CAMPI ELETTRICI
E MAGNETICI
VARIABILI NEL TEMPO

17

The Laws of Induction

17–1 The physics of induction

In the last chapter we described many phenomena which show that the effects of induction are quite complicated and interesting. Now we want to discuss the fundamental principles which govern these effects. We have already defined the emf in a conducting circuit as the total accumulated force on the charges throughout the length of the loop. More specifically, it is the tangential component of the force per unit charge, integrated along the wire once around the circuit. This quantity is equal, therefore, to the total work done on a single charge that travels once around the circuit.

We have also given the "flux rule," which says that the emf is equal to the rate at which the magnetic flux through such a conducting circuit is changing. Let's see if we can understand why that might be. First, we'll consider a case in which the flux changes because a circuit is moved in a steady field.

In Fig. 17–1 we show a simple loop of wire whose dimensions can be changed. The loop has two parts, a fixed U-shaped part (a) and a movable crossbar (b) that can slide along the two legs of the U. There is always a complete circuit, but its area is variable. Suppose we now place the loop in a uniform magnetic field with the plane of the U perpendicular to the field. According to the rule, when the crossbar is moved there should be in the loop an emf that is proportional to the rate of change of the flux through the loop. This emf will cause a current in the loop. We will assume that there is enough resistance in the wire that the currents are small. Then we can neglect any magnetic field from this current.

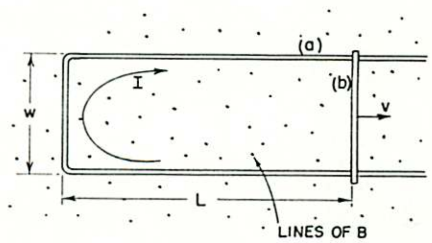


Fig. 17–1. An emf is induced in a loop if the flux is changed by varying the area of the circuit.

The flux through the loop is wLB, so the "flux rule" would give for the emf—which we write as $\mathcal{E}$—

$$\mathcal{E} = wB\frac{dL}{dt} = wBv,$$

where v is the speed of translation of the crossbar.

Now we should be able to understand this result from the magnetic $\boldsymbol{v} \times \boldsymbol{B}$ forces on the charges in the moving crossbar. These charges will feel a force, tangential to the wire, equal to vB per unit charge. It is constant along the length w of the crossbar and zero elsewhere, so the integral is

$$\mathcal{E} = wvB,$$

which is the same result we got from the rate of change of the flux.

The argument just given can be extended to any case where there is a fixed magnetic field and the wires are moved. One can prove, in general, that for any circuit whose parts move in a fixed magnetic field the emf is the time derivative of the flux, regardless of the shape of the circuit.

On the other hand, what happens if the loop is stationary and the magnetic field is changed? We cannot deduce the answer to this question from the same argument. It was Faraday's discovery—from experiment—that the "flux rule" is still correct no matter why the flux changes. The force on electric charges is given in complete generality by $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$; there are no new special "forces due to changing magnetic fields." Any forces on charges at rest in a stationary wire come from the $\boldsymbol{E}$ term. Faraday's observations led to the discovery that electric and magnetic fields are related by a new law: in a region where the magnetic field is changing with time, electric fields are generated. It is this electric field which drives the electrons around the wire—and so is responsible for the emf in a stationary circuit when there is a changing magnetic flux.

The general law for the electric field associated with a changing magnetic field is

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}. \tag{17.1}$$

We will call this Faraday's law. It was discovered by Faraday but was first written in differential form by Maxwell, as one of his equations. Let's see how this equation gives the "flux rule" for circuits.

Lorentz's law

$\boxed{\text{circuit moves}}$

$F/q = E + v \times B$

$\boxed{\text{flux rule}}$ *

$f_{e.m.} = -\frac{\partial \phi(B)}{\partial t}$

$\nabla \times E = -\frac{\partial B}{\partial t}$

$\boxed{\text{field changes}}$

Faraday's law

Using Stokes' theorem, this law can be written in integral form as

$$\oint_\Gamma \boldsymbol{E}\cdot d\boldsymbol{s} = \int_S (\nabla \times \boldsymbol{E})\cdot \boldsymbol{n}\, da = -\int_S \frac{\partial \boldsymbol{B}}{\partial t}\cdot \boldsymbol{n}\, da, \tag{17.2}$$

where, as usual, Γ is any closed curve and S is any surface bounded by it. Here, remember, Γ is a *mathematical* curve fixed in space, and S is a fixed surface. Then the time derivative can be taken outside the integral and we have

$$\begin{aligned}\oint_\Gamma \boldsymbol{E}\cdot d\boldsymbol{s} &= -\frac{\partial}{\partial t}\int_S \boldsymbol{B}\cdot\boldsymbol{n}\, da \\ &= -\frac{\partial}{\partial t}\,(\text{flux through } S).\end{aligned} \tag{17.3}$$

Applying this relation to a curve Γ that follows a *fixed* circuit of conductor, we get the "flux rule" once again. The integral on the left is the emf, and that on the right is the negative rate of change of the flux linked by the circuit. So Eq. (17.1) applied to a fixed circuit is equivalent to the "flux rule."

* with exceptions:

So the "flux rule"—that the emf in a circuit is equal to the rate of change of the magnetic flux through the circuit—applies whether the flux changes because the field changes or because the circuit moves (or both). The two possibilities—"circuit moves" or "field changes"—are not distinguished in the statement of the rule. Yet in our explanation of the rule we have used two completely distinct laws for the two cases—$\boldsymbol{v} \times \boldsymbol{B}$ for "circuit moves" and $\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$ for "field changes."

a) flux does not change and $emf \neq 0$

b) flux changes and $e.m.f = 0$

We know of no other place in physics where such a simple and accurate general principle requires for its real understanding an analysis in terms of *two different phenomena*. Usually such a beautiful generalization is found to stem from a single deep underlying principle. Nevertheless, in this case there does not appear to be any such profound implication. We have to understand the "rule" as the combined effects of two quite separate phenomena.

We must look at the "flux rule" in the following way. In general, the force per unit charge is $\boldsymbol{F}/q = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$. In moving wires there is the force from the second term. Also, there is an $\boldsymbol{E}$-field if there is somewhere a changing magnetic field. They are independent effects, but the emf around the loop of wire is always equal to the rate of change of magnetic flux through it.

17–2 Exceptions to the "flux rule"

We will now give some examples, due in part to Faraday, which show the importance of keeping clearly in mind the distinction between the two effects responsible for induced emf's. Our examples involve situations to which the "flux rule" cannot be applied—either because there is no wire at all or because the *path* taken by induced currents moves about within an extended volume of a conductor.

We begin by making an important point: The part of the emf that comes from the $\boldsymbol{E}$-field does not depend on the existence of a physical wire (as does the $\boldsymbol{v} \times \boldsymbol{B}$ part). The $\boldsymbol{E}$-field can exist in free space, and its line integral around any imaginary line fixed in space is the rate of change of the flux of $\boldsymbol{B}$ through that line. (Note that this is quite unlike the $\boldsymbol{E}$-field produced by static charges, for in that case the line integral of $\boldsymbol{E}$ around a closed loop is always zero.)

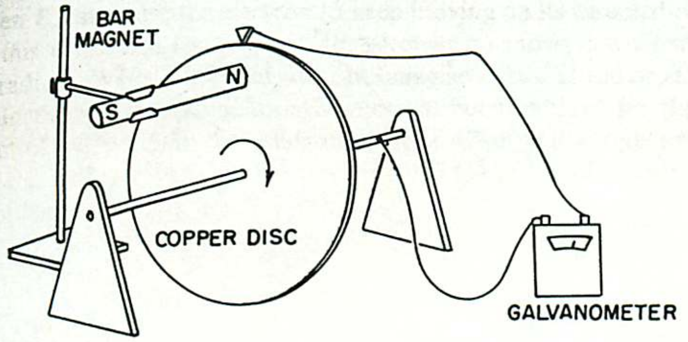


Fig. 17–2. When the disc rotates there is an emf from $\boldsymbol{v} \times \boldsymbol{B}$, but with no change in the linked flux.

Now we will describe a situation in which the flux through a circuit does not change, but there is nevertheless an emf. Figure 17–2 shows a conducting disc which can be rotated on a fixed axis in the presence of a magnetic field. One contact is made to the shaft and another rubs on the outer periphery of the disc. A circuit is completed through a galvanometer. As the disc rotates, the "circuit," in the sense of the place in space where the currents are, is always the same. But the part of the "circuit" in the disc is in material which is moving. Although the flux through the "circuit" is constant, there is still an emf, as can be observed by the deflection of the galvanometer. Clearly, here is a case where the $v \times B$ force in the moving disc gives rise to an emf which cannot be equated to a change of flux.

a

Now we consider, as an opposite example, a somewhat unusual situation in which the flux through a "circuit" (again in the sense of the place where the current is) changes but where there is *no* emf. Imagine two metal plates with slightly curved edges, as shown in Fig. 17–3, placed in a uniform magnetic field perpendicular to their surfaces. Each plate is connected to one of the terminals of a galvanometer, as shown. The plates make contact at one point P, so there is a complete circuit. If the plates are now rocked through a small angle, the point of contact will move to P'. If we imagine the "circuit" to be completed through the plates on the dotted line shown in the figure, the magnetic flux through this circuit changes by a large amount as the plates are rocked back and forth. Yet the rocking can be done with small motions, so that $v \times B$ is very small and there is practically no emf. The "flux rule" does not work in this case. It must be applied to circuits in which the *material* of the circuit remains the same. When the material of the circuit is changing, we must return to the basic laws. The *correct* physics is always given by the two basic laws

b

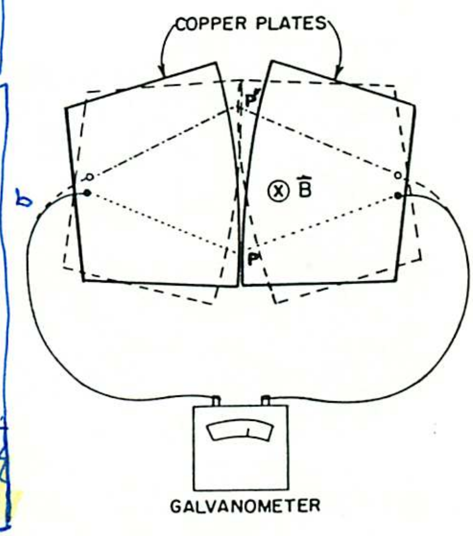


Fig. 17–3. When the plates are rocked in a uniform magnetic field, there can be a large change in the flux linkage without the generation of an emf.

$$F = q(E + v \times B),$$

$$\nabla \times E = -\frac{\partial B}{\partial t}\cdot$$

17–3 Particle acceleration by an induced electric field; the betatron

We have said that the electromotive force generated by a changing magnetic field can exist even without conductors; that is, there can be magnetic induction without wires. We may still imagine an electromotive force around an arbitrary mathematical curve in space. It is defined as the tangential component of E integrated around the curve. Faraday's law says that this line integral is equal to the rate of change of the magnetic flux through the closed curve, Eq. (17.3).

