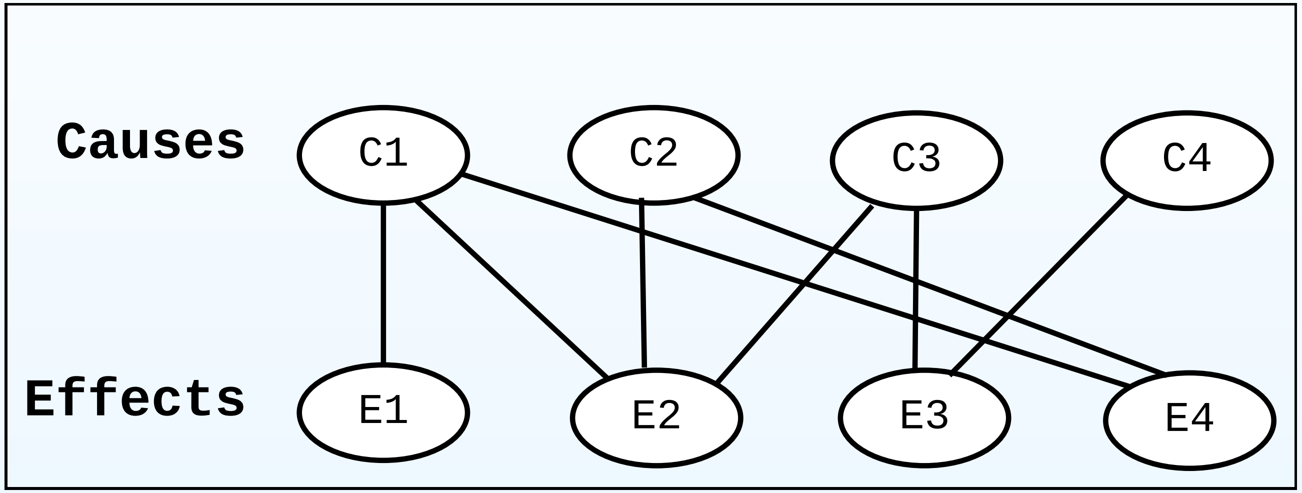
$$\begin{aligned}\frac{P(\text{HIV}\,|\,\text{Pos})}{P(\overline{\text{HIV}}\,|\,\text{Pos})} &= \frac{P(\text{Pos}\,|\,\text{HIV})}{P(\text{Pos}\,|\,\overline{\text{HIV}})}\cdot\frac{P_\circ(\text{HIV})}{P(\overline{\text{HIV}})}\\ &= \frac{\approx 1}{0.002}\times\frac{0.1/60}{\approx 1} = 500\times\frac{1}{600} = \frac{1}{1.2}\\ \Rightarrow P(\text{HIV}\,|\,\text{Pos}) &= 45.5\%\,.\end{aligned}$$

There are some advantages in expressing Bayes theorem in terms of odd ratios:

- There is no need to consider all possible hypotheses (how can we be sure?)
  We just make a comparison of any couple of hypotheses!
- Bayes factor is usually much more inter-subjective, and it is often considered an 'objective' way to report how much the data favor each hypothesis.

