Esercizi proposti (raccolta provvisoria) del corso di ONDE NON LINEARI E SOLITONI

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1 Propagazione ondosa lineare e non lineare

1.1 Onde dispersive lineari [1, 5, 16]

1) Given the Cauchy problem

$$u_t + i\omega(-i\partial_x)u = 0, \quad u(x,0) \text{ given}, \;\; x \in \mathbb{R}, \;\; t \geq 0, \tag{1}$$

1. show that the Fourier integral representation of its solution is

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}_0(k)e^{i(kx-\omega(k)t)}dk, \tag{2}$$

where $\hat{u}_0(k)$ is the Fourier transform of the initial condition $u(x,0)$:

$$\hat{u}_0(k) = \int_{\mathbb{R}} e^{-iky}u(y,0)dy. \tag{3}$$

and $\omega(k)$ is an entire function of k (so that, f.i., if $\omega(k) = k^2$, then $\omega(-i\partial_x) = (-i\partial_x)^2 = -\partial_x^2$).
2. Show that (2) can be written as a convolution integral, in the suggestive form:

$$u(x,t) = \int_{\mathbb{R}} S(x-y,t)u(y,0)dy, \tag{4}$$

where $S(x,t)$ is the "fundamental" solution of the PDE, defined as:

$$S(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(kx-\omega(k)t)}dk. \tag{5}$$

3. If $\omega(k) = k^n$, then $S(x,t)$ is the following similarity solution of the PDE:

$$\begin{aligned} S(x,t) &= \tfrac{1}{t^{1/n}} f\left(\tfrac{x}{t^{1/n}}\right), \\ f(\xi) &= \tfrac{1}{2\pi}\textstyle\int_{\mathbb{R}} e^{i(k\xi-k^n)}dk. \end{aligned} \tag{6}$$

4. Show that, if $u \in \mathbb{R}$ and $\omega(-k) = -\omega(k)$, then:

$$\begin{aligned} \overline{\hat{u}_0(k)} &= \hat{u}_0(-k), \quad k \in \mathbb{R} \\ u(x,t) &= \tfrac{1}{\pi} Re \int_0^{\infty} \hat{u}_0(k)e^{i(kx-\omega(k)t)}dk \end{aligned} \tag{7}$$

If, in addition, $\hat{u}_0(k)$ can be prolongued outside the real axis, then

$$\overline{\hat{u}_0(k)} = \hat{u}_0(-\bar{k}). \tag{8}$$

2) Given the following linear PDEs:

$$\begin{array}{ll} i) \;\; iu_t + u_{xx} = 0, & \text{free particle Schrödinger equation,} \\ ii) \;\; u_t + u_{xxx} = 0, & \text{linearized KdV equation,} \\ iii) \;\; u_{tt} - u_{xx} + u = 0, & \text{Klein - Gordon equation,} \end{array} \tag{9}$$

1. Construct the fundamental similarity solution (6) (only for i) and ii)).
2. Study the longtime behavior, for $t >> 1,\ x/t = O(1)$, of the solutions of their Cauchy problem using the stationary phase, Laplace, or saddle point methods, depending on the situation, and estimate the error.
3. Study of the relevance of the exact similarity solution in the longtime behavior (only for i) and ii)).

Solution:

i) Free particle Schrödinger equation:

$$\begin{array}{l} S(x,t) = \frac{1}{2\sqrt{\pi t}} e^{i(\frac{x^2}{4t} - \frac{\pi}{4})}, \\ u(x,t) = S(x,t)\left(A(\xi) + \frac{1}{t}B(\xi) + O(t^{-2})C(\xi)\right), \quad \xi = \frac{x}{2t} = O(1), \quad t >> 1 \\ A(\xi) = \hat{u}_0(\xi), \quad B(\xi) = -\frac{i}{4}A_{\xi\xi} \end{array} \tag{10}$$

ii) Linear KdV. For $x/t > 0$, the lines of constant $v(k)$ are the imaginary axis and the hyperbola ${k_R}^2 - 3{k_I}^2 + x/t = 0$. The steepest descent contour passing through the critical point $i\sqrt{\frac{x}{3t}}$ is the upper branch of the hyperbola, while the steepest descent contour passing through the critical point $-i\sqrt{\frac{x}{3t}}$ is the imaginary axis. The asymptotics is obtained replacing the integration real line by the steepest descent contour passing through $i\sqrt{\frac{x}{3t}}$.

$$\begin{array}{l} S(x,t) = \frac{1}{(3t)^{1/3}} Ai\left(\frac{x}{(3t)^{1/3}}\right), \\ u(x,t) \sim \frac{\hat{u}_0(|x/3t|^{1/2})}{\sqrt{4\pi|3x/t|^{1/2}t}} e^{-i2|x/3t|^{3/2}t + i\pi/4} + \text{c.c.}, \quad \frac{x}{3t} = O(1) < 0, \quad t >> 1, \\ u(x,t) \sim \frac{\hat{u}_0(i|x/3t|^{1/2})}{\sqrt{12\pi|3x/t|^{1/2}t}} e^{-2|x/3t|^{3/2}t}, \quad \frac{x}{3t} = O(1) > 0, \quad t >> 1, \\ u(x,t) \sim \frac{\hat{u}_0(0)}{2\pi(3t)^{1/3}} Ai\left(\frac{x}{(3t)^{1/3}}\right) - \frac{i\hat{u}'_0(0)}{2\pi(3t)^{2/3}} Ai'\left(\frac{x}{(3t)^{1/3}}\right), \quad \frac{x}{(3t)^{1/3}} = O(1), \quad t >> 1, \\ u(x,t) \sim \frac{\hat{u}_0(0)}{2\pi} S(x,t), \quad \frac{x}{(3t)^{1/3}} = O(1), \quad t >> 1, \end{array} \tag{11}$$

where $Ai(\xi)$ is the Airy function

$$Ai(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(k\xi - k^3)} dk, \tag{12}$$

solution of the ODE: $f''(\xi) - \xi f(\xi) = 0$.
iii) Klein-Gordon equation. The dispersion relation is two-valued (since the PDE is second order in t):

$$\omega^{\pm}(k) = \pm\sqrt{k^2+1}; \tag{13}$$

therefore the phase velocity is greater than the light speed 1, while the group velocity is less than 1:

$$\frac{\omega}{k} = \frac{\sqrt{k^2+1}}{k} > 1, \quad \frac{d\omega}{dk} = \frac{k}{\sqrt{k^2+1}} < 1 \tag{14}$$

The Fourier representation of the real solution reads:

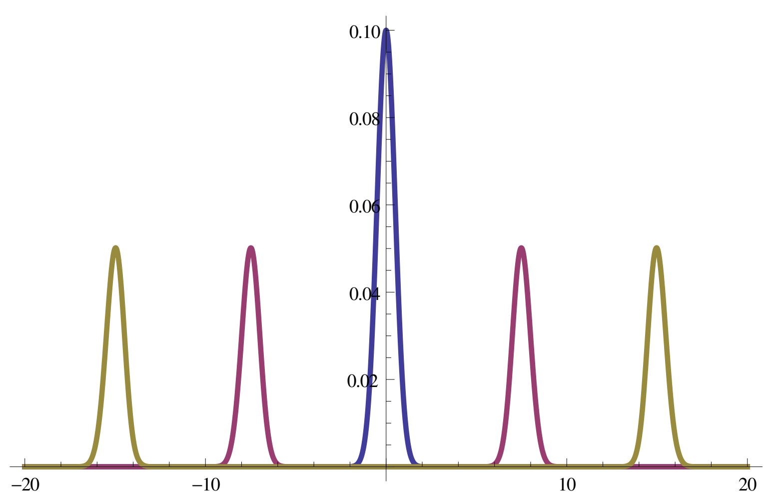
$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} B(k) e^{i(kx+\sqrt{k^2+1}t)} dk + \frac{1}{2\pi}\int_{\mathbb{R}} \overline{B(-k)} e^{i(kx-\sqrt{k^2+1}t)} dk, \tag{15}$$

