Note su “Propagazione Quadratica” dell’incertezza

Giustificazione della "somma in quadratura" degli "errori".

x, y, z, … misurate direttamente, indipendenti ed affette solo da errori casuali, stimati mediante misure ripetute *n* volte.

g = g(x, y, z, …) … funzione delle grandezze *x*, *y*, *z*, …

$\Delta x_i = (x_i - \langle x \rangle)$ … scarto *i* - esimo

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \langle x \rangle)^2}{N-1}} = \sqrt{\frac{\sum_{i=1}^{N}(\Delta x_i)^2}{N-1}}, \quad \ldots \textit{idem per le altre } \text{variabili}$$

$$\Rightarrow \sigma_g = \sqrt{\frac{\sum_{i=1}^{N}(g_i - \langle g \rangle)^2}{N-1}} = \sqrt{\frac{\sum_{i=1}^{N}(\Delta g_i)^2}{N-1}} = ???$$

$$\delta g \approx \left|\frac{\partial g}{\partial x}\right|_{(\bar{x}, \bar{y}, \bar{z}, \ldots)} \delta x + \left|\frac{\partial g}{\partial y}\right|_{(\bar{x}, \bar{y}, \bar{z}, \ldots)} \delta y + \left|\frac{\partial g}{\partial z}\right|_{(\bar{x}, \bar{y}, \bar{z}, \ldots)} \delta z$$

$$(\Delta g_i)^2 \approx \left(\left|\frac{\partial g}{\partial x}\right|_{(\bar{x}, \bar{y}, \bar{z}, \dots)} \Delta x_i + \left|\frac{\partial g}{\partial y}\right|_{(\bar{x}, \bar{y}, \bar{z}, \dots)} \Delta y_i + \left|\frac{\partial g}{\partial z}\right|_{(\bar{x}, \bar{y}, \bar{z}, \dots)} \Delta z_i + \dots \right)^2$$

$(\sigma_g)^2 = \dots$

$$\frac{1}{N-1}\sum_{i=1}^{N}(\Delta g_i)^2 \approx \frac{1}{N-1}\left[\sum_{i=1}^{N}\left[\left(\frac{\partial g}{\partial x}\right)^2_{(\bar{x}, \bar{y}, \bar{z}, \dots)}(\Delta x_i)^2 + \left(\frac{\partial g}{\partial y}\right)^2_{(\bar{x}, \bar{y}, \bar{z}, \dots)}(\Delta y_i)^2 + \left(\frac{\partial g}{\partial z}\right)^2_{(\bar{x}, \bar{y}, \bar{z}, \dots)}(\Delta z_i)^2 + \dots\right]\right]$$

$$\sigma_g^2 \approx \left(\frac{\partial g}{\partial x}\right)^2_{(\bar{x}, \bar{y}, \bar{z}, \dots)} \sigma_x^2 + \left(\frac{\partial g}{\partial y}\right)^2_{(\bar{x}, \bar{y}, \bar{z}, \dots)} \sigma_y^2 + \left(\frac{\partial g}{\partial z}\right)^2_{(\bar{x}, \bar{y}, \bar{z}, \dots)} \sigma_z^2$$

… le grandezze misurate direttamente (x, y, z, ...) devono essere non solo casuali… ma devono anche essere indipendenti tra di loro… altrimenti bisogna tenere conto anche del termine di "covarianza"

$$[\sigma(g)]^2 \approx \frac{\sum_{i=1}^{N}(g(x_i, y_i) - g(\overline{x}, \overline{y}))^2}{N-1} \approx \frac{\sum_{i=1}^{N}[(\frac{\partial g}{\partial x})_{\overline{x},\overline{y}}(x_i - \overline{x}) + (\frac{\partial g}{\partial y})_{\overline{x},\overline{y}}(y_i - \overline{y}) + \ldots]^2}{N-1} =$$

$$= \frac{\sum_{i=1}^{N}[(\frac{\partial g}{\partial x})_{\overline{x},\overline{y}}(\Delta x_i) + (\frac{\partial g}{\partial y})_{\overline{x},\overline{y}}(\Delta y_i)]^2}{N-1} =$$