As an example of the effect of such an induced electric field, we want now to consider the motion of an electron in a changing magnetic field. We imagine a magnetic field which, everywhere on a plane, points in a vertical direction, as shown in Fig. 17–4. The magnetic field is produced by an electromagnet, but we will not worry about the details. For our example we will imagine that the magnetic field is symmetric about some axis, i.e., that the strength of the magnetic field will depend only on the distance from the axis. The magnetic field is also varying with time. We now imagine an electron that is moving in this field on a path that is a circle of constant radius with its center at the axis of the field. (We will see later

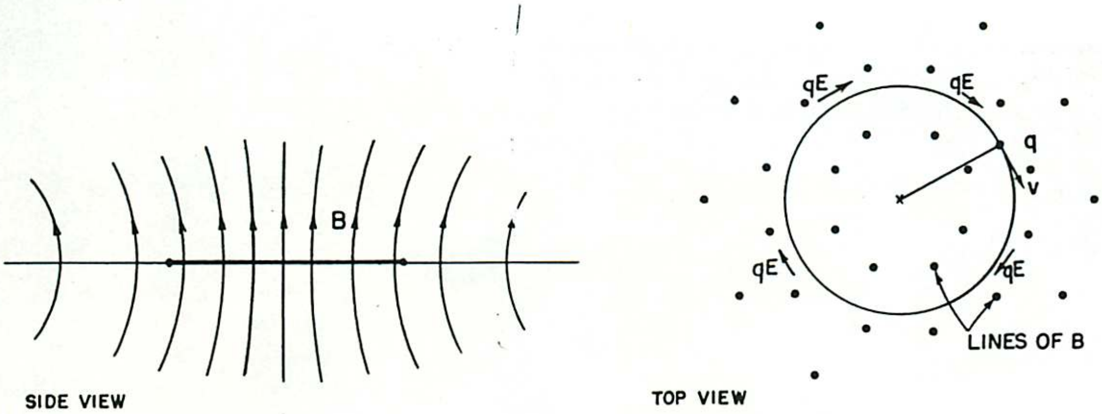


Fig. 17–4. An electron accelerating in an axially symmetric, time-varying magnetic field.

how this motion can be arranged.) Because of the changing magnetic field, there will be an electric field $\boldsymbol{E}$ tangential to the electron's orbit which will drive it around the circle. Because of the symmetry, this electric field will have the same value everywhere on the circle. If the electron's orbit has the radius r, the line integral of $\boldsymbol{E}$ around the orbit is equal to the rate of change of the magnetic flux through the circle. The line integral of $\boldsymbol{E}$ is just its magnitude times the circumference of the circle, $2\pi r$. The magnetic flux must, in general, be obtained from an integral. For the moment, we let B_{av} represent the average magnetic field in the interior of the circle; then the flux is this average magnetic field times the area of the circle. We will have

$$2\pi rE = \frac{\partial}{\partial t}(B_{\text{av}} \cdot \pi r^2).$$

Since we are assuming r is constant, E is proportional to the time derivative of the average field:

$$E = \frac{r}{2}\frac{dB_{\text{av}}}{dt}. \tag{17.4}$$

The electron will feel the electric force $q\boldsymbol{E}$ and will be accelerated by it. Remembering that the relativistically correct equation of motion is that the rate of change of the momentum is proportional to the force, we have

$$qE = \frac{dp}{dt}. \tag{17.5}$$

For the circular orbit we have assumed, the electric force on the electron is always in the direction of its motion, so its total momentum will be increasing at the rate given by Eq. (17.5). Combining Eqs. (17.5) and (17.4), we may relate the rate of change of momentum to the change of the average magnetic field:

$$\frac{dp}{dt} = \frac{qr}{2}\frac{dB_{\text{av}}}{dt}. \tag{17.6}$$

Integrating with respect to t, we find for the electron's momentum

$$p = p_0 + \frac{qr}{2}\Delta B_{\text{av}}, \tag{17.7}$$

where p_0 is the momentum with which the electrons start out, and ΔB_{av} is the subsequent change in B_{av}. The operation of a *betatron*—a machine for accelerating electrons to high energies—is based on this idea.

To see how the betatron operates in detail, we must now examine how the electron can be constrained to move on a circle. We have discussed in Chapter 11 of Vol. I the principle involved. If we arrange that there is a magnetic field $\boldsymbol{B}$ at the orbit of the electron, there will be a transverse force $q\boldsymbol{v} \times \boldsymbol{B}$ which, for a suitly chosen B, can cause the electron to keep moving on its assumed orbit. In the atron this transverse force causes the electron to move in a circular orbit of nstant radius. We can find out what the magnetic field at the orbit must be by ng again the relativistic equation of motion, but this time, for the transverse npc nt of the force. In the betatron (see Fig. 17–4), B is at right angles to v, so transverse force is qvB. Thus the force is equal to the rate of change of the transse component p_t of the momentum:

$$qvB = \frac{dp_t}{dt}. \tag{17.8}$$

ien a particle is moving in a *circle*, the rate of change of its transverse momentum qual to the magnitude of the total momentum times ω, the angular velocity of ation (following the arguments of Chapter 11, Vol. I):

$$\frac{dp_t}{dt} = \omega p, \tag{17.9}$$

ere, since the motion is circular,

$$\omega = \frac{v}{r}. \tag{17.10}$$

ting the magnetic force equal to the transverse acceleration, we have

$$qvB_{\text{orbit}} = p\,\frac{v}{r}, \tag{17.11}$$

ere B_{orbit} is the field at the radius r.

As the betatron operates, the momentum of the electron grows in proportion B_{av}, according to Eq. (17.7), and if the electron is to continue to move in its per circle, Eq. (17.11) must continue to hold as the momentum of the electron reases. The value of B_{orbit} must increase in proportion to the momentum p. nparing Eq. (17.11) with Eq. (17.7), which determines p, we see that the follow- relation must hold between B_{av}, the average magnetic field *inside* the orbit he radius r, and the magnetic field B_{orbit} at the orbit:

$$\Delta B_{\text{av}} = 2\,\Delta B_{\text{orbit}}. \tag{17.12}$$

correct operation of a betatron requires that the average magnetic field inside orbit increase at twice the rate of the magnetic field at the orbit itself. In these umstances, as the energy of the particle is increased by the induced electric d the magnetic field at the orbit increases at just the rate required to keep the ticle moving in a circle.

The betatron is used to accelerate electrons to energies of tens of millions of ts, or even to hundreds of millions of volts. However, it becomes impractical for acc ration of electrons to energies much higher than a few hundred million ts for several reasons. One of them is the practical difficulty of attaining the uired high average value for the magnetic field inside the orbit. Another is that (17.6) is no longer correct at very high energies because it does not include the of energy from the particle due to its radiation of electromagnetic energy so-called synchrotron radiation discussed in Chapter 36, Vol. I). For these sons, the acceleration of electrons to the highest energies—to many billions of tron volts—is accomplished by means of a different kind of machine, called a *chrotron.*

17–4 A paradox

We would now like to describe for you an apparent paradox. A paradox is a ation which gives one answer when analyzed one way, and a different answer en analyzed another way, so that we are left in somewhat of a quandary as to ially what should happen. Of course, in physics there are never any real para- es because there is only one correct answer; at least we believe that nature will

$i = n e v \qquad v = \frac{i}{n e}$

$$L = \frac{m r i}{n e} \sim \frac{9.1\,10^{-31} \cdot 0.01 \cdot 1}{1.6\,10^{-19} \cdot 10^{21}} = 10^{-34} \sim \hbar$$

$L_{tot} = L \cdot N_d$

L

act in only one way (and that is the *right way*, naturally). So in physics a parad· is only a confusion in our own understanding. Here is our paradox.

Imagine that we construct a device like that shown in Fig. 17–5. There is thin, circular plastic disc supported on a concentric shaft with excellent bearin so that it is quite free to rotate. On the disc is a coil of wire in the form of a sho solenoid concentric with the axis of rotation. This solenoid carries a steady curre I provided by a small battery, also mounted on the disc. Near the edge of the di and spaced uniformly around its circumference are a number of small metal spher insulated from each other and from the solenoid by the plastic material of the dis Each of these small conducting spheres is charged with the same electrostat charge Q. Everything is quite stationary, and the disc is at rest. Suppose now th by some accident—or by prearrangement—the current in the solenoid is inte rupted, without, however, any intervention from the outside. So long as the curre continued, there was a magnetic flux through the solenoid more or less paral to the axis of the disc. When the current is interrupted, this flux must go to zer There will, therefore, be an electric field induced which will circulate around circles centered at the axis. The charged spheres on the perimeter of the disc w all experience an electric field tangential to the perimeter of the disc. This electi force is in the same sense for all the charges and so will result in a net torque on t disc. From these arguments we would expect that as the current in the soleno disappears, the disc would begin to rotate. If we knew the moment of inertia the disc, the current in the solenoid, and the charges on the small spheres, we cou compute the resulting angular velocity.

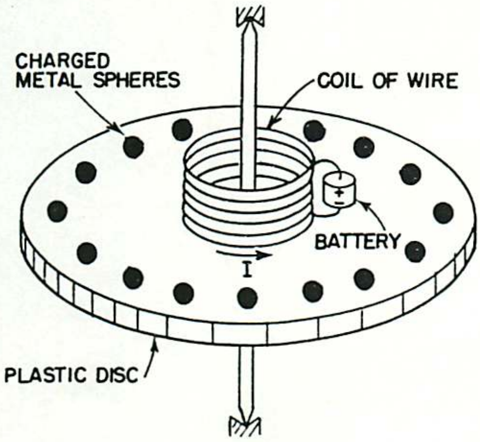


Fig. 17–5. Will the disc rotate if the current I is stopped?

But we could also make a different argument. Using the principle of the co servation of angular momentum, we could say that the angular momentum of t disc with all its equipment is initially zero, and so the angular momentum of t assembly should remain zero. There should be no rotation when the current stopped. Which argument is correct? Will the disc rotate or will it not? We w leave this question for you to think about.

We should warn you that the correct answer does not depend on any no essential feature, such as the asymmetric position of a battery, for example. fact, you can imagine an ideal situation such as the following: The solenoid made of superconducting wire through which there is a current. After the disc h been carefully placed at rest, the temperature of the solenoid is allowed to rise slowl When the temperature of the wire reaches the transition temperature betwe superconductivity and normal conductivity, the current in the solenoid will brought to zero by the resistance of the wire. The flux will, as before, fall to zer and there will be an electric field around the axis. We should also warn you that t solution is not easy, nor is it a trick. When you figure it out, you will have di covered an important principle of electromagnetism.

17–5 Alternating-current generator

In the remainder of this chapter we apply the principles of Section 17–1 analyze a number of the phenomena discussed in Chapter 16. We first look in mo detail at the alternating-current generator. Such a generator consists basically of coil of wire rotating in a uniform magnetic field. The same result can also achieved by a fixed coil in a magnetic field whose direction rotates in the mann described in the last chapter. We will consider only the former case. Suppose v have a circular coil of wire which can be turned on an axis along one of its dian eters. Let this coil be located in a uniform magnetic field perpendicular to the ax of rotation, as in Fig. 17–6. We also imagine that the two ends of the coil a brought to external connections through some kind of sliding contacts.

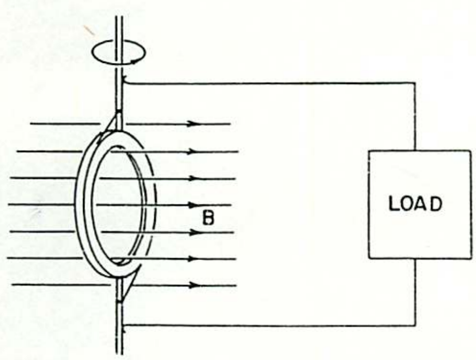
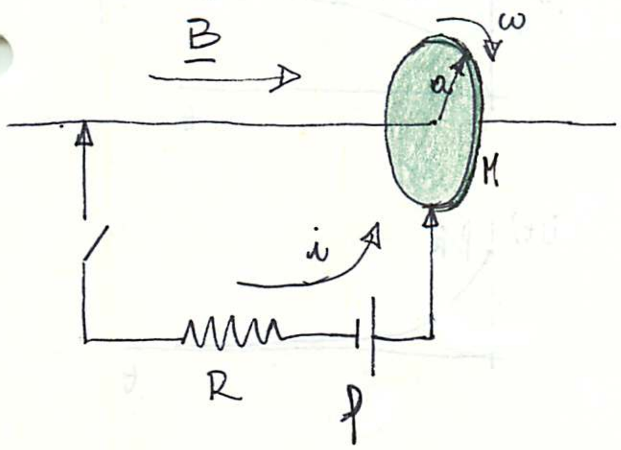


Fig. 17–6. A coil of wire rotating in a uniform magnetic field—the basic idea of the ac generator.

Due to the rotation of the coil, the magnetic flux through it will be changin The circuit of the coil will therefore have an emf in it. Let S be the area of the co and θ the angle between the magnetic field and the normal to the plane of the coil

* Now that we are using the letter A for the vector potential, we prefer to let S star for a Surface area.



