Exercise 1:

In the frame S a rod has length L and moves with a speed u along the x axis. What is the length of the rod in the S' system which moves with a speed V along the x axis?

Solution:

Let L_0 be the proper length of the rod, Then $L = \frac{L_0}{\gamma_u}$ and $L_0 = \gamma_u L$. In the frame S' the rod moves with velocity:

$$u' = \frac{u - V}{1 - \frac{uV}{c^2}} \tag{1}$$

Therefore, the length of the rod in this frame is $L' = \frac{L_0}{\gamma_{u'}} = \frac{\gamma_u}{\gamma_{u'}} L$. It can be shown that:

$$\gamma_{u'} = \gamma_u \gamma_V \left(1 - \frac{uV}{c^2}\right) \tag{2}$$

and hence:

$$L' = \frac{L}{\gamma_V \left(1 - \frac{uV}{c^2}\right)} \tag{3}$$

Exercise 2:

A massive mirror moves perpendicularly to its plane with speed V. Find the angle of reflection θ_2 for a light ray from such mirror if its angle of incidence is θ_1.

Solution:

We can write, for the light ray velocity components in the laboratory frame:

$$\begin{aligned} c_x &= c \cos\theta_1 \\ c_y &= c \sin\theta_1 \end{aligned} \tag{4}$$

and hence, in the rest frame of the mirror:

$$\begin{aligned} c'_x &= \frac{c_x - V}{1 - \frac{c_x V}{c^2}} = \frac{c\cos\theta_1 - V}{1 - \beta\cos\theta_1} \\ c'_y &= \frac{c_y}{\gamma\left(1 - \frac{c_x V}{c^2}\right)} = \frac{c\sin\theta_1}{\gamma(1 - \beta\cos\theta_1)} \end{aligned} \tag{5}$$

In this frame, the reflection angle equals the incident angle, hence the component after reflection are:

$$\begin{aligned} c'^{(2)}_x &= -c'_x \\ c'^{(2)}_y &= c'_y \end{aligned} \tag{6}$$

Going back to the laboratory frame:

$$c \sin\theta_2 = c_y^{(2)} = \frac{c_y'^{(2)}}{\gamma\left(1 + \frac{c_x'^{(2)} V}{c^2}\right)} \tag{7}$$

and, with the substitutions (6), the transformations (5) and some algebra:

$$\sin\theta_2 = \frac{1}{\gamma^2} \frac{\sin\theta_1}{1 + \beta^2 - 2\beta\cos\theta_1} \tag{8}$$

Exercise 3:
Calculate the π^+ lifetime in a reference frame where it moves with momentum 100 GeV/c [$m_\pi = 139.6$ MeV/c; $\tau_0 = 2.6 \times 10^-8$ s].

Solution:

$$\tau = \gamma\tau_0 = \frac{E}{m}\tau_0 = \frac{\sqrt{p^2 + m_\pi^2}}{m_\pi}\tau_0 = 716\ \tau_0 = 18.6\ \mu\text{s} \tag{9}$$

Exercise 4:

1. At what velocity the kinetic energy of a particle equals its rest energy?
2. At what velocity a cannon ball of 1 kg has a kinetic energy equal to the one of a proton with $\gamma = 10^11$?

Solution:

1. $mc^2 = K = (\gamma - 1)mc^2 \ \Rightarrow\ \gamma = 2 \ \Rightarrow\ \beta = 0.86 \ \Rightarrow\ v = 258000$ km/s
2. The proton kinetic energy is $K = (\gamma - 1)mc^2 \sim \gamma mc^2 \sim 10^{20}$ eV ~ 16 J. The cannon ball is non-relativistic and:

$$K = \frac{1}{2}mv^2 = 16 \text{ J} \ \Rightarrow\ v = 5.6m/s \tag{10}$$