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Optimal Monitoring of Position in Nonlinear Quantum Systems

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We discuss a model of repeated measurements of position in a quantum system which is monitored for a finite amount of time with a finite instrumental error. In this framework we recover the optimum monitoring of a harmonic oscillator proposed in the case of an instantaneous collapse of the wave function into an infinite-accuracy measurement result. We also establish numerically the existence of an optimal measurement strategy in the case of a nonlinear system. This optimal strategy is completely defined by the spectral properties of the nonlinear system.

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Improvement in the precision of measurements brings us to consider the ultimate limits of sensitivity imposed by quantum mechanics and to develop measurement strategies overcoming such limits [1]. One proposed example of these strategies, also called quantum nondemolition (QND) measurements, was the stroboscopic measurement of position in a harmonic oscillator. A series of ideal infinite precision and instantaneous measurements performed each half period of a harmonic oscillator represents an optimal measurement strategy with perfectly predictable results [2–4]. In a realistic scenario it is compulsory to study a strategy based on measurements which are affected by an instrumental error and which last a finite amount of time. Besides this generalization, as outlined in [1], quantum measurement models for nonlinear systems, i.e., systems which are not a harmonic oscillator, are still missing. In this Letter we study optimal strategies for measuring position in nonlinear systems monitored for a finite time with finite accuracy. By using the path-integral approach to quantum measurements [5, 6] we quantitatively recover the results for the QND stroboscopic measurements of a harmonic oscillator and we establish the existence of an optimal monitoring for a nonlinear system.

The standard quantum limit in a continuous measurement of position for nonlinear systems has been already analyzed in the framework of the path-integral approach [7]. The measuring system is schematized by an arbitrary measurement output $a(t)$ and an instrumental error Δa. The effect of the measurement modifies the path integral giving privilege to the paths close to the output $a(t)$. The propagator of a system in which the position is measured includes the influence of the measurement through a weight functional $w_{[a]}[x]$:

$$K_{[a]}(x'',\tau;x',0) = \int d[x] \exp\left\{\frac{i}{\hbar}\int_0^\tau L(x,\dot{x},t)dt\right\} w_{[a]}[x]. \tag{1}$$

The quantity $K_{[a]}(x'',\tau;x',0)$, called measurement amplitude hereafter, can be interpreted in two alternative ways. If the measurement output a is known, this is a transition amplitude from the point x' at time $t=0$ to the point x'' at time $t=\tau$ for the system undergoing the measurement with output $a(t)$. On the other side, if x', x'' are known, the same expression can be understood as an amplitude for the measurement to give the output

$a(t)$ with the above boundary conditions. If the system is initially in a pure state described by the wave function $\psi(x,0)$, according to the first interpretation of $K_{[a]}$, the quantity

$$P_{[a]} = \frac{|\langle\psi_{[a]}(\tau)|\psi_{[a]}(\tau)\rangle|^2}{\int |\langle\psi_{[a]}(\tau)|\psi_{[a]}(\tau)\rangle|^2 d[a]}, \quad (2)$$

where

$$\psi_{[a]}(x'',\tau) = \int K_{[a]}(x'',\tau;x',0)\psi(x',0)dx', \quad (3)$$

can be interpreted as a probability functional for the measurement output. Due to the influence of the measurement an effective position uncertainty arises:

$$\Delta a_{\rm eff}^2 = 2\frac{\int \tau^{-1}\int_0^\tau [a(t)-\bar{a}(t)]^2 dt\ P_{[a]}d[a]}{\int P_{[a]}d[a]}, \quad (4)$$

where $\bar{a}(t)$ is the most probable path which makes $P_{[a]}$ extremal. The effective uncertainty $\Delta a_{\rm eff}$ is greater than the instrumental error Δa unless the system is monitored in a classical regime, i.e., when $\Delta a \gg \sigma$ where σ is the width of the initial wave function $\psi(x,0)$, or in a QND way [6].

