Note sul metodo dei minimi quadrati

Metodo dei minimi quadrati

→ Due grandezze x ed y sono legate tra loro da una dipendenza funzionale di forma nota dove compaiono m parametri incogniti

$\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_m$

cioè:

$$\boxed{y = g(x, \lambda_1, \lambda_2, \ldots, \lambda_m)}$$

Sono state eseguite N coppie di misure sulle due G.F. x ed y:

x_i, y_i con $i = 1, 2, \ldots, N$

Le incertezze statistiche sulle x_i sono percentualmente trascurabili rispetto a quelle sulle y_i :

$$\boxed{\frac{\sigma(x_i)}{x_i} \ll \frac{\sigma(y_i)}{y_i}}$$

Il campione è stato estratto da una distribuzione di probabilità limite di tipo normale (GAUSS) centrata attorno al valore vero della y con un parametro di larghezza $\sigma(y_i)$. La generica misura i-esima sarà perciò governata dalla solita

$$\frac{1}{\sqrt{2\pi}\,\sigma(y_i)} \exp\left[-\frac{1}{2(\sigma(y_i))^2}\left(y_i - g(x_i, \lambda_1, \ldots, \lambda_m)\right)^2\right]$$

→ Il principio della massima verosimiglianza mi permette di determinare i migliori parametri λ_i in modo che risulti massima la probabilità di avere le N misure fatte (x_i, y_i $i = 1, 2, \ldots, N$)

$$P(y_1, y_2, \dots, y_N) = P(y_1) \cdot P(y_2) \cdot \dots \cdot P(y_N)$$

$$\propto \prod_{i=1}^{N} \frac{1}{\sigma_i} \exp\left\{ - \frac{[y_i - g(x_i, \lambda_1, \lambda_2, \dots, \lambda_m)]^2}{2[\sigma(y_i)]^2} \right\}$$

$$\boxed{P(y_1, y_2, \dots, y_N) = max \quad \longrightarrow \quad \sum_{1}^{N}{}_i \frac{[y_i - g(x_i, \lambda_1, \dots, \lambda_m)]^2}{2[\sigma(y_i)]^2} = min}$$

"massima verosimiglianza" "minimi quadrati"

$$\boxed{\text{residuo} = r_i = y_i - g(x_i, \lambda_1, \lambda_2, \dots, \lambda_m)}$$

Osservazioni

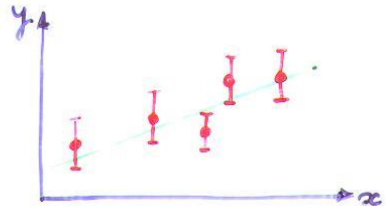
1) Nell'ipotesi che ciascuna y_i sia distribuita in modo "gaussiano", il problema è trattabile rigorosamente con il metodo della massima verosimiglianza

altrimenti....... tentativamente uso ugualmente lo stesso metodo dei minimi quadrati per qualunque tipo di funzione di distribuzione per le misure della variabile y..... i parametri incogniti saranno noti con una dev. std. più grande .....

2) La migliore stima dei parametri λ_i sarà perciò quella che si ottiene minimizzando la $\sum_1^N{}_i \frac{r_i^2}{\sigma_i^2}$ che è una funzione degli m λ_i

$$(y)_{y=y_i} = a(x)_{x=x_i} + b \qquad r_i = y_i^{exp} - y_i^{fit} = \left(y_i^{exp} - (a x_i + b)\right)$$

Metodo dei minimi quadrati nel caso di una retta



assenza di incertezze sperimentali

$\Downarrow$

ogni (x_i, y_i) $\quad i = 1, 2, 3, \dots, N$
deve giacere **esattamente** sulla **retta**: $\quad y = ax + b$

ci sono incertezze sperimentali

$\Downarrow$

vogliamo la **retta** $y = ax + b$
che **approssima meglio** l'insieme delle misure
$(x_i; y_i)$ $\quad i = 1, 2, 3, \dots, N$

$\rightarrow$ ulteriore **semplificazione**: tutte le $\sigma(y_i)$ sono **costanti** per tutti gli N punti

$$\boxed{\phi = \sum_{1}^{N}{}_i \left(y_i - a x_i - b\right)^2} = \text{somma dei residui al quadrato}$$

cerco di minimizzare ϕ variando
i due parametri a e b $\quad \boxed{\frac{\partial \phi}{\partial a} = 0, \ \frac{\partial \phi}{\partial b} = 0}$

$$\begin{cases} \dfrac{\partial \phi}{\partial a} = \sum_i 2(y - ax - b)(-x) = -2 \sum (xy - ax^2 - bx) = \\ \quad = -2 \left\{ \sum_i xy - a \sum x^2 - b \sum x \right\} = 0 \\ \dfrac{\partial \phi}{\partial b} = \sum_i 2(y - ax - b)(-1) = -2 \left\{ \sum y - a \sum x - bN \right\} = 0 \end{cases}$$

Sistema di:

$\Rightarrow$ **"Equazioni Normali"**

Stime migliori per i due parametri della retta $y = ax + b$

$$\boxed{\begin{cases} a\sum x^2 + b\sum x = \sum xy \\ a\sum x + bN = \sum y \end{cases}}$$

$$\begin{cases} \Delta = \begin{vmatrix} \sum x^2 & \sum x \\ \sum x & N \end{vmatrix} = N\sum x^2 - \left(\sum x\right)^2 = N^2\underbrace{\left[\langle x^2\rangle - (\langle x\rangle)^2\right]}_{\text{VARIANZA DELLA } x} \\ a = \dfrac{1}{\Delta}\begin{vmatrix} \sum xy & \sum x \\ \sum y & N \end{vmatrix} = \dfrac{N\sum xy - (\sum x)(\sum y)}{N\sum x^2 - (\sum x)^2} = \dfrac{\langle xy\rangle - \langle x\rangle\langle y\rangle}{\langle x^2\rangle - \langle x\rangle^2} \\ b = \dfrac{1}{\Delta}\begin{vmatrix} \sum x^2 & \sum xy \\ \sum x & \sum y \end{vmatrix} = \dfrac{(\sum x^2)(\sum y) - (\sum x)(\sum xy)}{N\sum x^2 - (\sum x)^2} = \dfrac{\langle x^2\rangle\langle y\rangle - \langle x\rangle\langle xy\rangle}{\langle x^2\rangle - \langle x\rangle^2} \end{cases}$$

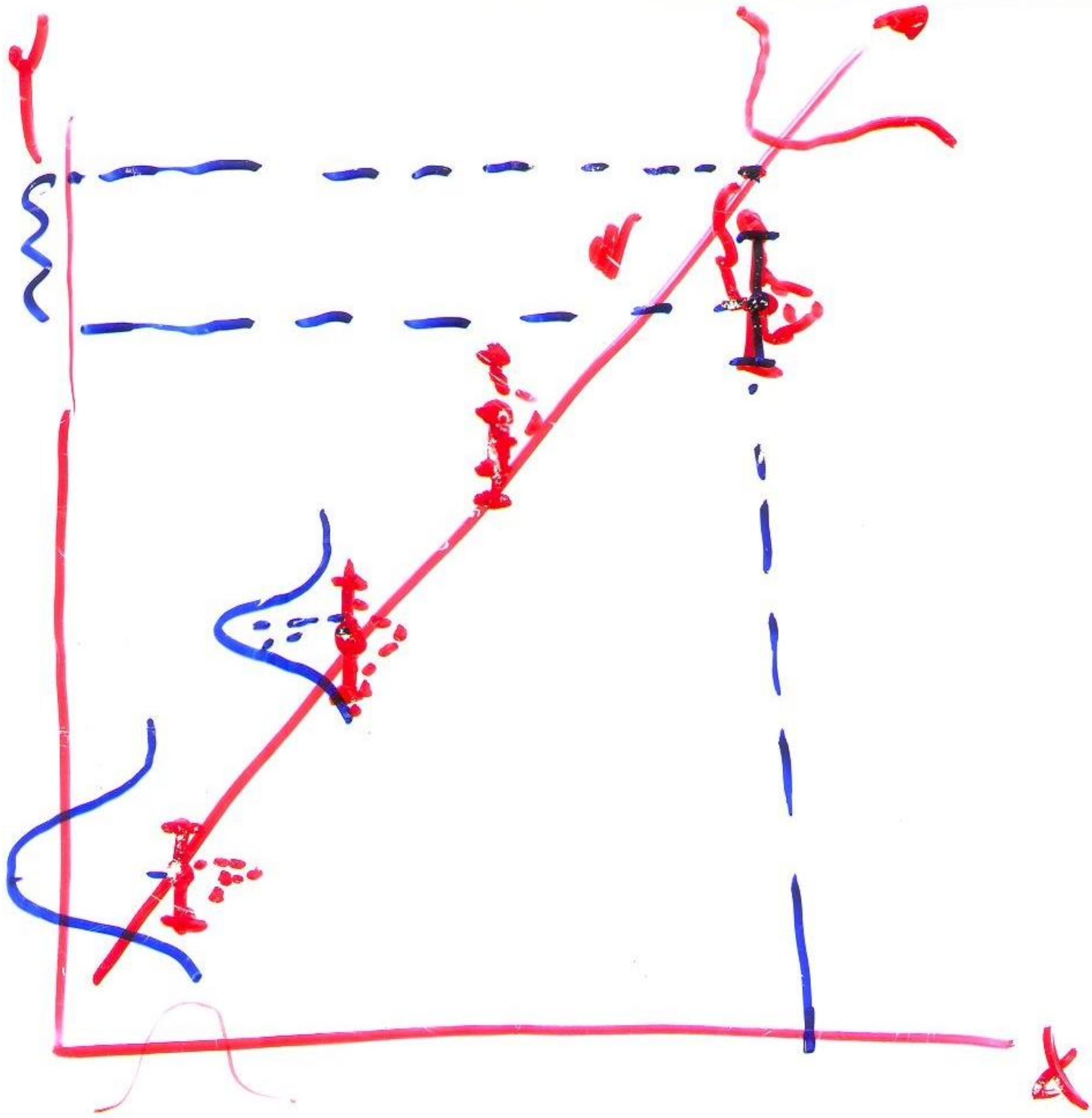
→ La retta dei minimi quadrati passa per il punto $(\langle x\rangle ; \langle y\rangle)$:

$$y(x = \langle x\rangle) = a\cdot\langle x\rangle + b = \frac{\langle xy\rangle - \langle x\rangle\langle y\rangle}{\langle x^2\rangle - \langle x\rangle^2}\cdot\langle x\rangle + \frac{\langle x^2\rangle\langle y\rangle - \langle x\rangle\langle xy\rangle}{\langle x^2\rangle - \langle x\rangle^2} =$$

$$= \frac{\langle xy\rangle\langle x\rangle - \langle x\rangle^2\langle y\rangle + \langle x^2\rangle\langle y\rangle - \langle x\rangle\langle xy\rangle}{\langle x^2\rangle - \langle x\rangle^2} =$$

$$= \frac{\langle y\rangle\left(\langle x^2\rangle - \langle x\rangle^2\right)}{\left(\langle x^2\rangle - \langle x\rangle^2\right)} = \langle y\rangle$$

$$\Longrightarrow \boxed{y(x = \langle x\rangle) = \langle y\rangle}$$


y
x

- $\boxed{\text{Sfruttando il punto } (\langle x \rangle ; \langle y \rangle)}$:

→ $\Big[y(x) \Big]_{y=\langle y \rangle} = a \cdot \Big[x \Big]_{x=\langle x \rangle} + b$

→ Poichè la retta dei m.q. passa per il baricentro dei punti $(x_i ; y_i)$:

$$y = ax + b$$

$$\langle y \rangle = a \langle x \rangle + b$$

sottraendo le due equazioni tra loro

$$y - \langle y \rangle = a \left(x - \langle x \rangle \right)$$

$$\hookrightarrow \quad y = a x + \underbrace{\left(\langle y \rangle - a \langle x \rangle \right)}_{b}$$

Ho due possibilità:

calcolo a, $\langle x \rangle$, $\langle y \rangle$ oppure a, b

→ Se $\langle x \rangle = 0$:

$$a = \frac{N \sum xy - \cancel{\sum x \cdot \sum y}}{N \sum x^2 - \cancel{(\sum x)^2}} \rightarrow \frac{\sum xy}{\sum x^2}$$

$$y = \langle y \rangle + a \cdot x$$

$$\rightarrow \underset{\sim}{S_e} \quad \begin{cases} x \rightarrow \\ y \rightarrow \end{cases} \boxed{\begin{array}{l} \xi = x - \langle x \rangle \\ \eta = y - \langle y \rangle \end{array}} \quad \& \begin{cases} \Sigma \xi = 0 \\ \Sigma \eta = 0 \end{cases} \quad (*)$$

$$y - \langle y \rangle = a\,(x - \langle x \rangle) \rightarrow \underset{\sim}{\eta = a\,\xi}$$

$$a = \frac{N\Sigma xy - \Sigma x\,\Sigma y}{N\cdot\Sigma x^2 - (\Sigma x)^2} = \frac{N\Sigma(\xi+\langle x\rangle)(\eta+\langle y\rangle) - \Sigma(\xi+\langle x\rangle)\Sigma(\eta+\langle y\rangle)}{N\cdot\Sigma(\xi+\langle x\rangle)^2 - \left[\Sigma(\xi+\langle x\rangle)\right]^2} =$$

$$= \frac{N\Sigma\left(\xi\eta + \xi\langle y\rangle + \eta\langle x\rangle + \langle x\rangle\langle y\rangle\right) - \left(\cancel{\Sigma\xi} + N\langle x\rangle\right)\left(\cancel{\Sigma\eta} + N\langle y\rangle\right)}{N\Sigma\left(\xi^2 + \langle x\rangle^2 + 2\langle x\rangle\xi\right) - \left(\cancel{\Sigma\xi} + N\langle x\rangle\right)^2}$$

$$= \frac{N\Sigma\xi\eta + N\langle y\rangle\cancel{\Sigma\xi} + N\langle x\rangle\cancel{\Sigma\eta} + \underset{\sim}{N^2\langle x\rangle\langle y\rangle} - \underset{\sim}{N^2\langle x\rangle\langle y\rangle}}{N\Sigma\xi^2 + \underset{\sim}{N^2\langle x\rangle^2} + 2\langle x\rangle N\cancel{\Sigma\xi} - \underset{\sim}{N^2\langle x\rangle^2}} =$$

$$= \frac{\cancel{N}\,\Sigma\,\xi\eta}{\cancel{N}\,\Sigma\,\xi^2} = \frac{\Sigma(\xi\eta)}{\Sigma(\xi^2)} \Longrightarrow \boxed{\eta = \left(\frac{\Sigma\xi\eta}{\Sigma\xi^2}\right)\xi}$$

