Ann Oper Res (2014) 215:245–256
DOI 10.1007/s10479-013-1385-5

The electoral system for the Italian Senate: an analogy with deterministic chaos?

An analysis via characteristic polynomials

G. Pontuale · F. Dalton · S. Genovese · E. La Nave · A. Petri

Published online: 8 May 2013
© Springer Science+Business Media New York 2013

Abstract The electoral system adopted for the allocation of seats in the Italian Senate utilizes a complex mechanism of awards at a regional level with the aim of strengthening, when necessary, the winning coalition and so improve overall government stability. The results presented here demonstrate that in a significant number of cases, the effect of the mechanism is opposite to that desired, *to wit*, weakening the resultant government by awarding more seats to the minority coalition. Indeed the award to the minority can even be such that the minority coalition becomes the majority and wins the election. The application of the award mechanism is strongly unpredictable as it depends crucially on the precise number of seats independently obtained in each region, and that each adjustment thereof can be positive, zero or negative; a characteristic that closely resembles the behaviour of a chaotic dynamical system whose trajectory, although purely deterministic, depends on infinitely precise details and is therefore unpredictable. To perform the systematic numerical analysis of the award effectiveness, we introduce characteristic polynomials, one for each electoral district, which carry information about all possible outcomes and award applications. Their product yields a polynomial containing the dependence of the result at national level on each of the regional awards.

G. Pontuale (✉) · F. Dalton · A. Petri
CNR—Istituto dei Sistemi Complessi, via del Fosso del Cavaliere 100, 00133 Rome, Italy
e-mail: giorgio.pontuale@isc.cnr.it

F. Dalton
e-mail: fergal.dalton@isc.cnr.it

A. Petri
e-mail: alberto.petri@isc.cnr.it

S. Genovese
Ufficio legislativo Gruppo PD, Senato della Repubblica Italiana, Rome, Italy
e-mail: simona.genovese@senato.it

E. La Nave
CNR—Istituto dei Sistemi Complessi, Dipartimento di Fisica, Università Sapienza, P.le A. Moro 5, 00189 Rome, Italy
e-mail: emilia.lanave@roma1.infn.it

Keywords Electoral systems · Generating functions · Chaotic systems

1 Introduction

The science behind generic voting and electoral schemes has been the subject of mathematical study for quite some time (see for example Penrose 1946, 1952). Recently however, there has been a drive to focus more on specific scenarios and electoral contests with a view to prediction and evaluation of the actual mechanisms employed (e.g. Grilli di Cortona et al. 1999 and refs. therein). Outcomes of representative electoral systems have been analyzed in some cases by means of direct simulations, e.g. computing the output of different systems starting from a given set of electoral preferences of the citizens (Fragnelli and Ortona 2006).

One achievement of such quantitative approaches to electoral problems has been to show that intuition in this field is often misleading. For example, Zyczkowski and Slomczynski (2004, 2012), Zyczkowski et al. (2006) examined the problem of how voting weight should be distributed to each member of the European Union's (EU) Council of Ministers, and further considered how to establish a "fair" majority threshold. Intuitively, one would assume that the weight of ministers' votes should be proportional to the number N of voters they represent however, by exploiting the mathematics of random walks (Feller 1968), the authors conclude that a fair weight would be proportional to the *square-root* of N.[1] In this paper we will consider the current electoral system for the Italian Senate, in particular the *premio di maggioranza* ("majority prize") mechanism which awards additional seats in each region to the corresponding majority coalition of that region, and evaluate its consequences at the national level.

Since its approval in 2005, the electoral law currently operating in the Italian national Senate, the so-called "Porcellum",[2] has been applied three times, in 2006, 2008 and 2013. In both 2006 and 2008 only two coalitions passed the electoral threshold contained in the law, and so obtained access to the allocation of the 315 senatorial seats. On each occasion, the final results were at odds with the criteria which originally guided the development of this complex mechanism of regional (district) awards,[3] those being, to give a small additional national majority to any coalition winning with only a weak majority, in order to improve the government's stability. In 2006 the result after applying the "prize" seats was exactly null: before the application of the prize, the two coalitions were in a situation of substantial equality, and this was unchanged at a national level after the application of the prize. The second time, in 2008, before the application of the prize, one coalition already enjoyed a significant advantage. With the prize, that coalition gained an additional 3 seats.

The third time, in 2013, the result was even more surprising and contrary to the early objectives of the law's proponents. Four coalitions passed the threshold and were allocated seats. This scenario amplified the unpredictable effects of the mechanism; a short analysis of this curious outcome will be given in a dedicated paragraph at the end of this section.