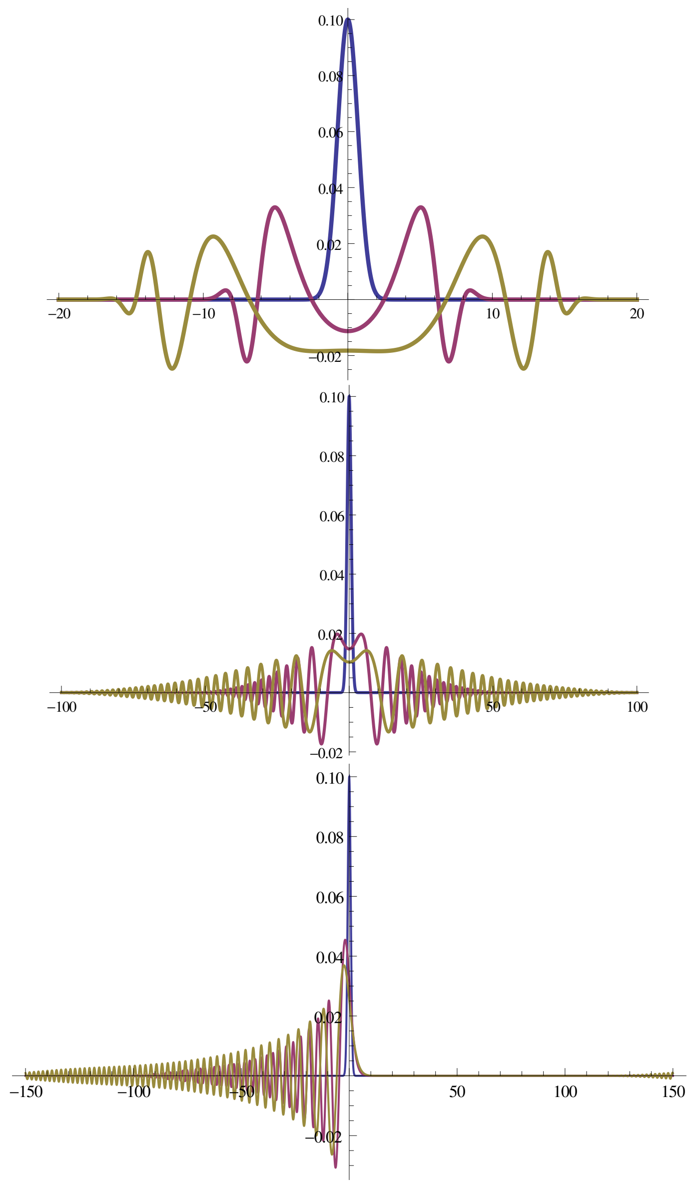
where

$$B(k) = \frac{1}{2}\left(\hat{u}_0(k) - i\frac{\hat{u}_0'(k)}{\sqrt{k^2+1}}\right), \tag{16}$$

where $\hat{u}_0(k)$ and $\hat{u}_0'(k)$ are the Fourier transforms of respectively $u(x,0)$ and $u_t(x,0)$. For $|x/t| < 1$ (inside the light cone) and $t >> 1$:

$$u \sim \frac{1}{\sqrt{2\pi t}}\left(1 - \left(\frac{x}{t}\right)^2\right)^{-3/4} B\left(-\frac{x}{\sqrt{t^2-x^2}}\right) e^{i\sqrt{t^2-x^2}+i\pi/4} + c.c. \tag{17}$$





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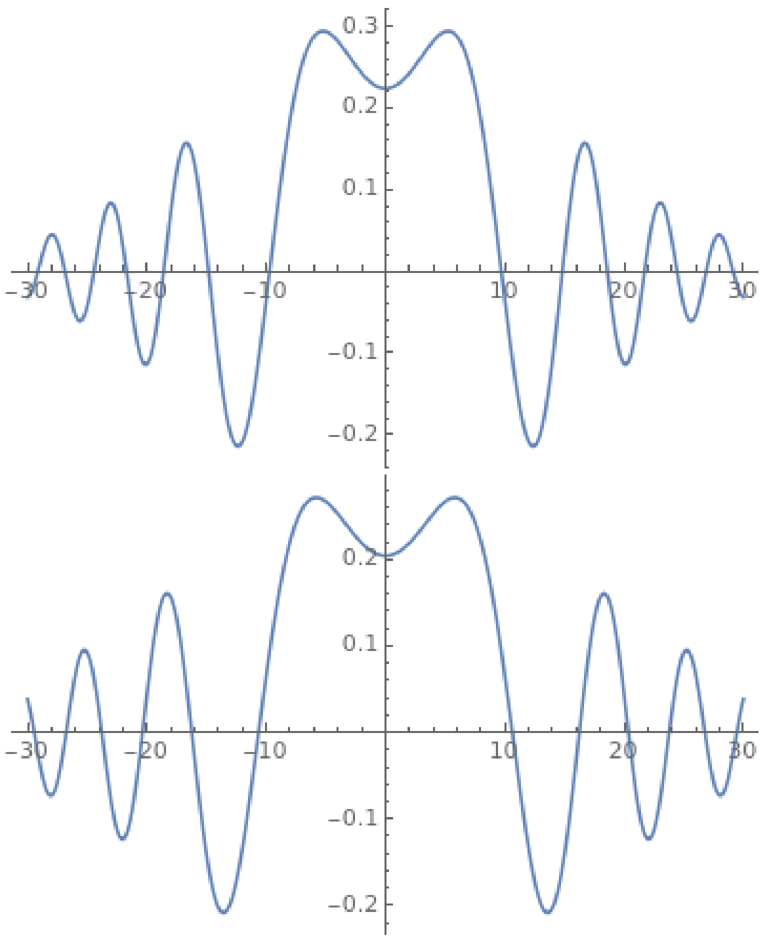


Figures 1. Three time steps ($t = 0$, $t = T/2$, $t = T$) of the numerical evolution of a gaussian initial condition according to, respectively, the wave, the Klein-Gordon, the linear Schrödinger (in this case we plot the real part of the complex wave function), and the linear KdV equations. The last two plots are the graphs of the analytic formula describing the asymptotics of the real part of the solution of the linear Schrödinger equation for a gaussian initial condition at $t = 10$ and at $t = 12$, to show that i) at fixed t, increasing x the wave-length decreases; ii) as t increases, if I travel with a crest, the distance of the neighboring crests increases; iii) if I travel with a certain wave number (a certain constant speed), crests overtake me.

3) Study the longtime behavior, for $t >> 1$, $x/t = O(1)$, of the Fourier integral

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}_0(k) e^{i(kx-\omega(k)t)} dk \tag{18}$$

under the hypothesis that there exists a unique stationary phase point $k_0(x/t) \in \mathbb{R}$, and that $\omega''(k_0) = 0$, $\omega'''(k_0) \neq 0$.