Bayesian networks

Let consider again our causes-effects *network*



Bayesian networks

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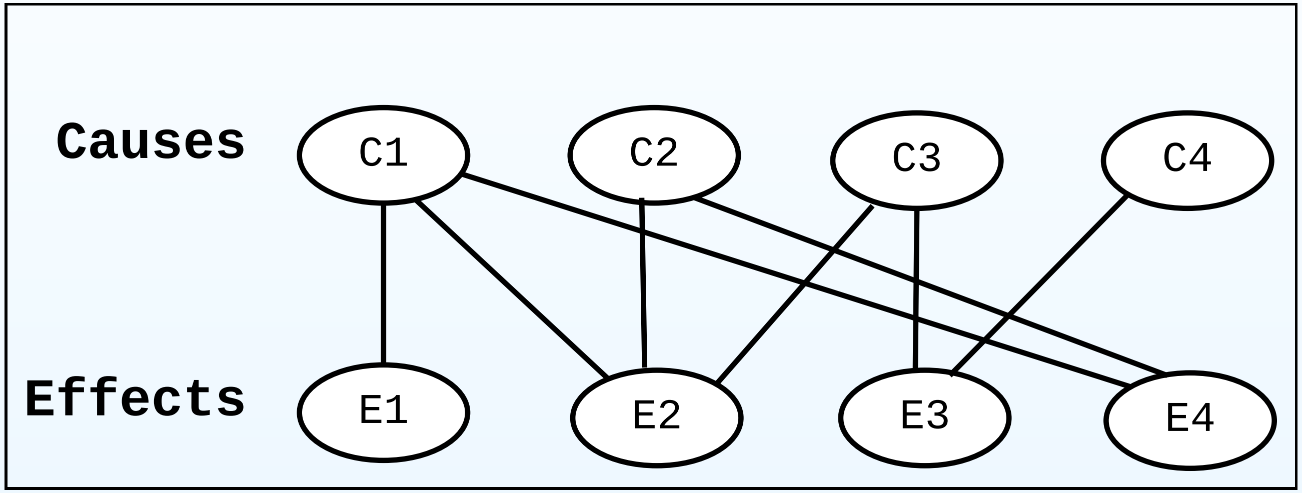
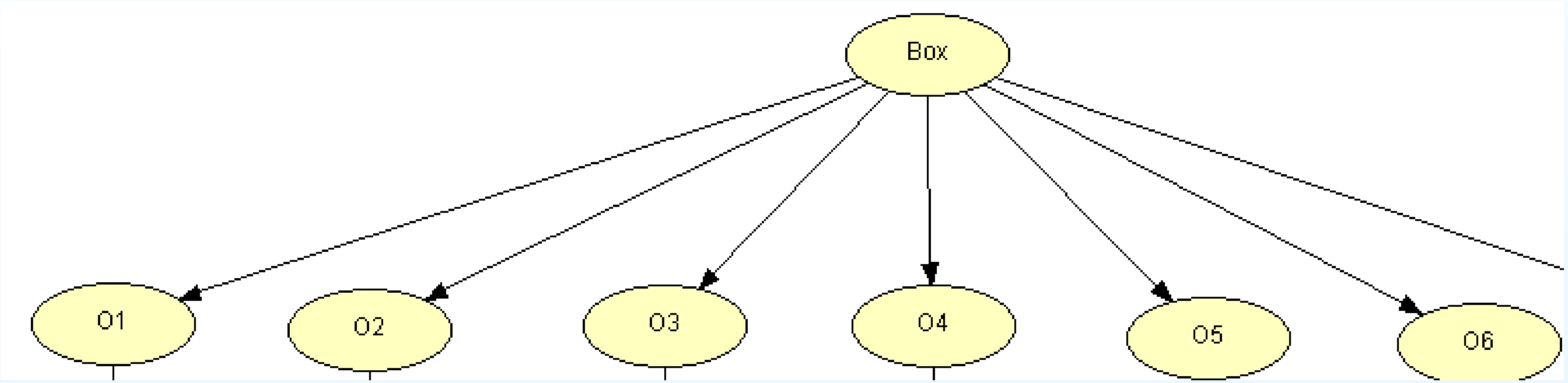