$$= \frac{\sum_{i=1}^{N}[(\frac{\partial g}{\partial x})_{\overline{x},\overline{y}}(\Delta x_i)]^2 + \sum_{i=1}^{N}[(\frac{\partial g}{\partial y})_{\overline{x},\overline{y}}(\Delta y_i)]^2 + 2\sum_{i=1}^{N}[(\frac{\partial g}{\partial x})_{\overline{x},\overline{y}}(\Delta x_i)][(\frac{\partial g}{\partial y})_{\overline{x},\overline{y}}(\Delta y_i)]}{N-1} =$$

$$= \frac{\sum_{i=1}^{N}[(\frac{\partial g}{\partial x})_{\overline{x},\overline{y}}(\Delta x_i)]^2}{N-1} + \frac{\sum_{i=1}^{N}[(\frac{\partial g}{\partial y})_{\overline{x},\overline{y}}(\Delta y_i)]^2}{N-1} + 2\frac{\sum_{i=1}^{N}[(\frac{\partial g}{\partial x})_{\overline{x},\overline{y}}(\Delta x_i)][(\frac{\partial g}{\partial y})_{\overline{x},\overline{y}}(\Delta y_i)]}{N-1}$$

$$\Rightarrow [\sigma(g)]^2 = (\frac{\partial g}{\partial x})^2_{\overline{x},\overline{y}}[\sigma(x)]^2 + (\frac{\partial g}{\partial y})^2_{\overline{x},\overline{y}}[\sigma(y)]^2 + 2(\frac{\partial g}{\partial x})_{\overline{x},\overline{y}}(\frac{\partial g}{\partial y})_{\overline{x},\overline{y}}[\sigma(x, y)]$$

$$\mathrm{cov}(x, y) \equiv \sigma(x, y) \equiv \frac{\sum_{i=1}^{N}[(x_i - \overline{x})][(y_i - \overline{y})]}{N-1} = \frac{N}{N-1}\frac{\sum_{i=1}^{N}[(x_i y_i - x_i\overline{y} - y_i\overline{x} + \overline{x}\,\overline{y})]}{N}$$

$$\Rightarrow [\sigma(x, y)] = \frac{N}{N-1}\left(\overline{xy} - \overline{x}\,\overline{y}\right)$$

Precisazione sulla frase "somma in quadratura".

La **Varianza** della GF derivata **$g = g(x,y)$**
e' percio' la somma delle varianze delle
GF indipendenti x, y pesate ciascuna con la
derivata al quadrato della g
(**rapidità di variazione**)
rispetto alla variabile specifica.

Propagazione degli errori casuali relativi nel caso di GF g=g(x,y) con legge di potenza:

$$g(x,y) = Ax^a y^b$$

$$\Rightarrow \sigma^2{}_g \approx \left(\frac{\partial(Ax^a y^b)}{\partial x}\right)^2 \sigma^2{}_x + \left(\frac{\partial(Ax^a y^b)}{\partial y}\right)^2 \sigma^2{}_y + 2\left(\frac{\partial(Ax^a y^b)}{\partial x}\right)\left(\frac{\partial(Ax^a y^b)}{\partial y}\right)\sigma(x,y) =$$

$$= (Aax^{a-1}y^b)^2 \sigma^2{}_x + (Ax^a by^{b-1})^2 \sigma^2{}_y + 2(Aax^{a-1}y^b)(Ax^a by^{b-1})\sigma(x,y)$$

$$\Rightarrow \left(\frac{\sigma_g}{g}\right)^2 = \frac{(Aax^{a-1}y^b)^2}{(Ax^a y^b)^2}\sigma^2{}_x + \frac{(Ax^a by^{b-1})^2}{(Ax^a y^b)^2}\sigma^2{}_y + 2\frac{(Aax^{a-1}y^b)(Ax^a by^{b-1})}{(Ax^a y^b)^2}\sigma(x,y) =$$

$$\Rightarrow \left(\frac{\sigma_g}{g}\right)^2 = a^2 \frac{\sigma^2{}_x}{x^2} + b^2 \frac{\sigma^2{}_y}{y^2} + 2ab\frac{\sigma(x,y)}{xy}$$