4) Given the linear PDE $\mathcal{P}(\partial_t, \nabla_{\vec{x}})u(\vec{x},t) = 0$, $\vec{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$ in $(n+1)$ dimensions, with $u \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$,
i) show that the solution of its Cauchy problem:

$$\mathcal{P}(\partial_t, \nabla_{\vec{x}})u(\vec{x},t) = 0, \quad u(\vec{x},0) = u_0(\vec{x}) \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n) \tag{19}$$

is given by the Fourier integral:

$$\begin{array}{l} u(\vec{x},t) = \int_{\mathbb{R}^n} \hat{u}_0(\vec{k}) e^{i(\vec{k}\cdot\vec{x}-\omega(\vec{k})t)} \frac{d\vec{k}}{(2\pi)^n} \\ \hat{u}_0(\vec{k}) = \int_{\mathbb{R}^n} u_0(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} d\vec{x} \end{array} \tag{20}$$

where $\omega(\vec{k})$ is obtained solving the equation $\mathcal{P}(-i\omega, i\vec{k}) = 0$ wrt ω.
ii) Show that, under the hypothesis that the vector equation for $\vec{k}$

$$\frac{\vec{x}}{t} = \nabla_{\vec{k}}\omega(\vec{k}) \tag{21}$$

admits a unique real solution $\vec{k}_0 = \vec{k}_0(\vec{x}/t) \in \mathbb{R}^n$, the extension of the stationary phase method for multiple integrals gives the following longtime behavior:

$$\begin{array}{l} u \sim \left(\frac{1}{2\pi t}\right)^{n/2} \left(det\left(\frac{\partial^2\omega(\vec{k}_0)}{\partial k_i \partial k_j}\right)\right)^{-1/2} \hat{u}_0(\vec{k}_0) e^{i(\vec{k}_0\cdot\vec{x}-\omega(\vec{k}_0)t+m\frac{\pi}{4})}, \\ m \equiv -\sum\limits_{j=1}^{n} \text{sign}(\lambda_j) \end{array} \tag{22}$$

where λ_j, $j = 1,..,n$ are the (real) eigenvalues of symmetric matrix $\left(\frac{\partial^2\omega(\vec{k}_0)}{\partial k_i \partial k_j}\right)$.

5) Let $\Gamma(z)$ be the Euler Γ function:

$$\Gamma(z) = \int\limits_0^\infty e^{-t} t^{z-1} dt, \quad Re\ z > 0. \tag{23}$$

i) Show that it is the generalization of the factorial: $\Gamma(n+1) = n!$, $n \in \mathbb{N}$.
ii) Use the Laplace method to construct the Stirling formula:

$$n! = n^n e^{-n}\sqrt{2\pi n}\left(1 + O(n^{-1})\right), \quad n >> 1. \tag{24}$$

6) Given the Airy function $Ai(x)$, defined by

$$Ai(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(kx-k^3)} dk, \quad x \in \mathbb{R}, \tag{25}$$

i) use the saddle point method to show that

$$\begin{array}{l} Ai(x) = \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}}(1 + O(x^{-3/2})), \quad x \gg 1, \\ Ai(x) = \frac{1}{\pi|x|^{1/4}} \cos\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4}\right)(1 + O(x^{-3/2})), \quad x \ll -1. \end{array} \tag{26}$$

ii) Use the above asymptotics to show that the longtime behavior of the solutions of the Cauchy problem for the linearized KdV equation in the region $|x|/t^{1/3} = O(1), \ t \gg 1$, matches well with the asymptotics in the left and right regions $|x|/t = O(1), \ t \gg 1$, $x < 0$ and $x > 0$.

7) Given the integral

$$f(x,t) = \int_a^b g(k) e^{i(kx-k^2t)} dk, \quad t \gg 1, \quad x/t = O(1), \tag{27}$$

where $g(k)$ is analytic in the domain delimited by the real axis and the line bisecting the second and fourth quadrants, i) show that the saddle point is $x/2t$ and the saddle point contour is given by the straight line passing through $x/2t$ and parallel to the line bisecting the second and fourth quadrants. ii) Show that, if $a < x/2t$ and $b > x/2t$, including the cases $a = -\infty$ and $b = \infty$, the asymptotics of $f(x,t)$ are given by the saddle point formula

$$f(x,t) = \frac{1}{\sqrt{4\pi t}} g\left(\frac{x}{2t}\right) e^{i\frac{x^2}{4t} - i\frac{\pi}{4}}(1 + O(1/t)), \quad t \gg 1, \quad x/t = O(1). \tag{28}$$

iii) Show that, if $a < x/2t$ and $b \in \mathbb{C}$, with $0 < \arg b < \pi/2$, the leading asymptotics is given, instead, by the integration by parts formula:

$$f(x,t) = \frac{g(b) e^{i(bx/t - b^2)t}}{2\pi i(x/t - 2b)t}(1 + O(1/t)), \quad t \gg 1, \quad x/t = O(1). \tag{29}$$

1.2 Onde iperboliche e la catastrofe del gradiente[18, 9, 7, 20, 8]

1) Show that the following linear PDE for the field $\rho(x,t)$:

$$\rho_t + c(x,t)\rho_x + a(x,t)\rho = b(x,t) \tag{30}$$

is equivalent to the system of two ODEs for the fields $(\tilde{\rho}(t), \tilde{x}(t))$:

$$\begin{array}{l} \frac{d\tilde{\rho}}{dt} + a(\tilde{x},t)\rho = b(\tilde{x},t), \\ \frac{d\tilde{x}}{dt} = c(\tilde{x},t). \end{array} \tag{31}$$

2) Find the general solution of the following linear PDEs:

$$\begin{array}{l} u_t + t^2 u_x + xu = 0, \quad \left(u = F(x - t^3/3)e^{-(t^4/12 + t(x - t^3/3))}\right), \\ i\gamma u_t + y u_x - x u_y = 0, \quad (...), \\ y u_x - x u_y = 0, \quad \left(u = F(x^2 + y^2)\right), \\ y u_x + x u_y = 0, \quad \left(u = F(x^2 - y^2)\right), \\ x u_x + y u_y = 0, \quad (u = F(y/x)), \\ x u_x - y u_y = 0, \quad (u = F(xy)), \\ x u_x + y u_y = x^2, \quad \left(u = x^2/2 + F(y/x)\right), \\ x u_x + y u_y = u, \quad (u = xF(y/x)), \\ x u_x + y u_y + z u_z = 0, \quad (u = F(y/x, z/x)), \\ g_y u_x - g_x u_y = 0, \quad g(x,y) \text{ given}, \quad (u = F(g(x,y))) \end{array} \tag{32}$$

3) Find the general solution of the following quasi-linear PDEs:

$$\begin{array}{l} i) \;\; u_t + c(u) u_x = 0, \quad u = F(x - c(u)t), \\ ii) \;\; u_t + c(u) u_x = 1, \\ c(u) = u \quad \Rightarrow \quad u = t + F(x - ut + t^2/2), \\ c(u) = u^2 \quad \Rightarrow \quad u = t + F(x - u^2 t + u t^2 - t^3/3) \end{array} \tag{33}$$

4) Given the two Cauchy problems for the Hopf equation:

$$\begin{array}{l} u_t + u u_x = 0, \quad u = u(x,t), \quad x \in \mathbb{R}, \; t \geq 0, \\ i) \;\; u(x,0) = e^{-x^2}, \\ ii) \;\; u(x,0) = (x^2+1)^{-1}, \end{array} \tag{34}$$

i) draw the 1-parameter family of characteristic curves; ii) find the first characteristic parameter ζ_b and the first breaking point (x_b, t_b).
A. i) $\zeta_b = 1/\sqrt{2},\; t_b = \sqrt{e/2},\; x_b = \sqrt{2}$. ii) $\zeta_b = 1/\sqrt{3},\; t_b = 8\sqrt{3}/9,\; x_b = \sqrt{3}$.

