Exercise 1 (charge exchange):

For the charge exchange reaction:

$$\pi^- + p \to n + \pi^0 \tag{1}$$

with $\pi^0 \to \gamma\gamma$, determine the relation between energy and relative $\gamma\gamma$ angle θ, assuming the pion and proton in the initial state to be at rest (pion capture).

Solution:

Given the 4-momenta $p_\pi = (m_\pi, 0)$ and $p_p = (m_p, 0)$, energy and momentum conservations give:

$$\begin{aligned} \vec{p}_n + \vec{p}_{\pi^0} &= 0 \Longrightarrow |\vec{p}_n|^2 = |\vec{p}_{\pi^0}|^2 \Longrightarrow E_n^2 - m_n^2 = E_{\pi^0}^2 - m_{\pi^0}^2 & (2) \\ E_n + E_{\pi^0} &= m_\pi + m_p & (3) \end{aligned}$$

and combining them:

$$E_{\pi^0} = \frac{m_\pi^2 + m_{\pi^0}^2 + m_p^2 - m_n^2 + 2m_\pi m_p}{2(m_p + m_{\pi^0})} = 138 \text{ MeV} \tag{4}$$

For the π^0 decay in two photons of 4-momentum p_1 and p_2 we can write:

$$m_{\pi^0}^2 = (p_1 + p_2)^2 = 2\vec{p}_1 \cdot \vec{p}_2 = 2E_1E_2(1 - \cos\theta) = 4E_1E_2 \sin^2\frac{\theta}{2} \tag{5}$$

but $E_1 + E_2 = E_{\pi^0}$, so:

$$m_{\pi^0}^2 = 4E_1(E_{\pi^0} - E_1)\sin^2\frac{\theta}{2} \tag{6}$$

With some algebra:

$$\begin{aligned} E_1 &= \frac{E_{\pi^0}}{2} \pm \sqrt{\left(\frac{E_{\pi^0}}{2}\right)^2 - \frac{m_{\pi^0}^2}{4\sin^2\frac{\theta}{2}}} & (7) \\ E_2 &= \frac{E_{\pi^0}}{2} \mp \sqrt{\left(\frac{E_{\pi^0}}{2}\right)^2 - \frac{m_{\pi^0}^2}{4\sin^2\frac{\theta}{2}}} & (8) \end{aligned}$$