(*) Infatti:

$$\begin{cases} \Sigma\,\xi = \Sigma\left(x - \frac{\Sigma x}{N}\right) = \Sigma x - N\left(\frac{\Sigma x}{N}\right) = 0 \\ \Sigma\,\eta = \Sigma\left(y - \frac{\Sigma y}{N}\right) = \Sigma y - N\left(\frac{\Sigma y}{N}\right) = 0 \end{cases}$$

Le due deviazioni standard

$\sigma(a) \qquad\qquad \sigma(b)$

sui due parametri del fit lineare

$y = a x + b$

provengono dalla propagazione statistica delle incertezze sulle misure y_i .... avendo ipotizzato

$$\frac{\sigma(x)}{x} \ll \frac{\sigma(y)}{y}$$

$$\Longrightarrow \sigma^2(a) = \sum_{i=1}^{N} \left[\frac{\partial}{\partial y_i} a(\dots y_i \dots) \right]^2 \sigma^2(y_i)$$

$$\Longrightarrow \sigma^2(b) = \sum_{i=1}^{N} \left[\frac{\partial}{\partial y_i} b(\dots y_i \dots) \right]^2 \sigma^2(y_i)$$

.... per il momento trascuro la covarianza $\sigma(a,b)$.....

$\boxed{y = ax + b}$

Stima delle incertezze statistiche su a, b

$$\sigma^2(b) = \sum_{1\,j}^{N}\left(\frac{\partial}{\partial y_j} b\right)^2 \cdot \sigma^2(y) = \sum_{1\,j}^{N}\left(\frac{\partial}{\partial y_j}\, \frac{\Sigma x^2 \Sigma y - \Sigma x \Sigma xy}{N \Sigma x^2 - (\Sigma x)^2}\right)^2 \cdot \sigma^2(y) =$$

$$= \sum_{1\,j}^{N}\left(\frac{\Sigma x^2 \cdot \frac{\partial}{\partial y_j}\sum_{1}^{N}{}_i\, y_i - (\Sigma x)\frac{\partial}{\partial y_j}\sum_{1}^{N}{}_i\, x_i y_i}{\underbrace{N\Sigma x^2 - (\Sigma x)^2}_{\Delta}}\right)^2 \sigma^2(y) =$$

$$= \left(\frac{\sigma(y)}{\Delta}\right)^2 \cdot \sum_{1\,j}^{N}\left((\Sigma x^2)\cdot 1 - (\Sigma x)\cdot x_j\right)^2 =$$

$$= \left(\frac{\sigma(y)}{\Delta}\right)^2 \cdot \sum_{1\,j}^{N}\left((\Sigma x^2)^2 + (\Sigma x)^2 x_j^2 - 2\,(\Sigma x^2)(\Sigma x)\, x_j\right) =$$

$$= \left(\frac{\sigma(y)}{\Delta}\right)^2 \cdot \underbrace{\left[N(\Sigma x^2)^2 + \underbrace{(\Sigma x)^2(\Sigma x^2) - 2(\Sigma x^2)(\Sigma x)^2}_{-(\Sigma x)^2(\Sigma x^2)} \right]}_{(\Sigma x^2)\cdot\Delta}$$

$$= \frac{\sigma^2(y)}{\Delta^2}(\Sigma x^2)\,\Delta$$

$$\boxed{\sigma^2(b) = \frac{\sigma^2(y)\left(\sum_{1}^{N}{}_i\, x_i^2\right)}{N\left(\sum_{1}^{N}{}_i\, x_i^2\right) - \left(\sum_{1}^{N}{}_i\, x_i\right)^2} = \frac{\sigma^2(y)\,\langle x^2\rangle}{N\left(\langle x^2\rangle - \langle x\rangle^2\right)}}$$

$$\boxed{y = a x + b}$$

$$\sigma^2(a) = \sum_{1}^{N}{}_J \left(\frac{\partial}{\partial y_J} a\right)^2 \cdot \sigma^2(y) = \sum_{1}^{N}{}_J \left(\frac{\partial}{\partial y_J} \frac{N\Sigma xy - \Sigma x \Sigma y}{\underbrace{N\Sigma x^2 - (\Sigma x)^2}_{\Delta}}\right)^2 \sigma^2(y) =$$

$$= \frac{\sigma^2(y)}{\Delta^2} \sum_{1}^{N}{}_J \left(N \cdot \frac{\partial}{\partial y_J} \sum_{1}^{N}{}_i x_i y_i - \Sigma x \frac{\partial}{\partial y_J} \sum_{1}^{N}{}_i y_i\right)^2 =$$

$$= \frac{\sigma^2(y)}{\Delta^2} \sum_{1}^{N}{}_J \left(N \cdot x_J - (\Sigma x) \cdot 1\right)^2 =$$

$$= \frac{\sigma^2(y)}{\Delta^2} \sum_{1}^{N}{}_J \left(N^2 x_J^2 + (\Sigma x)^2 - 2N(\Sigma x) x_J\right) =$$

$$= \frac{\sigma^2(y)}{\Delta^2} \underbrace{\left(N^2 (\Sigma x^2) + \underbrace{(\Sigma x)^2 N - 2N(\Sigma x)^2}_{-N(\Sigma x)^2}\right)}_{N \cdot \Delta} =$$

$$= \frac{\sigma^2(y)}{\Delta^2} \cdot N \cdot \Delta$$

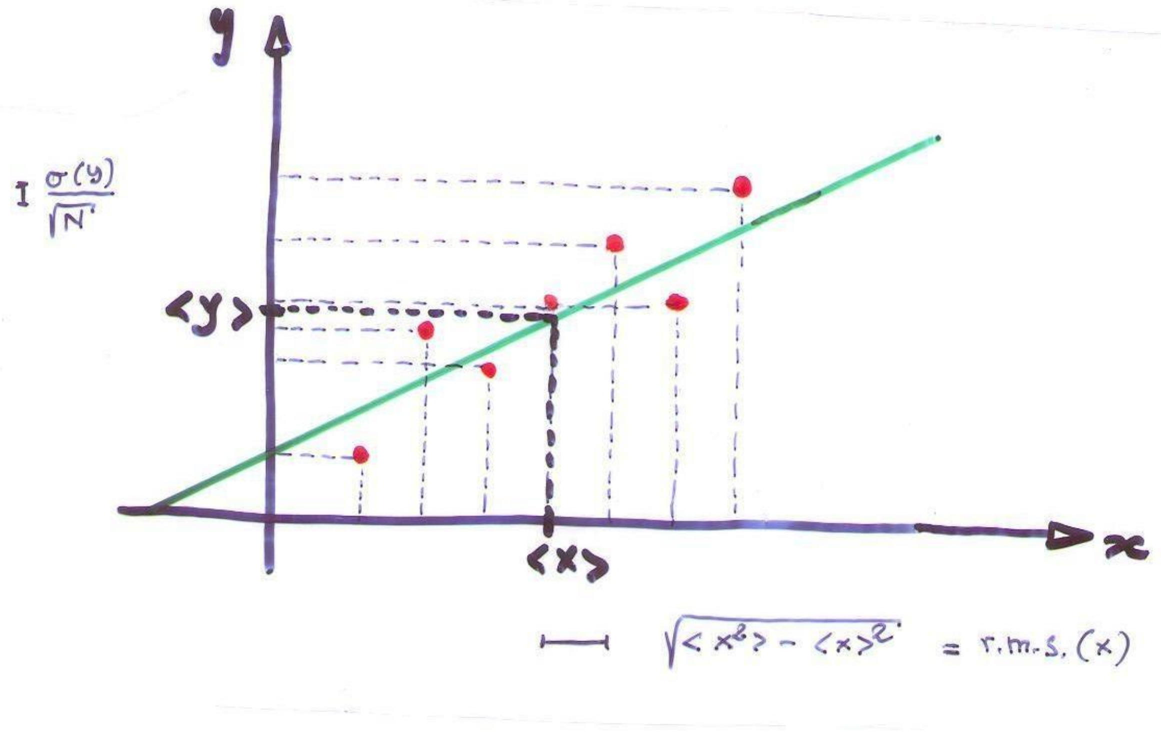
$$\boxed{\sigma^2(a) = \frac{\sigma^2(y)\, N}{N\left(\sum_{1}^{N}{}_i x_i^2\right) - \left(\sum_{1}^{N}{}_i x_i\right)^2} = \frac{\sigma^2(y)}{N\left(\langle x^2 \rangle - (\langle x \rangle)^2\right)}}$$

$\sigma^2(a)$, $\sigma^2(b)$ sono "intuitivi"?

$$y = ax + b$$

$$\begin{cases} \sigma^2(a) \overset{!?}{\approx} \left(\dfrac{\Delta y}{\Delta x}\right)^2 \approx \left(\dfrac{\sigma(y)/\sqrt{N}}{\sqrt{\langle x^2\rangle - \langle x\rangle^2}}\right)^2 = \dfrac{\sigma^2(y)/N}{\langle x^2\rangle - \langle x\rangle^2} \\ \sigma^2(b) \overset{!?}{\approx} \left(\dfrac{\Delta y}{\Delta x}\right)^2 \cdot (\Delta x)^2 \approx \dfrac{\sigma^2(y)/N}{\langle x^2\rangle - \langle x\rangle^2} \cdot \langle x^2\rangle \end{cases}$$

$$\boxed{\sigma^2(a) = \frac{\sigma^2(b)}{\langle x^2\rangle} \quad ; \quad \sigma^2(b) = \sigma^2(a)\,\langle x^2\rangle}$$



Sommario: $\boxed{y = ax + b}$

- $\Delta = N^2 \sum x^2 - \left(\sum x\right)^2 = N^2\left(\langle x^2\rangle - \langle x\rangle^2\right) = N^2 \cdot Var(x)$

- $\sigma^2(a) = \dfrac{\sigma_y^2/N}{Var(x)}$; $\sigma^2(b) = \dfrac{\sigma_y^2/N}{Var(x)} \cdot \langle x^2\rangle$;

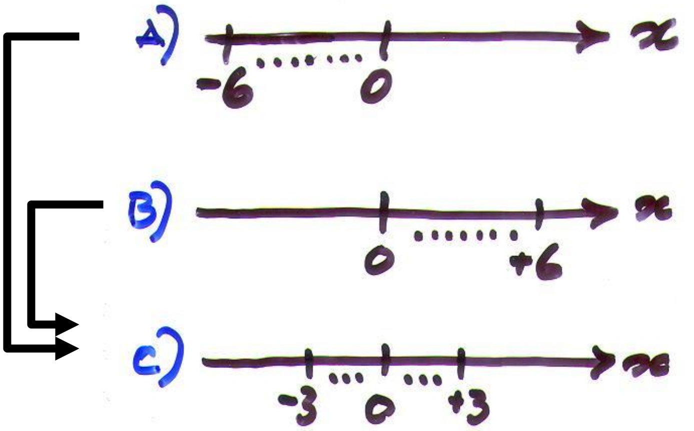
- $$\begin{cases} \sigma^2(a) = \dfrac{\sigma^2(b)}{\langle x^2\rangle} \\[2ex] \sigma^2(b) = \sigma^2(a) \cdot \langle x^2\rangle \end{cases}$$

$\Downarrow$ variando un solo parametro alla volta ....