5) Compression and rarefaction waves.
Consider the Cauchy problem:

$$\begin{array}{l} u_t + u u_x = 0, \quad u = u(x,t), \quad x \in \mathbb{R}, \; t \geq 0, \\ u(x,0) = a_2 H(-l - x) + a_1 H(x - l) + H(l^2 - x^2)\left(\frac{a_1+a_2}{2} - \frac{a_2-a_1}{2}x\right), \end{array} \tag{35}$$

in the two cases

$$
\begin{array}{ll}
i) \;\; a_2 > a_1 > 0, & \text{compression wave,} \\
ii) \;\; a_1 > a_2 > 0 & \text{rarefaction wave.}
\end{array}
\tag{36}
$$

Solve it explicitely, draw the characteristic curves and show that they describe respectively a compression and a rarefaction wave. Indicate if there is wave breaking and, if so, find ζ_b and (x_b, t_b).
A. For the compression wave:

$$
u(x,t) = \begin{cases} a_2, & x < a_2 t - l, \\ -\frac{a_2-a_1}{2l}\frac{x-\frac{a_2+a_1}{2}t}{1-\frac{a_2-a_1}{2l}t} + \frac{a_2+a_1}{2}, & -l + a_2 t < x < l + a_1 t, \\ a_1, & x > l + a_1 t. \end{cases}
\tag{37}
$$

There is wave breaking:

$$
t_b = \frac{2l}{a_2 - a_1}, \quad x_b = \frac{a_1 + a_2}{a_2 - a_1} l, \quad |\zeta_b| < 1
\tag{38}
$$

6) Consider the Cauchy problem

$$
\begin{array}{l}
u_t + u u_x = 0, \\
u(x,0) = f(x),
\end{array}
\tag{39}
$$

where f describes a single bump, and study analytically the behavior of the solution near breaking (immediately before, at, and immediately after breaking).
A. See section 3 of Appunti 1.

7) More on rarefaction waves.
i) Show that the solution of the Cauchy problem

$$
u_t + u u_x = 0, \quad u(x,0) = a_2 H(-x) + a_1 H(x), \quad a_2 < a_1
\tag{40}
$$

is given by

$$
u = \begin{cases} a_2, & x < a_2 t, \\ x/t, & a_2 t < x < a_1 t, \\ a_1, & x > a_1 t \end{cases}
\tag{41}
$$

Hint. Observe that this Cauchy problem can be viewd as the $l \to 0$ limit of that of the previous problem. But there are other ways of doing it ...
ii) Show that the solution of the Cauchy problem

$$
u_t + c(u) u_x = 0, \quad u(x,0) = a_2 H(-x) + a_1 H(x), \quad a_2 < a_1
\tag{42}
$$

is given by

$$
u = \begin{cases} a_2, & x < c(a_2)t, \\ A(x/t), & c(a_2)t < x < c(a_1)t, \\ a_1, & x > c(a_1)t \end{cases} \tag{43}
$$

where $A(\xi)$ is the inverse of function $c(u)$.

8) Given the following system of PDEs, establish if they are hyperbolic and, if so, write them in characteristic form.

i) The wave equation $u_{tt} - c^2 u_{xx} = 0$.
ii) The Klein - Gordon equation $u_{tt} - c^2 u_{xx} + u = 0$.
iii) The system

$$
\begin{array}{l} u_t + c(u,v)u_x = 0, \\ v_t + c(u,v)v_x = u \end{array} \tag{44}
$$

iv) The system

$$
\begin{array}{l} u_t + c(u)u_x = 0, \\ v_t + c(u)v_x + c'(u)vu_x = 0 \end{array} \tag{45}
$$

v) The gas dynamics equations

$$
\begin{array}{l} \rho_t + u\rho_x + \rho u_x = 0, \\ u_t + uu_x + \frac{p_x}{\rho} = 0, \\ S_t + uS_x = 0, \end{array} \tag{46}
$$

where $p = p(\rho, S)$.
R. i)

$$
\begin{array}{lll} \frac{d}{dt}(w - cv) = 0, & \frac{dx}{dt} = c, \;\; \Rightarrow & w - cv = A(x - ct), \\ \frac{d}{dt}(w + cv) = 0, & \frac{dx}{dt} = -c, \;\; \Rightarrow & w + cv = B(x + ct), \\ v \equiv u_x, \;\; w \equiv u_t & & \end{array} \tag{47}
$$

implying the well-known result $u = f(x - ct) + g(x + ct)$, with

$$
f'(\cdot) = -\frac{1}{2c}A(\cdot), \quad g'(\cdot) = \frac{1}{2c}B(\cdot). \tag{48}
$$

ii)

$$
\begin{array}{l} \varphi_t - c\varphi_x + u = 0, \\ u_t + cu_x - \varphi = 0, \\ \varphi \equiv u_t + cu_x. \end{array} \tag{49}
$$

iii) it is already in characteristic form, with the single characteristic $dx/dt = c(u,v)$ and two different characteristic forms (two different eigenvectors $(1,0)$

and $(0,1)$).
iv) The first equation is in characteristic form for the single field u; the second one cannot be put in characteristic form; therefore the system is not hyperbolic. Nevertheless it can be solved solving first the first equation, hyperbolic, on the characteristic $dx/dt = c(u)$, and then solving the second one on that characteristic (do it!).
v) Rewrite (46) in the form

$$\begin{array}{l} p_t + up_x + \rho a^2 u_x = 0, \\ u_t + uu_x + \frac{p_x}{\rho} = 0, \\ S_t + uS_x = 0, \end{array} \tag{50}$$

where $a^2(\rho) = \partial p/\partial \rho > 0$, obtaining the following eigenvalues and eigenvectors:

$$\begin{array}{ll} c_0 = u \text{ (gas speed)}, & \underline{L}_0 = (0,0,1), \\ c_\pm = u \pm a \text{ (sound speeds)}, & \underline{L}_\pm = (1, \pm a\rho, 0). \end{array} \tag{51}$$

Therefore the system in characteristic form reads:

$$\begin{array}{l} \frac{dp}{dt} \pm \rho a \frac{du}{dt} = 0, \quad \frac{dx}{dt} = u \pm a, \\ \\ \frac{dS}{dt}, \quad \frac{dx}{dt} = u. \end{array} \tag{52}$$

Verify that, in the linear limit in which we study small perturbations of the constant solution:

$$\begin{array}{l} \rho = \rho_0 + \varepsilon \rho_1(x,t) + O(\varepsilon^2), \quad p = p_0 + \varepsilon p_1(x,t) + O(\varepsilon^2), \\ u = \varepsilon u_1(x,t) + O(\varepsilon^2), \quad S = S_0 + \varepsilon S_1(x,t) + O(\varepsilon^2), \end{array} \tag{53}$$

we obtain

$$\begin{array}{l} p = p_0 + \varepsilon[f_-(x - a_0 t) + f_+(x + a_0 t)] + O(\varepsilon^2), \\ u = \frac{\varepsilon}{a_0 \rho_0}[f_-(x - a_0 t) - f_+(x + a_0 t)] + O(\varepsilon^2), \\ S = S_0 + \varepsilon g(x) + O(\varepsilon^2), \end{array} \tag{54}$$

where $a_0 = \sqrt{\partial p(\rho_0, S_0)/\partial \rho}$, and the functions $f_\pm$ and g are arbitrary.