Il disco conduttore di raggio a e massa M può ruotare senza attrito. All'istante t=0 il circuito viene chiuso con il disco fermo. Trovare ω(t)

(Ruota di Barlow)

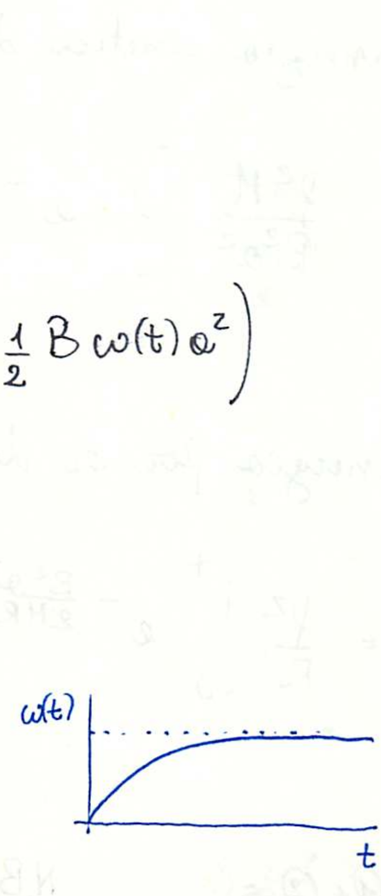
$\omega(0) = 0$

$$\begin{cases} f - f_i(t) = i(t)\,R \\ f_i(t) = \frac{1}{2} B\,\omega(t)\,a^2 = \int_0^a B\cdot\omega x\,dx = \int_0^a \underline{E}\cdot d\underline{l} = \int_0^a (\underline{v}\times\underline{B})\cdot d\underline{l} \\ I\,\dot{\omega}(t) = \frac{1}{2}\,i(t)\,B\,a^2 \qquad I = \frac{1}{2} M a^2 \end{cases}$$

$$\longrightarrow \int_0^a x\;\;i\,B\,dx$$

$$I\,\dot{\omega}(t) = \frac{1}{2} B a^2\,\frac{f - f_i(t)}{R} = \frac{1}{2}\frac{B a^2}{R}\left(f - \frac{1}{2} B\,\omega(t)\,a^2\right)$$

$$\boxed{\dot{\omega}(t) + \frac{B^2 a^4}{4RI}\,\omega(t) = \frac{B a^2 f}{2RI}}$$

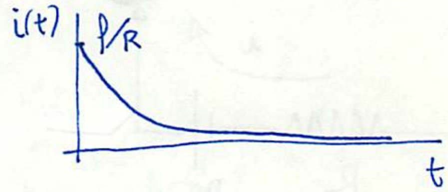


$$\omega(t) = A\,e^{-\frac{B^2 a^4}{4RI}t} + \frac{2f}{B a^2}$$

$$\omega(t) = \frac{2f}{B a^2}\left[1 - e^{-\frac{B^2 a^4}{4RI}t}\right] = \frac{2f}{B a^2}\left[1 - e^{-\frac{B^2 a^2}{2MR}t}\right]$$

$$f_i(t) = f\left[1 - e^{-\frac{B^2a^2t}{2MR}}\right]$$

$$i(t) = \frac{f}{R}\, e^{-\frac{B^2a^2}{2MR}t}$$



① energia dissipata Joule al tempo $t =$

$$= \int_0^t i(t')^2 R\, dt' = \frac{f^2}{R}\int_0^t e^{-\frac{B^2a^2}{MR}t'}\, dt' = \frac{f^2 M}{B^2a^2}\left[1 - e^{-\frac{B^2a^2t}{MR}}\right]$$

② energia cinetica del disco $= \frac{1}{2} I\, \omega(t)^2 =$

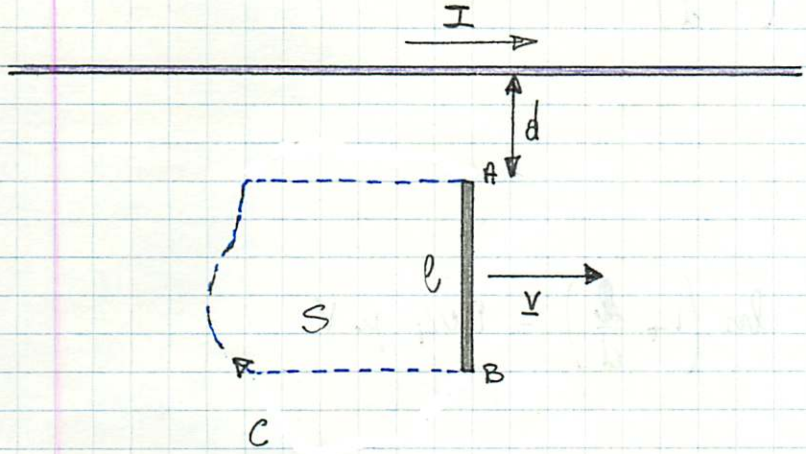
$$= \frac{f^2 M}{B^2a^2}\cdot\left[1 + e^{-\frac{B^2a^2t}{MR}} - 2\, e^{-\frac{B^2a^2t}{2MR}}\right]$$

③ energia fornita dal generatore $= \int_0^t f\, i(t')\, dt' =$

$$= \frac{f^2}{R}\int_0^t e^{-\frac{B^2a^2}{2MR}t'}\, dt' = 2\,\frac{f^2 M}{B^2a^2}\left[1 - e^{-\frac{B^2a^2t}{2MR}}\right]$$

① + ② = ③ N.B.: questo è un esempio di eccezione alla regola del flusso (ϕ è costante) la quale può essere applicata solo allo stesso circuito materiale. Le leggi sempre corrette sono $\underline{F} = q(\underline{E} + \underline{v}\times\underline{B})$ $\underline{\nabla}\times\underline{E} = -\frac{\partial B}{\partial t}$

Una barra metallica di lunghezza $l = 10$ cm si muove di moto traslatorio uniforme con velocità $v = 1\ \text{ms}^{-1}$ mantenendosi perpendicolare ad un filo rettilineo indefinito percorso da corrente $I = 5A$ a distanza $d = 5.5$ cm da esso. Calcolare la d.d.p. ΔV tra gli estremi della sbarra.



Si consideri l'equazione di Maxwell $\underline{\nabla} \times \underline{E} = -\frac{\partial}{\partial t}\underline{B}$

e se ne faccia il flusso attraverso una qualsiasi superficie che abbia la barra come tratto di contorno

$$\int_S \underline{\nabla} \times \underline{E} \cdot \hat{n}\, dS = -\int_S \frac{\partial \underline{B}}{\partial t} \cdot \hat{n}\, dS = -\frac{d}{dt}\int_S \underline{B} \cdot \hat{n}\, dS$$

per il teorema di Stockes e osservando che $\underline{E}$ è nullo al di fuori della barra :

$$\int_S \underline{\nabla} \times \underline{E} \cdot \hat{n}\, dS = \int_C \underline{E} \cdot d\underline{l} = \int_A^B \underline{E} \cdot d\underline{l} = V_B - V_A$$

si noti che $\underline{E}$ è un campo elettromotore e vale $V_B - V_A = +\int_A^B \underline{E} \cdot d\underline{l}$

Nel tempo infinitesimo dt si ha:

$$d\int_S \underline{B}\cdot\hat{n}\, dS = \int_d^{d+l} \frac{\mu_0 I}{2\pi x}\, v\, dt\, dx =$$

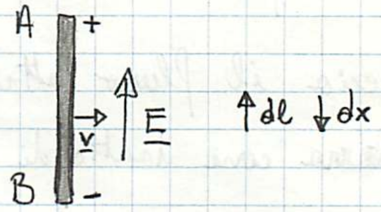
$$= \frac{\mu_0 I}{2\pi}\, v\, dt\, \ln\left(1+\frac{l}{d}\right) \quad > 0$$

quindi:

$$V_A - V_B = \frac{\mu_0 I v}{2\pi} \ln\left(1+\frac{l}{d}\right) = 1.04\ \mu V$$

alternativamente il campo elettromotore è dato da:

$$\underline{E} = \underline{v} \times \underline{B}$$



$$V_A - V_B = \int_B^A \underline{E}\cdot d\underline{l} = \int_{d+l}^{d} \frac{\mu_0 I v}{2\pi x}(-dx) = -\frac{\mu_0 I v}{2\pi} \ln\left(\frac{d}{d+l}\right)$$

$$= \frac{\mu_0 I v}{2\pi} \ln\left(1+\frac{l}{d}\right) = 1.04\ \mu V$$

the flux rule (integral statement) $\mathcal{E} = -\frac{d}{dt}\Phi$ applies to both the situations "circuit moves" (differential statement $\underline{E} = \underline{F}/q = \underline{v}\times\underline{B}$). and "field changes" (differential statement $\underline{\nabla}\times\underline{E} = -\frac{\partial B}{\partial t}$)

(but the material circuit must be unchanged!)

Un anello conduttore di raggio a è formato da due semianelli di resistività ρ_1 e ρ_2 per unità di lunghezza. Esso è immerso in un campo di induzione magnetica perpendicolare al piano dell'anello e variabile nel tempo con legge $B = \beta \cdot t$. Determinare la d.d.p. tra i punti di giunzione dei due semianelli

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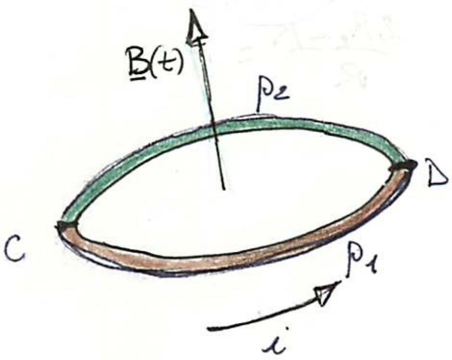


$$\underline{\nabla} \times \underline{E} = -\frac{\partial B}{\partial t} \qquad \text{Faraday's law}$$

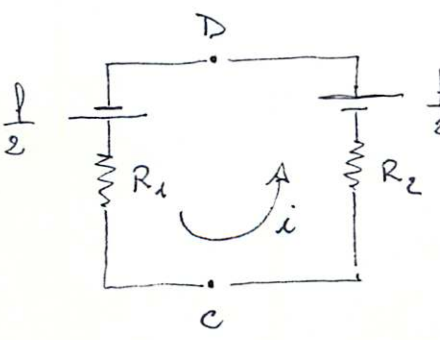
$$f.e.m. = \int_C \underline{E} \cdot d\underline{\ell} = \int_S \underline{\nabla} \times \underline{E} \cdot \hat{n}\, dS = -\int_S \frac{\partial B}{\partial t} \cdot \hat{n}\, dS$$

$$|f.e.m.\ indotta| = |f| = \frac{d\,\phi(B)}{dt} = \pi a^2 \beta$$

$$\text{corrente circolante} = i = \frac{f}{R} = \frac{\pi a^2 \beta}{\pi(\rho_1 + \rho_2)a}$$

$$f = \oint \underline{E}_i \cdot d\underline{\ell} = \int_D^C \underline{E}_i \cdot d\underline{\ell} + \int_C^D \underline{E}_i \cdot d\underline{\ell} = \frac{f}{2} + \frac{f}{2} \quad \text{per simmetria}$$

circuito equivalente per determinare $V_C - V_D$



$R_1 = \pi a \rho_1$

$R_2 = \pi a \rho_2$

usando il ramo sinistro:

$$V_C - V_D = \frac{f}{2} - iR_1 = \frac{f}{2} - f\,\frac{R_1}{R} = \frac{f}{2}\,\frac{R-2R_1}{R} =$$

$$= \frac{f}{2}\,\frac{\rho_2-\rho_1}{\rho_1+\rho_2} = \frac{\pi a^2 B}{2}\,\frac{\rho_2-\rho_1}{\rho_2+\rho_1}$$

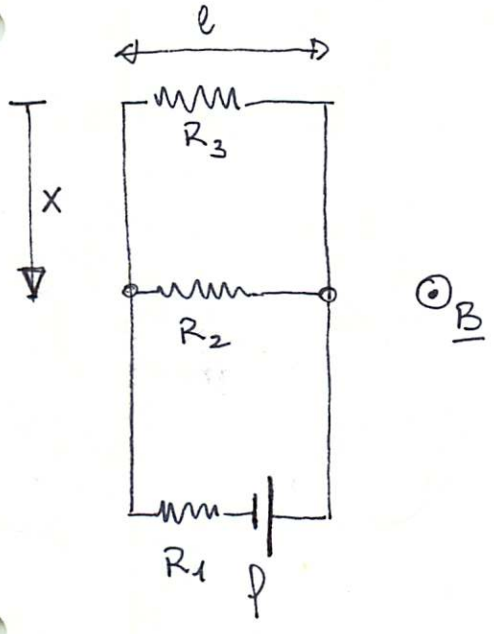
alternativamente usando quello destro:

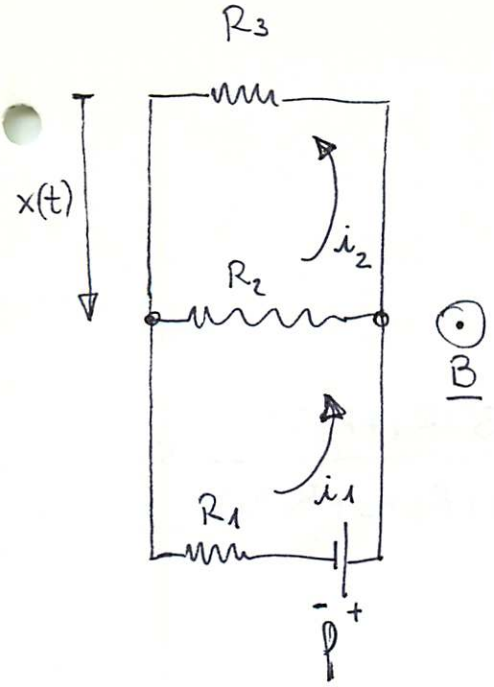
$$V_C - V_D = -\frac{f}{2} + iR_2 = -\frac{f}{2} + f\,\frac{R_2}{R} = \frac{f}{2}\,\frac{2R_2-R}{R} =$$

$$= \frac{f}{2}\,\frac{\rho_2-\rho_1}{\rho_1+\rho_2}$$

Il circuito mostrato in figura è immerso in un campo di induzione magnetica $\underline{B}$ uniforme perpendicolare al piano del circuito e uscente dal foglio. La resistenza R_2 ha massa m e può scorrere senza attrito lungo la direzione verticale $\hat{x}$. Discutere il moto di R_2 al variare del valore della f.e.m. f supponendo che all'istante iniziale sia $x(0) = x_0$ ed $\dot{x}(0) = 0$.