- $\left\{\begin{matrix} N \nearrow \\ \sigma_y^2 \searrow \\ Var(x) \nearrow \end{matrix}\right\}$ $\qquad \sigma^2(a) \searrow \qquad \sigma^2(b) \searrow$

  $\left\{\langle x^2\rangle \nearrow\right\}$ $\qquad \sigma^2(a) \searrow \qquad \sigma^2(b) \nearrow$

$N=7$ con passo 1



	$\langle x\rangle$	$\langle x^2\rangle$	$Var(x)$
A)	-3	13	4
B)	$+3$	13	4
C)	0	$\boxed{4}$	4

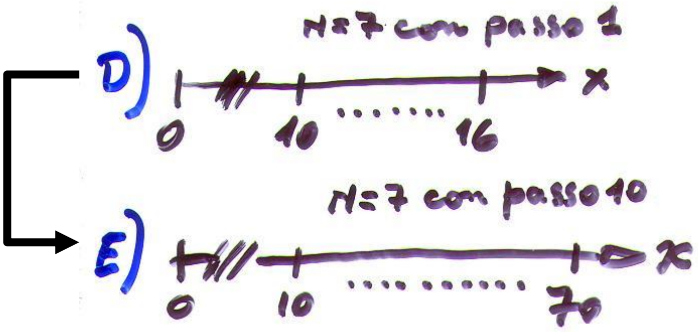
- passare dalle situazioni A) o B) alla C) comporta $\langle x^2\rangle \searrow$ pertanto:

$\sigma^2(a)$ non cambia ma $\sigma^2(b) \searrow$

pendenza　　　　　Termine noto

$N=7$ con passo 1

$N=7$ con passo 10



	$\langle x\rangle$	$\langle x^2\rangle$	$Var(x)$
D)	13	173	4
E)	40	$2'000$	$\boxed{400}$

- passare dal caso D) al caso E) comporta:

$Var(x) \nearrow$ pertanto $\sigma^2(a) \searrow$ "Pendenza"

ma anche $\langle x^2\rangle \nearrow$ perciò $\sigma^2(b) \propto \frac{\langle x^2\rangle}{Var(x)} \searrow$ "Termine noto"

$$\sigma^2(b) \propto \frac{\langle x^2\rangle}{\langle x^2\rangle - \langle x\rangle^2} =$$

$$= \frac{1}{1 - \frac{\langle x\rangle^2}{\langle x^2\rangle}} \sim 1 + \dots$$

< 1

$$\langle x\rangle^2 \gg \langle x^2\rangle$$

$$\boxed{\sigma^2(b) \searrow \quad \langle x\rangle^2 \ll \langle x^2\rangle}$$

Termine Noto

- Migliore stima dell'incertezza sulle misure y_i in base alla migliore retta ricavata con il metodo dei minimi quadrati

$$\boxed{\sigma_{fit}(y) = \sqrt{\frac{\sum_{i=1}^{N} (y_i - a x_i - b)^2}{N-2}}}$$

nell'ipotesi che $\begin{cases} \dfrac{\sigma(x_i)}{x_i} \ll \dfrac{\sigma(y_i)}{y_i} & i = 1, 2, \ldots, N \\ \sigma(y_i) = \text{costante per } i = 1, 2, \ldots, N \end{cases}$

→ varianza del fit $= [\sigma_{fit}(y)]^2$

→ Il termine N-2 anziché N tiene conto dei 2 parametri a, b ricavati dal "fit" cioè il "numero dei gradi di libertà" ovvero il numero di misure indipendenti è N-2

convenzionalmente si chiamano:

$\sigma(y_i)$ = errore interno ai dati

$\sigma_{fit}(y)$ = " esterno " "

⟶ mi aspetto che $\sigma(y_i) \approx \sigma_{fit}(y)$

⟶ se ho una sola serie di N coppie (x_i, y_i) ovvero se non conosco $\sigma(y_i)$ al suo posto in $\sigma(a)$ e $\sigma(b)$ non potrò che usare $\sigma_{fit}(y)$ ma, attenzione al Test del χ^2!

→ Sommario sui minimi quadrati

$$\boxed{y = a + bx}$$

con solo σ_y di tipo costante per ogni y

- $$a = \frac{(\Sigma x^2)(\Sigma y) - (\Sigma x)(\Sigma xy)}{N(\Sigma x^2) - (\Sigma x)^2} = \frac{\langle x^2\rangle\langle y\rangle - \langle x\rangle\langle xy\rangle}{\langle x^2\rangle - \langle x\rangle^2}$$

- $$b = \frac{N(\Sigma xy) - (\Sigma x)(\Sigma y)}{N(\Sigma x^2) - (\Sigma x)^2} = \frac{\langle xy\rangle - \langle x\rangle\langle y\rangle}{\langle x^2\rangle - \langle x\rangle^2}$$

$$\Delta = N(\Sigma x^2) - (\Sigma x)^2 = N^2(\langle x^2\rangle - \langle x\rangle^2)$$

- $$\sigma_{fit} = \sqrt{\frac{1}{N-2}\sum_{1}^{N}{}_i [y_i - (a + bx_i)]^2}$$

- $$\sigma_a = \sigma_y \sqrt{\frac{\Sigma x^2}{\Delta}} = \frac{\sigma_y}{\sqrt{N}} \cdot \sqrt{\frac{\langle x^2\rangle}{(\langle x^2\rangle - \langle x\rangle^2)}}$$

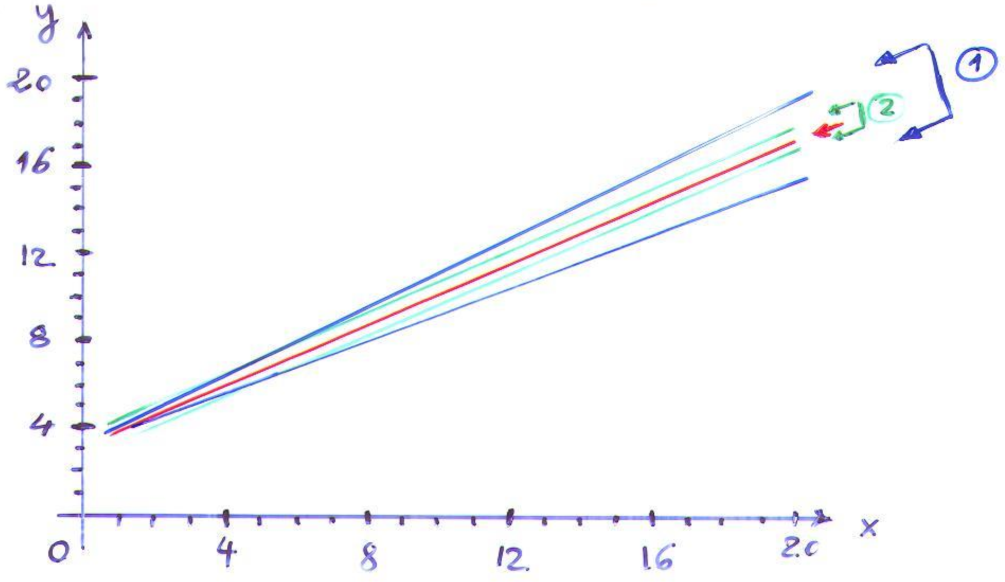
- $$\sigma_b = \sigma_y \sqrt{\frac{N}{\Delta}} = \frac{\sigma_y}{\sqrt{N}} \cdot \frac{1}{\sqrt{(\langle x^2\rangle - \langle x\rangle^2)}}$$

- $$\sigma_{(a,b)} = -\frac{\sigma^2(y) \cdot \sum_1^N{}_i\, x_i}{N \cdot \sum_1^N{}_i\, x_i^2 - \left(\sum_1^N{}_i\, x_i\right)^2} = -\frac{\sigma^2(y)}{N} \cdot \frac{\langle x\rangle}{\langle x^2\rangle - \langle x\rangle^2}$$

$$|\sigma_{(a,b)}| < \sigma_a \cdot \sigma_b$$

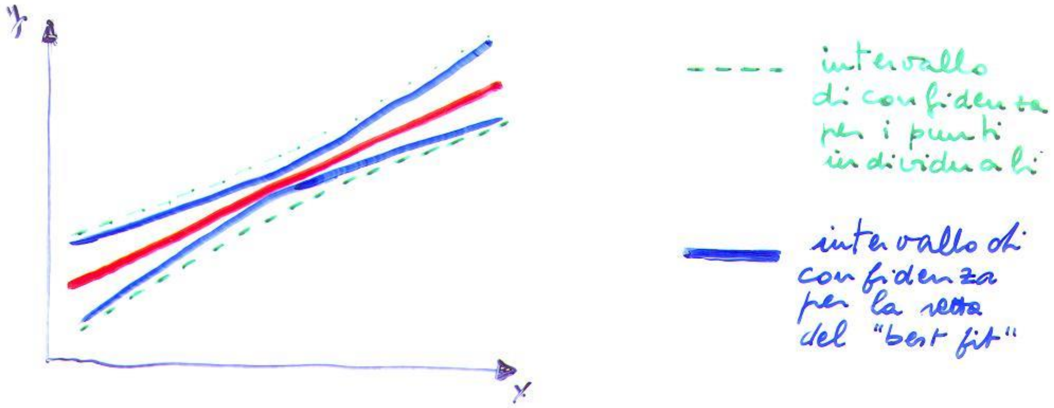
- $$\sigma(y = a + bx) = \sqrt{x^2 \cdot \sigma_b^2 + \sigma_a^2 + 2x\sigma_{(a,b)}}$$

$$\boxed{y = ax + b}$$



① $y = [a \pm \sigma(a)]\,x + b$

② $y = a \cdot x + [b \pm \sigma(b)]$



$$\sigma_{fit}(y) = \sqrt{\frac{\sum_1^N i\,(y_i - a x_i - b)^2}{N-2}} \cong \sqrt{\frac{1{,}6429}{7-2}} \cong 0{,}5732$$

$\boxed{\text{Test del } \chi^2}$ per giustificare la relazione funzionale $y = ax + b$

.... in realtà dovrei provare più tipi di dipendenze funzionali, per ciascuna calcolare il χ^2, quindi potrei decidere confrontando i diversi χ^2.

1

$$\chi^2_{fit} = \sum_1^N i\, \frac{\left(y_i^{fit} - y_i^{exp}\right)^2}{\sigma^2_{fit}} = \frac{\sum_1^N i\left(a_{fit}\, x_{i_{exp}} + b_{fit} - y_{i_{exp}}\right)^2}{\frac{\sum_1^N i\,\left(y_i - a_{fit}\, x_{i_{exp}} - b_{fit}\right)^2}{N-2}} = N-2 \quad ??$$

Non ho quindi alcuna informazione utile calcolando il χ^2_{fit} mediante $\sigma_{fit}(y)$: otterrei un $\chi^2_{fit} = N-2$ indipendentemente dalla "bontà" del fit!

2 Dovrei avere più serie di N coppie di valori per x, y da cui estrarre $\sigma(y)$. $\xrightarrow{se}$ $\sigma(y) = 0{,}5$?

$$\chi^2_{fit} = \sum_1^N i \left(\frac{y_i^{fit} - y_i^{exp}}{\sigma(y)}\right)^2 \cong 6{,}6 \quad \text{con } \nu = 7 - 2 = 5 \rightarrow P\left(\chi^2 \geq 6{,}6 \atop \nu = 5\right) \cong 29\%$$

$\gg 5\%$ soglia tipica

3 Se ho solo una serie di N coppie di misure x_i, y_i ed una retta del fit, ma non ho $\sigma(y)$:

$$\left[\frac{\text{dispersione max degli } N\ y_i \text{ attorno alla retta del fit}}{\text{numero di } M \text{ intervallini (ciascuno pari a } \Delta)}\right] \rightarrow \Delta$$

$F^{exp}_{(i)}$ numero di differenze $y_i^{exp} - y_i^{fit}$ cadono nell'intervallo i-esimo

$F^{gauss}_{(i)} = N \cdot f_{gauss}\left(i, \langle\rangle = 0\ \&\ \sigma = \sigma_{fit}(y)\right) \cdot \Delta$

$$\chi^2_{fit} = \sum_{1}^{M^* \geq M} i \left(\frac{F^{exp}_{(i)} - F^{gauss}_{(i)}}{\sqrt{F^{gauss}_{(i)}}}\right)^2 ; \quad \nu = M^* - \text{numero parametri ricavati dal fit} - 1 \text{ (cond. norm.)}$$

"Test del χ^2 sulla distribuzione dei residui"

i	x_i^{exp}	y_i^{exp}	y_i^{fit}	$y_i^{exp} - y_i^{fit}$
1	2	5	4.63	0.3571
2	3	4.5	5.34	-0.8572
3	4	6.0	6.05	-0.0715
4	5	7.5	6.76	0.7142
5	6	7.5	7.47	0.0001
6	7	8.5	8.18	0.2856
7	8	8.5	8.89	-0.4287

scelgo $\Delta = 0,5$ con $N = 7$:

estremi intervallo ($y^{ex} - y^{fit}$)		centro intervallo (ε)	F^{exp}	F^{gauss}	$\frac{(F^{gauss} - F^{exp})^2}{F^{gauss}}$
-3,0	-2,5	-2,75	0	0,0000	
-2,5	-2,0	-2,25	0	0,0011	0,0011
-2,0	-1,5	-1,75	0	0,0230	0,0230
-1,5	-1,0	-1,25	0	0,2259	0,2259
-1,0	-0,5	-0,75	1	1,0349	0,0012
-0,5	0	-0,25	2	2,2150	0,0209
0	0,5	+0,25	3	2,2150	0,2782
0,5	1,0	0,75	1	1,0349	0,0012
1,0	1,5	1,25	0	0,2259	0,2259
1,5	2,0	1,75	0	0,0230	0,0230
2,0	2,5	2,25	0	0,0011	0,0011
2,5	3,0	2,75	0	0,0000	

4 bin
6 bin
10 bin

$$\chi^2_{fit} = \sum_{10\,bin}' \frac{(F^{gauss} - F^{exp})^2}{F^{gauss}} \simeq 0,8$$

$$\nu = 10 - 2 - 1 = 7$$

$$P(\chi^2 \geq 0,8,\ \nu = 7) = 99\%$$

$\sigma_{fit} = 0.5732$

$$f_{GAUSS} = \frac{e^{-\varepsilon^2/2\sigma_{fit}^2}}{\sqrt{2\pi}\,\sigma_{fit}} \simeq$$

$$\simeq 0.6960\, e^{-1.5218\,\varepsilon^2}$$

$$F^{th} = \Delta \cdot N \cdot f_{GAUSS}$$

..... HO CONTEGGI TROPPO BASSI NEI VARI CANALI DELL' ISTOGRAMMA .... 10 BIN SONO TROPPI

χ^2 = Σ_{4bin}[(F^G-F^{EXP})2/F^G] =
(0.0012+0.02782+0.0012) = 0.3015
E(χ^2) = ν = 4 -3 = 1