9) Show that i) the Riemann invariants of the wave equation $u_{tt} - c^2 u_{xx} = 0$, $c > 0$ are given by $r_\pm = w \mp cv$, where $v = u_x$ and $w = u_t$, so that the PDE is written as the system of ODEs in characteristic form:

$$\frac{dr_\pm}{dt} = 0, \quad \frac{dx}{dt} = \pm c. \tag{55}$$

ii) The Riemann invariants of the gas dynamics equations (46) (under the constant entropy S hypothesis) are given by

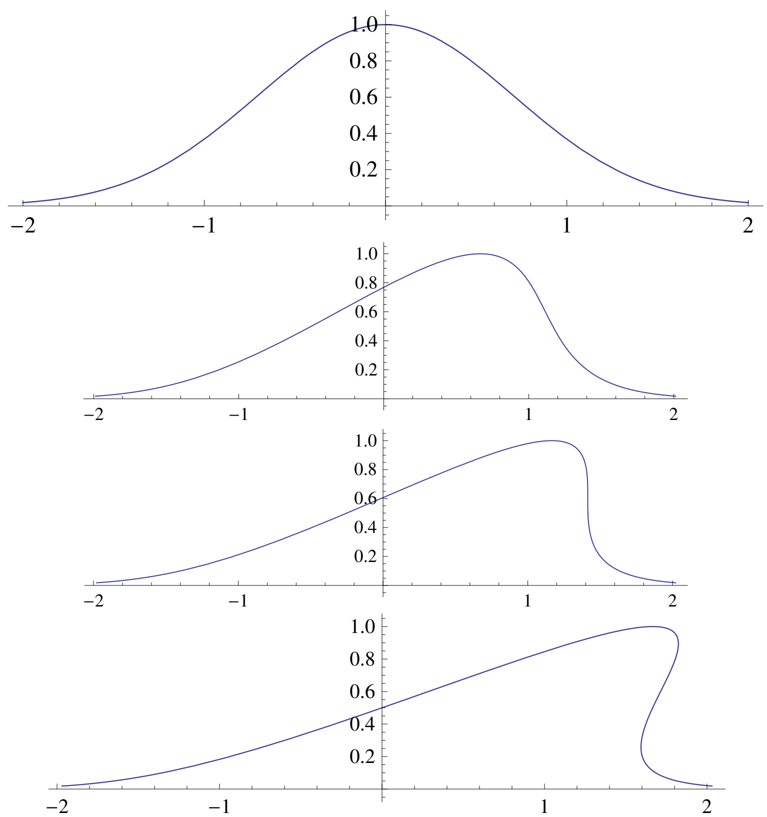
$$r_{\pm} = \int^{\rho} \frac{a(\rho')}{\rho'} d\rho' \pm u, \tag{56}$$

where $a^2(\rho) = p'(\rho) > 0$, so that the system (52) decouples as follows:

$$\frac{dr_{\pm}}{dt} = 0, \quad \frac{dx}{dt} = u \pm a(\rho). \tag{57}$$

Show that, for an adiabatic process ($p = \kappa\rho^{\gamma}$),

$$\begin{array}{l} a^2 = \kappa\gamma\rho^{\gamma-1}, \\ r_{\pm} = \frac{2\sqrt{\kappa\gamma}}{\gamma-1}\rho^{\frac{\gamma-1}{2}} \pm u = \frac{2a}{\gamma-1} \pm u. \end{array} \tag{58}$$



Figures 2. The evolution of a gaussian according to the Hopf equation (through the numerical inversion of the analytic solution).

1.3 Regolarizzazione dissipativa e l'equazione di Burgers. Regolarizzazione dispersiva e l'equazione KdV; funzioni ellittiche

Regolarizzazione dissipativa

1) Regularize the compression wave of problem 5) of section 2.1.2

2) What happens if we look for discontinuous solutions of $u_t + uu_x = 0$ in the form $u = H(s(t) - x)u^-(x, t) + H(x - s(t))u^+(x, t)$, where $H(x)$ is the Heaviside step function and $u^\pm(x, t)$ are smooth functions?

3) Consider the Cauchy problem

$$\begin{aligned} &u_t + uu_x = 0, \\ &u(x, 0) = f(x), \end{aligned} \tag{59}$$

where $f(x)$ describes a single bump, and study the behavior of the regularized (shock) solution near breaking.
A. See section 4 of Appunti 1.

4) Given the Cauchy problem

$$\begin{aligned} &u_t + c(u)u_x = 0, \quad c(u) = Q'(u), \\ &u(x, 0) = f(x), \end{aligned} \tag{60}$$

where $f(x)$ describes a single bump,
i) construct the shock condition

$$\dot{s} = \frac{Q(u_2) - Q(u_1)}{u_2 - u_1} \tag{61}$$

and show that it is equivalent of placing the vertical shock to cut equal area lobi of the three valued solution.
ii) Show that, if $c(u) = u$, $Q(u) = u^2/2$, the shock equations involving $s(t), \eta_1(t), \eta_2(t)$ can be reformulated as cutting equal area lobi on the initial profile:

$$\int_{\eta_1}^{\eta_2} f(\eta)d\eta = \frac{1}{2}(\eta_1 - \eta_2)(f(\eta_1) + f(\eta_2)) \tag{62}$$

5) Given the Burgers equation $u_t + uu_x = \nu u_{xx}, \ \nu > 0$,
i) Show that, for localized solutions in $\mathbb{R}$, $dM/dt = 0$ and $dE/dt < 0$, where

M is the mass and E is the energy:

$$M = \int_{\mathbb{R}} u(x,t)dx, \quad E = \int_{\mathbb{R}} u^2(x,t)dx. \tag{63}$$

ii) find its traveling wave solution satisfying the boundary conditions $u(x,t) \to u_\pm, \ x \to \pm\infty$, where $u_\pm$ are constants, and discuss the shock structure.
iii) Find its similarity solutions.