Si trascuri il fenomeno di autoinduzione





indicate con i_1 ed i_2 le correnti nelle due maglie le relative equ. di Kirchhoff sono

$$\begin{cases} f + f_{i1} = i_1 (R_1 + R_2) - i_2 R_2 \\ f_{i2} = -i_1 R_2 + i_2 (R_2 + R_3) \end{cases}$$

se $x(t)$ è la posizione di R_2 al tempo t ed $\dot{x}(t)$ la sua velocità le f.e.m. indotte nelle maglie 1 e 2 sono

$$f_{i1} = l B \dot{x}(t) \qquad f_{i2} = - l B \dot{x}(t)$$

campo di Lorentz

$\underline{E} = \underline{v} \times \underline{B}$

f.e.m. $= \int \underline{E} \cdot d\underline{l} = \pm l v B$

(f_{i1} è concorde con f se $\dot{x} > 0$)

si ha quindi

$$\begin{cases} f = i_1 R_1 + i_2 R_3 \\ l B \dot{x} = i_1 R_2 - i_2 (R_2 + R_3) \end{cases}$$

con soluzione

$$i_1 = \frac{f (R_2 + R_3) + l B R_3 \dot{x}}{R_1 R_2 + R_2 R_3 + R_3 R_1} \qquad i_2 = \frac{f R_2 - l B R_1 \dot{x}}{R_1 R_2 + R_2 R_3 + R_3 R_1}$$

l'eq. del moto di R_2 è :

$$m\ddot{x} = mg - (i_1 - i_2)\, l B$$

$$\ddot{x} = g - \frac{lBfR_3}{m(R_1R_2+R_2R_3+R_3R_1)} - \frac{l^2B^2(R_1+R_3)}{m(R_1R_2+R_2R_3+R_3R_1)}\dot{x}$$

posto $\quad G \equiv g - \dfrac{lBR_3 f}{m(R_1R_2+R_2R_3+R_3R_1)}$

$$K \equiv \frac{l^2B^2(R_1+R_3)}{m(R_1R_2+R_2R_3+R_3R_1)}$$

si ha $\quad \ddot{x}(t) + K\dot{x}(t) = G$

$$\dot{x}(t) = \frac{G}{K}\left(1 - e^{-Kt}\right)$$

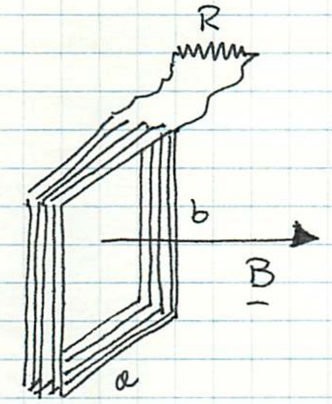
$$x(t) = x_0 + \frac{G}{K}t - \frac{G}{K^2}\left(1 - e^{-Kt}\right)$$

se $G > 0$ i.e. $f < \dfrac{mg(R_1R_2+R_2R_3+R_3R_1)}{lBR_3}$ $\quad$ R_2 scende

se $G = 0$ i.e. $f =$ $\quad$ " $\quad$ R_2 rimane ferma

se $G < 0$ i.e. $f >$ $\quad$ " $\quad$ R_2 sale

Una bobina costituita da $N = 100$ spire rettangolari di lati $a = 2$ cm e $b = 3$ cm è posta in un campo $B = 0.1\,T$ perpendicolare al piano delle spire. La bobina è collegata ad una resistenza $R = 10\,\Omega$. Determinare la carica totale che fluisce attraverso la resistenza quando la bobina viene portata in una regione di spazio in cui $B = 0$.



durante lo spostamento $\phi(B)$ varia e si ha una corrente circolante

$$i(t) = \frac{f(t)}{R} = \frac{1}{R}\frac{d}{dt}\phi(B(t))$$

la carica che attraversa la resistenza è:

$$q = \int_0^\infty i(t)\,dt = \frac{1}{R}\int_0^\infty \frac{d\phi}{dt}\,dt = \frac{1}{R}\Delta\phi$$

dove $\Delta\phi = 0 - BNab$

$$|q| = \frac{BNab}{R} = 6 \cdot 10^{-4}\,C$$

la potenza Joule è:

$$P_J(t) = f(t)\cdot i(t) = \frac{f(t)^2}{R} = \frac{(2\pi\nu B l^2 N)^2}{R}\sin^2(2\pi\nu t)$$

la corrispondente potenza media è:

$$\langle P_J\rangle = \frac{1}{T}\int_0^T P_J(t)\,dt = \nu\int_0^{\frac{1}{\nu}} \frac{(2\pi\nu B l^2 N)^2}{R}\sin^2(2\pi\nu t)\,dt =$$

$$= \nu\,\frac{(2\pi\nu B l^2 N)^2}{R}\,\frac{1}{2\pi\nu}\int_0^{2\pi}\sin^2\theta\,d\theta =$$

$$= \frac{2\pi^2\nu^2 B^2 l^4 N^2}{R} = 2\,\frac{(\pi\nu B N l^2)^2}{R} = 4.9\ 10^4\ W$$

All'istante t la bobina subisce un momento meccanico

$$|\underline{m}\times\underline{B}| = N l^2\, i(t)\; B\sin\theta(t) = \frac{N^2 l^4 B^2 2\pi\nu}{R}\sin^2(2\pi\nu t)$$

Il lavoro che occorre compiere per far eseguire alla bobina un giro completo a frequenza ν costante è:

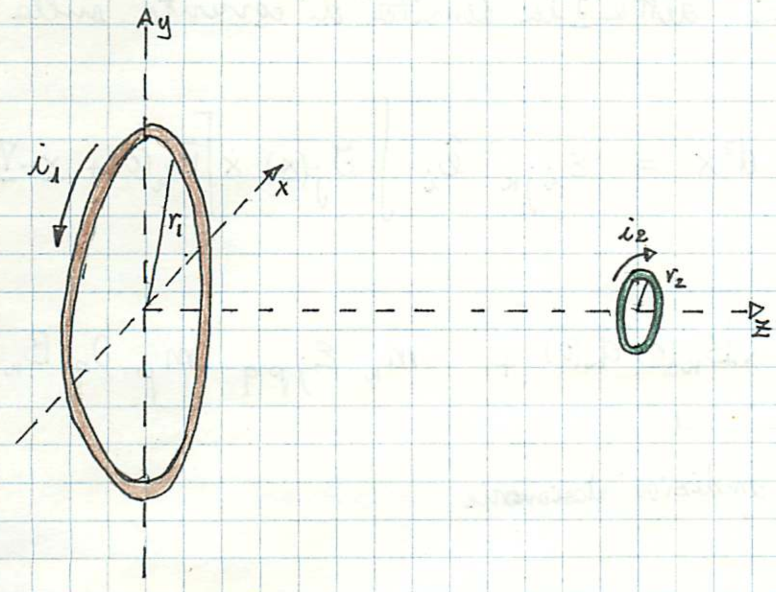
$$L = \int_0^{2\pi}|\underline{m}\times\underline{B}|\,d\theta = N l^2\,\frac{N\,2\pi\nu B l^2}{R}\,B\int_0^{2\pi}\sin^2\theta\,d\theta =$$

$$= \frac{\pi^2 2\nu N^2 B^2 l^4}{R}$$

la potenza meccanica media è:

$$\langle P_m\rangle = \frac{L}{T} = L\nu = 2\,\frac{(\pi\nu B N l^2)^2}{R} = \langle P_J\rangle$$

Una spira di raggio r_1 è percorsa da corrente i_1. Una seconda spira di raggio $r_2 \ll r_1, z$ e resistenza R si avvicina alla prima con velocità v. Le due spire giacciono su piani paralleli ed hanno lo stesso asse. Si determini la corrente i_2 che circola nella seconda spira in funzione della sua distanza dalla prima spira, la forza magnetica risentita dalla seconda spira e quella necessaria per mantenerla in moto con v costante, l'equivalenza tra lavoro meccanico ed effetto Joule.



Poiché $r_2 \ll r_1, z$ il flusso del campo $\underline{B}$ generato dalla prima spira sulla seconda è:

$$\phi(\underline{B}) \simeq \pi r_2^2 \cdot \frac{\mu_0}{2} \frac{i_1 r_1^2}{(r_1^2 + z^2)^{3/2}}$$

essendo $z(t)$ la distanza tra i centri delle due spire

La corrente $i_2(z)$ è:

$$i_2(z) = \frac{|f(z)|}{R} = \frac{1}{R}\left|\frac{d\phi(\underline{B})}{dt}\right| = \frac{1}{R}\left|\frac{d\phi(\underline{B})}{dz}\frac{dz}{dt}\right| = \frac{3}{2}\frac{\mu_0 \pi i_1 v r_1^2 r_2^2}{R} \cdot \frac{z}{(z^2 + r_1^2)^{5/2}}$$

e circola in verso opposto a i_1

la forza risentita dalla spira (opposta alla forza che si deve applicare per mantenerla a velocità costante) vale trascurando l'autoinduzione; e in condizioni quasistazionarie:

$$\underline{F} = -\underline{\nabla}\left(-\underline{m}\cdot\underline{B}\right)$$

$$\underline{m} = -i_2 \pi r_2^2 \hat{z} \qquad \mathcal{U} = -\underline{m}\cdot\underline{B} = -\frac{\mu_0\pi}{2}\frac{i_1 r_1^2 r_2^2}{(r_1^2+z^2)^{3/2}}\, i_2(z)$$

infatti detta $\underline{J}$ la densità di corrente sulla spira 2

$$\underline{F} = \int \underline{J}(\underline{x}) \times \underline{B}(\underline{x})\, d^3x = \varepsilon_{ijk}\, \hat{e}_i \int J_j(\underline{x}) \left[B_k(0) + \underline{x}\cdot\underline{\nabla}B_k(0) + \dots\right] d^3x$$

$$= \varepsilon_{ijk}\, \hat{e}_i \left\{ \underbrace{\int J_j(\underline{x})\, d^3x}_{\substack{\parallel \\ 0 \text{ in condizioni stazionarie}}} B_k(0) + \varepsilon_{jpq}\, m_p\, \partial_q B_k(0) + \dots \right\}$$

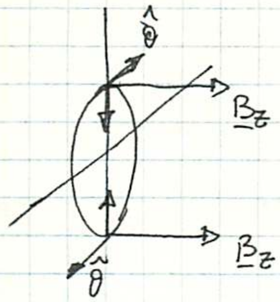
$$F_i = m_k\, \partial_i B_k(0)$$

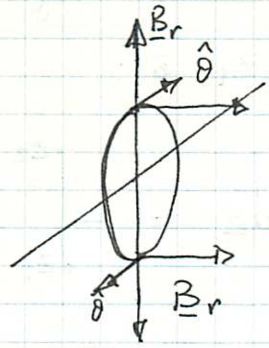
$$F_x = F_y = 0 \qquad F_z = -i_2 \pi r_2^2 \frac{dB}{dz} = \frac{9}{4}\pi^2\mu_0^2 \frac{r_1^4 r_2^4 v i_1^2}{R} \frac{z^2}{(z^2+r_1^2)^5}$$

alternativamente usando coordinate cilindriche $r\theta z$ centrate sulla spira 1

$$\underline{F} = \int \underline{J}(\underline{x}) \times \underline{B}(\underline{x})\, d^3x = \int_0^{2\pi} \hat{\theta}\, i_2\, r_2 d\theta \times \underline{B} =$$

$$= \hat{z}\; 2\pi r_2\, i_2\, B_r(r_2, \theta, z) \qquad \text{con } \frac{\partial}{\partial\theta} B_r(r,\theta,z) = 0$$





$$\int_0^{2\pi} \hat{\theta} \times B_z \, d\theta = 0 \qquad\qquad \int_0^{2\pi} \hat{\theta} \times \underline{B}_r \, d\theta \propto \hat{z}$$