..... Provo un bin con $\Delta = 1,0$
anziché con $\Delta = 0,5$

i	estremi intervallo differenze $y^{exp} - y^{fit}$	centro intervallo ε	F^{exp}	F^{th}	$\frac{(F^{th} - F^{exp})^2}{F^{th}}$
1	$-3.0 \div -2.0$	-2.5	0	0.00	0.00
2	$-2.0 \div -1.0$	-1.5	0	0.02	0.02
3	$-1.0 \div 0$	-0.5	3	3.33	0.03
4	$0 \div 1.0$	0.5	4	3.33	0.13
5	$1.0 \div 2.0$	1.5	0	0.02	0.02
6	$2.0 \div 3.0$	2.5	0	0.00	0.00

$$F^{th} = \underbrace{f(\varepsilon)}_{\text{GAUSS}} \cdot \Delta \cdot N$$

$$N = \sum F^{exp} = 7$$

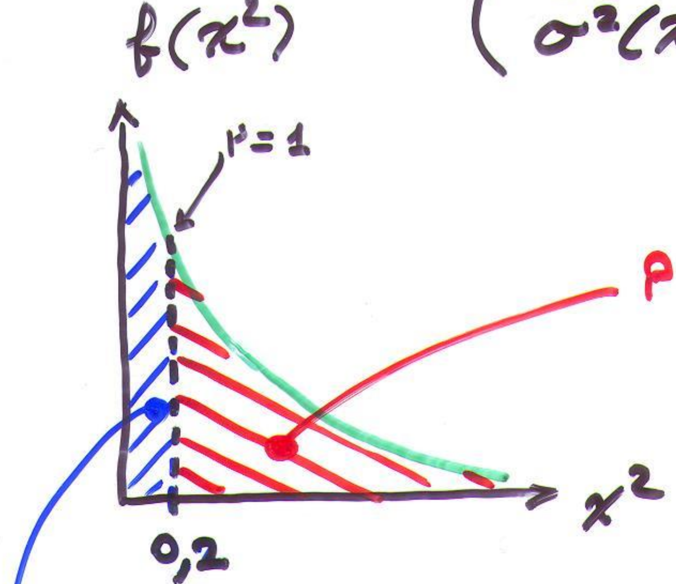
$$\Delta = 1,0$$

$$f(\varepsilon) = \frac{e^{-\varepsilon^2/2\sigma_{fit}^2}}{\sqrt{2\pi}\,\sigma_{fit}}$$

$$\left.\begin{aligned} \chi^2_{fit} &= \sum_{2}^{5}{}_i \frac{(F^{exp} - F^{th})^2}{F^{th}} = 0.2 \\ \nu &= 4 - 2 - 1 = 1 \end{aligned}\right\} \quad P\left(\chi^2_{\nu=1} \geq 0.2\right) = 74.7\%$$

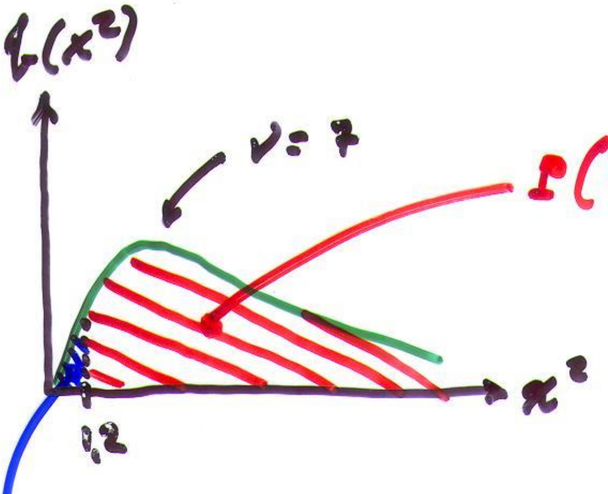
$$\boxed{E(\chi^2) = \nu = 1}$$

$$\left(\begin{array}{l} E(\chi^2)=\nu \\ \sigma^2(\chi^2)=2\nu \end{array}\right)$$



$$P(\chi^2 \geq 0{,}2\,;\ \nu=1) = 74{,}7\ \%$$

$$P(\chi^2 \leq 0{,}2\,;\ \nu=1) = 1 - P(\chi^2 \geq 0{,}2\,;\ \nu=1) = \underline{\underline{25{,}3\ \%}}$$



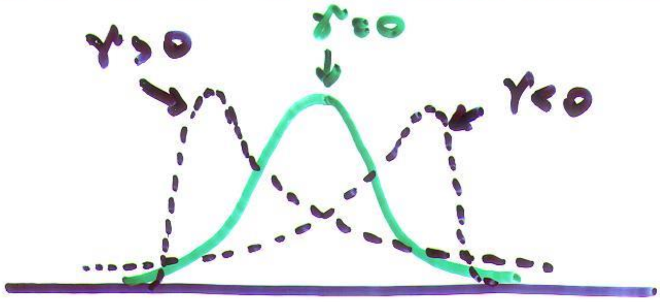
$$P(\chi^2 \geq 1{,}2\,;\ \nu=7) = 99\ \%$$

$$P(\chi^2 \leq 1{,}2\,;\ \nu=7) = 1 - P(\chi^2 \geq 1{,}2\,;\ \nu=7) = \underline{\underline{1\ \%}}$$

• **"SKEWNESS"**

$$\gamma \equiv \frac{E\left[(x - E(x))^3\right]}{\left\{E(x - E(x))^2\right\}^{3/2}} = \frac{\mu_3}{\sigma^3}$$

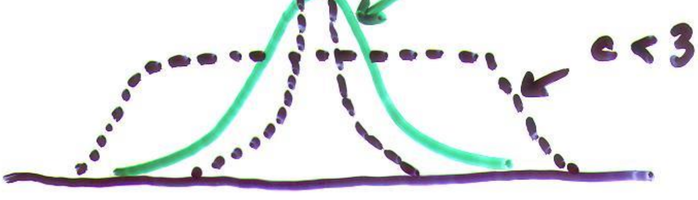
"misura la asimmetria della funzione di distribuzione"

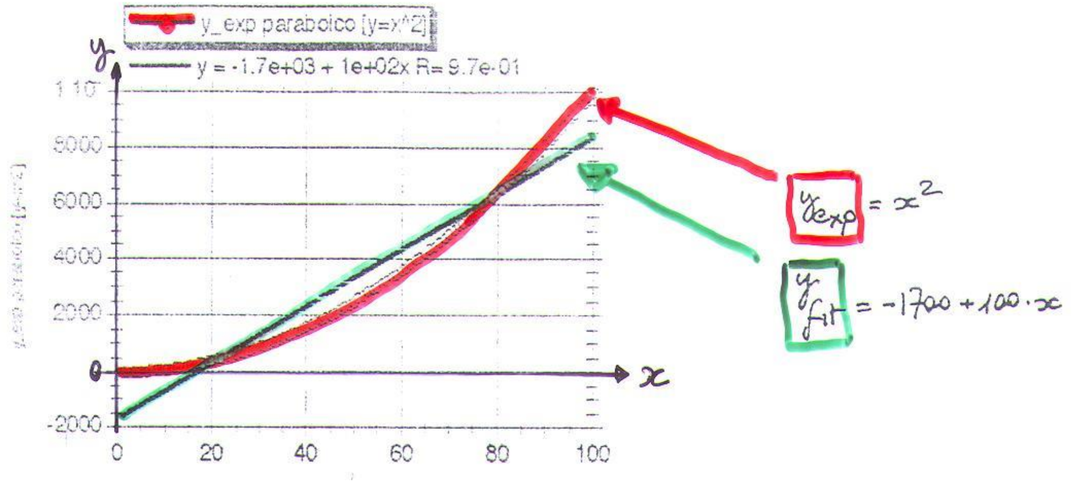


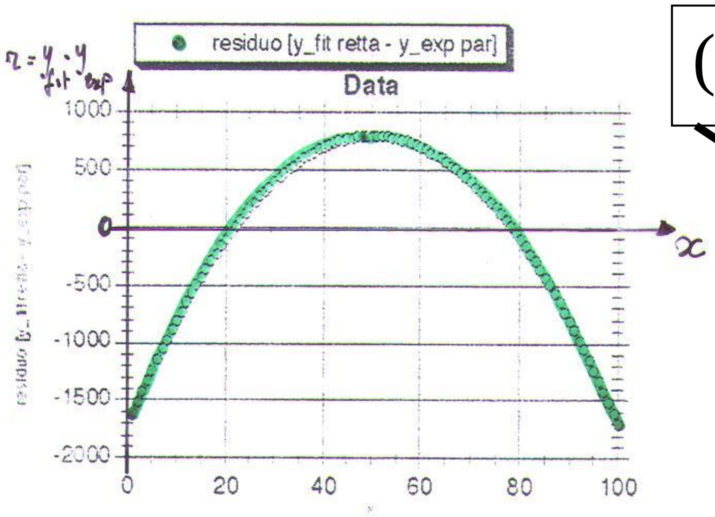
• **"KURTOSIS"** o **"FLATNESS"**

$$c \equiv \frac{E\left[(x - E(x))^4\right]}{\left\{E(x - E(x))^2\right\}^2} = \frac{\mu_4}{\sigma^4}$$

"misura l'importanza relativa delle code rispetto alla parte centrale nella funzione di distribuzione"

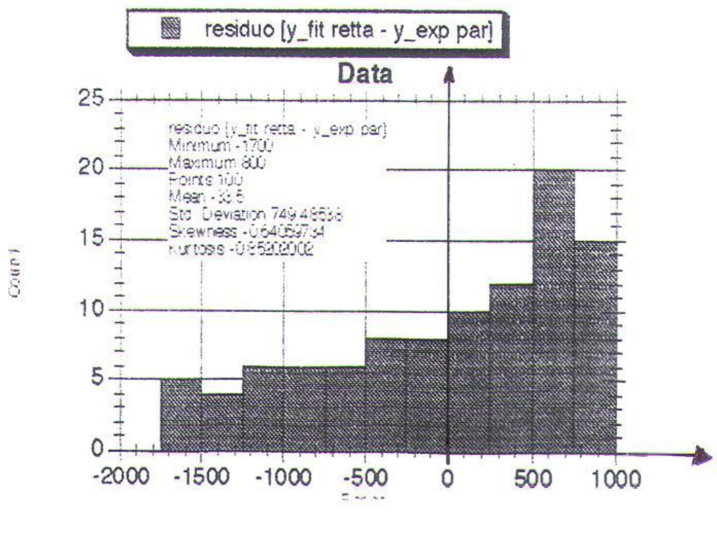






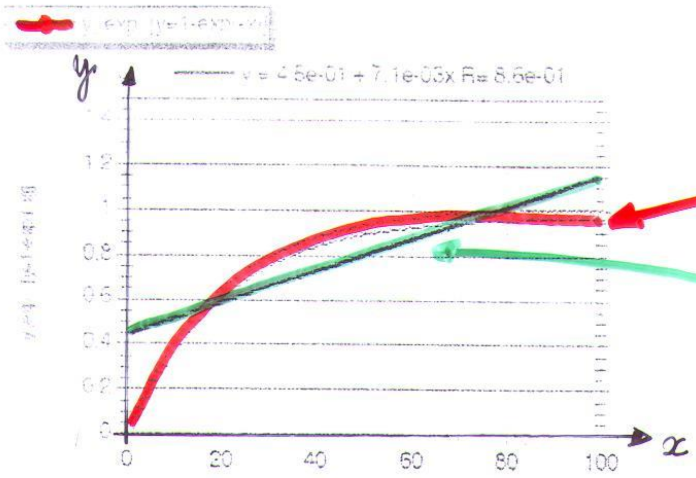
$(Y_{\text{fit,retta}} - Y_{\text{exp,parabola}})$

residui vs. x



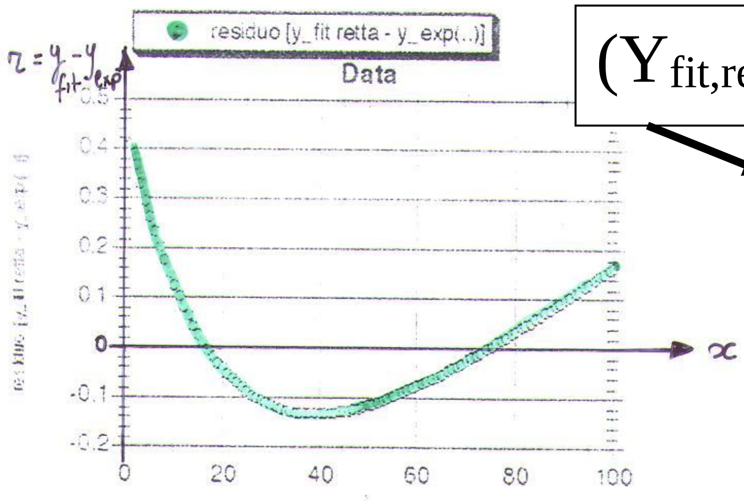
γ (SKEWNESS) $\simeq -0{,}64 < 0$

c (KURTOSIS) $\simeq -0{,}85 < 3$



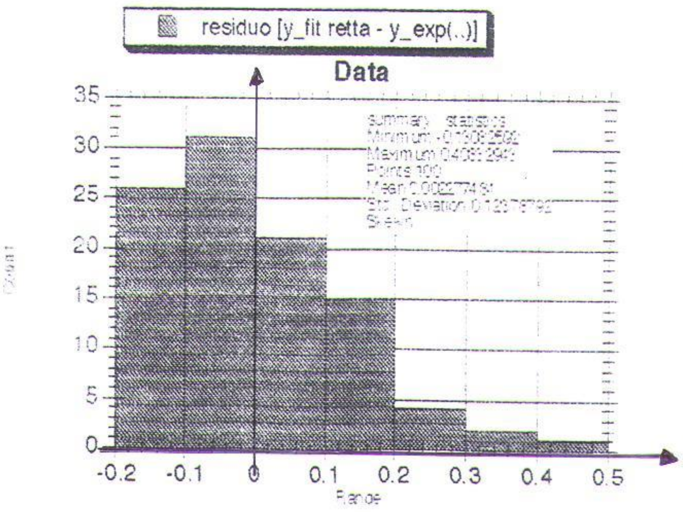
$y_{exp} = 1 - \exp(-x/20)$

$y_{fit} = 0{,}45 + 0{,}0071 \cdot x$



$(Y_{fit,retta} - Y_{exp,esponenziale})$

residui vs x



γ (SKEWNESS) $\simeq 1{,}06$ **> 0**

c (KURTOSIS) $\simeq 0{,}77$ **< 3**

$$\boxed{y = ax + b}$$

$$\sigma^2(y) = \left(\frac{\partial}{\partial a} y\right)^2 \sigma^2(a) + \left(\frac{\partial}{\partial b} y\right)^2 \sigma^2(b) +$$

$$+ 2 \frac{\partial y}{\partial a} \cdot \frac{\partial y}{\partial b} \cdot \sigma(a,b)$$