6) Show that the solution of the Cauchy problem for the Burgers equation $u_t + uu_x = \nu u_{xx}$ with initial condition $u(x,0) = f(x)$ is given by

$$u(x,t) = \frac{\int_{\mathbb{R}} \frac{x-\eta}{t} e^{-\frac{G(x,\eta,t)}{2\nu}} d\eta}{\int_{\mathbb{R}} e^{-\frac{G(x,\eta,t)}{2\nu}} d\eta} \tag{64}$$

where

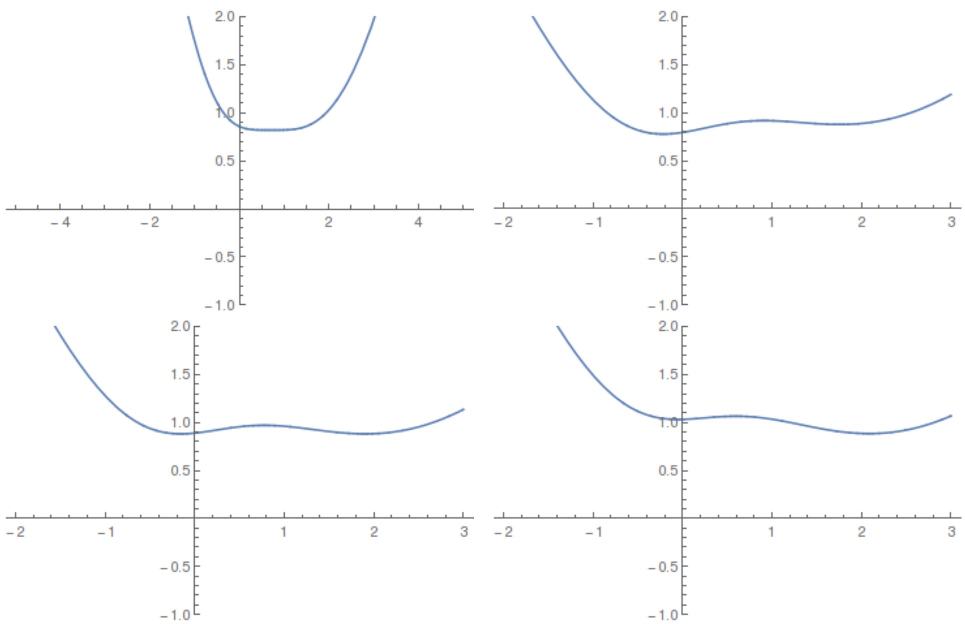
$$G(x,\eta,t) = \int_0^\eta f(\eta')d\eta' + \frac{(x-\eta)^2}{2t} \tag{65}$$

7) Consider the Cauchy problem for the Burgers equation $u_t + uu_x = \nu u_{xx}$ with Gaussian initial condition $u(x,0) = f(x) = e^{-x^2}$, and let $\eta_b = 1/\sqrt{2} \sim 0.71, \ x_b = \sqrt{2} \sim 1.41, \ t_b = \sqrt{e/2} \sim 1.16$ be the breaking parameters of the Hopf equation $u_t + uu_x = 0$ corresponding to the above Gaussian initial condition (see a previous excercise).
7a) Study the function

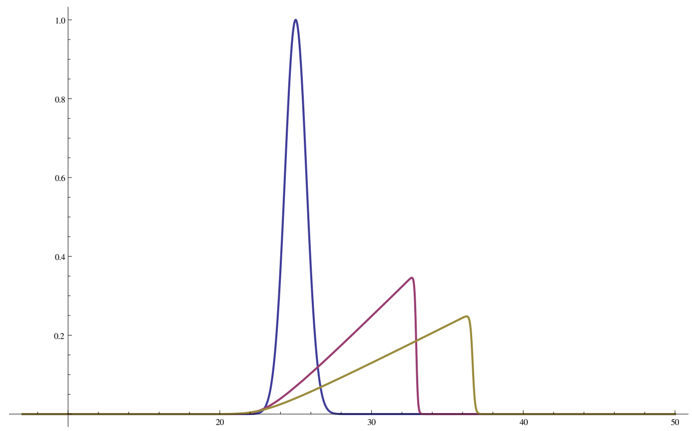
$$G(x,\eta,t) = \int_0^\eta f(\eta')d\eta' + \frac{(x-\eta)^2}{2t} \tag{66}$$

as function of the variable η, with $x \in \mathbb{R}, \ t > 0$ parameters in the following way. i) Show that, for $\eta \to \pm\infty$, $G(x,\eta,t)$ behaves as a parabola: $G \sim \eta^2/2t$. ii) Show that, for $0 < t < t_b$, $G(x,\eta,t)$ possesses just one extremal point, a global minimum η_0. iii) Show that, for $t > t_b$, there is a finite interval $x \in (x_-, x_+)$ in which $G(x,\eta,t)$ possesses three extremal points $\eta_2 < \eta_0 < \eta_1$ such that η_1, η_2 are local minima and η_0 is a local maximum. iv) Show that: if $x \in (x_-, x_+)$ and is close to x_-, the global minimum is η_2; if it is close to x_+, the global minimum is η_1; there is an intermediate value of $x \in (x_-, x_+)$ for which η_1, η_2 give the same value of G: $G(x,\eta_1,t) = G(x,\eta_2,t)$ and are

then global minima. v) Show that, if $x \notin (x_-, x_+)$, then there is only one extremal point, a global minimum η_0. vi) Make plottings of all the above cases (see Figures 3).
7b) Use the above results to investigate the solution (64) of the Cauchy problem for the Burgers equation $u_t + uu_x = \nu u_{xx}$ with Gaussian initial condition $u(x,0) = f(x) = e^{-x^2}$, when $0 < \nu \ll 1$ (small dissipation), showing that such solution tends, for $\nu \to 0$, to the shock solution of the Hopf equation, for the same initial condition.



Figures 3. Plots of the function $G(x, \eta, t)$ vaying η, for the Gaussian initial condition $f(\eta) = e^{-\eta^2}$, and for the following choices of (x, t): (x_b, t_b), $(x_b + 0.440, t_b + 1)$, $(x_b + 0.547, t_b + 1)$, $(x_b + 0.700, t_b + 1)$. We remark that, at (x_b, t_b), $G(x, \eta, t)$ has the global minimum at the triple point $\eta = \eta_b$; at $t = t_b + 1$, varying x in a suitable interval, the global minimum changes: if $x = x_b + 0.440$, the global minimum is for $\eta = \eta_2 < 0 < \eta_1$; if $x \sim x_b + 0.547$, the first η_2 and third η_1 local minima give rise to approximately the same value of $G = 0.8807$ and are global minima; if $x = x_b + 0.700$, the global minimum is for $\eta = \eta_1$.



Figures 4. Three time steps ($t = 0$, $t = T/2$, $t = T$) of the evolution of a gaussian initial condition according to the Burgers equation with small dissipation (numerical solution).

1.4 Regolarizzazione dispersiva e l'equazione KdV; funzioni ellittiche

Quadratures and elliptic functions

1) Show that the one dimensional Newton equation $\ddot{x} = -dV(x)/dx$ is integrated to the quadrature $t - t_0 = \int\limits_0^x \frac{dy}{\sqrt{2(E-V(y))}}$, where $E = \dot{x}^2/2 + V(x)$ is the constant energy.

