0.1 Formulario di analisi complessa

Analisi complessa

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$x, y \in \mathbb{R},\ r = |z| > 0,\ \theta = \arg z \in [0, 2\pi)\ \ (\text{oppure } \theta = \arg z \in (-\pi, \pi])\,. \tag{1}$$

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \quad \begin{cases} r = \sqrt{x^2 + y^2}, \\ \theta = \arg z = \arctan\frac{y}{x}, \end{cases} \tag{2}$$

$$\text{Arg} z = \arg z + 2\pi n, \quad n \in \mathbb{Z} \tag{3}$$

$$e^{2\pi i n} = 1,\ n \in \mathbb{Z}; \quad |e^{i\alpha}| = 1,\ \alpha \in \mathbb{R} \tag{4}$$

Somme notevoli:

$$\begin{aligned} &(a+b)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a^k b^{n-k}, \quad \begin{pmatrix} n \\ k \end{pmatrix} = \tfrac{n!}{k!(n-k)!}, \\ &\sum_{k=0}^{n} z^k = \tfrac{1-z^{n+1}}{1-z} \quad \Rightarrow \quad \begin{cases} \sum_{k=0}^{n} e^{ik\theta} = \frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} = e^{in\frac{\theta}{2}} \frac{\sin\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}}, \\ \sum_{k=0}^{\infty} z^k = \frac{1}{1-z},\ |z| < 1, \end{cases} \\ &\sum_{k=0}^{n} k = \tfrac{n(n+1)}{2}, \quad \sum_{k=0}^{n} k^2 = \tfrac{n(n+1)(2n+1)}{6}, \end{aligned} \tag{5}$$

Equazioni di Cauchy - Riemann per funzioni $f(z) = u(x, y) + iv(x, y)$ analitiche

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \qquad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}. \tag{6}$$

Formule integrali di Cauchy

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z')}{z' - z} dz', \quad z \in \text{dominio interno a } \gamma \tag{7}$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z')}{(z' - z)^{n+1}} dz', \quad z \in \text{dominio interno a } \gamma \tag{8}$$

Serie di Taylor:

$$\begin{aligned} &f(z) = \sum_{n=0}^{\infty} f_n (z - z_0)^n, \quad |z - z_0| < R, \quad f_n = \tfrac{f^{(n)}(z_0)}{n!} \\ &\tfrac{1}{R} = \lim_{n\to\infty} \sup |f_n|^{1/n} = \lim_{n\to\infty} \sup \left| \tfrac{f_{n+1}}{f_n} \right| \end{aligned} \tag{9}$$

Formula di Stirling

$$n! = n^n e^{-n} \sqrt{2\pi n}(1 + o(1)), \quad n \to \infty. \tag{10}$$

0.2 Formulario di analisi funzionale

Norme discrete e continue.

$$||\underline{x}||_\infty = sup_k |x_k|, \quad ||\underline{x}||_p = \left(\sum_k |x_k|^p \right)^{1/p}, \ 1 \le p < \infty$$

$$||f||_\infty = sup_{t\in[a,b]} |f(t)|, \ \ ||f||_p = \left(\int_a^b |f(t)|^p dt \right)^{1/p}$$

Disuguaglianze importanti.