[1] This result emerges by considering the probability of a random walk returning to the origin after exactly N steps, corresponding to the situation where N votes sum to zero and so each individual vote becomes crucial in determining the outcome. Only in this way can each single vote of the EU have an equal weight irrespective of the state to which it belongs. Furthermore, this choice of weighting leads naturally to the determination of a fair majority threshold under which each voter holds the same power.

[2] Italian Law n. 270 21/12/2005.

[3] In the following, the terms "prize" and "award" will be used as synonymous.

Table 1 Results of 2013 elections

Coalitions	*A*	*B*	*C*	*D*
Before prize seats	98	98	74	29
After prize seats	112	115	54	18
Award	+14	+17	−11	−20

In this article we report a detailed numerical analysis of the award mechanism, and evaluate the dependence of its effect and its efficiency on the electoral outcome. The analysis will consider a system with only two competing coalitions, this assumption being strongly representative of the situation observed in Italy in the 2006 and 2008 elections, and as such permits us to clarify the practical, and historically realized, drawbacks of the award mechanism.

In order to evaluate all the possible outcomes at national level, we will introduce a characteristic polynomial for each region which contains information about all the possible electoral outcomes and all possible applications of the prize in that region. The product of all the regional characteristic polynomials generates a global polynomial equation (1) that contains the same information on the national scale.

It will be shown that the regional application of the award mechanism can produce unexpected effects, contrary to the aims of the law itself, on the final allocation of seats amongst the two contending parties, even so far as to give a net award to the losing coalition and a net penalty to the winning. Without knowing in advance the precise outcome in each region, it is impossible to predict the overall net effect of the award. Such behaviour is strongly reminiscent of chaotic deterministic dynamical systems whose trajectories, although perfectly predictable in principle, vary wildly with the slightest change in initial conditions, as better described in the following.

Finally, an analysis of the award "efficiency", as a function of the initial difference between the two coalitions, will show how the ability to strengthen a weakly leading coalition is reduced to zero as the initial difference approaches zero but becomes more effective when one contender already enjoys a considerable advantage. This is clearly opposite to the desired aim of strengthening weak majorities while leaving strong majorities unaltered.

2013 elections In the recent 2013 elections for the first time a *multipole* (four coalitions) scenario occurred which amplified the unpredictable effects of the award mechanism, and largely belied the widespread belief that, whatever the electoral result, a law of this kind would guarantee that the dominant coalitions could be realistically expected to form a stable government. In Table 1, we observe that the two main opposing coalitions start from a situation of equality before the prize (118 seats each). The coalition with the highest popular vote *A* gained 14 seats (4 % of the total) and the second most popular *B* gained 17 seats, both at the expense of the two minor coalitions *C* and *D*. It is clear, then, that the application of the award has not guaranteed to any list or coalition an absolute majority and has, moreover, excessively penalized the minority coalitions—indeed, the weakest coalition lost an astounding 38 % of its seats.

Such circumstances of three or more coalitions could, in principle, permit a coalition with a clear minority of seats (33.4 % in the case of three coalitions, 25.1 % in the case of four, and so on) to obtain 55 % of seats after the application of the award, in clear contradiction of voter expression. Indeed, from the analysis presented here one may expect even more interesting outcomes such as the 2013 result which derived, in part, from recent upheavals in

Table 2 Number of seats currently allocated to the 21 regional districts. (*) the award mechanism does not apply by law. (#) the award mechanism has no effect in a two-party system

District	Seats	District	Seats	District	Seats
Valle d'Aosta*	1	Molise*	2	Trentino Alto Adige*	7
Friuli Venezia Giulia#	7	Umbria#	7	Abruzzo#	7
Basilicata#	7	Liguria	8	Marche	8
Sardegna#	9	Calabria	10	Toscana	18
Emilia Romagna	21	Puglia	21	Piemonte	22
Veneto	24	Sicilia	26	Lazio	27
Campania	30	Lombardia	47	Esteri*	6

the Italian political landscape However, the case with more than two coalitions is extremely complex, and highly demanding even from a numerical point of view.