For simplicity we represent an actual measurement with instrumental error Δa lasting a time τ through a weight functional $w_{[a]}[x]$:

$$w_{[a]}[x] = \exp\left\{-\frac{1}{2\Delta a^2\tau}\int_0^\tau [x(t)-a(t)]^2 dt\right\}. \quad (5)$$

As shown in [7], the evaluation of the path integral can be overcome by writing an effective Schrödinger equation which takes into account the influence of the measurement. This equation can be solved analytically in the case of the harmonic oscillator or numerically for a generic system. In the former situation the effective Lagrangian corresponds to a forced linear oscillator

$$L_{\rm eff} = \frac{m}{2}\dot{x}^2 - \frac{m\omega_r^2}{2}x^2 - \frac{i\hbar}{\tau\Delta a^2}a(t)x + \frac{i\hbar}{2\tau\Delta a^2}a(t)^2 \quad (6)$$

with renormalized complex frequency

$$\omega_r^2 = \omega^2 - \frac{i\hbar}{m\tau\Delta a^2}. \quad (7)$$

Since we are interested in a finite but small value of τ, we choose to approximate the measurement results with constant values $a(t) = \epsilon$ which are the set of all the arbitrary measurement outputs in the limit $\tau \to 0$. The probability functional of the measurement path $P_{[a]}$ is then reduced to a function of the amplitude ϵ. When the initial state is chosen to be Gaussian of width σ,

$$\psi(x,0) = \left(\frac{1}{\pi\sigma^2}\right)^{1/4}\exp\left(-\frac{x^2}{2\sigma^2}\right), \quad (8)$$

the probability $P(\epsilon)$ is also a Gaussian function,

$$P(\epsilon) = \frac{1}{\sqrt{\pi}\Delta a_{\rm eff}}\exp\left(-\frac{\epsilon^2}{\Delta a_{\rm eff}^2}\right) \quad (9)$$

with an effective uncertainty

$$\Delta a_{\rm eff}^{-2}(\tau) = \frac{1}{\Delta a^2}{\rm Re}\left[1 + \frac{\sigma^2}{\Delta a^2}\left(i\frac{2\beta+1}{\alpha\omega_r\tau} - \beta^2\gamma\right)\right] - \frac{\sigma^2}{\Delta a^4}\left\{{\rm Re}\left[\beta\left(1 - \frac{i\alpha\gamma}{\sin(\omega_r\tau)}\right)\right]\right\}^2 \times\left\{{\rm Re}\left[\frac{1+i\alpha\tan(\omega_r\tau)}{1+\frac{i}{\alpha}\tan(\omega_r\tau)}\right]\right\}^{-1} \quad (10)$$

having introduced $\alpha = m\omega_r\sigma^2/\hbar$, $\beta = [\cos(\omega_r\tau) - 1]/[\omega_r\tau\sin(\omega_r\tau)]$, and $\gamma = 1/[1 - i\alpha\cot(\omega_r\tau)]$. Under the influence of the measurement the initial state collapses into a state localized around the measurement result. If, for simplicity, we suppose that this measurement result is most probably compatible with (8), i.e., $a(t) = 0$, the initial Gaussian state just changes its width to

$$\sigma(\tau) = \sigma\left\{{\rm Re}\left[\frac{\alpha^2\sin(\omega_r\tau) - i\alpha\cos(\omega_r\tau)}{\sin(\omega_r\tau) - i\alpha\cos(\omega_r\tau)}\right]\right\}^{-\frac{1}{2}}. \quad (11)$$

After the measurement the state evolves according to the dynamical law of the free, i.e., unmeasured system. For the harmonic oscillator the state remains a Gaussian having a width oscillating in time,

$$\sigma(t+\tau) = \sigma(\tau)\left(\frac{1 + [\hbar/m\omega\sigma(\tau)^2]^2\tan^2(\omega t)}{1+\tan^2(\omega t)}\right)^{1/2}. \quad (12)$$

Equations (10) and (12) allow us to study quantitatively a measurement strategy which consists of a sequence of measurements of duration τ equally spaced by a quiescent time ΔT in which no measurement is performed. The repeated collapses of the wave function during the measurements determine an asymptotic effective uncertainty. This is evident in Fig. 1 where we show the dependence of the effective uncertainty upon the number of measurements. After a few measurements the effective uncertainty reaches an asymptotic value which does not depend on the initial state of the system.