& $\dfrac{\partial y}{\partial a} = x$, $\dfrac{\partial y}{\partial b} = 1$

$$\boxed{\sigma^2(y) = x^2 \sigma^2(a) + \sigma^2(b) + 2x \cdot \sigma(a,b)}$$

La varianza sulla y calcolata usando il fit dipenderà dalle varianze dei due parametri ricavati dal fit e dalla covarianza tra questi parametri

Nel caso limite di $N=2$:

$$a(y_1, y_2) \simeq a(\bar{y}_1, \bar{y}_2) + \frac{\partial a}{\partial y_1}(y_1-\bar{y}_1) + \frac{\partial a}{\partial y_2}(y_2-\bar{y}_2)$$

$$b(y_1, y_2) \simeq b(\bar{y}_1, \bar{y}_2) + \frac{\partial b}{\partial y_1}(y_1-\bar{y}_1) + \frac{\partial b}{\partial y_2}(y_2-\bar{y}_2)$$

$\downarrow$

$$(a-\bar{a})(b-\bar{b}) \simeq \frac{\partial a}{\partial y_1}\frac{\partial b}{\partial y_1}(y_1-\bar{y}_1)^2 + \frac{\partial a}{\partial y_2}\frac{\partial b}{\partial y_2}(y_2-\bar{y}_2)^2 +$$

$$+ \frac{\partial a}{\partial y_1}\frac{\partial b}{\partial y_2}(y_1-\bar{y}_1)(y_2-\bar{y}_2) +$$

$$+ \frac{\partial a}{\partial y_2}\frac{\partial b}{\partial y_1}(y_1-\bar{y}_1)(y_2-\bar{y}_2)$$

$\downarrow$

$$E\left[(a-\bar{a})(b-\bar{b})\right] \simeq \frac{\partial a}{\partial y_1}\frac{\partial b}{\partial y_1}\sigma^2(y_1) + \frac{\partial a}{\partial y_2}\frac{\partial b}{\partial y_2}\sigma^2(y_2) +$$

$$+ \frac{\partial a}{\partial y_1}\frac{\partial b}{\partial y_2}\sigma(y_1, y_2) +$$

$$+ \frac{\partial a}{\partial y_2}\frac{\partial b}{\partial y_1}\sigma(y_1, y_2)$$

$$\Rightarrow \sigma(a, b) \equiv E\left[(a-E(a))(b-E(b))\right] =$$

$$= \sum_{i=1}^{N}\frac{\partial a}{\partial y_i}\frac{\partial b}{\partial y_i}\sigma^2(y_i)$$

Covarianza tra a e b in $\boxed{y = ax + b}$

poiché gli stessi dati (x_i, y_i) $i = 1, 2, 3, \dots, N$
sono usati per stimare
entrambi i parametri a e b, questi parametri
risultano correlati pur essendo scorrelate
le N misure di partenza.

$$\mathrm{cov}(a,b) = \sum_i \left[\frac{\partial a}{\partial y_i}\right]\left[\frac{\partial b}{\partial y_i}\right] \sigma^2(y_i) =$$

$$= \sum_i \left[\frac{\partial}{\partial y_i}\left(\frac{N\Sigma xy - \Sigma x \Sigma y}{\underbrace{N\Sigma x^2 - (\Sigma x)^2}_{\Delta}}\right)\right]\left[\frac{\partial}{\partial y_i}\left(\frac{\Sigma x^2 \Sigma y - \Sigma x \Sigma xy}{\underbrace{N\Sigma x^2 - (\Sigma x)^2}_{\Delta}}\right)\right]\sigma^2(y_i)$$

$$= \sum_i \left[\frac{1}{\Delta}(N x_i - \Sigma x) \cdot \frac{1}{\Delta}\left(\Sigma x^2 - (\Sigma x) x_i\right)\right]\sigma^2(y)$$

$$= \frac{\sigma^2(y)}{\Delta^2} \sum_i \Big(N \Sigma x^2 \cdot x_i - N \Sigma x \, x_i^2 + {} - \Sigma x \, \Sigma x^2 + (\Sigma x)^2 x_i \Big) =$$

$$= \frac{\sigma^2(y)}{\Delta^2}\Big(N \cdot \Sigma x^2 \cdot \Sigma x - N \cdot \Sigma x \cdot \Sigma x^2 \underbrace{- N \cdot \Sigma x \cdot \Sigma x^2 + (\Sigma x)^2 (\Sigma x)}_{\begin{subarray}{c} -(\Sigma x)(N\Sigma x^2 - (\Sigma x)^2) = \\ = -(\Sigma x)\cdot\Delta \end{subarray}} \Big)$$

$$\boxed{\mathrm{cov}(a,b) = -\frac{\sigma^2(y)\cdot \sum_1^N {}_i\, x_i}{N \cdot \sum_1^N {}_i\, x_i^2 - \left(\sum_1^N {}_i\, x_i\right)^2} = -\frac{\sigma^2(y)}{N} \cdot \frac{\langle x \rangle}{\langle x^2 \rangle - \langle x \rangle^2}}$$

→ Se scelgo come origine dell'asse delle x il punto medio $\langle x \rangle$, allora $\mathrm{cov}(a,b) = 0$!!

Interpolazione ed estrapolazione usando la retta dei minimi quadrati: $\begin{pmatrix} \text{data } x \text{ ricavo } y \\ \text{o viceversa} \end{pmatrix}$

$y = ax+b$

→ $E(y) = E(ax+b) = ax+b$
cioè x è un parametro mentre a e b sono variabili casuali

→ $\sigma^2(y) = \sigma^2(ax+b) =$

$$= \left(\frac{\partial}{\partial a}(ax+b)\right)^2 \sigma^2(a) + \left(\frac{\partial}{\partial b}(ax+b)\right)^2 \sigma^2(b) +$$

$$+ 2 \cdot \frac{\partial(ax+b)}{\partial a} \cdot \frac{\partial(ax+b)}{\partial b} \cdot \mathrm{cov}(a,b) =$$

$$= x^2 \cdot \boxed{\sigma^2(a)} + \boxed{\sigma^2(b)} + 2 \cdot x \cdot \boxed{\mathrm{cov}(a,b)} =$$

$$= x^2 \cdot \boxed{\frac{\sigma^2(y) N}{\Delta}} + \boxed{\frac{\sigma^2(y)\left(\sum_1^N {}_i x_i\right)^2}{\Delta}} + 2x \boxed{\left(-\frac{\sigma^2(y) \sum_1^N {}_i x_i}{\Delta}\right)}$$

Se ho scelto l'origine dell'asse delle x in modo che coincida con $\langle x \rangle$, allora $\mathrm{cov}(a,b) = 0$ e

$$\sigma^2(y) = \left(\frac{\partial}{\partial a} y\right)^2 \sigma^2(a) + \left(\frac{\partial y}{\partial b}\right)^2 \sigma^2(b)$$

$x \longrightarrow x - \langle x \rangle = ?$

$$\boxed{\begin{aligned}&\sigma^2(ax+b) = x^2\sigma^2(a) + \sigma^2(b) + 2x\,\sigma(a,b);\\ &\sigma^2(a) = \frac{(\sigma_y^2/N)}{\langle x^2\rangle - \langle x\rangle^2};\\ &\sigma^2(b) = \frac{(\sigma_y^2/N)\,\langle x^2\rangle}{\langle x^2\rangle - \langle x\rangle^2};\\ &\sigma(a,b) = -\frac{(\sigma_y^2/N)\,\langle x\rangle}{\langle x^2\rangle - \langle x\rangle^2};\end{aligned}}$$

$$\Rightarrow\quad \sigma^2(ax+b) = x^2\frac{\sigma_y^2}{N}\frac{1}{\langle x^2\rangle - \langle x\rangle^2} + \frac{\sigma_y^2}{N}\frac{\langle x^2\rangle}{\langle x^2\rangle - \langle x\rangle^2} - 2x\frac{\sigma_y^2}{N}\frac{\langle x\rangle}{\langle x^2\rangle - \langle x\rangle^2} =$$

$$= \frac{\sigma_y^2}{N}\frac{1}{(\langle x^2\rangle - \langle x\rangle^2)}\left(x^2 + \langle x^2\rangle - 2x\langle x\rangle\right)$$

$$\mathrm{var}(x) = \sigma^2(x) = \langle x^2\rangle - \langle x\rangle^2 \rightarrow \langle x^2\rangle = \sigma^2(x) + \langle x\rangle^2$$

$$\sigma^2(ax+b) = \frac{\sigma_y^2}{N}\cdot\frac{1}{\sigma^2(x)}\left(x^2 + \sigma^2(x) + \langle x\rangle^2 - 2x\langle x\rangle\right)$$

$$\sigma^2(ax+b) =$$

$$= \frac{\sigma_y^2}{N} \cdot \frac{1}{\sigma^2(x)} \left(\sigma^2(x) + (x - \langle x \rangle)^2 \right)$$

$$\Rightarrow \boxed{\sigma^2(ax+b) = \left(\frac{\sigma_y^2}{N}\right) \left(1 + \frac{(x - \langle x \rangle)^2}{\sigma^2(x)} \right)}$$

$$\boxed{\sigma^2(x) = \langle x^2 \rangle - \langle x \rangle^2}$$

$$y = ax + b \longrightarrow \boxed{x = \frac{1}{a}y - \frac{b}{a}}$$

$$y(x = x_0) = y_0 = ? \quad ; \quad x(y = y_0) = x_0 = ?$$

$$\rightarrow E(x) = E\left(\frac{y}{a} - \frac{b}{a}\right) \simeq \frac{y}{a} - \frac{b}{a}$$

$$\rightarrow \sigma^2(x) = \left[\frac{\partial}{\partial a}\left(\frac{y}{a} - \frac{b}{a}\right)\right]^2 \sigma^2(a) + \left[\frac{\partial}{\partial b}\left(\frac{y}{a} - \frac{b}{a}\right)\right]^2 \sigma^2(b) +$$

$$+ 2 \frac{\partial}{\partial a}(\dots) \frac{\partial}{\partial b}(\dots) \operatorname{cov}(a,b) =$$

$$= \left(\frac{y}{-a^2} - \frac{b}{-a^2}\right)^2 \sigma^2(a) + \left(\frac{y}{a} - \frac{1}{a}\right)^2 \sigma^2(b) +$$

$$+ 2\left(\frac{y}{-a^2} - \frac{b}{-a^2}\right)\left(\frac{y}{a} - \frac{1}{a}\right) \operatorname{cov}(a,b) =$$

$$= \frac{(y-b)^2}{a^4} \underbrace{\sigma^2(a)}_{\frac{\sigma^2(y)\,N}{\Delta}} + \frac{(y-1)^2}{a^2} \underbrace{\sigma^2(b)}_{\frac{\sigma^2(y)(\Sigma x)^2}{\Delta}} - 2\,\frac{(y-b)(y-1)}{a^3} \underbrace{\operatorname{cov}}_{-\frac{\sigma^2(y)\,\Sigma x}{\Delta}}$$