2) Study the inversion, in the complex plane, of the quadrature

$$w(z) = \int\limits_0^z \frac{dt}{\sqrt{1-t^2}}, \tag{67}$$

that arises in the solution of the harmonic oscillator equation, providing an alternative definition of the sine function, and infer the basic properties of $\sin w$ from such inversion:
i) $\sin w$ is entire in $\mathbb{C}$; ii) $\sin w$ has simple zeroes in $w = n\pi,\ n \in \mathbb{N}$, iii) $\sin w$ is odd and satisfies the following periodicity properties:

$$\sin(w + 2\pi) = \sin w; \quad \sin(w + \pi) = -\sin w. \tag{68}$$

3) i) Study the inversion, in the complex plane, of the quadrature

$$w(z,\kappa)=\int\limits_0^z \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}},\quad 0<\kappa<1 \tag{69}$$

defining the elliptic sine function $z=sn(w,\kappa)$ [15]. Show, in particular, that i) $w(z,\kappa))$ maps the half plane Im $z>0$ into the rectangle of the w complex plane of vertices $-K(\kappa), K(\kappa), K(\kappa)+iK'(\kappa), -K(\kappa)+iK'(\kappa)$, where

$$\begin{array}{l} K(\kappa)=\int\limits_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2t^2)}}, \\ K'(\kappa)=K(\kappa'),\quad {\kappa'}^2=1-\kappa^2. \end{array} \tag{70}$$

ii) using the Schwarz reflection principle [15], show that the elliptic sine function can be analytically extended into the whole complex w plane as a meromorphic doubly periodic function with periods $4K(\kappa)$ and $2iK'(\kappa)$:

$$sn(w+4K(\kappa),\kappa)=sn(w,\kappa),\quad sn(w+2iK'(\kappa),\kappa)=sn(w,\kappa), \tag{71}$$

possessing the simple zeroes $2mK(\kappa)+i2nK'(\kappa)$ and the simple poles $2mK(\kappa)+i(2n+1)K'(\kappa)$, for $m,n\in\mathbb{Z}$. iii) Show the additional properties

$$\begin{array}{l} sn(w)=\overline{sn(2K-\bar{w}))}, \\ sn(w+2K(\kappa),\kappa)=-sn(w,\kappa),\quad sn(-w,\kappa)=-sn(w,\kappa). \end{array} \tag{72}$$

4) Basic properties of elliptic functions Having defined as elliptic function a doubly periodic complex function $f(z)$ of complex variable z, with the two independent periods $2\omega_1, 2\omega_2\in\mathbb{C}$:

$$f(z+2m_1\omega_1+2m_2\omega_2)=f(z),\quad m_1,m_2\in\mathbb{Z}, \tag{73}$$

let Π_{00} the fundamental parallelogramme generated by the two periods. Show that i) if $f(z)$ is entire, then it is constant. ii) The order of its poles inside Π_{00} is ≥ 2. iii) The order of its poles inside Π_{00} is equal to the number ν of the zeroes (counted with their moltiplicity) of $(f(z)-A)$, with $A\in\mathbb{C}$, inside Π_{00}. Verify these properties for $sn(w)$.

5) Other elliptic functions.
The definition

$$x=sn(u,\kappa)=\sin\varphi(u,\kappa) \tag{74}$$

suggests the introduction of other elliptic functions:

$$\begin{aligned} cn(u,\kappa) &\equiv \cos\varphi(u,\kappa), \\ dn(u,\kappa) &\equiv \sqrt{1-\kappa^2\sin^2\varphi(u,\kappa)}. \end{aligned} \tag{75}$$

Show that

$$sn^2u + cn^2u = 1, \quad dn^2u + \kappa^2 sn^2u = 1 \tag{76}$$

and that

$$\begin{aligned} \tfrac{d\varphi(u)}{du} &= \left(\tfrac{du}{d\varphi}\right)^{-1} = \sqrt{1-\kappa^2\sin^2\varphi(u)} = dn(u), \\ \tfrac{dsn(u)}{du} &= \tfrac{d\sin\varphi(u)}{du} = \cos\varphi(u)\tfrac{d\varphi(u)}{du} = cn(u)dn(u), \\ \tfrac{dcn(u)}{du} &= -sn(u)dn(u), \\ \tfrac{ddn(u)}{du} &= -\kappa^2 sn(u)dn(u). \end{aligned} \tag{77}$$

6) Elliptic integral of second type
Introduced the elliptic integral of second type:

$$E(\varphi,\kappa) = \int_0^{\varphi} \sqrt{1-\kappa^2\sin^2\varphi}\,d\varphi = \int_0^{x} \sqrt{\frac{1-\kappa^2t^2}{1-t^2}}dt \tag{78}$$

where the second integral follows from the change of variables $x = \sin\varphi$, and the complete elliptic integral of second type:

$$E(\kappa) \equiv E(\pi/2,\kappa) = \int_0^{\pi/2} \sqrt{1-\kappa^2\sin^2\varphi}\,d\varphi = \int_0^{1} \sqrt{\frac{1-\kappa^2t^2}{1-t^2}}dt, \tag{79}$$

i) show that

$$E(\varphi,\kappa) = \int_0^{u} dn^2(u)du \tag{80}$$

ii) show that sn^2 and dn^2 are periodic of period $2K(\kappa)$, and show that their average over that period are:

$$\overline{dn^2} = \frac{E(\kappa)}{K(\kappa)}, \quad \overline{sn^2} = \frac{1}{\kappa^2}\frac{K(\kappa)-E(\kappa)}{K(\kappa)}. \tag{81}$$

7) Show that

$$\begin{aligned} &K(\kappa) \to \infty, \;\; E(\kappa) \to 1, \quad \text{as } \kappa \to 1, \\ &K(\kappa) = \tfrac{\pi}{2} + \tfrac{\pi}{8}\kappa^2 + O(\kappa^4), \quad E(\kappa) = \tfrac{\pi}{2} - \tfrac{\pi}{8}\kappa^2 + O(\kappa^4), \quad \kappa << 1. \end{aligned} \tag{82}$$

8) Rectification of the ellipse.
Given the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0, \tag{83}$$

show that the length of the ellipse arc having as boundary points $P(0)$ and $P(\varphi)$ is

$$s = aE(\varphi, \kappa^2), \quad \kappa \equiv \frac{\sqrt{a^2 - b^2}}{a}, \tag{84}$$

and infer that the ellipse length is $4aE(\kappa)$.

9) The simple pendulum.
i) Show that the general solution of the simple pendulum equation $\ddot{\theta} + \frac{g}{L}\sin\theta = 0$ is expressed in terms of the elliptic sine function in the following way:

$$\theta(t) = 2\sin^{-1}\left(\kappa sn\left(\omega(t - t_0), \kappa\right)\right), \quad \omega = \sqrt{\frac{g}{L}}, \tag{85}$$

and the period of oscillations is

$$T = \frac{4K(\kappa)}{\omega} \tag{86}$$

where $sn(z, \kappa)$ is the Jacobi elliptic sine function, $\kappa = \sqrt{\frac{1+E/\omega^2}{2}}$, and $E = \frac{\dot{\theta}^2}{2} - \omega^2\cos\theta$ is the constant energy of the system.
ii) Show that, if $\bar{\theta}$ is the inversion angle, then:

$$\begin{array}{l} \cos\bar{\theta} = -\frac{E}{\omega^2}, \quad \kappa^2 = \sin^2\frac{\bar{\theta}}{2}, \\ \theta(t) \sim \bar{\theta} - \omega^2\kappa\sqrt{1-\kappa^2}(t - \frac{T}{4})^2, \quad t \sim \frac{T}{4}. \end{array} \tag{87}$$

iii) Show that, in the case of small oscillations $|\bar{\theta}| << 1$, one has $\kappa << 1$, and

$$\begin{array}{l} \omega(t - t_0) \sim \int\limits_0^x \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} x \quad \Rightarrow \quad sn(\omega(t - t_0)) \sim \sin(\omega(t - t_0)), \\ K(\kappa) \sim \int\limits_0^1 \frac{dt}{\sqrt{1-t^2}} = \pi/2, \\ T(\kappa) \sim \frac{2\pi}{\omega}, \\ \theta \sim 2\sin^{-1}\left(\kappa\sin(\omega(t - t_0))\right) \sim 2\kappa\sin(\omega(t - t_0)) = \bar{\theta}\sin(\omega(t - t_0)), \\ \kappa \sim \bar{\theta}/2. \end{array} \tag{88}$$