The measurement strategy we have described can be optimized by choosing the duration τ of each measurement and the quiescent time ΔT between two consecutive measurements. As we show in Fig. 2 the asymptotic $\Delta a_{\rm eff}$ has minima when ΔT is a multiple of half period of the harmonic oscillator $T \equiv 2\pi/\omega$, i.e., in coincidence with the minima of Eq. (12). The minima of $\Delta a_{\rm eff}$ reach the instrumental error Δa if $\tau \lll \tau_c$, where τ_c is the critical value,

$$\frac{1}{\tau_c} = \frac{\hbar}{m}\left(\frac{1}{\Delta a^2} + \frac{1}{\sigma^2}\right). \quad (13)$$

Indeed for this impulsive regime the effective uncertainty

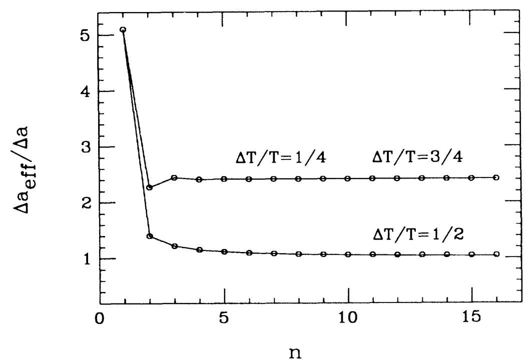

FIG. 1. Effective uncertainty $\Delta a_{\rm eff}$ vs the number n of repeated measurements in the case of a harmonic oscillator. Three different quiescent times ΔT are shown: circles are numerical results and solid lines are the analytical result of Eqs. (10)–(12). Note that the cases $\Delta T/T = 1/4$ and $\Delta T/T = 3/4$ coincide. We put $2m = \hbar = \omega = 1$, $\Delta a = 1$, $\sigma = 5$, and $\tau/T = 10^{-5}$.

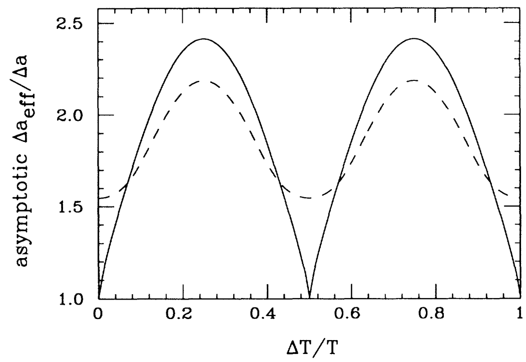

FIG. 2. Dependence of the asymptotic effective uncertainty $\Delta a_{\rm eff}$ on the quiescent time ΔT for the harmonic oscillator. The different curves are relative to different measurement times τ: two solid coincident lines are for $\tau = 0$ and $\tau = 10^{-5}T$; the dashed line is for $\tau = 10^{-1}T \sim \tau_c$.

and the width of the collapsed wave function are simply written as

$$\lim_{\tau\to 0} \Delta a_{\rm eff}(\tau) = \sqrt{\Delta a^2 + \sigma^2}\,, \tag{14}$$

$$\lim_{\tau\to 0} \sigma(\tau) = \sqrt{\frac{\sigma^2 \Delta a^2}{\sigma^2 + \Delta a^2}}. \tag{15}$$

In the limit of an infinite number of measurements the wave function asymptotically collapses to a δ function, $\Delta a_{\rm eff}$ approaches Δa, and an ideal QND stroboscopic strategy is obtained. It is worthwhile to observe that only for $\tau \sim \tau_c$ the optimal effective uncertainty significantly departs from Δa while for $\tau \ll \tau_c$ the ideal situation $\Delta a_{\rm eff} = \Delta a$ is very well approximated. In other words τ_c is the time scale which defines a quasistroboscopic behavior of the measurement.