2 The regional award mechanism

Article 57 of the Italian Constitution specifies that the Senate shall be elected on a regional basis. Each Italian administrative region, 21 in total, is an electoral district and is assigned seats according to its population. The 2001 ISTAT census resulted in the number of regional seats being assigned as shown in Table 2 (Decreto del Presidente della Repubblica 11/02/2006), for a total of 315 seats. Amongst these 21 regions (including a special "foreign" region), the electoral law specifies that an award mechanism shall be applied in 17 regions, governing the allocation of 299 of the total 315 seats. Of the remaining 16 seats, 10 are assigned to regions in which the award does not apply (Valle d'Aosta, Molise and Trentino Alto Adige) and 6 to the recently introduced foreign electoral jurisdiction for Italians living abroad, which therefore is equivalent to a 21st region.

The award mechanism which operates in the 17 regions can be described as follows. Seats in each of the regions are initially allocated to the competing coalitions according to a strict proportional representation (Hare method of "*natural quotients and the highest remainders*"). If the winning coalition then has less than 55 % of the regional seats, it is awarded additional seats, at the expense of the losing coalition(s), to bring its total number of seats to 55 % (fractions are rounded up), the remaining seats being proportionally allocated to the remaining coalition(s) as before. A winning coalition with an initial majority of 55 % or more of seats in the region is not awarded any prize (i.e. the winner in each regional is guaranteed to have at least 55 % of the seats for that region). Thus, for example, from data in Table 2, in Lombardia a minimum of 26 seats are attributed to the winning coalition; in Campania a minimum of 17, and so on.

With this scenario, the total number of configurations that may result is very high and depends on the number of participating coalitions in each region. The results presented here refer to the case in which only two coalitions are presented for election. This is sufficiently close to the actual case of the 2006 and 2008 Italian political elections, showing that our analysis is historically valid.

It is worth noting that, in this case of only two coalitions, the award has no practical effect in 5 of the 17 regions where the mechanism applies (Friuli Venezia Giulia, Umbria, Abruzzo, Basilicata and Sardegna) where a small odd number of seats (7 or 9) has been assigned. In these regions, the winning coalition will always exceed the 55 % threshold even if it wins

Table 3 Comparison of the possible outcomes of the election in Liguria before and after the application of the prize

Before	0	1	2	3	4	4	5	6	7	8
After (k)	0	1	2	3	3	5	5	6	7	8
Prize (h)	0	0	0	0	-1	$+1$	0	0	0	0

by only one seat: e.g. $4/7 = 0.571 > 55\ \%$. Thus the majority prize is applicable in only 12 out of 21 regions in this two-party system.

3 Computing award effects in a bipolar scenario

The most striking possible effect of the majority award mechanism is that a coalition which initially globally loses the election can, region-by-region, gain enough "prize seats" to be transformed into the national winner (henceforth we will consider the outcome for only one of the competing coalitions. In this bipolar scenario, the other coalition obviously enjoys or endures the complementary outcome). As an example, consider the case of a hypothetical Senate with only two regions, Lombardia and Puglia, expressing a total of 68 senators. A coalition with 12 initial seats from Puglia (where it wins), and 23 in Lombardia (where it loses), would have an initial majority of 35 to 33 in this Senate of 68 seats. After the application of the award, it would elect 12 senators in Puglia and 21 in Lombardia, with a total of $12 + 21 = 33$, thus losing at a national level. Considering the real Senate with all 17 relevant regions, rather than the imaginary 2-region Senate, this "rollover" can occur in a significant percentage of cases. We now analyze in detail in which, and how many, cases a favorable result for a coalition becomes unfavorable after the application of the regional bonus.

As described above, in each region the award gives rise to a non-linear correspondence between the number of votes and the number of seats a coalition obtains. The case of Liguria, described in Table 3, is shown graphically in the inset of Fig. 1 as an example: the final result jumps from 3 to 5 seats due to the action of the award mechanism.

Even in this simplified bipolar case, the total number of possible election outcomes is very high. This is easy to demonstrate by observing that, for example, in Valle d'Aosta (Table 2) a coalition can win 0 or 1 seats, corresponding to 2 possibilities; in Trentino from 0 to 7, corresponding to $7 + 1 = 8$ possibilities. For a region with n seats there are, in principle, $n + 1$ possible outcomes; when the number of seats is even and the prize is applicable (see for example the Liguria case in Table 3), there are $n + 2$ possibilities (the award can go either way depending on which coalition has the greater number of votes). The criteria adopted for distinguishing between "different" election outcomes is that the number of seats won by a coalition should be different in at least one of the 21 regions. So the total number of possible outcomes is the product of these possibilities in all 21 districts: $2 \cdot 3 \cdot 7 \cdot \ldots \cdot 32 \cdot 48 = 1.2 \cdot 10^{23}$, a number comparable to Avogadro's number (the number of atoms in 1 gram of Hydrogen). Among these, the prize applies in $9.9 \cdot 10^{22}$ cases, about 82 % of the total. Fortunately, it is not necessary to undertake the daunting task of individually enumerating all these cases; even a state-of-the-art supercomputer would take several years to do so. Instead we introduce a technique that greatly facilitates the calculation in advance, and tells us the number and type of different regional configurations that produce an equivalent national result. The technique we introduce derives from the use of generating functions in probability theory (see Appendix). One of its biggest advantages is that the probability distribution for the sum

