Continuous measurements in quantum systems

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During a continuous measurement, quantum systems can be described by a stochastic Schrödinger equation which, in the appropriate limit, reproduces the von Neumann wavefunction collapse. The average behavior on the ensemble of all measurement results is described by a master equation obtained from a general model of measurement apparatus consisting of an infinite set of degrees of freedom linearly interacting with the measured system and in contact with a reservoir at high temperature.

In ordinary quantum mechanics measurements are taken into account by postulating the wave function collapse [1]. This approach has the unpleasant feature of introducing an extra assumption in the theory and deals only with instantaneous and perfect measurements. It is more realistic and satisfactory to recognize that a measured system is not isolated but in interaction with a measurement apparatus. Of course, a detailed description of a particular measurement apparatus would have limited utility. One needs an effective quantum equation which, while reproducing the main features of a measurement process, does not explicitly contain the degrees of freedom of the meter.

During the past 20 years several approaches have been attempted in the above mentioned direction. It turns out that, apart from a difference in language and points of view, a mathematical equivalence can be established among all them [2]. Here, we briefly review some aspects of the corresponding modified quantum mechanics which includes the effects of a continuous measurement.

Let us consider a system initially prepared in the state $|\psi(0)\rangle$. During the measurement of an observable corresponding to the operator $\hat{A}$ this state should evolve, preserving its norm, into one corresponding to a definite eigenvalue of $\hat{A}$. The evolution must have a stochastic character since different measurements with the same initial state $|\psi(0)\rangle$ may give different results. All these requirements are satisfied by the nonlinear quantum state diffusion equation, proposed in [3–5],

$$
\begin{aligned}
\mathrm{d}\big|\psi_{[\xi]}(t)\big\rangle = \bigg\{ &-\frac{\mathrm{i}}{\hbar}\widehat{H} - \frac{1}{2}\kappa\big[\hat{A} - \big\langle\psi_{[\xi]}(t)\big|\,\hat{A}\,\big|\psi_{[\xi]}(t)\big\rangle\big]^2 \bigg\}\big|\psi_{[\xi]}(t)\big\rangle\mathrm{d}t \\
&+ \sqrt{\kappa}\big[\hat{A} - \big\langle\psi_{[\xi]}(t)\big|\,\hat{A}\,\big|\psi_{[\xi]}(t)\big\rangle\big]\big|\psi_{[\xi]}(t)\big\rangle\xi(t)\,\mathrm{d}t.
\end{aligned}
\tag{1}
$$

The stochastic process $\xi(t)$ can be chosen as a real white noise

$$\int \mathrm{d}[\xi]\xi(t) = 0, \qquad \int \mathrm{d}[\xi]\xi(t)\xi(s) = \delta(t - s) \tag{2}$$

with respect to the functional integration measure

$$\mathrm{d}[\xi] = \lim_{N\to\infty} \prod_{n=1}^{N} \mathrm{d}\xi^{(n)} \sqrt{\frac{t}{2\pi N}} \exp\left(-\frac{t\xi^{(n)2}}{2N}\right). \tag{3}$$

The constant κ measures the strength of the coupling with the (infinite) degrees of freedom of the measurement apparatus represented by $\xi(t)$. Equation (1) reduces to the ordinary Schrödinger equation for $\kappa = 0$.

The stochastic process $\langle\psi_{[\xi]}(t)|\hat{A}|\psi_{[\xi]}(t)\rangle$ is the result of a single continuous measurement. If many measurements are repeated with the measured system always prepared in the same initial state $|\psi(0)\rangle$, we define an average result and an associated variance by averaging over the Gaussian stochastic process $\xi(t)$:

$$\overline{a(t)} = \int \mathrm{d}[\xi]\langle\psi_{[\xi]}(t)|\,\hat{A}\,|\psi_{[\xi]}(t)\rangle, \tag{4}$$

$$\Delta a(t)^2 = \int \mathrm{d}[\xi]\langle\psi_{[\xi]}(t)|\left[\hat{A} - \overline{a(t)}\right]^2|\psi_{[\xi]}(t)\rangle. \tag{5}$$

These and other averaged quantities can be evaluated in a more direct way by considering a dynamical equation for the averaged density matrix operator

$$\hat{\rho}(t) = \int \mathrm{d}[\xi]|\psi_{[\xi]}(t)\rangle\langle\psi_{[\xi]}(t)|. \tag{6}$$

By using Ito algebra and eq. (1), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = -\frac{\mathrm{i}}{\hbar}\left[\widehat{H}, \hat{\rho}(t)\right] - \frac{1}{2}\kappa\left[\hat{A}, \left[\hat{A}, \hat{\rho}(t)\right]\right] \tag{7}$$

which reduces to the standard quantum master equation for $\kappa = 0$. Equation (7) is of Lindblad class and this ensures the positivity of $\hat{\rho}(t)$ [6,7]. Since no realizations of the stochastic process $\xi(t)$ have now to be selected, we call *nonselective measurement* the process described by eq. (7). *Selective measurements* are those described by eq. (1). According to the usual quantum rules, we finally write

$$\overline{a(t)} = \mathrm{Tr}\left[\hat{A}\hat{\rho}(t)\right], \tag{8}$$

$$\Delta a(t)^2 = \mathrm{Tr}\left[\left(\hat{A} - \overline{a(t)}\right)^2\hat{\rho}(t)\right] \tag{9}$$

and similarly for other averaged quantities.