In Fig. 1 we also compare the analytical results of Eqs. (10) and (12) (solid curves) with the numerical integration of the effective Schrödinger equation (dots). This allows us to check the accuracy of a numerical method (the error is less than 0.1%) we use to study nonlinear systems. We focus our attention on a system described by the Lagrangian

$$L = \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2 - \frac{\lambda}{4}x^4\,. \tag{16}$$

Also in this case the measurement strategy discussed for a harmonic oscillator gives rise to an asymptotic effective uncertainty. As shown in Fig. 3 the asymptotic $\Delta a_{\rm eff}$ does not depend on the initial state but is a function of the measurement and quiescent times. Figure 4 shows that in the impulsive regime $\tau \ll \tau_c$ the asymptotic $\Delta a_{\rm eff}$ is an approximatively periodic function of the quiescent time ΔT. The nature of these oscillations is understood in terms of the energy eigenvalues E_i of the nonlinear oscillator. Indeed these eigenvalues dictate the time evolution of the wave function during the quiescent intervals according to characteristic periods $T_{ij}/T = \hbar\omega/|E_i - E_j|$. Since after each measurement the wave function collapses around the measurement result, again chosen as $a(t) = \bar{a}(t) = 0$, the relevant characteristic periods are those corresponding to the smallest even eigenstates. A WKB evaluation of the first two relevant terms gives $T_{20}/T = 0.225$ and $T_{40}/T = 0.098$. The fundamental time T_{20} corresponds to the principal minima shown in Fig. 4 and T_{40} corresponds to the other secondary minima. When the quiescent time ΔT is close to a multiple of both T_{20} and T_{40} an absolute minimum is expected. This is what we observe in Fig. 4 at $\Delta T \simeq 3T_{20} \simeq 7T_{40}$. Unlike the case of the harmonic oscillator, the general incommensurability of the characteristic periods T_{ij} forbids reaching an optimal strategy

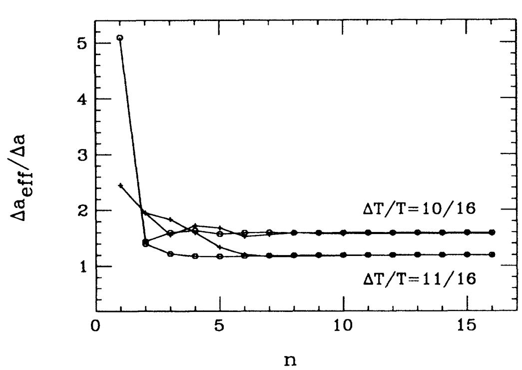

FIG. 3. Effective uncertainty $\Delta a_{\rm eff}$ vs the number n of repeated impulsive measurements for the anharmonic oscillator with $\lambda = 4$. Two different quiescent times ΔT are shown. Circles correspond to an initial Gaussian state with $\sigma = 5$ and crosses are relative to a double peaked initial state. The solid lines are an eye guide. We put $\tau/T = 10^{-5}$.

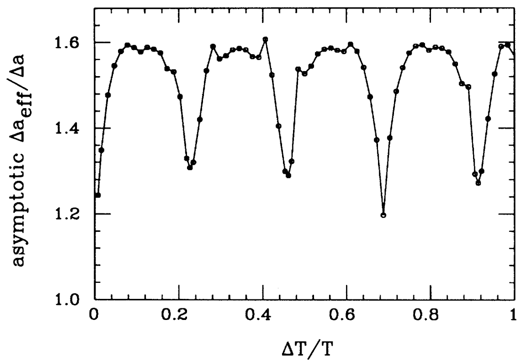

FIG. 4. Dependence of the asymptotic effective uncertainty $\Delta a_{\rm eff}$ on the quiescent time ΔT for the anharmonic oscillator with $\lambda = 4$.

with an asymptotic $\Delta a_{\rm eff} = \Delta a$ also in the impulsive regime.

Two problems recently under investigation also from a phenomenological point of view may take advantage of our approach. First, it has been suggested that the hypothesis of realism underlying classical mechanics can be confronted in the macroscopic domain with quantum predictions, namely, the existence of macroscopic distinguishable states, measuring the magnetic flux in a rf SQUID [8–10]. In this proposal there is also the assumption of a so-called noninvasive measurement whose role has been criticized due to a potential incompatibility with limitations in the accuracy of any measurement dictated by the uncertainty principle [11–13]. Second, a quantum Zeno effect has been proposed to account for an experiment involving inhibition of optical transitions between quantum states due to the measurement process [14] but some debate in the literature followed on the validity of such an interpretation [15, 16]. A quantitative study of both these problems is possible within the framework we propose here.

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