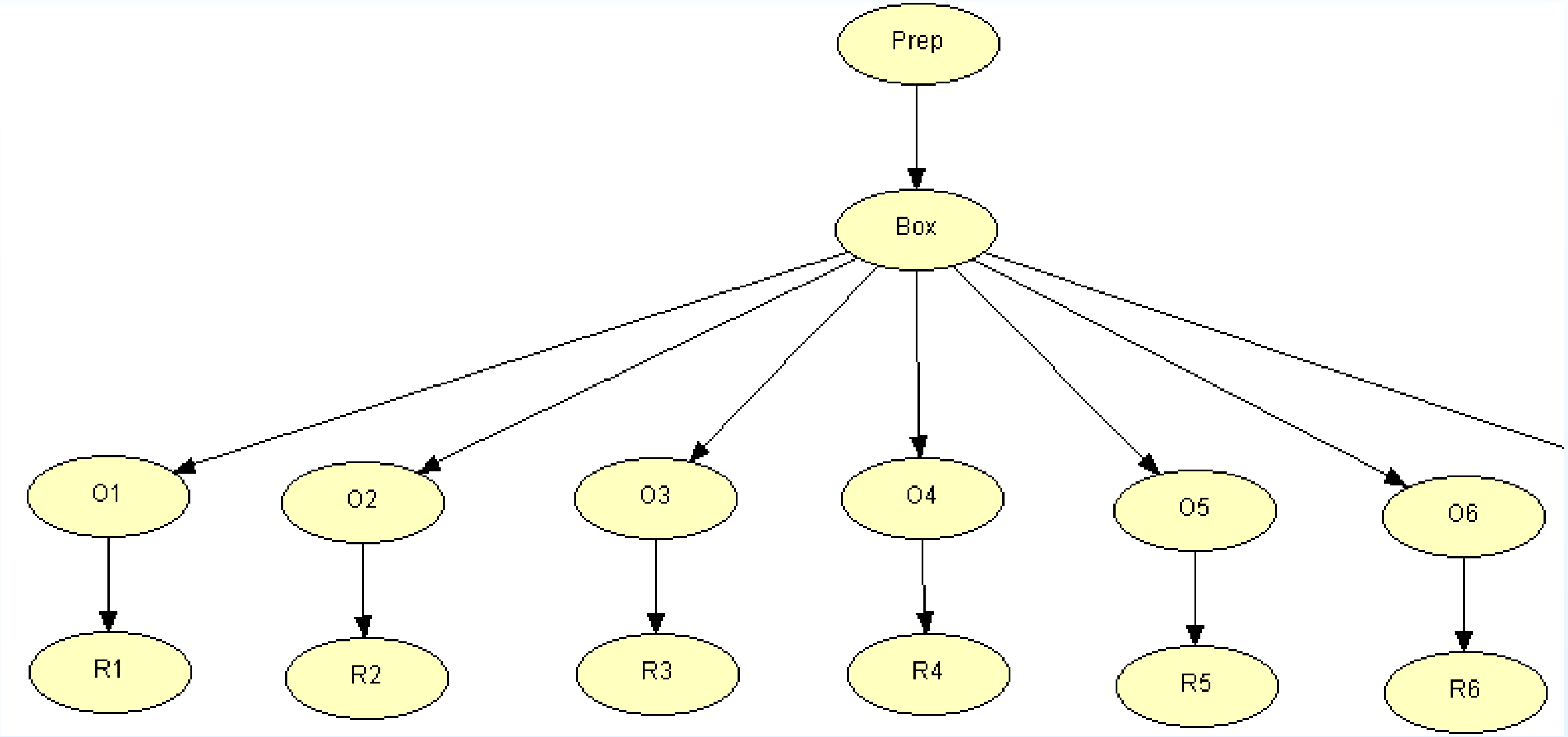
In complex, real live situations the effects themselves can be considered as causes of other effects, and so on.

Bayesian networks

Basic network:

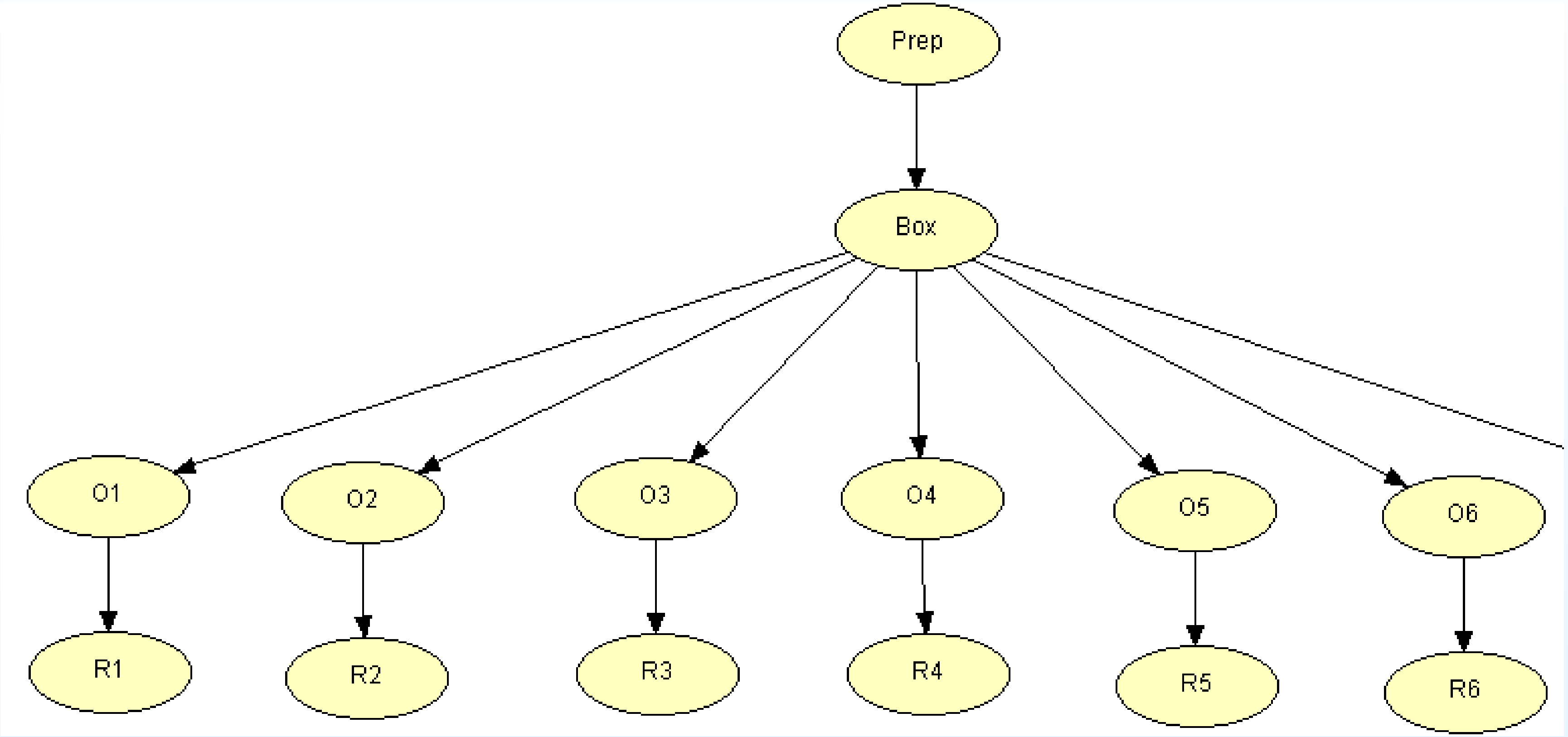


Bayesian networks



- The ball color is told by a reporter who might lie. (Devices might err!)
- We are not sure about the way the box was prepared.