Sommario sui minimi quadrati:

$$\boxed{y = ax + b} \quad \text{con } \sigma(y) \text{ costante } \forall y_i$$

$$a = \frac{N(\sum xy) - (\sum x)(\sum y)}{N(\sum x^2) - (\sum x)^2} = \frac{\langle xy \rangle - \langle x \rangle \langle y \rangle}{\langle x^2 \rangle - \langle x \rangle^2}$$

$$b = \frac{(\sum x^2)(\sum y) - (\sum x)(\sum xy)}{N(\sum x^2) - (\sum x)^2} = \frac{\langle x^2 \rangle \langle y \rangle - \langle x \rangle \langle xy \rangle}{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\Delta = N(\sum x^2) - (\sum x)^2 = N^2(\langle x^2 \rangle - \langle x \rangle^2)$$

$$\sigma_{fit}(y) = \sqrt{\frac{1}{N-2}\sum_1^N{}_i (y_i - a x_i - b)^2}$$

$$\sigma^2(a) = \frac{\sigma^2(y)\, N}{\Delta} = \frac{\sigma^2(y)}{N} \cdot \frac{1}{(\langle x^2 \rangle - \langle x \rangle^2)}$$

$$\sigma^2(b) = \frac{\sigma^2(y)\,(\sum x^2)}{\Delta} = \frac{\sigma^2(y)}{N} \cdot \frac{\langle x^2 \rangle}{(\langle x^2 \rangle - \langle x \rangle^2)}$$

$$\sigma(a,b) = -\frac{\sigma^2(y)\,(\sum x)}{\Delta} = -\frac{\sigma^2(y)}{N} \cdot \frac{\langle x \rangle}{(\langle x^2 \rangle - \langle x \rangle^2)}$$

$$\sigma^2(y = ax + b) = x^2 \cdot \sigma^2(a) + \sigma^2(b) + 2 \cdot x \cdot \sigma(a,b)$$

$$|\sigma(a,b)| < \sigma(a) \cdot \sigma(b)$$

	x	y	SIGMA(a*x+b)	Yfit - SIGMA(a*x+b)	Yfit + SIGMA(a*x+b)	Yexp - Yfit	(Yexp - Yfit) / Yfit	SIGMA(a*x+b) / Yfit
0	2.0000	5.0000	0.39057	4.2523	5.0335	0.35712	0.076918	0.084122
1	3.0000	4.5000	0.30639	5.0508	5.6636	-0.85717	-0.16000	0.057193
2	4.0000	6.0000	0.24223	5.8292	6.3137	-0.071460	-0.011770	0.039896
3	5.0000	7.5000	0.21666	6.5691	7.0024	0.71425	0.10526	0.031928
4	6.0000	7.5000	0.24223	7.2578	7.7423	-4.0000e-05	-5.3333e-06	0.032297
5	7.0000	8.5000	0.30639	7.9079	8.5207	0.28567	0.034777	0.037300
6	8.0000	8.5000	0.39057	8.5380	9.3192	-0.42862	-0.048005	0.043744

$$y_{fit} = \underbrace{(0{,}71 \pm 0{,}11)}_{\pm 15\%}\,x + \underbrace{(3{,}21 \pm 0{,}58)}_{\pm 18\%}$$

$$\sigma_{fit} = 0{,}57 \longrightarrow \sigma_{fit}/\langle y \rangle \simeq 0{,}57/6{,}5 \simeq 8{,}8\%$$

$$Y_{exp} - Y_{fit} \longrightarrow -0.86 \;\text{—}\; +0{,}71$$

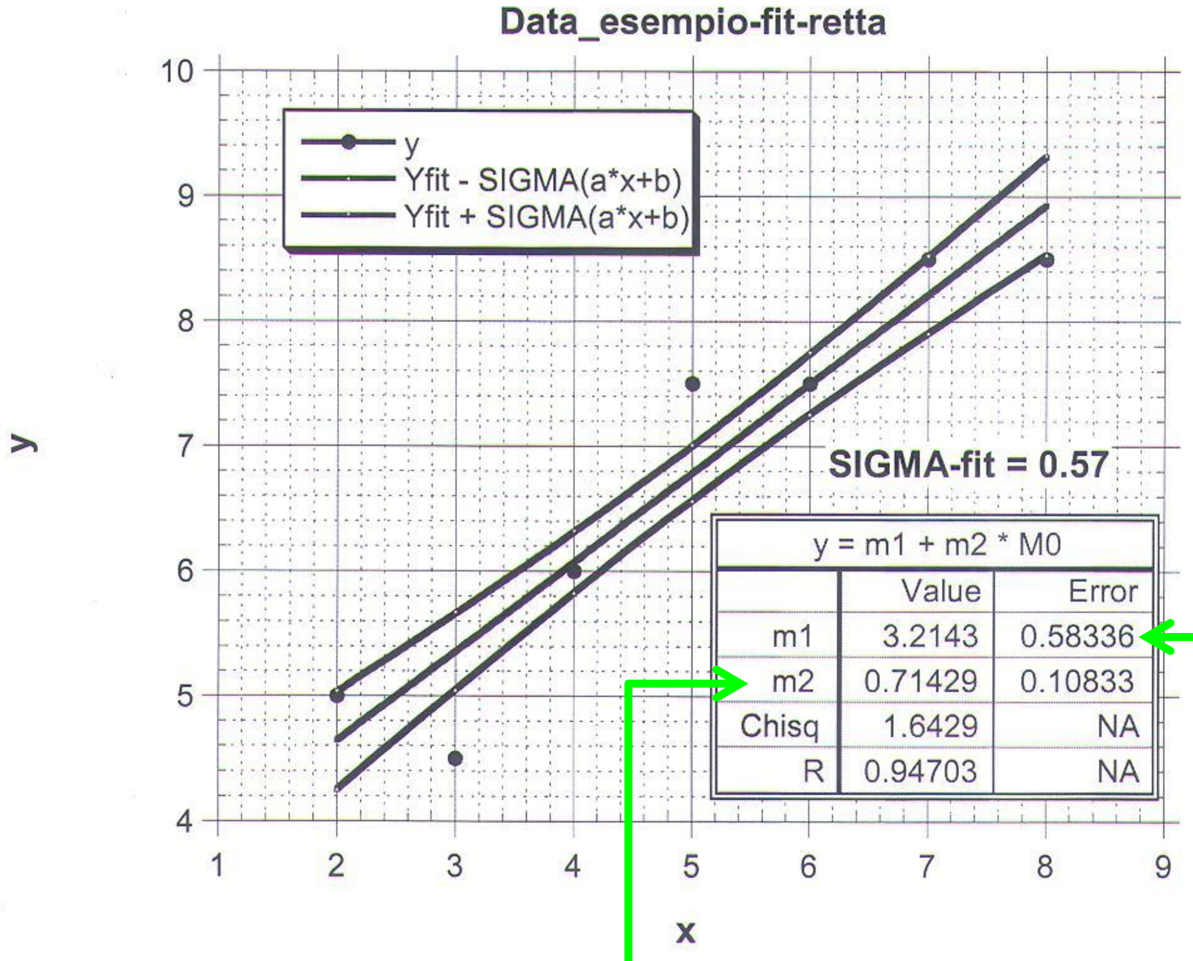
$$\langle\rangle = 0.00 \;;\; \sigma_{\langle\rangle} = 0{,}20$$

$$\frac{Y_{exp} - Y_{fit}}{Y_{fit}} \longrightarrow -16\% \;\text{—}\; +10\%$$

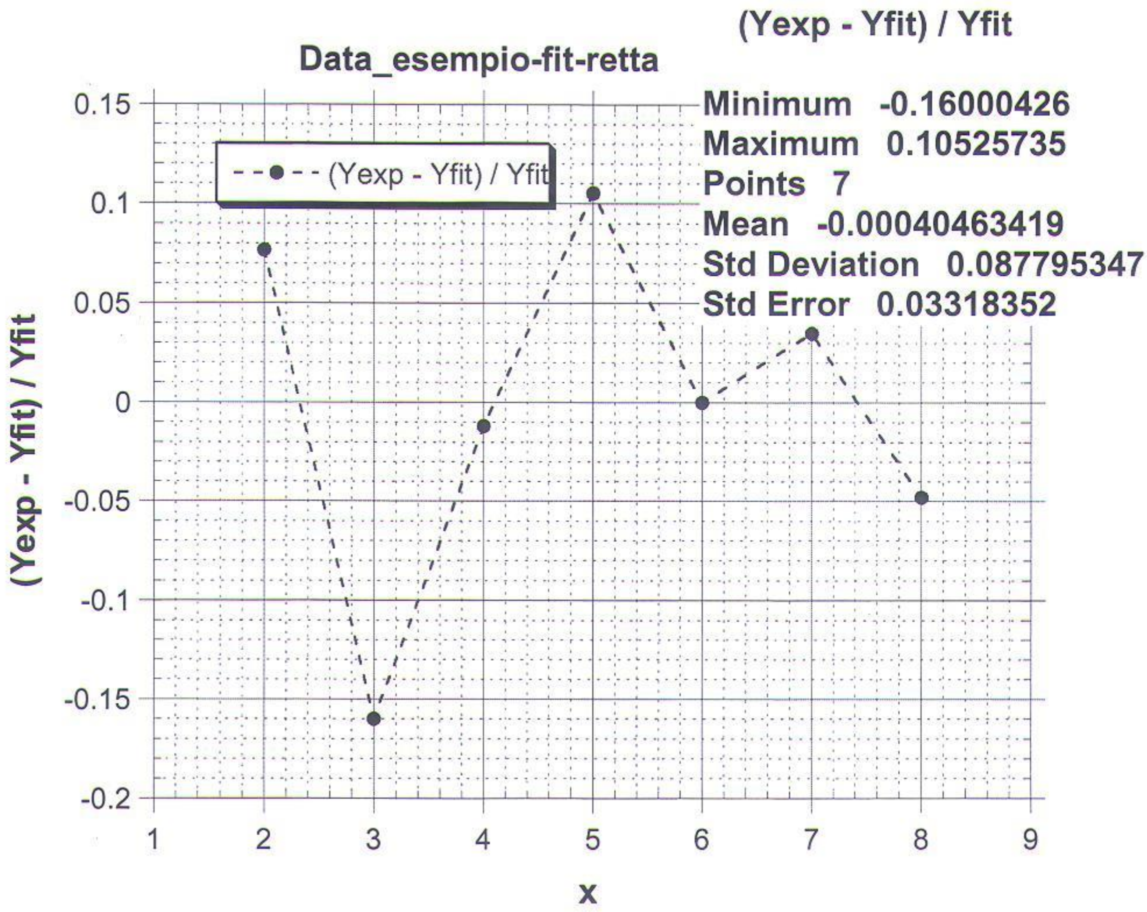
$$\langle\rangle = 0.000 \;;\; \sigma_{\langle\rangle} = 0.033$$

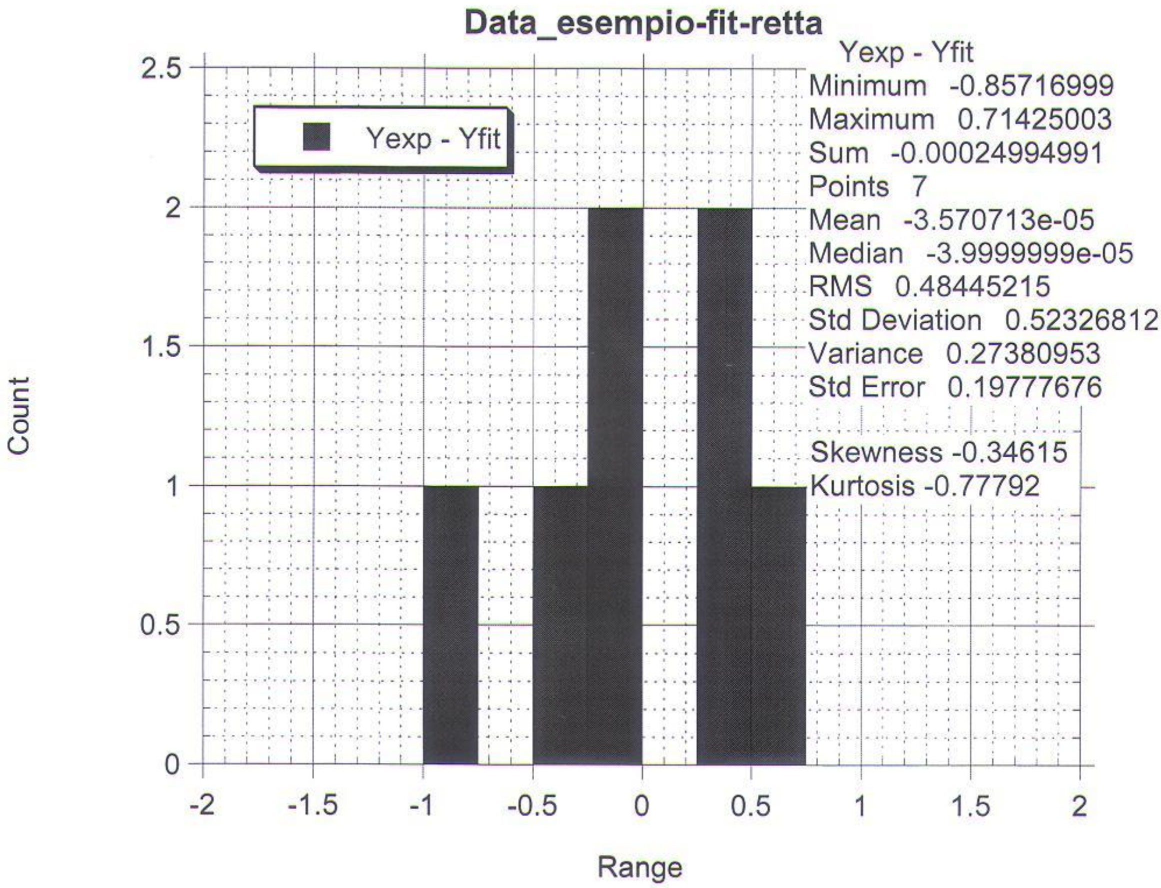
$$\frac{\sigma(ax+b)}{Y_{fit}} \longrightarrow 3.2\% \;\text{—}\; 8.4\%$$