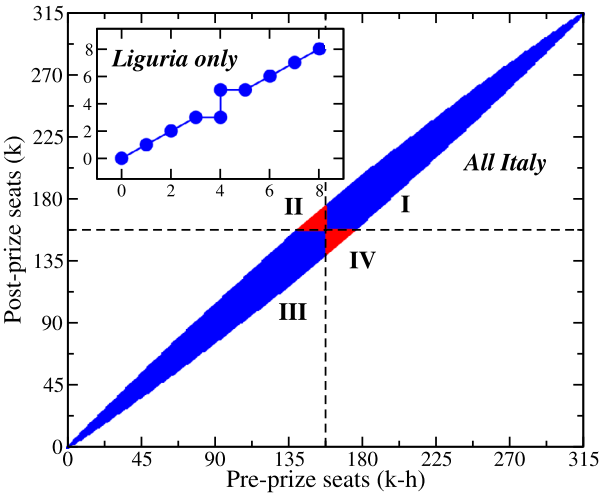

Fig. 1 Schematic of the effect of the regional award. In both *ordinate* and *abscissa* the values refer to the national number of seats. For each possible outcome before the award application reported on the abscissa, the ordinate shows the possible outcome after the award application. *Inset*: the effect on the region Liguria. *Main body*: the possible effect resulting at national level by combining all the regional effects. The *grey central region* (*red* online) shows where the would-be winning coalition before the award application, can be overturned

of two, or more, variables can be computed by simple products instead of by convolution sums. In analogy we first construct for every region the table of all the possible outcomes (before and after the award), and obtain the national outcome by taking their product. The case of Liguria is shown in Table 3. We then introduce two auxiliary variables a and z to construct a regional polynomial $F_{\mathrm{Reg}}(z)$ containing terms $a^h z^k$ where k indicates the number of seats obtained after the prize (in Table 3 for Liguria, row two: $0 \le k \le 8$) and h indicates the prize obtained (in Table 3 for Liguria, row three: $h \in \{-1, 0, +1\}$). So if k seats are obtained but no prize is awarded, the exponent $h = 0$ and so the coefficient of z^k is $a^0 = 1$. If a single seat is awarded as a prize, then the coefficient of the relative z^k is a^1; two prize seats yields $a^2 z^k$ and so on (note that no value is ever ascribed to the variables a and z—they are simply symbols which keep track of the number of seats obtained and allow us to distinguish all the different ways in which a coalition can arrive at a given final number of seats).

For example, for Val d'Aosta where a coalition can obtain zero or one seats, the polynomial reads $F_{\mathrm{VdA}}(z) = z^0 + z^1 = 1 + z$. There is no prize in Val d'Aosta and so no terms with coefficients a^j, $j \neq 0$. For Liguria, from the data in Table 3, the polynomial reads:

$$F_{\mathrm{Lig}}(z) = 1 + z + z^2 + z^3 + a^{-1}z^3 + a^1 z^5 + z^5 + z^6 + z^7 + z^8.$$

So a coalition in Liguria can obtain 5 final seats through two trajectories: simply winning five (z^5), or winning four plus one prize seat ($a^1 z^5$).

To obtain the possible outcomes at national level we now multiply the characteristic polynomials of all regions together resulting in a global polynomial of degree 315, the total number of seats in the Senate:

$$\begin{aligned} F(z) &= \prod_{\mathrm{Reg}} F_{\mathrm{Reg}}(z) = 1 + 21z + 230z^2 + 1749z^3 + \left(10372 + 2a + 2a^{-1}\right)z^4 \\ &\quad + \left(51088 + 41a + 41a^{-1}\right)z^5 + \cdots + 21z^{314} + z^{315} \\ &= \sum_k \left(\sum_h A_{kh} a^h \right) z^k. \end{aligned} \tag{1}$$

Each term in z is characterized by an exponent k associated with the final score. Of course, the mixing of results from the different regions can produce different trajectories to the same

final score k. Thus the coefficient of z^k is generally a polynomial in a where each exponent h represents the net contribution of the prize to the score k.