$$E[g(x,y)] = \overline{g(x,y)} = ??$$

$$g(x,y) \approx g(\bar{x},\bar{y}) + \left(\frac{\partial g}{\partial x}\right)_{\bar{x},\bar{y}}(x-\bar{x}) + \left(\frac{\partial g}{\partial y}\right)_{\bar{x},\bar{y}}(y-\bar{y}) + \ldots$$

$$g_i = g(x_i, y_i) \approx g(\bar{x},\bar{y}) + \left(\frac{\partial g}{\partial x}\right)_{\bar{x},\bar{y}}\Delta x_i + \left(\frac{\partial g}{\partial y}\right)_{\bar{x},\bar{y}}\Delta y_i + \ldots$$

$$\Downarrow$$

$$\bar{g} = \frac{1}{N}\left[\sum_{i=1}^{N} g_i\right] \approx \frac{1}{N}\sum_{i=1}^{N}\left[g(\bar{x},\bar{y}) + \left(\frac{\partial g}{\partial x}\right)_{\bar{x},\bar{y}}\Delta x_i + \left(\frac{\partial g}{\partial y}\right)_{\bar{x},\bar{y}}\Delta y_i + \ldots\right]$$

$$\bar{g} \approx g(\bar{x},\bar{y}) + \ldots \quad \textit{poichè} \quad \sum_{i=1}^{N}\Delta x_i = 0 \quad , \quad \sum_{i=1}^{N}\Delta y_i = 0$$

OSSERVAZIONI:

… sviluppo in serie di **Taylor** della funzione *g* **troncato al primo ordine**, per valori delle variabili intorno ai rispettivi valori medi.

… pertanto i **risultati** che si ottengono saranno **validi**

- nel limite in cui è corretto avere trascurato termini di ordine superiore al primo,
- perché effettivamente la funzione *g* è lineare nelle variabili.

Generalizzando, *g* dovrebbe essere lineare o linearizzabile in un intervallo delle variabili dell'ordine delle relative deviazioni standard. Altrimenti, … il **valor medio di una funzione** potrà essere anche molto **diverso** dal valore della funzione calcolato in corrispondenza del valor medio dell'argomento.

$$g(x,y) \approx g(\bar{x},\bar{y}) + \left(\frac{\partial g}{\partial x}\right)_{\bar{x},\bar{y}}(\Delta x) + \left(\frac{\partial g}{\partial y}\right)_{\bar{x},\bar{y}}(\Delta y) + \frac{1}{2}\left(\frac{\partial^2 g}{\partial x^2}\right)_{\bar{x},\bar{y}}(\Delta x)^2 + \frac{1}{2}\left(\frac{\partial^2 g}{\partial y^2}\right)_{\bar{x},\bar{y}}(\Delta y)^2 + \ldots$$

$$\Rightarrow \quad \bar{g} = \frac{1}{N}\left[\sum_{i=1}^{N} g_i\right] \approx$$

$$\approx \frac{1}{N}\sum_{i=1}^{N}\left[g(\bar{x},\bar{y}) + \left(\frac{\partial g}{\partial x}\right)_{\bar{x},\bar{y}}\Delta x_i + \left(\frac{\partial g}{\partial y}\right)_{\bar{x},\bar{y}}\Delta y_i + \frac{1}{2}\left(\frac{\partial^2 g}{\partial x^2}\right)_{\bar{x},\bar{y}}(\Delta x)^2 + \frac{1}{2}\left(\frac{\partial^2 g}{\partial y^2}\right)_{\bar{x},\bar{y}}(\Delta y)^2 + \ldots\right]$$

$$\bar{g} \approx g(\bar{x},\bar{y}) + \frac{1}{2}\frac{N-1}{N}\left(\frac{\partial^2 g(x,y)}{\partial x^2}\right)_{\bar{x},\bar{y}}(\sigma_x)^2 + \frac{1}{2}\frac{N-1}{N}\left(\frac{\partial^2 g(x,y)}{\partial y^2}\right)_{\bar{x},\bar{y}}(\sigma_y)^2$$

$$\Rightarrow \quad \bar{g} - g(\bar{x},\bar{y}) \approx \frac{1}{2}\frac{N-1}{N}\left(\frac{\partial^2 g(x,y)}{\partial x^2}\right)_{\bar{x},\bar{y}}(\sigma_x)^2 + \frac{1}{2}\frac{N-1}{N}\left(\frac{\partial^2 g(x,y)}{\partial y^2}\right)_{\bar{x},\bar{y}}(\sigma_y)^2 \quad !!!$$

… sistematica apparente !!!

