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- linearization formulae should be used with care$^{(*)}$
  [$^{(*)}$ *Yet other prescriptions*]
- ...and possibly avoided!
- But, nevertheless, when used correctly they offer useful insights in the dependence of the final result on the *input quantities* in terms of relative uncertainties.

Linearization of monomial forms

The coefficients of the linear expansion around the expected values acquire a very simple and useful form

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In the case of several Y's the elements c_{ki} of the tranformation matrix $\mathbf{C}$ are then

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Linearization of monomial forms: single Y

Special subcase: variance from independent variables

In the case of a single Y and independent $\underline{X}$, we get

$$\sigma^2(Y) \;\approx\; \sum_i \alpha_i^2 \left(\left.\frac{Y}{X_i}\right|_{\mathrm{E}(\underline{X})} \right)^2 \sigma^2(X_i)$$

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r_Y^2 &\approx \sum_i \alpha_i^2 \cdot r_{X_i}^2 \,, \qquad \left[\longrightarrow \sum_{ij} \rho_{ij}\, \alpha_i\, \alpha_j\, r_i\, r_j \right]
\end{aligned}$$

having indicated with r the relative (standard) uncertainties $\sigma()/|E()|$.

Linearization of monomial expressions of independent variables

Exercise

Imagine we want to measure g with a pendulum:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

from which it follows

$$g = (2\pi)^2\, l\, T^{-2}$$

Q.: How precisely we have to measure l and T if we require they contribute equally to r_g, that we want to keep $\leq 1\,\%$?

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Try. . .

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If the determinations of l and T are independent, then, in *percentages* (p):

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$$p_g \;\leq\; 1$$

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Last remarks on linearization

- Linerization formulae rely on the fact that the trasformations are 'enough' linear in the regions where the probability mass of the input quantities are concentrated (around their expected values).

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  [Imagine for example $Y = 1 + 2 * X$, with $X \sim \mathcal{N}(0, 0.05)$, having an 'infinite relative uncertainty', but resulting in a practically normal distribution of Y around 1, with a 10% relative uncertainty, that is $Y \sim \mathcal{N}(1, 0.1)$]

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- Always check by Monte Carlo!