For example, terms of the type a^6 and a^{-2} in the coefficient of z^{159} mean that a number of seats after the prize of 159 can be obtained by means of an overall positive prize of 6 seats (obtained starting from an initial score of 153 seats), or by means of 2 negative seats (obtained starting from 161 seats). Furthermore, for a given k, there are many ways of obtaining the same h: for instance a coalition might obtain one award seat in Campania and zero in Lazio, or zero in Campania and one in Lazio, contributing twice to the term a^1. Thus the coefficient A_{kh} of $a^h z^k$ indicates how many different outcomes yield a prize h and the post-award number of seats k.

4 Space of results and unpredictable trajectories

We are now in a position to illustrate the overall effect of the award mechanism on all the possible national election results. As explained above, in Eq. (1) each term of $F(z)$ consists of terms $a^h z^k$ where k corresponds to the final number of seats obtained and h to the award seats. $k - h$, therefore, equals the number of seats obtained *prior* to the award mechanism; e.g. row one in Table 3. We plot k as a function of $k - h$ in Fig. 1.

The first observation concerns the degeneracy of the final result: each point of the abscissa corresponds a multiplicity of points on the ordinate and vice versa. The prize may affect virtually any electoral outcome though its effect naturally diminishes towards the extremities.

In Fig. 1, at half the seats available, or $157\frac{1}{2}$, two straight lines demarcate four quadrants of the graph. Points located in quadrants I and III represent cases where the prize does not alter the winner: the winner before the prize is still the winner after the prize. Quadrants II and IV, however, represent cases where the effect of the prize is such that the loser becomes the winner and vice-versa, effectively overturning the result (gray triangles, red online). Figure 2 shows in detail the "critical" zone of Fig. 1, in which the effect of the prize may reverse the result (given the symmetry between the winner and loser quadrant II is equivalent to IV, as III is to I).

However, even when the application of the prize does not overturn the result, the prize can nonetheless weaken the winning coalition, defeating the very *raison d'etre* of the law itself. This happens in all cases of quadrant I which are below the bisector (thick dot-dashed line, red online). This paradoxical result, at its extreme, can even weaken a strong winner with a comfortable majority of 175 seats, to the bare minimum 1-seat majority of 158 seats (green spot). Symmetrically, it can strengthen a weak opposition which obtains only 140 seats, right up to the minimum 1-seat minority of 157 seats (III quadrant). Again, quadrants II and IV in which the award mechanism overturns the initial election result, can, in extremis, cause a strong winner with 174 seats (55.2 % of seats) to actually lose and become a minority while the minority with 141 seats obtains an overall relative award of 12 %, to become the majority with 158 seats. The mechanism can theoretically lead to a maximum displacement of 17 senators. Given that the law was introduced *specifically* to strengthen an initially weak majority, then all points in quadrant IV and those under the bisector in quadrant I represent a failure of the law in which the resulting effect is opposite to that desired. In addition, we can also say that points in quadrant I above the bisector which are close to the midpoint $(157\frac{1}{2}, 157\frac{1}{2})$ are also failures insofar as there is a weak winner which is poorly strengthened by the electoral law.

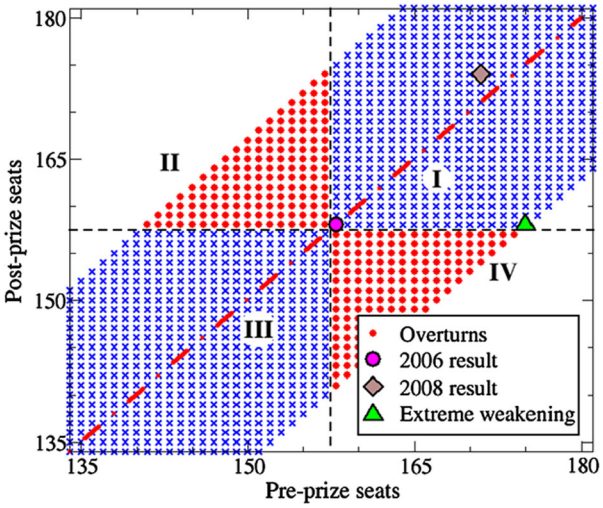

Fig. 2 Details of the critical zone delimited in Fig. 1. For those cases falling in the II and IV quadrants (*red points* online) application of the regional awards results in a tipping of the winning coalitions. The actual results of the Italian political elections of 2006 and 2008 are indicated by *larger spots*