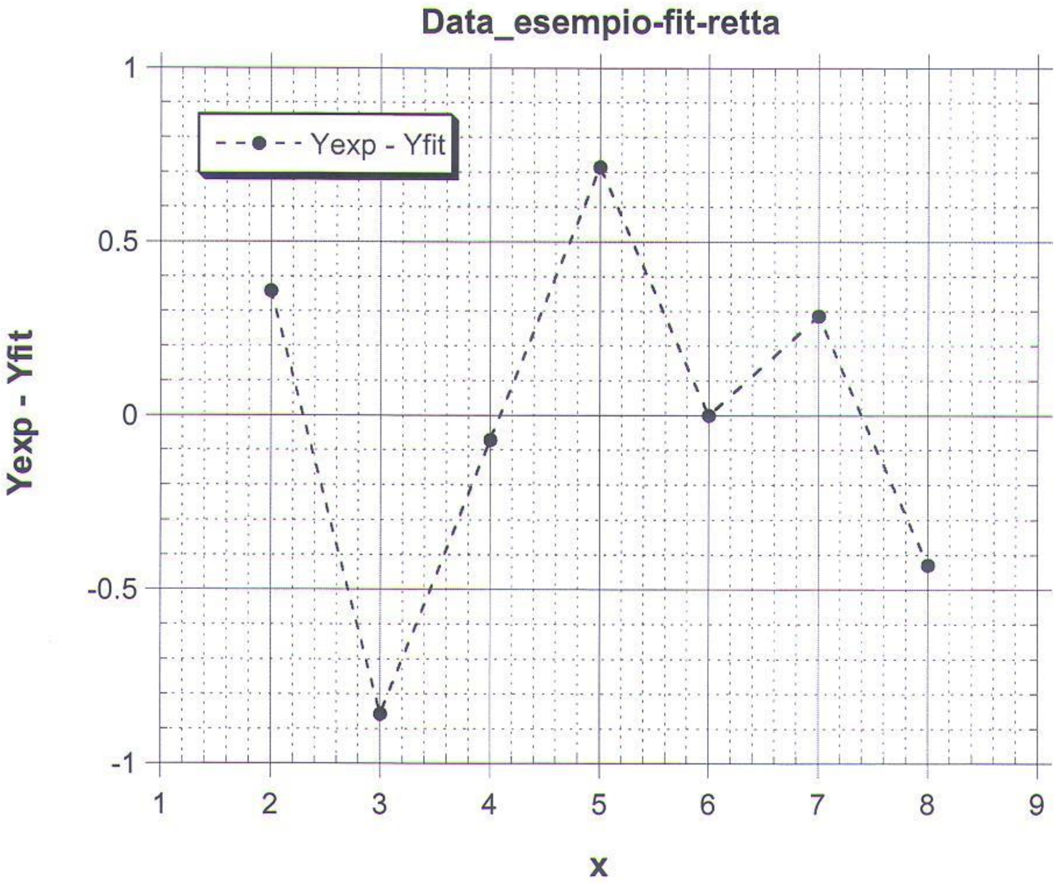
$$\langle\rangle = 0.0497 \;;\; \sigma_{\langle\rangle} = 0.0070 \longrightarrow \frac{\sigma_{\langle\rangle}}{\langle\rangle} = 14\%$$


Data_esempio-fit-retta
y
Yfit - SIGMA(a*x+b)
Yfit + SIGMA(a*x+b)
SIGMA-fit = 0.57
y = m1 + m2 * M0
Value Error
m1 3.2143 0.58336
m2 0.71429 0.10833
Chisq 1.6429 NA
R 0.94703 NA
x
y

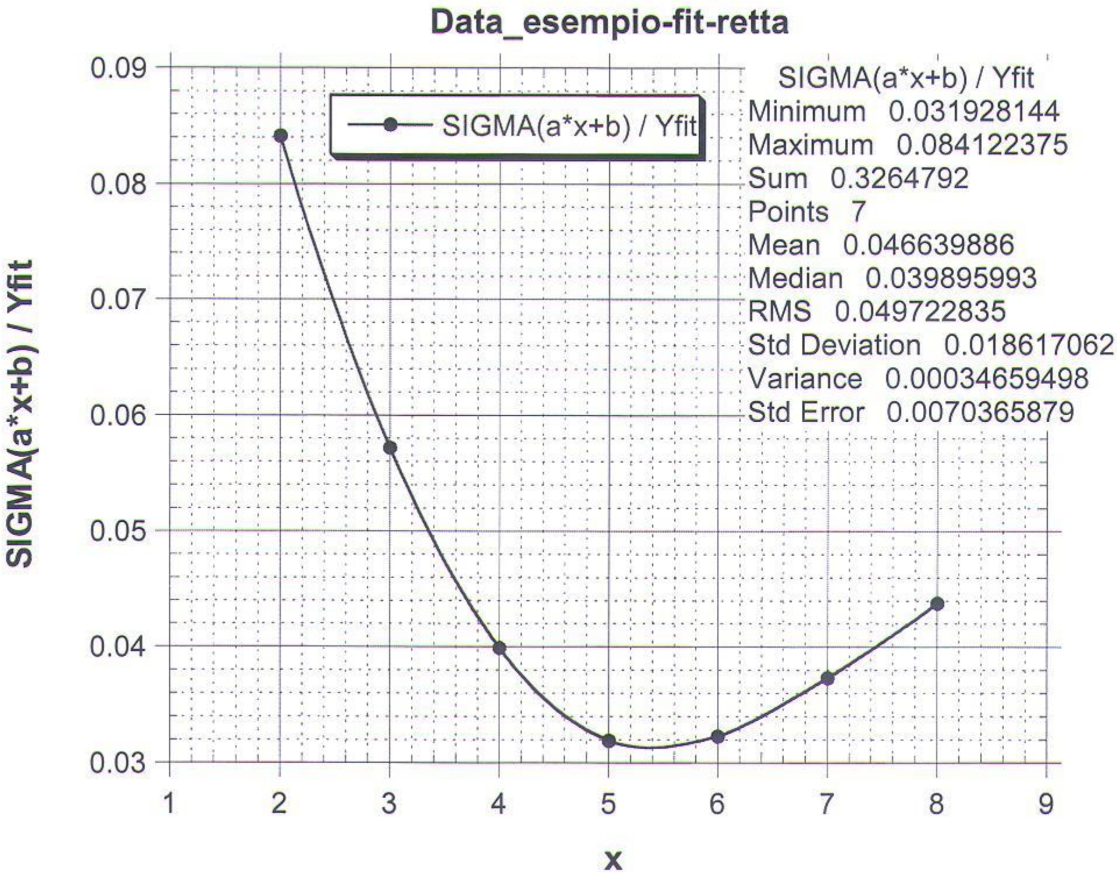
$$Y = aX + b$$

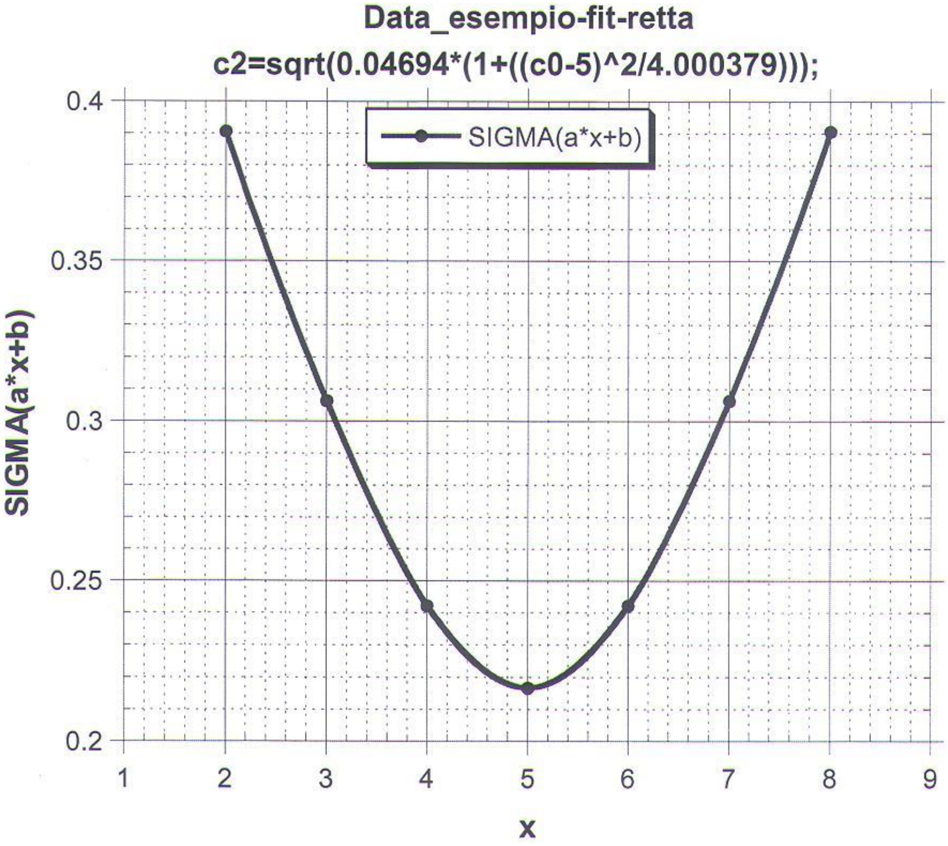

(Yexp - Yfit) / Yfit
Data_esempio-fit-retta
Minimum -0.16000426
Maximum 0.10525735
Points 7
Mean -0.00040463419
Std Deviation 0.087795347
Std Error 0.03318352
(Yexp - Yfit) / Yfit
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
1
2
3
4
5
6
7
8
9
x
(Yexp - Yfit) / Yfit


Data_esempio-fit-retta
Yexp - Yfit
Minimum -0.85716999
Maximum 0.71425003
Sum -0.00024994991
Points 7
Mean -3.570713e-05
Median -3.9999999e-05
RMS 0.48445215
Std Deviation 0.52326812
Variance 0.27380953
Std Error 0.19777676
Skewness -0.34615
Kurtosis -0.77792
Yexp - Yfit
Count
Range
2.5
2
1.5
1
0.5
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2



Data_esempio-fit-retta
Yexp - Yfit
1
0.5
0
-0.5
-1
1
2
3
4
5
6
7
8
9
x
Yexp - Yfit


Data_esempio-fit-retta
SIGMA(a*x+b) / Yfit
SIGMA(a*x+b) / Yfit
Minimum 0.031928144
Maximum 0.084122375
Sum 0.3264792
Points 7
Mean 0.046639886
Median 0.039895993
RMS 0.049722835
Std Deviation 0.018617062
Variance 0.00034659498
Std Error 0.0070365879
SIGMA(a*x+b) / Yfit
x
0.09
0.08
0.07
0.06
0.05
0.04
0.03
1
2
3
4
5
6
7
8
9



$$\sigma(ax+b) = \sqrt{\frac{\sigma_y^2}{N}\left(1 + \frac{(x - \langle x \rangle)^2}{\sigma^2(x)}\right)}$$

- Applicazione del metodo dei m.q. al caso di una dipendenza funzionale di tipo non lineare

→ $\boxed{h(t) = h(0) + \frac{1}{2} g t^2}$ ho N coppie misurate (t_i, h_i)

lo scopo finale può essere quello di avere una stima dell'accelerazione di gravità g usando il metodo visto del "fit" lineare

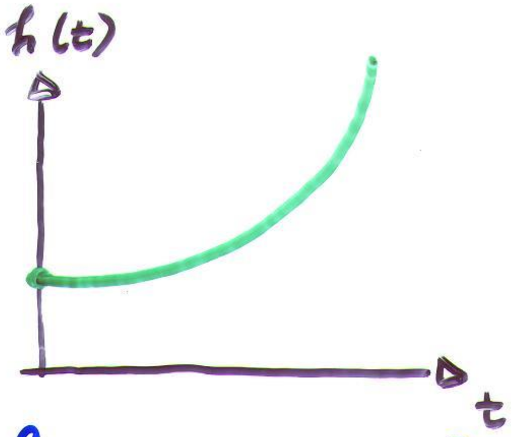
$$y = ax + b \qquad \begin{cases} h(t) \longrightarrow y(x) \\ t^2 \longrightarrow x \\ \frac{1}{2} g \longrightarrow a \\ h(0) \longrightarrow b \end{cases}$$