… sistematica apparente: generalizzazione

$$g = x^{\alpha} \cdot y^{\beta}$$

$$\frac{\partial g}{\partial x} = \alpha x^{\alpha-1} y^{\beta} \rightarrow \frac{\partial^2 g}{\partial x^2} = \alpha(\alpha-1)\, x^{\alpha-2} y^{\beta}$$

$$\frac{\partial g}{\partial y} = x^{\alpha} \beta y^{\beta-1} \rightarrow \frac{\partial^2 g}{\partial y^2} = x^{\alpha} \beta(\beta-1)\, y^{\beta-2}$$

$$\bar{g} - g(\bar{x},\bar{y}) = \frac{1}{2}\frac{N-1}{N}\left(\alpha(\alpha-1)\, x^{\alpha-2} y^{\beta} \sigma_x^2 + \beta(\beta-1)\, x^{\alpha} y^{\beta-2} \sigma^2_y\right)$$

$g = x \cdot y$ per $\alpha = 1$ & $\beta = 1$ $\rightarrow \bar{g} - g(\bar{x},\bar{y}) = 0$ ✓

$g = \frac{x}{y}$ per $\alpha = 1$ & $\beta = -1$ $\rightarrow \bar{g} - g(\bar{x},\bar{y}) = \frac{1}{2}\frac{N-1}{N}\left(0 + (-1)(-2)\, x^{1} y^{-3} \sigma^2_y\right) =$

$$= \frac{1}{2}\frac{N-1}{N} \cdot \frac{2x}{y^3}\sigma^2_y$$ ✓

Supponiamo di avere effettuato su di un corpo N misurazioni, sia della sua massa M_i che del suo volume V_i. Si possono così calcolare N valori della sua densità ρ_i dai rapporti M_i/V_i.

$M_1, M_2, \ldots, M_i, \ldots, M_N$ ➔ $\langle M \rangle = \Sigma Mi/N$

$V_1, V_2, \ldots, V_i, \ldots, V_N$ ➔ $\langle V \rangle = \Sigma Vi/N$

$\ldots, \rho_i = M_i/V_i, \ldots$ ➔ $\langle \rho \rangle = \Sigma \rho i/N$

➔ **$E(\rho) = \langle \rho \rangle = ??$**

➔ **$E[(\rho - E(\rho))^2]$ = varianza(ρ) = ??**

➔ E(ρ)

$$\Rightarrow \quad \bar{g} - g(\bar{x},\bar{y}) \approx \frac{1}{2}\frac{N-1}{N}\left(\frac{\partial^2 g(x,y)}{\partial x^2}\right)_{\bar{x},\bar{y}}(\sigma_x)^2 + \frac{1}{2}\frac{N-1}{N}\left(\frac{\partial^2 g(x,y)}{\partial y^2}\right)_{\bar{x},\bar{y}}(\sigma_y)^2$$

$$\rho = \frac{M}{V}$$

$$\frac{\partial(M/V)}{\partial M} = \frac{1}{V} \quad \Rightarrow \frac{\partial^2(M/V)}{\partial M^2} = 0 \quad ; \quad \frac{\partial(M/V)}{\partial V} = \frac{-M}{V^2} \quad \Rightarrow \frac{\partial^2(M/V)}{\partial V^2} = 2\frac{M}{V^3}$$