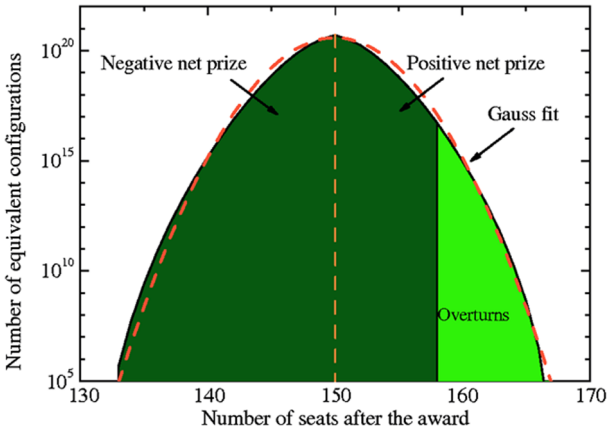

Fig. 3 Multiplicity of the different outcomes for effect of the award, given 150 initial seats before the award

We must emphasize that our analysis does not attempt to ascribe any probability to any electoral outcome, or make any predictive analysis of the possible outcomes of a given election. In our model, each outcome has its own multiplicity, that is, the number of trajectories leading to it expressed by $\sum_h A_{kh}$ from Eq. (1). As an example, Fig. 3 shows the multiplicity of a single initial outcome in which one coalition obtains 150 seats before the award mechanism; this is effectively a section perpendicular to the plane of Figs. 1 and 2 (note the vertical logarithmic scale). The region on the right shows the number of outcomes where this losing coalition gains seats in virtue of the prize. In the light green zone (right) the prize overturns the electoral result, implying that this initially losing coalition wins the election. The shape of such curves approaches a Gaussian as the number of pre-prize seats approaches $157\frac{1}{2}$, showing that the total number of seats after the award behaves similarly to the sum of uncorrelated random numbers. A growing skewness is observed as the point where the section is taken departs from the pre-prize tie.

Table 4 summarizes the different types of effect the award mechanism can have; the "overturning" between winner and loser occurs in $2.2 \cdot 10^{21}$ cases, almost 2 % of the total. If we consider only the initial national results which *could* potentially lead to an overturn,

Table 4 Possible cases summary

Total cases	$1.201 \cdot 10^{23}$	100 %
Award application	$9.87 \cdot 10^{22}$	82 %
No award application	$2.14 \cdot 10^{22}$	18 %
Overturn	$2.2 \cdot 10^{21}$	1.85 %
Weakening (without overturn)	$4.1 \cdot 10^{22}$	33.7 %

it is found that it occurs in 4.5 % of cases (i.e. all cases where the initial result is from 141 to 173 seats). Even more significant is the number of cases in which the leading coalition is weakened by the prize: $4.0 \cdot 10^{22}$, more than one-third of the total.

The award algorithm can therefore alter two identical national results in many, often opposite, ways, and recalls to mind the behaviour of deterministic chaotic dynamical systems (Strogatz 1994).

In fact, the evolution of a deterministic chaotic system is in principle perfectly predictable, on condition that its initial conditions are perfectly known, *i.e.* to infinite precision, which is obviously impossible in practice. Similarly in our case, to predict the overall effect of the award mechanism requires the knowledge of the *exact* initial number of seats awarded to each party in each region. Clearly, the regions are not infinite, rather they are relatively few, as is the required "precision" of the prediction, but the global uncertainty of regional outcomes within elections renders the outcome highly unpredictable (Cvitanović et al. 2012; Gleick 2011; Saari 2001). This is, for instance, the main reason for which long term weather forecast become more and more unreliable with increasing the time lag.

5 Efficiency

As described above, many different regional results can combine to produce the same national result, as the prize received from the award mechanism is not determined by the initial national result, but from the sum of the initial, independent, regional results which can naturally oppose one another. Therefore, the initial nation-wide difference of seats Δ_i between the two coalitions does not alone convey sufficient information to determine the final difference of seats Δ_f. We wish, therefore, to aggregate the performance of the award mechanism in such a way as to determine if it is achieving its ostensible objective, that is, to strengthen a weak majority.

For example, if all configurations yielding an initial result of, say, 157:156 seats, were to yield, on average, only one extra seat to the winner (i.e. an increase of only 2 in the majority), we would say that the award mechanism is very little effective.