The usual von Neumann collapse can be recovered for an impulsive and infinitely strong measurement. Let a_n and $|a_n\rangle$ be the eigenvalues and eigenvectors of $\hat{A}$, respectively (for simplicity, we assume a nondegenerate and discrete spectrum). According to eq. (1), after a very short time t and for κ so large that $\kappa t \gg 1$ the initial

wave function $|\psi(0)\rangle$ transforms into a state $|\psi_{\xi^{(0)}}(t)\rangle$ which depends on $\xi^{(0)}$, the value of the stochastic process at time $t = 0$, and has $\hat{A}$-representation

$$\langle a_n|\psi_{\xi^{(0)}}(t)\rangle \simeq \exp\left[-\kappa t\left(a_n - a(0) - \frac{\xi^{(0)}}{2\sqrt{\kappa}}\right)^2\right]\exp\left(\frac{\xi^{(0)2}t}{4}\right)\langle a_n|\psi(0)\rangle, \quad (10)$$

where $a(0) = \langle\psi(0)|\hat{A}|\psi(0)\rangle$. Provided that $a(0) \neq a_n$ for all n, in the limit $t \to 0$ and $\kappa t \to \infty$ eq. (10) shows an instantaneous collapse into the eigenstate of $\hat{A}$ with eigenvalue closest to $a(0) + \xi^{(0)}/\sqrt{\kappa}$. By averaging over all possible values of $\xi^{(0)}$, we get

$$\overline{\langle\psi_{\xi^{(0)}}(t)|\,\hat{A}\,|\psi_{\xi^{(0)}}(t)\rangle} = \int d\xi^{(0)}\sqrt{\frac{t}{2\pi}}\exp\left(-\frac{t\xi^{(0)2}}{2}\right)\sum_n a_n\big|\langle a_n|\psi_{\xi^{(0)}}(t)\rangle\big|^2$$

$$= \sum_n a_n\big|\langle a_n|\psi(0)\rangle\big|^2. \quad (11)$$

This is the result expected on the basis of the von Neumann postulate: in a measurement of $\hat{A}$ at time $t = 0$, the probability that the state $|\psi(0)\rangle$ collapses into the eigenstate $|a_n\rangle$ is $|\langle a_n|\psi(0)\rangle|^2$. Analogously, eq. (7) provides an instantaneous diagonalization of the density matrix in the $\hat{A}$-representation.

The basic eqs. (1) and (7) can be obtained by considering an explicit model of measurement apparatus. We let the measured system, described by the classical Hamiltonian $H(p, q)$, to interact with an infinite set of degrees of freedom (P_n, Q_n) via a harmonic potential. The total system is closed and the Hamiltonian is given by $H_{\text{tot}} = H + H_{\text{env}}$, where

$$H_{\text{env}} = \sum_n\left(\frac{P_n^2}{2M} + \frac{M\omega_n^2}{2}\big[Q_n - \lambda A(p, q)\big]^2\right). \quad (12)$$

The constant λ is a transduction factor. By choosing initial conditions corresponding to the environment variables at thermal equilibrium with temperature T around the instantaneous state of the measured system, the reduced density matrix of the measured system $\rho(q_1, q_2, t)$ can be exactly evaluated [2]. Moreover, for a proper continuous distribution of the frequencies ω_n, eq. (7) is recovered in the high temperature limit. Equation (1) is then obtained via a decomposition of the two-particle Green function associated to $\rho(q_1, q_2, t)$ into a couple of single-particle Green functions [2]. The phenomenological constant κ is expressed in terms of the parameters of the model.

The high temperature limit necessary for obtaining eqs. (1) and (7) from the above mentioned model has a definite physical meaning. In order to avoid paradoxical features, the measurement apparatus has to be classical with respect to an external observer [8] and in our model this is obtained for $k_B T \gg \hbar\omega_n$.

In the case of measurements of position, in [9] it has been shown that eqs. (1) and (7) provide a correct behavior also in the controversial case of measurements on macro-

scopic systems. Indeed, classical behavior is always established in the macroscopic limit due to an instantaneous convergence into properly defined coherent states.

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References

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