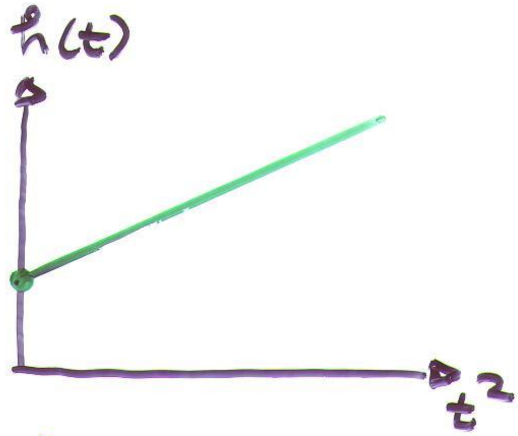
$$\begin{cases} a = \dfrac{N \sum xy - \sum x \sum y}{N \sum x^2 - (\sum x)^2} \longrightarrow \dfrac{1}{2} g = \dfrac{N \sum t^2 h - \sum t^2 \sum h}{N \sum t^4 - (\sum t^2)^2} \\[2ex] b = \dfrac{\sum x^2 \sum y - \sum x \sum xy}{N \sum x^2 - (\sum x)^2} \longrightarrow h(0) = \dfrac{\sum t^4 \sum h - \sum t^2 \sum t^2 h}{N \sum t^4 - (\sum t^2)^2} \end{cases}$$

$$\begin{cases} \sigma(a) = \sigma(y) \cdot \sqrt{\dfrac{N}{N \sum x^2 - (\sum x)^2}} \longrightarrow \sigma\left(\frac{1}{2} g\right) = \sigma(h) \sqrt{\dfrac{N}{N \sum t^4 - (\sum t^2)^2}} \\[2ex] \sigma(b) = \sigma(y) \cdot \sqrt{\dfrac{\sum x^2}{N \sum x^2 - (\sum x)^2}} \longrightarrow \sigma\big(h(0)\big) = \sigma(h) \sqrt{\dfrac{\sum t^4}{N \sum t^4 - (\sum t^2)^2}} \end{cases}$$

$$\sigma^2[f(x)] = \left(\frac{d f(x)}{d x}\right)^2 \sigma^2(x) \longrightarrow \sigma^2\left(\frac{1}{2} g\right) = \left(\frac{1}{2}\right)^2 \sigma^2(g)$$

$$\longrightarrow \boxed{2\left(\frac{1}{2} g\right) \pm 2\sigma\left(\frac{1}{2} g\right) = g \pm \sigma(g)}$$





$$h(t) = h(0) + \frac{1}{2} g t^2 \qquad h(t) = h(0) + \frac{1}{2} g (t^2)$$

$h(0) \pm \sigma[h(0)]$

$$h(0) = \frac{\Sigma t^4 \Sigma h - \Sigma t^2 \Sigma t^2 h}{N \Sigma t^4 - (\Sigma t^2)^2}$$

$$\sigma[h(0)] = \sigma(h) \sqrt{\frac{\Sigma t^4}{N t^4 - (\Sigma t^2)^2}}$$

$g \pm \sigma(g)$

$$g = 2 \cdot \left(\frac{1}{2} g\right) = 2 \cdot \frac{N \Sigma t^2 h - \Sigma t^2 \Sigma h}{N \Sigma t^4 - (\Sigma t^2)^2}$$

$$\sigma(g) = 2 \cdot \sigma\left(\frac{1}{2} g\right) = 2 \cdot \sigma(h) \sqrt{\frac{N}{N \Sigma t^4 - (\Sigma t^2)^2}}$$

$$\boxed{A(t) = A(0)\, e^{-t/\tau}}$$

ho N coppie misurate (t_i, A_i)

$$\ln A = \ln A(0) - \frac{t}{\tau} \longrightarrow y = ax + b$$

$\ln A$	y
$\ln A(0)$	b
t	x
$-\frac{1}{\tau}$	a

$$\begin{cases} a = \dfrac{N\sum xy - \sum x \sum y}{N\sum x^2 - (\sum x)^2} \longrightarrow -\dfrac{1}{\tau} = \dfrac{N\sum t\cdot \ln A - \sum t \sum \ln A}{N\sum t^2 - (\sum t)^2} \\[2ex] b = \dfrac{\sum x^2 \sum y - \sum x \sum xy}{N\sum x^2 - (\sum x)^2} \longrightarrow \ln A(0) = \dfrac{\sum t^2 \sum \ln A - \sum t \sum t \ln A}{N\sum t^2 - (\sum t)^2} \end{cases}$$

$$\begin{cases} \sigma(a) = \sigma(y)\sqrt{\dfrac{N}{N\sum x^2 - (\sum x)^2}} \longrightarrow \sigma\left(-\dfrac{1}{\tau}\right) = \sqrt{\dfrac{N}{N\sum t^2 - (\sum t)^2}}\,\sigma(\ln A) \\[2ex] \sigma(b) = \sigma(y)\sqrt{\dfrac{\sum x^2}{N\sum x^2 - (\sum x)^2}} \longrightarrow \sigma(\ln A(0)) = \sqrt{\dfrac{\sum t^2}{N\sum t^2 - (\sum t)^2}}\,\sigma(\ln A) \end{cases}$$

$$\sigma^2\left(-\frac{1}{\tau}\right) = \left(\frac{d}{d\tau}\left(-\frac{1}{\tau}\right)\right)^2 \sigma^2(\tau) = \frac{\sigma^2(\tau)}{\tau^4} \longrightarrow \sigma(\tau) = \tau^2 \cdot \sigma\left(-\frac{1}{\tau}\right)$$

$$\sigma^2(\ln A(0)) = \left(\frac{d}{dA(0)} \ln A(0)\right)^2 \sigma^2(A(0)) = \frac{\sigma^2(A(0))}{(A(0))^2} \longrightarrow \sigma(A(0)) = A(0)\cdot\sigma(\ln A(0))$$

idem per $\sigma(\ln A)$

$A(0) = \exp[\ln A(0)]$	$\tau = -1/(-1/\tau)$
$\sigma(A(0)) = A(0)\cdot\sigma(\ln A(0))$	$\sigma(\tau) = \tau^2 \cdot \sigma\left(-\frac{1}{\tau}\right)$

Esempi di <u>variazione nel tempo di tipo esponenziale</u>

→ popolazione di batteri

→ numero di nuclei radioattivi

$$\boxed{N(t) = N(0)\,e^{-t/\tau}} \quad \rightarrow \quad \ln N(t) = \ln N(0) - \frac{1}{\tau}\cdot t$$

→ $\tau \equiv$ vita media della popolazione

$$N(\tau) = N(0)/e \cong N(0)/2{,}718$$
$$= 0{,}368 \cdot N(0)$$

→ $t_{1/2} \equiv$ tempo di dimezzamento

$$N(t_{1/2}) = N(0)/2 = N(0)\,e^{-\frac{t_{1/2}}{\tau}}$$

$$\ln 1/2 = - t_{1/2}/\tau \qquad : \quad t_{1/2} = 0{,}693 \cdot \tau$$

i	t_i	t_i^2	$N(t_i)$	$\ln N(t_i)$	$t_i \cdot \ln N(t_i)$	$\sigma(N(t_i))$ $\sqrt{N(t_i)}$
1	0	0	153'000	11.938	0	391,152
2	1	1	137'000	11.828	11.828	370,135
3	2	4	128'000	11.760	23.520	357,771
	3	5		35.526	35.348	
	Σt	Σt^2		$\Sigma \ln N$	$\Sigma\, t \cdot \ln N$	

$$\boxed{\underbrace{y}_{\ln[N(t)]}(x) = \underbrace{a}_{-\frac{1}{\tau}}\,\underbrace{x}_{t} + \underbrace{b}_{\ln[N(0)]}}$$

$$\sigma(\ln N(t)) = \frac{d \ln N(t)}{dN}\,\sigma(N) = \frac{\sigma(N)}{N} = \frac{1}{\sqrt{N}}$$

i	$\sigma(\ln N(t_i))$
1	2.557×10^{-3}
2	2.702 "
3	2.795 "

$\langle \sigma \rangle \simeq 2{,}68 \times 10^{-3}$

$\langle \sigma \rangle \simeq 0.0027$ ← $\simeq$ costante per tutte le misure .....

$$\Delta = N \cdot \left(\sum_i t^2\right) - \left(\sum_i t\right)^2 = 3 \cdot 5 - (3)^2 = 6 \quad \text{giorni}^2$$

$$a = \left(-\frac{1}{\tau}\right) = \frac{N \cdot \sum_i \left(t \cdot \ln N(t)\right) - \left(\sum t\right)\left(\sum \ln N(t)\right)}{\Delta} =$$

$$= \frac{3 \cdot (35{,}348) - (3) \cdot (35{,}526)}{6} = -0{.}089 \ \text{giorni}^{-1}$$

$$\hookrightarrow \tau = 11{,}236 \ \text{giorni}$$

$$b = \ln[N(0)] = \frac{\left(\sum t^2\right)\left(\sum \ln N(t)\right) - \left(\sum t\right)\left(\sum t \ln N(t)\right)}{\Delta} =$$

$$= \frac{5 \cdot (35{,}526) - 3 \cdot (35{,}348)}{6} = 11{,}931$$

$$\hookrightarrow N(0) = e^{11{,}931} = 151\,903{,}389$$

$\sigma(a)$

$$\sigma\left(-\frac{1}{\tau}\right) = \sqrt{\frac{N}{\Delta}} \cdot \sigma\left(\ln(N(t))\right) = \sqrt{\frac{3}{6}} \cdot 2{,}68 \times 10^{-3} = 0{.}0019 \quad \text{giorni}^{-1}$$

$$\hookrightarrow \sigma(\tau) = \tau^2 \cdot \sigma\left(-\frac{1}{\tau}\right) = (11{,}236)^2 \cdot (0{,}0019) = 0{,}2397$$

da $a \pm \sigma(a)$ ..... ➡ $\boxed{\tau \pm \sigma(\tau) = (11{,}24 \pm 0{,}24) \ \text{giorni}}$

$$\frac{\sigma(\tau)}{\tau} = \frac{0{,}24}{11{,}24} \simeq 2\%$$

$\sigma(b)$

$$\sigma\left(\ln N(0)\right) = \sqrt{\frac{\sum t^2}{\Delta}} \cdot \sigma\left(\ln N\right) = \sqrt{\frac{5}{6}} \cdot 2{,}68 \times 10^{-3} = 2{,}446 \times 10^{-3}$$

$$\hookrightarrow \sigma(N(0)) = N(0) \cdot \sigma\left(\ln N(0)\right) = 151\,903{,}389 \cdot 2{,}446 \times 10^{-3} \simeq 372$$
$$= 370$$

da $b \pm \sigma(b)$ .... ➡ $\boxed{N(0) \pm \sigma(N(0)) = 151\,900 \pm 370}$

$$\frac{\sigma(N(0))}{N(0)} = \frac{370}{151\,900} \simeq 0{,}2\%$$

→ provando un fit lineare: $N(t) = a \cdot t + b$

$a \pm \sigma(a) : (12{,}5 \pm 2.0) \times 10^3$

$\sigma(a)/a : \quad 16\%$

$b \pm \sigma(b) : (151{,}8 \pm 2.6) \times 10^3$

$\sigma(b)/b : \quad 1.7\%$

$\sigma_{fit} : \quad 2.9 \times 10^3$

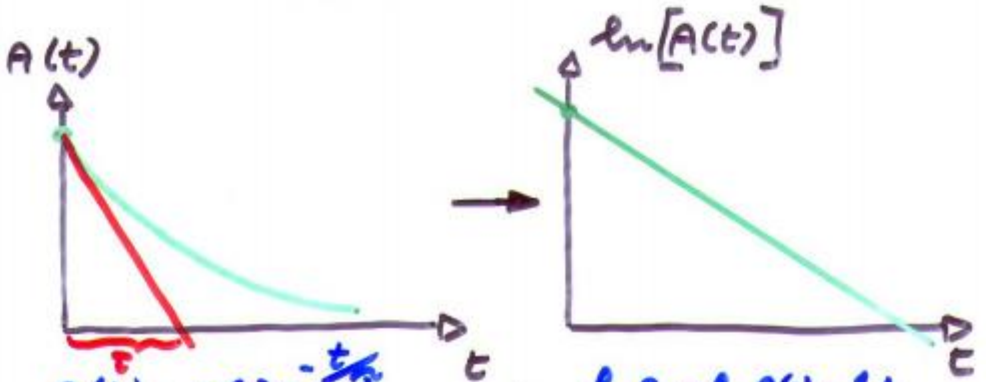
...... rimane sempre il problema che sono solo 3 punti!

→ confronto i due fit per $N(0)$:

$N(t) = N(0)\, e^{-t/\tau} \quad \rightarrow \quad N(0) = (151{,}90 \pm 0.37) \times 10^3$

$N(t) = a \cdot t + b \quad \rightarrow \quad N(0) = b = (151.8 \pm 2.6) \times 10^3$

$$\sqrt{\sigma^2(ax+b)} = \sqrt{\frac{\sigma_y^2}{N} \cdot \left(1 + \frac{(x - \langle x \rangle)^2}{\sigma^2(x)}\right)} \underset{@\,x=0}{\simeq} \sqrt{\frac{(2.9 \times 10^3)^2}{3} \cdot \frac{1}{0.667}} = 2{,}6 \times 10^3$$



$$A(t) = A(0)e^{-t/\tau} \longrightarrow \ln A = \ln A(0) - \frac{1}{\tau}t$$

$A(0) \pm \sigma[A(0)]$

$$A(0) = \exp\left[\ln A(0)\right] = \exp\left[\frac{\Sigma t^2 \Sigma \ln A - \Sigma t \Sigma t \ln A}{N\Sigma t^2 - (\Sigma t)^2}\right]$$

$$\sigma[A(0)] = A(0)\cdot\sigma\left(\ln[A(0)]\right) = \exp\left[\frac{\Sigma t^2 \Sigma \ln A - \Sigma t \Sigma t \ln A}{N\Sigma t^2 - (\Sigma t)^2}\right]\cdot\frac{\sigma(\ln A)\sqrt{\Sigma t^2}}{\sqrt{N\Sigma t^2 - (\Sigma t)^2}}$$

$\tau \pm \sigma(\tau)$

$$\tau = -1/(-1/\tau) = -1\Big/\left[\frac{N\Sigma t \Sigma \ln A - \Sigma t \Sigma \ln A}{N\Sigma t^2 - (\Sigma t)^2}\right]$$

$$\sigma(\tau) = \tau^2\cdot\sigma(-1/\tau) = \left(\frac{N\Sigma t^2 - (\Sigma t)^2}{N\Sigma t \Sigma \ln A - \Sigma t \Sigma \ln A}\right)^2\cdot\frac{\sigma(\ln A)\sqrt{N}}{\sqrt{N\Sigma t^2 - (\Sigma t)^2}}$$