Naturally, in a bipartite system, the maxim "one man's gain is another man's loss" holds true: if one party gains seats through the award, the other party loses seats. Therefore the change in the majority in the Senate is actually twice the value of the award received: $(\Delta_f - \Delta_i) = 2 \times h$ (where h is the award received in any given configuration, the coefficient of the variable a in the characteristic polynomial above). We define the "efficiency" of the award as the average of the quantity $(\Delta_f - \Delta_i)$ over all configurations which yield a given initial national result Δ_i:

$$e(\Delta_i) = \langle \Delta_f - \Delta_i \rangle. \tag{2}$$

This is a measure of how effective the mechanism is at strengthening a given coalition from its initial position. This curve is shown in Fig. 4.

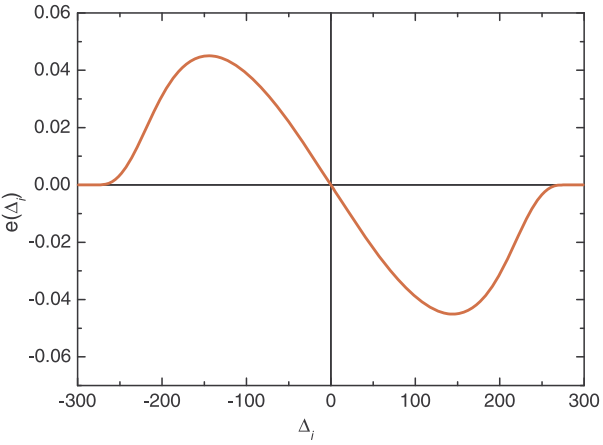

Fig. 4 The efficiency of the majority award as function of the number of seats is minimum when it should be maximum, close to the tie, and maximum in the presence of an initially robust majority, when not necessary

We note first that the absolute value of this "efficiency" is very small. This inherent feature does not bode well for the stated desire of increasing the stability of an initially weak government and is due to the fact that the various configurations produce positive and negative awards with almost perfect symmetry, coupled with the contributions from large numbers of configurations where no award is effected but which nonetheless contribute to the average. Moreover, the graph shows how the efficiency is reduced to its lowest when the initial difference between coalitions is close to zero: $e(\Delta_i)|_{\Delta_i=0} = 0$, then grows with Δ_i up to a maximum before decreasing as a large Δ_i pushes more and more regions beyond the realm of the award mechanism (i.e. above a 55 % initial majority). This overall behaviour near $\Delta_i = 0$ is precisely the opposite of what one might expect from a mechanism specifically implemented to enhance government stability when required: as it stands, the prize yields a weak effect when a strong effect is required (as in the 2006 elections) and a strong effect when a weak effect is required (as in 2008).

Therefore we not only have a mechanism which produces the opposite average effect to what is desired, but the fluctuations thereof are such that the final effect in a single electoral realization are far from obvious and may not only fail to strengthen a weak majority, but even transform a strong majority into a minority.

6 Possible corrective measures

Of the possible measures which might be introduced to correct for these undesirable effects, perhaps the most simple would be a nation-wide award mechanism, designed such that each elected senator is in any case a direct expression of the electorate in the corresponding region (thus satisfying the constitutional requirement described above). This might be accomplished by assuming an electoral mechanism which "regionalizes" the national award as follows: once the most-voted coalition is identified, it is assigned seats according to proportional representation; the extra seats required to reach the minimum reasonable majority (i.e. 55 %) is calculated and these seats are awarded to the winning coalition, divided amongst the various regions in proportion to their respective populations. These "award" seats are assigned throughout the regions to the coalition with most votes at a national level.

7 Conclusions

A detailed numerical analysis of the possible effects of the awards mechanism legislated for the Italian Senate has been performed, considering a simple bipolar system similar to the actual political situation in Italy for the 2006 and 2008 political elections. Within this framework we have developed an approach based on characteristic polynomials for the possible regional outcomes, that permit the computation of the effect of the local awards on the national outcome.

Our results indicate that this, apparently commonplace, electoral law hides some counter intuitive and surprising features, absolutely not obvious from a cursory analysis and, most likely, not evident to the lawmaker.

Without wishing to enter into a political discussion and assuming that the lawmaker's intention was to genuinely improve government stability subsequent to an election, this analysis has shown that the law may easily fail to achieve its objectives or, in a significant number of cases, actually achieve an opposite effect: in almost one-third of cases the law will weaken the winning coalition instead of strengthening it, and in a small number of cases can even overturn the result transforming the defeated into the winner. This is due to the inherent unpredictability of the precise election result, and yields a system reminiscent of chaotic dynamical systems.