$$\begin{aligned}
&\textstyle\sum_k |x_k y_k| \le \left(\sum_k |x_k|^p\right)^{1/p} \left(\sum_k |y_k|^q\right)^{1/q}, \ \ \frac{1}{p} + \frac{1}{q} = 1 \ \text{ Holder (discreta)}, \\
&\int_a^b |f(t)g(t)|dt \le \left(\int_a^b |f(t)|^p \right)^{1/p} \left(\int_a^b |g(t)|^q \right)^{1/q}, \ \ \frac{1}{p} + \frac{1}{q} = 1 \ \text{ Holder (continua)}, \\
&\textstyle\left(\sum_k |x_k + y_k|^p\right)^{1/p} \le \left(\sum_k |x_k|^p\right)^{1/p} + \left(\sum_k |y_k|^p\right)^{1/p}, \ \ \text{Minkowski (discreta)} \\
&\left(\int_a^b |f(t)+g(t)|^p dt \right)^{1/p} \le \left(\int_a^b |f(t)|^p dt \right)^{1/p} + \left(\int_a^b |g(t)|^p dt \right)^{1/p}, \ \ \text{Minkowski (continua)}; \\
&|\sum_{k=1}^n \bar{\xi}_k \eta_k|^2 \le \left(\sum_{k=1}^n |\xi_k|^2 \right) \left(\sum_{k=1}^n |\eta_k|^2 \right), \ \ \text{Cauchy-Schwartz discreta}, \\
&|\int_a^b \overline{f(t)} g(t) dt|^2 \le \left(\int_a^b |f(t)|^2 dt \right) \left(\int_a^b |g(t)|^2 dt \right), \ \ \text{Cauchy-Schwartz cont.}
\end{aligned} \tag{11}$$

Spazi di successioni.
l_f spazio delle successioni finite.
l_0 spazio delle successioni convergenti a 0.
l_p spazio delle successioni tali che $|| \cdot ||_p < \infty$.
l_∞ spazio delle successioni limitate.

Spazi di funzioni: $C_\infty[a,b] = (C_{[a,b]}, || \cdot ||_\infty)$, $C_p[a,b] = (C_{[a,b]}, || \cdot ||_p)$,
$L_\infty[a,b]$= spazio delle funzioni limitate in $[a,b]$,
$L_p[a,b] = \{f(t) : \ ||f||_p < \infty|\}$.

Serie di Fourier nell'intervallo $[-\pi, \pi]$:

$$f(x) = \sum_{n\in\mathcal{Z}} f_n e^{inx} = c_0 + \sum_{n=1}^\infty \{s_n \sin(nx) + c_n \cos(nx)\},$$

$$f_n = \int_{-\pi}^{\pi} \frac{dx}{2\pi} e^{-inx} f(x),$$

$$c_0 = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(x)dx, \quad c_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \cos(nx)f(x)dx, \quad s_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \sin(nx)f(x)dx,$$

Relazione di Parseval:

$$\int\limits_{-\pi}^{\pi} dx\bar{f}(x)g(x) = 2\pi\sum_{n\in\mathcal{Z}}\bar{f}_n g_n = 2\pi\bar{c}_0 c_0' + \pi\sum_{1}^{\infty}\{\bar{c}_n c_n' + \bar{s}_n s_n'\},$$

$f(x) = g(x)$:

$$\int\limits_{-\pi}^{\pi} |f(t)|^2 dt = 2\pi\sum_{n\in\mathbb{Z}}|f_n|^2 = 2\pi|c_0|^2 + \pi\sum_{n=1}^{\infty}(|c_n|^2 + |s_n|^2)$$

Trasformata di Fourier:

$$f(x) = \int\limits_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\hat{f}(k),$$

$$\hat{f}(k) = \int\limits_{-\infty}^{\infty} dx e^{-ikx}f(x),$$

Relazione di Parseval-Plancherel:

$$\int\limits_{-\infty}^{\infty} dx\bar{f}(x)g(x) = \int\limits_{-\infty}^{\infty}\frac{dk}{2\pi}\overline{\hat{f}(k)}\hat{g}(k)$$

$f(x) = g(x)$:

$$\int\limits_{-\infty}^{\infty} dx|f(x)|^2 = \int\limits_{-\infty}^{\infty}\frac{dk}{2\pi}|\hat{f}(k)|^2$$

Teorema di convoluzione:

$$R(x) = \int\limits_{-\infty}^{\infty} dx' G(x-x')I(x') \quad\Leftrightarrow\quad \hat{R}(k) = \hat{G}(k)\hat{I}(k)$$