Per calcolare $B_r(r,\theta,z)$ si consideri che in ogni punto

$$\underline{\nabla} \cdot \underline{B} = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(B_r \cdot r\right) + \frac{1}{r}\frac{\partial}{\partial \theta} B_\theta + \frac{\partial}{\partial z} B_z$$

per motivi di simmetria deve essere $B_\theta = 0$ e quindi

$$\frac{\partial}{\partial r}\left(r B_r\right) = - r \frac{\partial B_z}{\partial z}$$

integrando questa relazione tra $r=0$ ed $r=r_2$

$$r_2 B_r\left(r_2, \theta, z\right) - 0 = - \int_0^{r_2} r \frac{\partial B_z}{\partial z}(r,\theta,z)\, dr$$

$$\simeq - \frac{\partial B_z}{\partial z}\left(0, \theta, z\right) \frac{1}{2} r_2^2$$

$$B_r\left(r_2, \theta, z\right) \simeq - \frac{1}{2} r_2 \frac{\partial B_z}{\partial z}$$ che dà ancora

$$\underline{F} = - \hat{z} \, \pi r_2^2 \, i_2 \frac{dB_z}{dz}$$

Il lavoro meccanico eseguito sulla spira per farla spostare da z_1 a $z_2 < z_1$ è:

$$L = \int_{z_1}^{z_2} -\underline{F} \cdot d\underline{s} = -\int_{z_1}^{z_2} F_z \, dz = \mathcal{U}(z_2) - \mathcal{U}(z_1)$$

$$= \frac{9}{4} \mu_0^2 \pi^2 \frac{r_1^4 r_2^4 v \, i_1^2}{R} \int_{z_2}^{z_1} \frac{z^2}{(z^2+r_1^2)^5} \, dz$$

L'energia dissipata per effetto Joule nel medesimo intervallo di tempo è:

$$E_J = \int_{t_1}^{t_2} i_2(t)^2 R \, dt = \int_{z_1}^{z_2} i_2(t)^2 R \frac{dt}{dz} \, dz =$$

$$= + \int_{z_1}^{z_2} i_2(z)^2 \frac{R}{v} \, dz = \qquad \left(\frac{dz}{dt} = -v \Rightarrow \frac{dt}{dz} = -\frac{1}{v} \right)$$

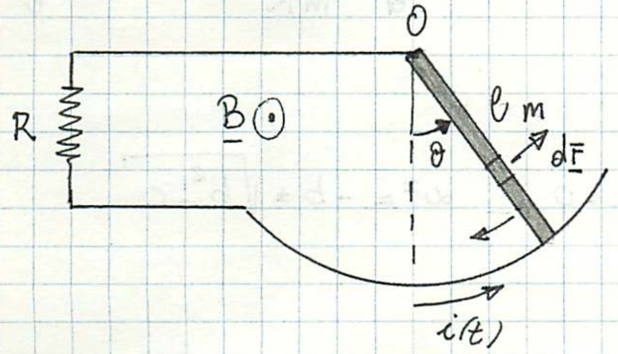
$$= + \int_{z_1}^{z_2} i_2 \, \pi r_2^2 \frac{dB}{dz} = \int_{z_1}^{z_2} d\mathcal{U} - dm_z B_z$$

$$d\mathcal{U} = -d\underline{m} \cdot \underline{B} - \underline{m} \cdot d\underline{B} = dm_z B_z + m_z \, dB_z$$

$$dL = d\mathcal{U}$$

$$d\mathcal{E}_J = d\mathcal{U} + d\underline{m} \cdot \underline{B} = d\mathcal{U} - (-d\underline{m} \cdot \underline{B})$$

Una asta metallica di lunghezza $l = 50$ cm e massa $m = 20g$ ha un estremo vincolato in un punto attorno al quale può oscillare in un piano verticale. L'altro estremo mediante un contatto strisciante chiude un circuito di resistenza $R = 80\,\Omega$ immerso in un campo $B = 1.5\,T$ normale al piano del circuito. Trascurando gli attriti e supponendo che l'asta viene lasciata cadere da ferma da un angolo $\theta_0 = 30°$ con la verticale calcolare la carica che attraversa una sezione dell'asta fino al primo passaggio per la verticale. L'andamento di $\theta(t)$ per $\theta_0 = 2°$ $\dot{\theta}_0 = 0$.



la corrente indotta è $i(t) = -\frac{1}{R}\frac{d}{dt}\phi(B) = -\frac{d}{dt}\left(\phi_0 + \frac{1}{2}\theta l^2 B\right)\frac{1}{R}$

$$= -\frac{1}{2}\dot{\theta}(t)\, l^2 \frac{B}{R}$$

in senso antiorario per B uscente dal foglio e $\dot{\theta} < 0$

la carica q è:

$$|q| = \int_0^{t^*} |i(t)|\,dt = \frac{1}{2} l^2 \frac{B}{R}\int_0^{t^*} \dot{\theta}(t)\,dt = \frac{1}{2} l^2 B \left|\theta(t^*) - \theta(0)\right| =$$

$$= \frac{1}{2}\frac{B}{R} l^2 \theta_0 = 1.2\,10^{-3} C$$

l'equazione del moto dell'asta è:

$$I\ddot{\theta} = -mg\frac{l}{2}\sin\theta + \int_0^l i\,dx\,B \cdot x$$

$$= -mg\frac{l}{2}\sin\theta \qquad -\frac{1}{2}\,l^2\frac{B^2}{R}\dot{\theta}(t)\,\frac{l^2}{2}$$

$$I = \text{momento di inerzia} = \int_0^l \frac{m}{l}dx\,x^2 = \frac{1}{3}ml^2$$

$$\ddot{\theta}(t) + 2b\,\dot{\theta}(t) + c\sin\theta(t) = 0 \qquad b = \frac{3}{8}\frac{B^2l^2}{mR} \qquad c = \frac{3}{2}\frac{g}{l}$$

per piccole oscillazioni $\sin\theta \simeq \theta$

$$\ddot{\theta} + 2b\,\dot{\theta} + c\theta = 0 \qquad \alpha^2 + 2b\alpha + c = 0 \qquad \alpha^{\pm} = -b \pm \sqrt{b^2 - c}$$

$$\theta(t) = A^+e^{\alpha^+t} + A^-e^{\alpha^-t}$$

con i dati del problema si ha $b^2 - c < 0 \qquad \alpha^{\pm} = -b \pm i\sqrt{c - b^2}$

$$\theta(t) = e^{-bt}\left[C\sin\left(\sqrt{c-b^2}\,t\right) + D\cos\left(\sqrt{c-b^2}\,t\right)\right]$$

$$\begin{cases} D = \theta_0 \\ -bD + \sqrt{c-b^2}\,C = 0 \end{cases} \qquad \begin{array}{l} D = \theta_0 \\ C = \dfrac{b}{\sqrt{c-b^2}}\theta_0 \end{array}$$

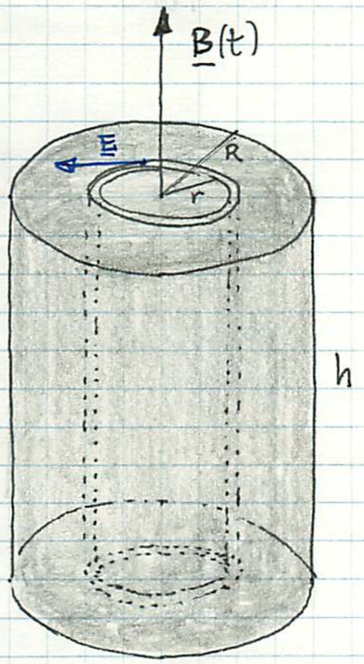
$$\theta(t) = \theta_0 e^{-bt}\left[\cos\left(\sqrt{c-b^2}\,t\right) + \frac{b}{\sqrt{c-b^2}}\sin\left(\sqrt{c-b^2}\,t\right)\right]$$

$b = 0.132$

$c = 29.4$

Un cilindro metallico di conduttività σ altezza h e raggio R si trova in un campo B uniforme parallelo al suo asse con $B(t) = B_0 \exp(-t/\tau)$

Si calcoli l'energia dissipata nel cilindro per effetto Joule a partire da $t=0$.



Si consideri la spira circolare compresa tra i raggi r e $r+dr$

In essa circola una corrente

$$di(t) = \mathcal{E}\, dG$$

dovuta alla f.e.m. $\mathcal{E} = -\pi r^2 \dfrac{dB}{dt}$

e dove $dG = \dfrac{\sigma h\, dr}{2\pi r}$ è la conduttanza infinitesima associata al "filo" di lunghezza $2\pi r$ e sezione $h dr$

La potenza sviluppata per effetto Joule nell'intero cilindro è

$$W(t) = \int dW = \int \mathcal{E}(t) \cdot di(t) = \int_0^R \left(\pi r^2 \frac{dB(t)}{dt}\right)^2 \frac{\sigma h\, dr}{2\pi r} =$$

$$= \frac{\pi}{2} \sigma h \left(\frac{dB(t)}{dt}\right)^2 \int_0^R r^3\, dr = \frac{\pi}{8} \sigma h R^4 \left(\frac{dB(t)}{dt}\right)^2 =$$

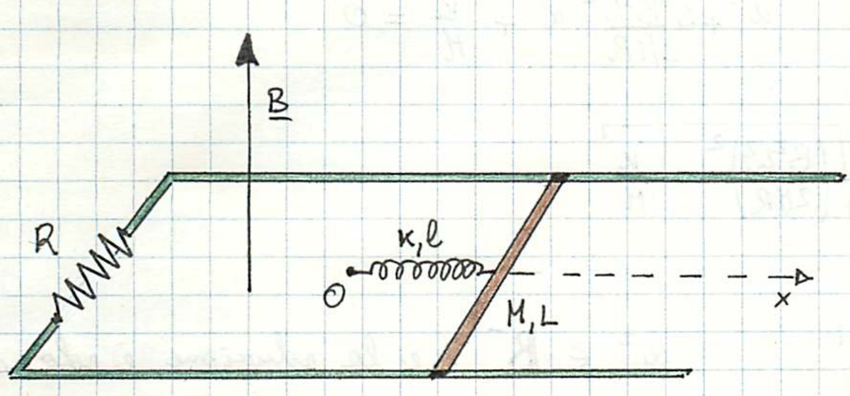
$$= \frac{\pi}{8} \sigma h R^4 \frac{B_0^2}{\tau^2} e^{-\frac{2t}{\tau}}$$

l'energia totale dissipata è:

$$E = \int_0^{\infty} W(t)\, dt = \frac{\pi}{8} \sigma h R^4 \frac{B_0^2}{\tau^2} \frac{\tau}{2} e^{-\frac{2t}{\tau}} \Big|_{\infty}^{0} =$$

$$= \frac{\pi}{16} \frac{\sigma h R^4 B_0^2}{\tau}$$

Una sbarretta conduttrice di massa M e lunghezza L ha le estremità incernierate su due guide parallele orizzontali chiuse su una resistenza R. Sulla sbarretta agisce una molla di costante k e lunghezza di riposo l. Il sistema è immerso in un campo $\underline{B}$ uniforme perpendicolare al piano del sistema. Si scriva l'equazione del moto della sbarretta e si dica per quali valori di B il moto è oscillatorio smorzato.