Even when considering the average behaviour, the mechanism fails to achieve its objective, delivering a small award when a large award is required, and vice-versa. The conclusion to draw is that the implementation of a fair electoral system, reflecting as much as possible the will of the voters, should avoid the emotional semi-empiricism usually employed by lawmakers, but should instead be built on a robust methodological basis; electoral procedures should be checked for their actual efficacy in producing the desired effects before being decreed in law. This can be done only if the electoral procedure is designed using appropriate tools based on a firm foundation of scientific and mathematical analyses.

Acknowledgements A.P., F.D. and G.P. commemorate Bruno Simeone and his profound thoughts and teachings on the science of electoral systems.

Appendix: Generating functions

Generating functions are a widely used tool in probability calculus. Given a random variable x that can assume non-negative integer values ℓ with probability p_ℓ, its generating $G(z)$ function is defined as $G(z) = \sum p_\ell z^\ell$, with $0 \le z < 1$. The generating function possesses a set of useful properties and in several cases makes calculations easier. Among the main properties there are

$$\begin{aligned} &\text{normalization:} \quad G(1) = 1, \\ &\text{expectation:} \quad \bar{x} = \left.\frac{\partial G}{\partial z}\right|_{z=1}. \end{aligned}$$

Given two variables identically distributed, x_1 and x_2, it is easily seen that the generating function $F(z)$ for the sum variable, $y = x_1 + x_2$, is $F(z) = G^2(z)$. Since $F(z) = \sum_\ell q_\ell z^\ell$, it is straightforward to derive the probabilities for y as

$$q_\ell = \frac{1}{\ell!} \left.\frac{\partial F^\ell}{\partial^\ell z}\right|_{z=0}.$$

In a similar fashion if $G_1(z), G_2(z), \ldots, G_N(z)$ are the generating functions of N (differently distributed) random variables $x_1, x_2, \ldots, x_N$, the generating function for $y = x_1 + x_2 + \cdots + x_N$ will be given by the product $F(z) = G_1(z) \cdot G_2(z) \ldots G_N(z)$.

References

Cvitanović, P., Artuso, R., Mainieri, R., Tanner, G., & Vattay, G. (2012). *Chaos: classical and quantum*. Copenhagen: Niels Bohr Institute.

Feller, W. (1968). *An introduction to probability theory and its applications* (Vol. 1, 3rd ed.). New York: Wiley. http://www.amazon.com/exec/obidos/redirect?tag=citeulike07-20&path=ASIN/0471257087.

Fragnelli, F., & Ortona, G. (2006). In B. Simeone & F. Pukelsheim (Eds.), *Mathematics and democracy* (pp. 65–81). Berlin and Heidelberg: Springer.

Decreto del Presidente della Repubblica 11/02/2006. *Gazzetta Ufficiale 36*, 13/02/2006.

Gleick, J. (2011). *Chaos: making a new science*. *Open road iconic ebooks*. http://www.amazon.com/exec/obidos/redirect?tag=citeulike07-20&path=ASIN/B004Q3RRPI.

Grilli di Cortona, P., Manzi, C., Pennisi, A., Ricca, F., & Simeone, B. (1999). *SIAM monographs on discrete mathematics and its applications: Evaluation and optimization of electoral systems*. Philadelphia: SIAM.

Penrose, L. S. (1946). *Journal of the Royal Statistical Society*, *109*(1), 53. http://www.jstor.org/stable/2981392.

Penrose, L. S. (1952). *On the objective study of crowd behaviour*. London: H.K. Lewis.

Saari, D. G. (2001). *Chaotic elections! A mathematician looks at voting*. Providence: Am. Math. Soc. http://www.amazon.com/exec/obidos/redirect?tag=citeulike07-20&path=ASIN/0821828479. (Here the word chaos is used in the general common sense.)

Strogatz, S. H. (1994). *Nonlinear dynamics and chaos: with applications in physics, biology, chemistry, and engineering*. Reading: Addison-Wesley.

Zyczkowski, K., & Slomczynski, W. (2004). arXiv:cond-mat/0405396v2.

Zyczkowski, K., & Slomczynski, W. (2012). arXiv:1104.5213v2.

Zyczkowski, K., Slomczynski, W., & Zastawniak, T. (2006). *Physics World*, *19*, 35–37.