Bayesian networks

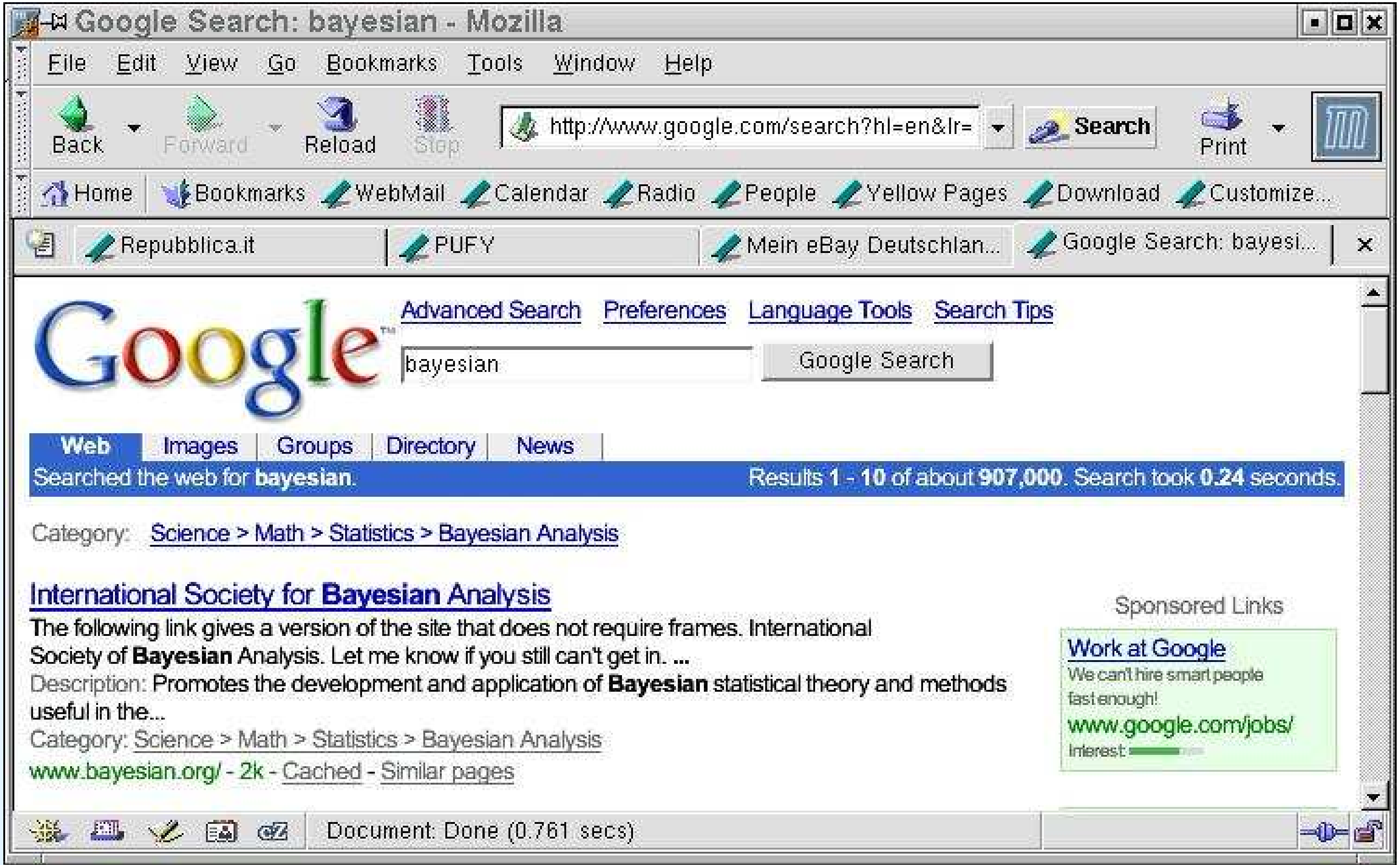


⇒ Let us play with the six boxes using **HUGIN Expert** software

Conclusions

- Subjective probability recovers intuitive idea of probability.
- Nothing negative in the adjective 'subjective'. Just recognize, honestly, that probability depends on the status of knowledge, different from person to person.
- Most general concept of probability that can be applied to a large variety of cases.
- The adjective Bayesian comes from the intense use of Bayes' theorem to update probability once new data are acquired.
- Subjective probability is foundamental in decision issues, if you want to base decision on the probability of different events, together with the gain of each of them.
- Bayesian networks are powerful conceptuals/mathematical/ software tools to handle complex problems with variables related by probabilistic links.

Are Bayesians ‘smart’ and ‘brilliant’?