Sia x(t) la coordinata della sbarretta all'istante t lungo le guide

se $\dot{x}(t)$ è la velocità della barretta allo stesso istante, la f.e.m. indotta nel circuito è

$$|f| = B L \dot{x}$$

cui corrisponde una corrente

$$i(t) = \frac{f(t)}{R} = \frac{BL}{R}\,\dot{x}(t)$$

che circola in verso orario se $\dot{x} > 0$, antiorario se $\dot{x} < 0$

La barretta è dunque sottoposta ad una forza magnetica (trascurando l'autoinduzione)

$$F_m = i\,\underline{L} \times \underline{B} = -\frac{B^2 L^2}{R}\,\dot{x}(t)$$

L'equ del moto della barretta è:

$$M\ddot{x}(t) = -\frac{B^2L^2}{R}\dot{x}(t) - k(x(t)-\ell)$$

$$\ddot{x} + \frac{B^2L^2}{MR}\dot{x} + \frac{k}{M}(x-\ell) = 0$$

posto $\xi = x - \ell$ $\qquad \ddot{\xi} + \frac{B^2L^2}{MR}\dot{\xi} + \frac{k}{M}\xi = 0$

eq. caratteristica: $\qquad \alpha^2 + 2\frac{B^2L^2}{2MR}\alpha + \frac{k}{M} = 0$

$$\alpha^{\pm} = -\frac{B^2L^2}{2MR} \pm \sqrt{\left(\frac{B^2L^2}{2MR}\right)^2 - \frac{k}{M}}$$

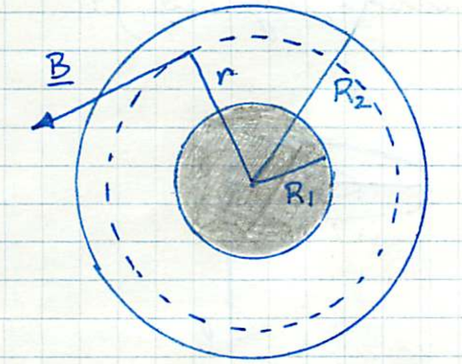
se $\frac{B^2L^2}{2MR} \geq \sqrt{\frac{k}{M}}$ $\qquad \alpha^{\pm} \in \mathbb{R}^-$ e la soluzione è fatta di soli esponenziali smorzanti.

se $\frac{B^2L^2}{2MR} < \sqrt{\frac{k}{M}}$ $\qquad \alpha^{\pm} = -\frac{B^2L^2}{2MR} \pm i\sqrt{\frac{k}{M} - \left(\frac{B^2L^2}{2MR}\right)^2}$

cioè $\alpha^{\pm} \in \mathbb{C}$ con $\alpha^+ = (\alpha^-)^*$ e la soluzione è del tipo:

$$x(t) = \left(A\cos\omega t + B\sin\omega t\right)e^{-\frac{B^2L^2}{2MR}t} \qquad \omega = \sqrt{\frac{k}{M} - \left(\frac{B^2L^2}{2MR}\right)^2}$$

Un cavo coassiale è costituito da un filo metallico cilindrico di raggio $R_1 = 1.5\ mm$ posto al centro di una superficie metallica (di spessore trascurabile) di raggio $R_2 = 7.5\ mm$.
I due conduttori sono percorsi da corrente di uguale intensità e verso opposto (la densità di corrente sulla sezione del conduttore interno è uniforme). Si calcoli il coefficiente di autoinduzione del cavo per unità di lunghezza.

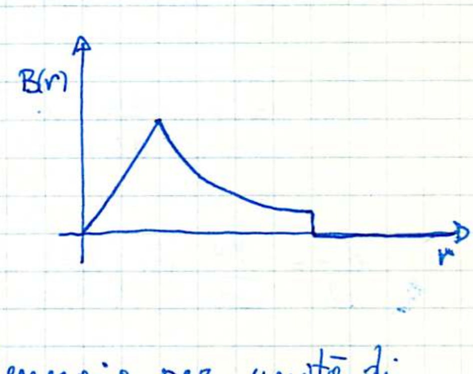


Applicando il teorema della circuitazione ad una circonferenza concentrica al cavo e di raggio r si ha:

$$\int_C \underline{B} \cdot d\underline{l} = 2\pi r B(r) = \mu_0 \int_S \underline{J} \cdot \hat{n}\, ds$$

$$= \mu_0 \begin{cases} i \dfrac{r^2}{R^2} & r < R_1 \\ i & R_1 < r < R_2 \\ i - i & r > R_2 \end{cases}$$

quindi $$B(r) = \frac{\mu_0 i}{2\pi} \begin{cases} \dfrac{r}{R_1^2} & r < R_1 \\ \dfrac{1}{r} & R_1 < r < R_2 \\ 0 & r > R_2 \end{cases}$$



la densità di energia è $w(r) = \dfrac{B(r)^2}{2\mu_0}$ e l'energia per unità di lunghezza è data da:

$$\frac{d\mathcal{E}}{dl} = \int_0^{\infty} \frac{B(r)^2}{2\mu_0} 2\pi r\, dr = \frac{\mu_0 i^2}{4\pi}\left[\int_0^{R_1} \left(\frac{r}{R_1^2}\right)^2 r\, dr + \int_{R_1}^{R_2} \frac{1}{r^2} r\, dr\right] =$$

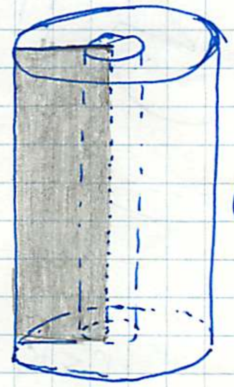
$$= \frac{\mu_0 i^2}{4\pi}\left[\frac{r^4}{4R_1^4}\Big|_0^{R_1} + \ln r\Big|_{R_1}^{R_2}\right] = \frac{\mu_0 i^2}{4\pi}\left[\frac{1}{4} + \ln\left(\frac{R_2}{R_1}\right)\right]$$

ponendo $\frac{dE}{dl} = \frac{1}{2}\frac{dL}{dl} i^2$ si trova il coefficiente di autoinduzione per unità di lunghezza: $\frac{dL}{dl} = \frac{\mu_0}{2\pi}\left[\frac{1}{4} + \ln\left(\frac{R_2}{R_1}\right)\right] =$

$= 0.37 \; 10^{-6} \; Hm^{-1}$

si noti che per $R_2 >> R_1$ $\quad \frac{dL}{dl} = \frac{\mu_0}{2\pi}\ln\left(\frac{R_2}{R_1}\right)$ che è il risultato che si troverebbe per $R_2>>R_1$, usando la relazione (non corretta, in generale):

$\phi(B) = L\, i$



l

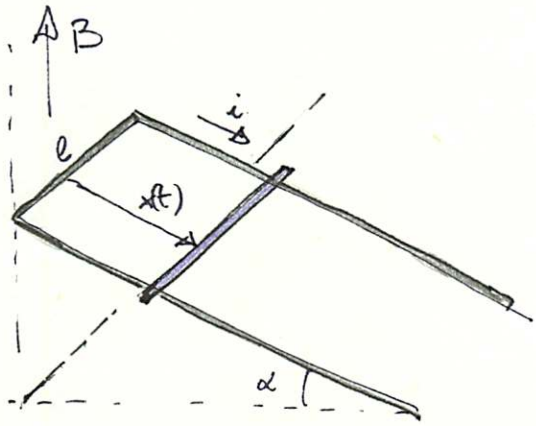
infatti il flusso di $\underline{B}$ concatenato con un piano di cavo lungo l vale

$$\phi(\underline{B}) = \int_0^{R_2} B(r)\; l\, dr = \frac{\mu_0 i}{2\pi} l \int_0^{R_1}\frac{r}{R_1^2}dr$$

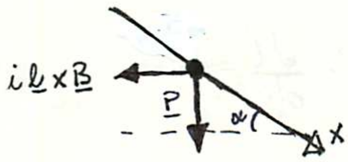
$$+ \frac{\mu_0 i}{2\pi} l \int_{R_1}^{R_2}\frac{dr}{r} = l\,\frac{\mu_0 i}{2\pi}\left[\ln\left(\frac{R_2}{R_1}\right) + \frac{1}{2}\right]$$

$$= l\,\frac{dL}{dl}\, i$$

$$\frac{dL}{dl} = \frac{\mu_0}{2\pi}\ln\left(\frac{R_2}{R_1}\right) + \frac{1}{2} \approx \frac{\mu_0}{2\pi}\ln\left(\frac{R_2}{R_1}\right) \quad \text{per} \quad R_2 >> R_1$$



Si determini la corrente nel circuito quando la sbarretta di massa m scivola senza attrito nei casi: a) la barretta ha resistenza R b) ha anche induttanza L.



a)
$$\begin{cases} m\ddot{x} = mg\sin\alpha - ilB\cos\alpha \\[2ex] i = \dfrac{\mathcal{E}}{R} = \dfrac{l\dot{x}B\cos\alpha}{R} \end{cases}$$

$$\ddot{x}(t) + \frac{l^2B^2\cos^2\alpha}{mR}\,\dot{x}(t) = g\sin\alpha$$

$$\dot{x}(t) = \frac{mRg\sin\alpha}{l^2B^2\cos^2\alpha}\left[1 - e^{-\frac{l^2B^2\cos^2\alpha}{mR}t}\right]$$

$$x(t) = \frac{mRg\sin\alpha}{l^2B^2\cos^2\alpha}\left\{t - \frac{mR}{l^2B^2\cos^2\alpha}\left[1 - e^{-\frac{l^2B^2\cos^2\alpha}{mR}t}\right]\right\}$$

$$i(t) = \frac{mg\sin\alpha}{lB\cos\alpha}\left[1 - e^{-\frac{l^2B^2\cos^2\alpha}{mR}t}\right]$$

b)

$$\begin{cases} \ddot{x} = g \sin\alpha - i \dfrac{lB}{m}\cos\alpha \\ l\dot{x}B\cos\alpha - L\dfrac{di}{dt} = iR \end{cases}$$

$$l\ddot{x}B\cos\alpha - L\frac{d^2 i}{dt^2} = R\frac{di}{dt}$$

$$lB\cos\alpha\, g\sin\alpha - \frac{l^2B^2\cos^2\alpha}{m} i - L\frac{d^2 i}{dt^2} - R\frac{di}{dt} = 0$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{l^2B^2\cos^2\alpha}{mL} i = \frac{lBg\sin\alpha\cos\alpha}{L}$$

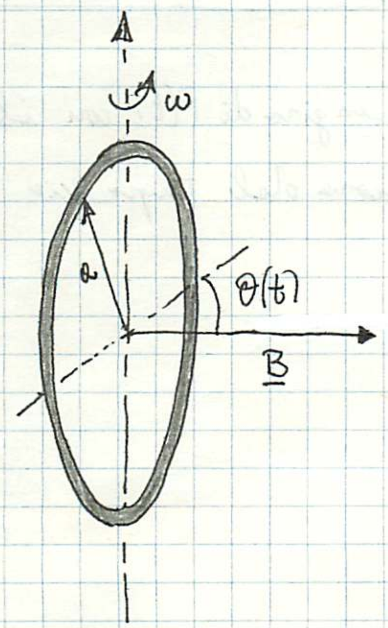
$$i(t) = C^+ e^{\beta^+ t} + C^- e^{\beta^- t} + \frac{mg\sin\alpha}{lB\cos\alpha}$$

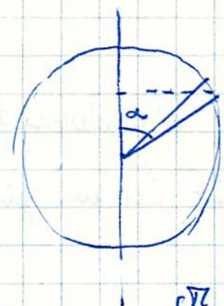
$$\beta^{\pm} = -\frac{R}{2L} \pm \frac{R}{2L}\sqrt{1 - \frac{4l^2B^2\cos^2\alpha}{mR^2}L}$$

$$\frac{di}{dt}(0) = 0 \qquad i(0) = 0$$

$$i(t) = \frac{mg\sin\alpha}{lB\cos\alpha}\left\{1 - \frac{1}{2}e^{\beta^+ t}\left[1 + \frac{1}{\sqrt{1 - \frac{4l^2B^2\cos^2\alpha L}{mR^2}}}\right] - \frac{1}{2}e^{\beta^- t}\left[1 - \frac{1}{\sqrt{1 - \frac{4l^2B^2\cos^2\alpha L}{mR^2}}}\right]\right\}$$

Un anello circolare di raggio a, costituito con un filo di rame di resistività $\rho = 1.7\ 10^{-8}\ \Omega\, m$ ruota intorno a un suo diametro in un campo $B = 2\ 10^{-2}\ T$ ortogonale all'asse di rotazione.
Detta ω_0 la velocità angolare a $t = t_0$ si calcoli, la potenza media dissipata in un giro nell'ipotesi $\omega(t) = \omega_0$, il tempo necessario affinché $\omega(t)$ si riduca di un fattore e rispetto ad ω_0.
Si ricorda che il momento d'inerzia dell'anello rispetto ad un asse diametrale è $I = \delta \pi a^3 S$ dove S è la sezione del filo e $\delta = 8.9\ g\ cm^{-3}$.