$$\Rightarrow \quad \bar{\rho} = \frac{\overline{M}}{\overline{\overline{V}}} + \frac{1}{2}\frac{N-1}{N}\left(2\frac{\overline{M}}{\overline{\overline{V}}^3}\right)(\sigma_V)^2 = \frac{\overline{M}}{\overline{\overline{V}}}\left(1+\frac{N-1}{N}\left(\frac{\sigma_V}{\overline{\overline{V}}}\right)^2\right) \approx \frac{\overline{M}}{\overline{\overline{V}}} \quad se \quad \frac{\sigma_V}{\overline{\overline{V}}} << 1$$

$$\Rightarrow \text{ ...numericamente: } \frac{\overline{M}}{\overline{\overline{V}}} \approx 2.70 \ g/cm^3 , \frac{\sigma_V}{\overline{\overline{V}}} \approx \frac{1\, cm^3}{100\, cm^3} \approx 10^{-2} , \frac{N-1}{N} \approx 1$$

$$\Rightarrow \bar{\rho} - \frac{\overline{M}}{\overline{\overline{V}}} \approx 0.00027 \ g/cm^3$$

➔ $E[(\rho-E(\rho))^2]$

$$\Rightarrow [\sigma(g)]^2 = \left(\frac{\partial g}{\partial x}\right)^2_{\bar{x},\bar{y}}[\sigma(x)]^2 + \left(\frac{\partial g}{\partial y}\right)^2_{\bar{x},\bar{y}}[\sigma(y)]^2 + 2\left(\frac{\partial g}{\partial x}\right)_{\bar{x},\bar{y}}\left(\frac{\partial g}{\partial y}\right)_{\bar{x},\bar{y}}[\sigma(x,y)]$$

$$\rho = \frac{M}{V}$$

$$\frac{\partial(M/V)}{\partial M} = \frac{1}{V} \quad ; \quad \frac{\partial(M/V)}{\partial V} = \frac{-M}{V^2}$$

$$\Rightarrow [\sigma(\rho)]^2 = \left(\frac{1}{\overline{V}}\right)^2[\sigma(M)]^2 + \left(\frac{-\overline{M}}{\overline{V}^2}\right)^2[\sigma(V)]^2 + 2\left(\frac{1}{\overline{V}}\right)\left(\frac{-\overline{M}}{\overline{V}^2}\right)[\sigma(M,V)] =$$

$$= \left(\frac{\overline{M}}{\overline{V}}\right)^2\left(\left[\frac{\sigma(M)}{\overline{M}}\right]^2 + \left[\frac{\sigma(V)}{\overline{V}}\right]^2 - 2\left[\frac{\sigma(M,V)}{\overline{M}\,\overline{V}}\right]\right) \approx \left(\frac{\overline{M}}{\overline{V}}\right)^2\left(\left[\frac{\sigma(M)}{\overline{M}}\right]^2 + \left[\frac{\sigma(V)}{\overline{V}}\right]^2\right)$$

$$\Uparrow$$

… … … … *se trascuro il contributo della* covarianza…

$$\sigma(M,V) = \frac{1}{N-1}\sum_{i=1}^{N}(M_i - \overline{M})(V_i - \overline{V})$$

Giustificazione della deviazione standard della media di N misurazioni di una GF soggetta a "errori casuali"

$$E(x) = \bar{x} = \frac{\sum_{i=1}^{N} x_i}{N} \quad \Leftrightarrow \quad \boxed{\bar{x}(x_1, x_2, \ldots x_i, \ldots x_N)}$$

$$\Rightarrow \quad [\sigma(\bar{x})]^2 = \sum_{i=1}^{N} \left(\frac{\partial \bar{x}}{\partial x_i}\sigma(x_i)\right)^2 = \sum_{i=1}^{N} \left(\frac{1}{N}\sigma(x_i)\right)^2$$

$$se \quad \sigma(x_i) = \sigma(x) \quad uguale \quad per \quad ogni \quad i$$

$$\Rightarrow \quad [\sigma(\bar{x})]^2 = N\left(\frac{\sigma(x)}{N}\right)^2 = \frac{[\sigma(x)]^2}{N}$$

$$\Rightarrow \boxed{\sigma(\bar{x}) = \frac{\sigma(x)}{\sqrt{N}}}$$

… la varianza della media $\sigma^2(\langle x \rangle)$ è 1/N volte la varianza $\sigma^2(x)$ della variabile