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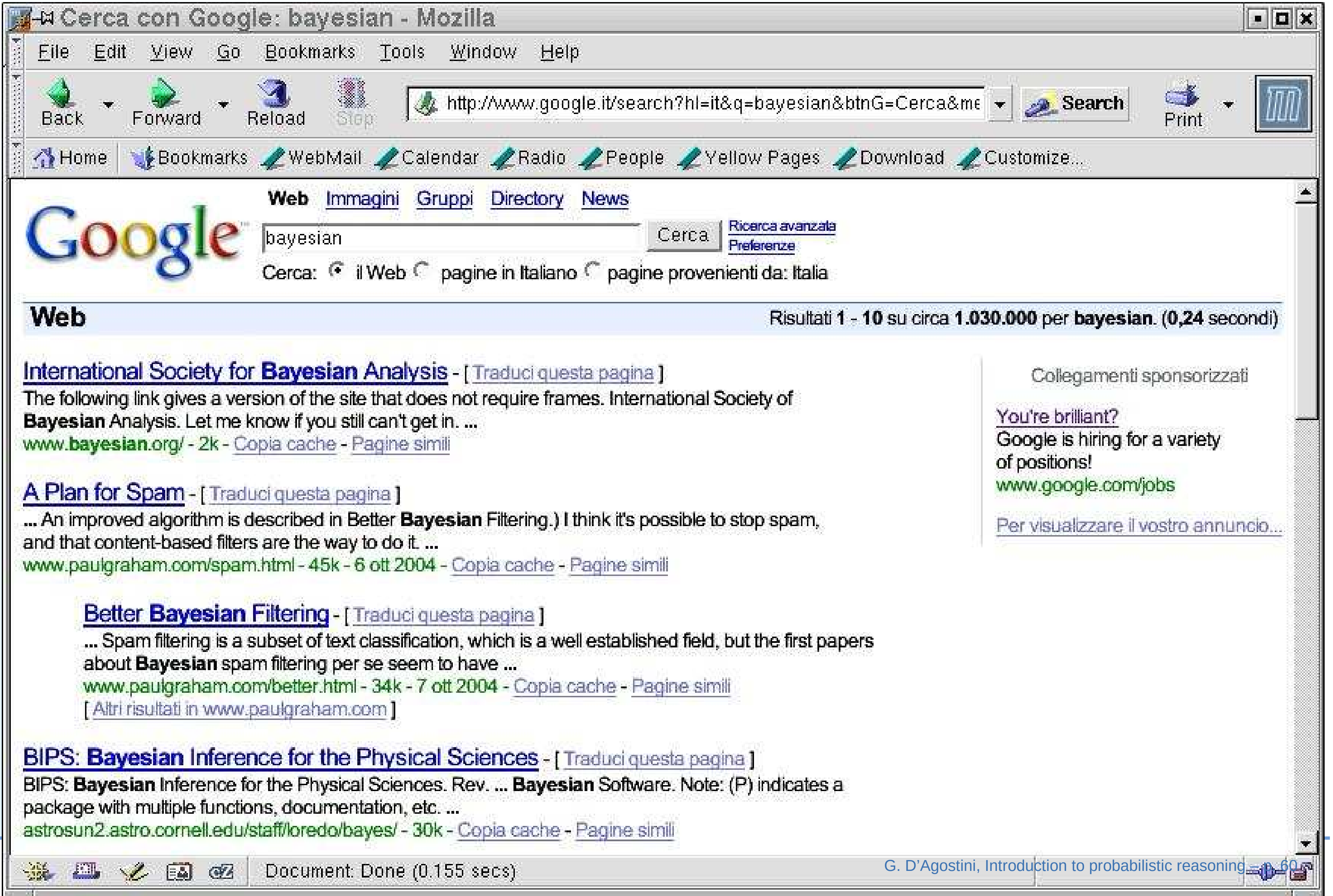

End of lecture

End of lecture

Notes

The following slides should be reached by hyper-links, clicking on words with the symbol †

Determinism/indeterminism

Pragmatically, as far as uncertainty and inference matter,
it doesn’t really matter.

“Though there be no such thing as Chance in the world; our ignorance of the real cause of any event has the same influence on the understanding, and begets a like species of belief or opinion” (Hume)

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Hume’s view about ‘combinatoric evaluation’

“There is certainly a probability, which arises from a superiority of chances on any side; and according as this superiority increases, and surpasses the opposite chances, the probability receives a proportionable increase, and begets still a higher degree of belief or assent to that side, in which we discover the superiority.”

Hume's view about 'combinatoric evaluation'

"There is certainly a probability, which arises from a superiority of chances on any side; and according as this superiority increases, and surpasses the opposite chances, the probability receives a proportionable increase, and begets still a higher degree of belief or assent to that side, in which we discover the superiority. If a dye were marked with one figure or number of spots on four sides, and with another figure or number of spots on the two remaining sides, it would be more probable, that the former would turn up than the latter; though, if it had a thousand sides marked in the same manner, and only one side different, the probability would be much higher, and our belief or expectation of the event more steady and secure." (David Hume)

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Hume’s view about ‘frequency based evaluation’

“Being determined by custom to transfer the past to the future, in all our inferences; where the past has been entirely regular and uniform, we expect the event with the greatest assurance, and leave no room for any contrary supposition.”

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“Being determined by custom to transfer the past to the future*, in all our inferences; where the past has been entirely regular and uniform, we expect the event with the greatest assurance, and leave no room for any contrary supposition. But where* different effects *have been found to* follow from causes*, which are to appearance exactly similar,* all these various effects must occur to the mind *in transferring the past to the future, and enter into our consideration, when we determine the probability of the event.”*

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Bet odds to express confidence

"The best way to explain it is, I'll bet you fifty to one that you don't find anything" (Feynman)

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“The best way to explain it is, I’ll bet you fifty to one that you don’t find anything” (Feynman)

“It is a bet of 11,000 to 1 that the error on this result (the mass of Saturn) *is not 1/100th of its value”* (Laplace)

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$\longrightarrow$ 99.99% confidence on the result

$\Longrightarrow$ Is a 95% C.L. upper/lower limit a ‘19 to 1 bet’?

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