$$I = \int dm\, h^2 = \int_0^{\pi} \frac{d\alpha}{2\pi} m\, (a \sin\alpha)^2$$

$$= \frac{1}{\pi} m a^2 \int_0^{\pi} \sin^2\alpha\, d\alpha = \frac{1}{2} m a^2$$

Sia $\theta(t)$ l'angolo formato dal piano dell'anello con la direzione di $\underline{B}$
$\theta_0 = \theta(t_0)$

Nel circuito dell'anello viene indotta una f.e.m.

$$f = -\frac{d}{dt}\phi(\underline{B}) = -\frac{d}{dt}\ \pi a^2 B \sin\theta(t) = -\pi a^2 B\, \dot\theta(t) \cos\theta(t)$$

La resistenza dell'anello è $R = \rho \dfrac{2\pi a}{S}$

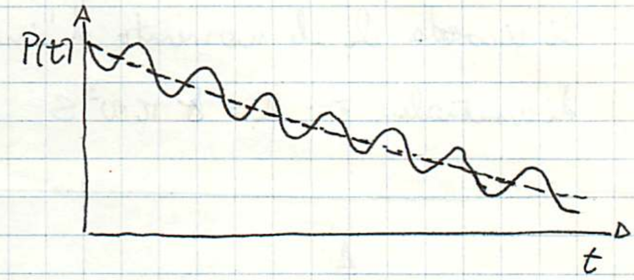
la potenza Joule istantanea vale $P(t) = \dfrac{f^2}{R} = \dfrac{\pi^2 a^4 B^2 \dot\theta^2 \cos^2\theta}{\rho \frac{2\pi a}{S}}$

Nell'ipotesi $\dot{\theta}(t) = \omega_0$ costante, la potenza media dissipata in un giro è

$$\overline{P} = \frac{\pi a^3 B^2 \omega_0^2 S}{4\rho}$$

L'energia cinetica dell'anello diminuisce a causa della dissipazione Joule

$$\frac{d}{dt}\left(\frac{1}{2} I \dot{\theta}(t)^2\right) = -P(t)$$



approssimando l'andamento oscillante in un giro di $P(t)$ con il suo valore medio in un giro (dipendente ancora dal tempo via $\dot{\theta}(t)^2$)

$$\dot{\theta}(t) \equiv \omega(t)$$

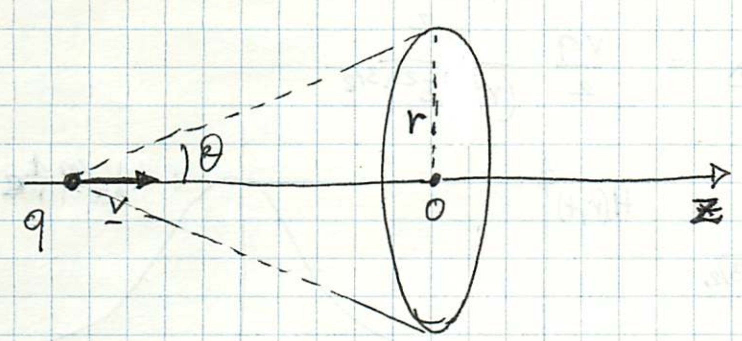
$$I \omega \dot{\omega} = -\frac{\pi a^3 B^2 \omega^2 S}{4\rho}$$

$$\dot{\omega} = -\omega \frac{\cancel{\pi}\cancel{a^3} B^2 \cancel{S}}{4\rho} \cdot \frac{1}{\delta \cancel{\pi}\cancel{a^3}\cancel{S}} = -\omega \frac{B^2}{4\rho\delta}$$

$$\omega(t) = \omega(t_0) \exp\left[-\frac{B^2}{4\rho\delta}(t - t_0)\right]$$

$$\omega(t) = \frac{1}{e}\,\omega(t_0) \quad \text{per} \quad t - t_0 = \frac{4\rho\delta}{B^2} = 1.5 \ \text{s}$$

Si consideri una particella di carica q in moto rettilineo uniforme con velocità v. Si calcoli l'intensità della corrente di spostamento attraverso un cerchio fisso ortogonale alla traiettoria e centrato su di essa e il campo magnetico prodotto. Si riottenga la legge di Biot Savart per cariche distribuite uniformemente con densità λ.



Si consideri un cerchio di raggio r centrato in un punto O della traiettoria della carica (asse z)

All'istante t in cui la particella si trova in $z(t)$ il flusso di $\underline{D}$ attraverso il cerchio è:

$$\phi(\underline{D}) = \int_0^r \frac{q}{4\pi(\rho^2+z^2)}\, 2\pi\rho\, d\rho\, \frac{-z}{\sqrt{\rho^2+z^2}} =$$

$$= -\frac{q}{2}\, z \int_0^r \frac{\rho\, d\rho}{(\rho^2+z^2)^{3/2}} = -\frac{q}{2}\, z \left(\rho^2+z^2\right)^{-\frac{1}{2}}\Big|_r^0 =$$

$$= -\frac{q}{2}\, z\left(\frac{1}{-z} - \frac{1}{\sqrt{r^2+z^2}}\right) = \frac{q}{2}\left(1-\cos\theta\right) = \frac{q}{2}\left(1+\frac{z}{\sqrt{r^2+z^2}}\right)$$

la corrente di spostamento attraverso il cerchio è

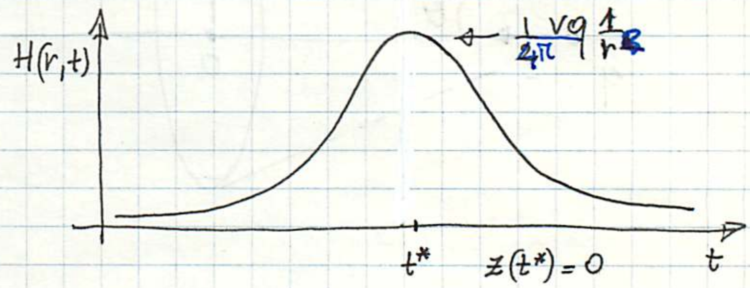
$$i_D = \frac{d}{dt}\phi(\underline{D}) = \frac{d}{dz}\phi(D)\,\frac{dz}{dt} = v\,\frac{q}{2}\,\frac{r^2}{(r^2+z^2)^{3/2}}$$

Per motivi di simmetria H è azimutale $\underline{H} = \hat{z}\times\hat{r}\,H(r)$

$$\int_{2\pi r} \underline{H}\cdot d\underline{l} = H(r)\,2\pi r = \int_{\pi r^2} \underline{\nabla}\times\underline{H}\cdot\hat{z}\,ds = \int \frac{\partial \underline{D}}{\partial t}\cdot\hat{z}\,dS =$$

$$= \frac{d}{dt}\phi(\underline{D}) = i_D = \frac{vq}{2}\,\frac{r^2}{(r^2+z^2)^{3/2}}$$

$$H(r,t) = \frac{1}{4\pi}\,qv\,\frac{r}{(r^2+z(t)^2)^{3/2}}$$



Se invece di una carica in moto si ha una distribuzione lineare con densità λ cioè una corrente $i = \frac{dq}{dt} = \frac{dq}{dz}\frac{dz}{dt} = \lambda v$

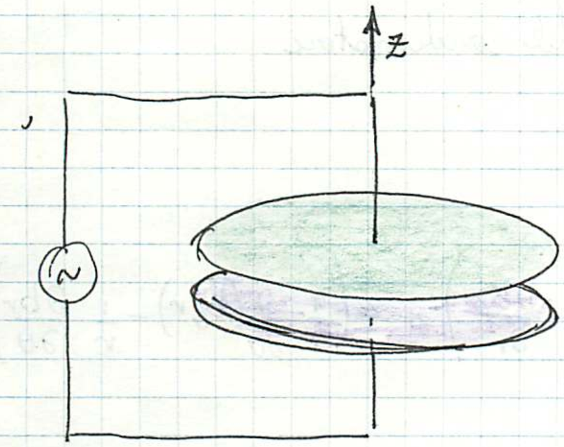
si ha: $dH = \frac{1}{4\pi}dq\;v\,\frac{r}{(r^2+v^2)^{3/2}}$ $\quad dq = \frac{dq}{dz}dz = \lambda\,dz$

$$H = \int_{-\infty}^{+\infty} \frac{1}{4\pi}\,v\,\lambda\,dz\,\frac{r}{(r^2+z^2)^{3/2}} = \frac{\lambda v r}{4\pi}\int_{-\infty}^{+\infty}\frac{dz}{(z^2+r^2)^{3/2}} =$$

$$= \frac{\lambda v r}{4\pi}\,\frac{z}{r^2\sqrt{z^2+r^2}}\Bigg|_{-\infty}^{+\infty} = \frac{\lambda v r}{4\pi}\,\frac{2}{r^2} = \frac{\lambda v}{2\pi r} = \frac{i}{2\pi r}$$

da la legge di Biot-Savart

Un condensatore piano ad armature circolari di raggio R distanti d è collegato ad un generatore di f. e.m. alternata $f(t) = f_0 \sin \omega t$
Trascurando gli effetti di bordo si calcoli $\underline{H}$ tra le armature.



la d.d.p. ai capi del condensatore è $f_0 \sin \omega t = f(t)$

detto $C = \dfrac{\varepsilon_0 \pi R^2}{d}$ la sua capacità

la carica sulle armature del condensatore è

$$q(t) = C f(t) = \frac{\varepsilon_0 \pi R^2}{d} f_0 \sin \omega t$$

l'induzione elettrica all'interno del condensatore è parallela a z e vale

$$D(t) = \sigma(t) = \frac{q(t)}{\pi R^2} = \frac{\varepsilon_0 f_0}{d} \sin \omega t \qquad E(t) = \frac{\sigma(t)}{\varepsilon_0} = \frac{D(t)}{\varepsilon_0}$$

Dall'eq. di Maxwell $\quad \underline{\nabla} \times \underline{H} - \dfrac{\partial \underline{D}}{\partial t} = \underline{j} \quad$ (Ampere-Maxwell)

per $\underline{j} = 0$ si ha:

$$(\underline{\nabla} \times \underline{H})_x = (\underline{\nabla} \times \underline{H})_y = 0 \qquad (\underline{\nabla} \times \underline{H})_z = \frac{\partial D}{\partial t} = \frac{\varepsilon_0 f_0 \omega}{d} \cos \omega t$$

Per motivi di simmetria $\underline{H}$ è azimutale all'interno del condensatore

$$\underline{H} = \hat{z} \times \hat{r} \; H(r) = \hat{\theta} \, H_\theta(r)$$

essendo r la distanza dal centro del condensatore

usando coordinate cilindriche in cui

$$\underline{\nabla} \times \underline{b} = \hat{r}\left(\frac{1}{r}\frac{\partial b_z}{\partial \theta} - \frac{\partial b_\theta}{\partial z}\right) + \hat{\theta}\left(\frac{\partial b_r}{\partial z} - \frac{\partial b_z}{\partial r}\right) + \hat{z}\left(\frac{1}{r}\frac{\partial}{\partial r}(b_\theta r) - \frac{1}{r}\frac{\partial b_r}{\partial \theta}\right)$$

$$\underline{\nabla} \times \underline{H} = \hat{z} \; \frac{1}{r}\left(H_\theta(r) + r\frac{\partial H_\theta(r)}{\partial r}\right) = \hat{z} \; \frac{\varepsilon_0 f_0 \omega}{d} \cos\omega t$$

$$\frac{1}{r} H_\theta(r) + \frac{d}{dr} H_\theta(r) = \frac{\varepsilon_0 f_0 \omega}{d} \cos\omega t$$

$$H_\theta(r) = \frac{r}{2} \frac{\varepsilon_0 f_0 \omega}{d} \cos\omega t$$

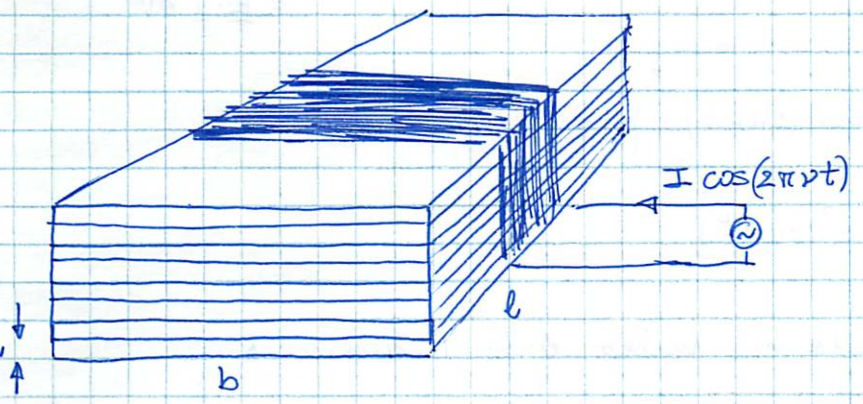
alternativamente

$$\int_{\pi r^2} \underline{\nabla} \times \underline{H} \cdot \hat{z} \, dS = \int \underline{H} \cdot d\underline{l} = 2\pi r H_\theta(r) = \int_{\pi r^2} \frac{\partial \underline{D}}{\partial t} \cdot \hat{z} \, dS =$$

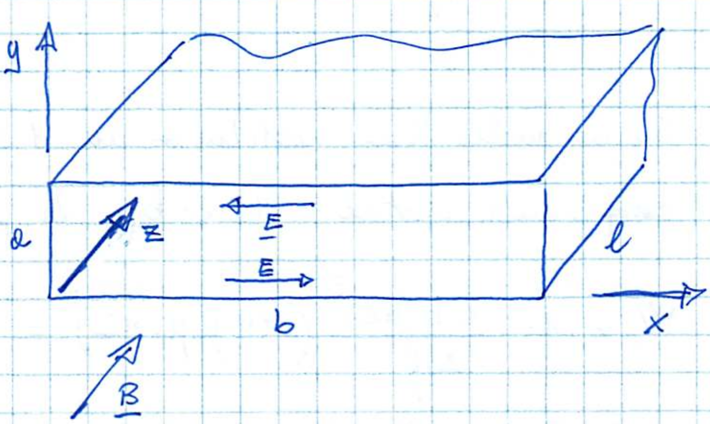
$$= \pi r^2 \frac{\varepsilon_0 f_0 \omega}{d} \cos\omega t$$

$$H_\theta(r) = \frac{r}{2} \frac{\varepsilon_0 f_0 \omega}{d} \cos\omega t$$

Per diminuire le perdite, il nucleo di un trasformatore è costituito di $N=100$ lamine di ferro ($\mu_r=1000$) elettricamente isolate l'una rispetto all'altra. Le dimensioni di ciascuna lamina sono $l=20$ cm $a=0.01$ cm $b=6$ cm e lo spessore dell'isolante è trascurabile. Intorno al nucleo sono ~~è~~ avvolte $n=4$ spire cm^{-1} di un filo percorso da corrente sinusoidale di ampiezza $I=5A$ e frequenza $\nu=50$ Hz. Sapendo che la resistività del ferro è $\rho=9\,10^{-6}\,\Omega$ cm si calcoli la potenza media dissipata per effetto delle correnti indotte nel nucleo. Se il nucleo fosse costituito da 1 sola lamina di spessore Na di quanto aumenterebbe la potenza media dissipata?



Si consideri una singola lamina e un sistema di riferimento come in figura:



Si ha $\underline{B}(t)=\left(0,0,A\cos(2\pi\nu t)\right)$

dove $A=\mu_0\mu_r nI$

Poiché $\underline{B}$ varia nel tempo si genera un campo $\underline{E}$ in base all'equazione $\underline{\nabla}\times\underline{E}=-\frac{\partial \underline{B}}{\partial t}$

Per $b\gg a$ la simmetria del problema implica $\underline{E}=\left(E(y),0,0\right)$ con $E\left(\frac{a}{2}\right)=0$ poiché $E\left(\frac{a}{2}+\delta y\right)=-E\left(\frac{a}{2}-\delta y\right)$

$$\underline{\nabla} \times \underline{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ E(y) & 0 & 0 \end{vmatrix} = \left(0, 0, -\partial_y E(y)\right) = - \quad \partial_t \underline{B} = \left(0, 0, \ 2\pi\nu A \sin(2\pi\nu t)\right)$$

$$\partial_y E(y) = -\ 2\pi\nu A \sin(2\pi\nu t)$$

$$E(y) = E(0) - 2\pi\nu A \sin(2\pi\nu t)\, y$$

imponendo $E\left(\frac{a}{2}\right) = 0 \Rightarrow E(0) = 2\pi\nu A \sin(2\pi\nu t) \frac{a}{2}$

$$E(y) = -2\pi\nu A \sin(2\pi\nu t) \left(y - \frac{a}{2}\right)$$

Dalla forma locale della legge di Ohm troviamo una densità di corrente indotta $\underline{J}(t) = \frac{1}{\rho} \underline{E}(t)$

La potenza dissipata in un volumetto dV è $dW = \rho J^2 dV = \frac{E^2}{\rho} dV$
prendendo $dV = l\, b\, dy$ $= \underline{E} \cdot \underline{J}\, dV$

$$W = \int_0^y l b\, dy \ \frac{1}{\rho} \left[2\pi\nu A \sin(2\pi\nu t)\right]^2 \left(y - \frac{a}{2}\right)^2 =$$

$$= l b \frac{a^3}{12} \frac{1}{\rho} (2\pi\nu A)^2 \sin^2(2\pi\nu t)$$

La potenza media dissipata nel nucleo delle N lamine è

$$\overline{W_N} = N \frac{l b a^3}{24\rho} \left(2\pi\nu \mu_0 \mu_r n I\right)^2 \simeq 0.346 \ W$$

Se il nucleo fosse costituito da 1 sola lamina di spessore Na, sempre nell'ipotesi $Na \ll b$, la potenza media dissipata sarebbe

$$\overline{W_1} = 1 \cdot \frac{l b (Na)^3}{24\rho} \left(2\pi\nu \mu_0 \mu_r n I\right)^2 = N^2 \ \overline{W_N}$$

cioè 10^4 volte più grande

Un filo conduttore di raggio a è a distanza d da una lamina conduttrice. Filo e lamina costituiscono un circuito in cui fluisce una corrente I. Determinare il campo $\underline{B}$, la densità di corrente J nel piano e l'induttanza del circuito per unità di lunghezza

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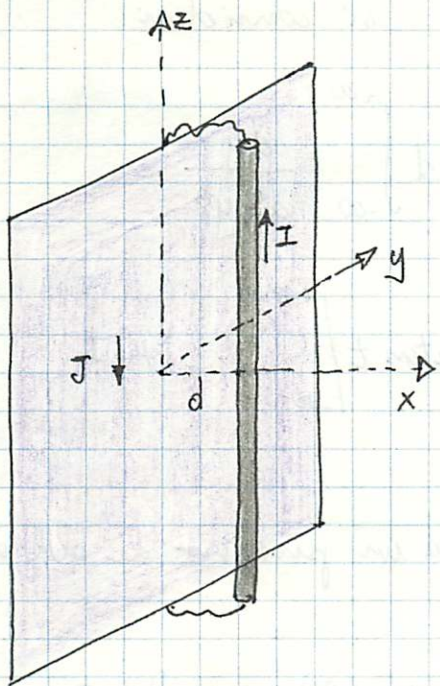
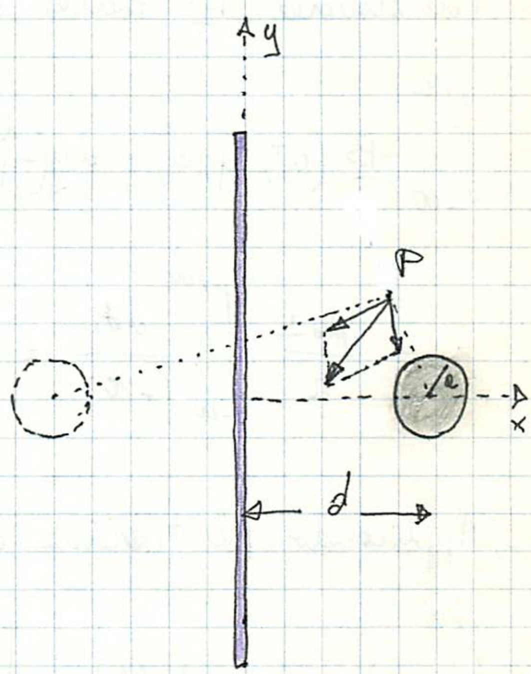




Nel limite di lastra e filo infiniti per motivi di simmetria deve essere:

$$\underline{B}(0, y, z) = - B(y)\,\hat{y}$$

Un campo così fatto sul piano $x=0$ è ottenuto matematicamente sostituendo la lastra con un filo immagine di raggio a nella posizione $z=-d$ in cui fluisce una corrente I in verso opposto al filo reale.

In un punto P qualsiasi del semispazio $x>0$ si ha

$$\underline{B}(x, y, z) = \frac{\mu_0 I}{2\pi}\left(\frac{\hat{z}\times(x-d, y, 0)}{(x-d)^2+y^2} - \frac{\hat{z}\times(x+d, y, 0)}{(x+d)^2+y^2}\right) =$$

$$= \frac{\mu_0 I}{2\pi}\left(\frac{-y\hat{x}+(x-d)\hat{y}}{(x-d)^2+y^2}+\frac{y\hat{x}-(x+d)\hat{y}}{(x+d)^2+y^2}\right)$$

che per $x=0$ da $\underline{B}(0,y,z) = -\frac{\mu_0 I}{\pi}\frac{d}{d^2+y^2}\hat{y}$

Per trovare $\underline{B}$ nella regione $x<0$ si consideri

$$\int_{-\infty}^{+\infty}\underline{B}(0^+,y,z)\cdot dy(-\hat{y}) = \frac{\mu_0 I}{\pi} d\int_{-\infty}^{+\infty}\frac{dy}{d^2+y^2} =$$

$$= \frac{\mu_0 I}{\pi}\int_{-\infty}^{+\infty}\frac{dt}{1+t^2} = \frac{\mu_0 I}{\pi}\arctan t\Big|_{-\infty}^{+\infty} = \mu_0 I$$

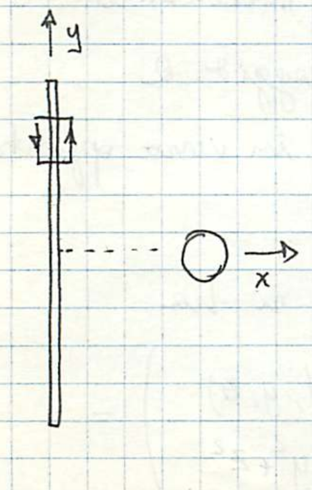
Applicando il teorema di Ampere ad un percorso che abbraccia la lamina:

$$\int_{-\infty}^{+\infty}\underline{B}(0^+,y,z)\cdot dy(-\hat{y}) + \int_{-\infty}^{+\infty}\underline{B}(0^-,y,z)\cdot dy\,\hat{y} + \text{infinitesimi} = \phi(\underline{J})$$

$$\mu_0 I + \int_{-\infty}^{+\infty}B_y(0^-,y,z)\,dy = \mu_0 I \quad \text{quindi}$$

$B_y(0^-,y,z)=0$

Date le proprietà di simmetria di $\underline{B}$ si ha $\underline{B}=0$ per $x<0$



Applicando il teorema di Ampere al circuito infinitesimo in figura si ha:

$$-\frac{\mu_0 I}{\pi}\frac{d}{d^2+y^2}dy + \text{infinitesimi di ordine superiore} =$$

$$= -J(y)\,dy\,\mu_0$$

$$\underline{J}(y) = -\hat{z} \quad \frac{I}{\pi} \quad \frac{d}{d^2+y^2}$$

$$\int_{-\infty}^{+\infty} \underline{J}(\underline{y}) \cdot (-\hat{z})\, dy = I \qquad \text{come deve}$$

Per calcolare l'induttanza si consideri il flusso di $\underline{B}$ attraverso ~~la~~ superficie che ha per lati gli assi $(x=d \;\; y=0)$ e $(x=0, y=0)$

$$\phi(\underline{B}) = \int_{a}^{d-a} dx \int_{z_1}^{z_2} dz \; -B_y(x,0,z) =$$

$$= (z_2 - z_1) \int_{a}^{d-a} \frac{\mu_0 I}{2\pi} \left(\frac{1}{d-x} + \frac{1}{d+x} \right) dx =$$

$$= (z_2 - z_1) \frac{\mu_0 I}{2\pi} \Big[-\ln(d-x) + \ln(d+x) \Big]_{a}^{d-a} =$$

$$= (z_2 - z_1) \frac{\mu_0 I}{2\pi} \ln \frac{2d-a}{a}$$

il coefficiente di autoinduzione per unità di lunghezza è

$$\frac{dL}{dz} = \frac{1}{z_2 - z_1} \cdot \frac{\phi(B)}{I} = \frac{\mu_0}{2\pi} \ln \frac{2d-a}{a} \approx \frac{\mu_0}{2\pi} \ln \frac{2d}{a}$$