10) Elliptic solution of KdV
Construct the traveling wave solution

$$u = U(\Theta), \quad \Theta := x - ct - x_0, \tag{89}$$

of the Korteweg - de Vries equation $u_t + uu_x + \varepsilon^2 u_{xxx} = 0$, through the quadrature

$$\frac{1}{\sqrt{3}} \frac{\zeta}{\varepsilon} = \int_\gamma^U \frac{dU}{\sqrt{P(U)}}, \tag{90}$$

where $P(U) = -(U - \alpha)(U - \beta)(U - \gamma)$, α, β, γ are three real arbitrary constant roots of $P(U)$, with $\alpha \le \beta \le \gamma$, and

$$c = \frac{\alpha + \beta + \gamma}{3}, \tag{91}$$

and show that the solution can be written in terms of the elliptic sine as:

$$\begin{aligned} U &= \gamma - (\gamma - \beta) sn^2 \left(\sqrt{\tfrac{\gamma-\alpha}{12}} \tfrac{\zeta}{\varepsilon}, \kappa \right), \\ \kappa &= \sqrt{\tfrac{\gamma-\beta}{\gamma-\alpha}}. \end{aligned} \tag{92}$$

11) Show that, if $\beta \to \alpha$ (the case of two coinciding roots), then $\kappa \to 1$, and

$$sn(u, \kappa) \to \tanh u. \tag{93}$$

Consequently, the travelling wave solution of KdV reduces to

$$U = \gamma - (\gamma - \alpha) \tanh^2 \left(\sqrt{\frac{\gamma - \alpha}{12}} \frac{\zeta}{\varepsilon} \right) = \alpha + \frac{\gamma - \alpha}{\cosh^2 \left(\sqrt{\frac{\gamma-\alpha}{12}} \frac{\zeta}{\varepsilon} \right)} \tag{94}$$

If, in addition, $\alpha = 0$, then the travelling wave solution reduces to the so-called 1-soliton solution of KdV

$$U = \frac{3c}{\cosh^2 \left(\frac{\sqrt{c}}{2} \frac{x - ct - x_0}{\varepsilon} \right)}, \tag{95}$$

an exponentially localized travelling wave whose velocity is proportional to the amplitude and inversely proportional to the $\sqrt{width}$.

Regolarizzazione dispersiva

In analogy with what done for the Burgers equation, in the $O(\varepsilon)$ region near the first breaking, the solution will be described by the traveling wave solution given in terms of elliptic functions found in the previous problem, where the constants α, β, γ are replaced by functions $\alpha(x,t), \beta(x,t), \gamma(x,t)$ having $O(1)$ variations in space-time (slow variations with respect to the fast $O(\varepsilon^{-1})$ oscillations of the elliptic functions):

$$\begin{array}{l} u \sim U(\Theta;\alpha(x,t),\beta(x,t),\gamma(x,t)) = U(\Theta;x,t), \\ \Theta = \frac{1}{\varepsilon}(x - c(x,t)t - x_0), \\ c(x,t) = \frac{\alpha(x,t)+\beta(x,t)+\gamma(x,t)}{3}. \end{array} \tag{96}$$

To find $\alpha(x,t), \beta(x,t), \gamma(x,t)$ one uses the following Witham method [20].

As we know, KdV has ∞-many conservation laws; verify that the first three are:

$$\begin{array}{l} u_t + \left(\frac{u^2}{2} + \varepsilon^2 u_{xx}\right)_x = 0, \quad \text{conservation of mass}, \\ \left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3} + \varepsilon^2 u u_{xx} - \varepsilon^2 \frac{u_x^2}{2}\right)_x = 0, \quad \text{conservation of energy}, \\ \left(\frac{u^3}{3} - \varepsilon^2 u_x^2\right)_t + \left(\frac{u^4}{4} - 2\varepsilon^4 u_x u_{xxx} + \varepsilon^4 u_{xx}^2 + \varepsilon^2 u^2 u_{xx} - 2\varepsilon^2 u u_x^2\right)_x = 0, \\ \text{conservation of momentum} \end{array} \tag{97}$$

Now we observe that

$$\partial_x \to D_x := \varepsilon^{-1}\partial_\Theta + \partial_x, \quad \partial_t \to D_t := -c(x,t)\varepsilon^{-1}\partial_\Theta + \partial_t, \tag{98}$$

and that the fast oscillations can be treated doing averages with respect to the fast variable Θ over the period L

$$\overline{F}(x,t) := \frac{1}{L}\int_0^L F(\Theta;x,t)d\Theta \tag{99}$$

(here $\overline{F}$ is the average of F, not its complex conjugate!). For instance:

$$\overline{D_x(F(u))} = \frac{1}{L}\int_0^L \left(\varepsilon^{-1}(F(u))_\Theta + (F(u))_x\right) d\Theta = \partial_x \frac{1}{L}\int_0^L F(u)d\Theta = \partial_x \overline{F(u)}, \tag{100}$$

since $\overline{(F(u))_\Theta} = 0$ by periodicity.

Applying this averaging procedure to the three conservation laws, verify that one obtains the following three PDEs with respect to the variables x, t only (the fast variable Θ has been averaged away):

$$
\begin{array}{l}
\overline{u}_t + \left(\frac{\overline{u^2}}{2}\right)_x = 0, \\
\left(\frac{\overline{u^2}}{2}\right)_t + \left(\frac{\overline{u^3}}{3} - \frac{3}{2}\overline{u_\Theta^2}\right)_x = 0, \\
\left(\frac{\overline{u^3}}{3} - \overline{u_\Theta^2}\right)_t + \left(\frac{\overline{u^4}}{4} + 3\overline{u_{\Theta\Theta}^2} - 4\overline{uu_\Theta^2}\right)_x = 0.
\end{array}
\tag{101}
$$

It is a first order hyperbolic system of three equations. While an hyperbolic system of two equations admits always two Riemann invariants, only in exceptional cases, like this one, it is possible to find three Riemann invariants:

$$
r_1 = \frac{\alpha+\beta}{2}, \quad r_2 = \frac{\alpha+\gamma}{2}, \quad r_3 = \frac{\beta+\gamma}{2},
\tag{102}
$$

with

$$
\alpha = r_1 + r_2 - r_3, \quad \beta = r_1 - r_2 + r_3, \quad \gamma = -r_1 + r_2 + r_3,
\tag{103}
$$

satisfying the celebrated Witham equations:

$$
\begin{array}{l}
r_{it} + v_i(\vec{r}) r_{ix} = 0, \quad i = 1,2,3, \\
\\
v_1(\vec{r}) = \frac{r_1+r_2+r_3}{3} - \frac{2}{3}(r_2 - r_1)\frac{K(\kappa^2)}{K(\kappa^2)-E(\kappa^2)}, \\
v_2(\vec{r}) = \frac{r_1+r_2+r_3}{3} - \frac{2}{3}(r_2 - r_1)\frac{(1-\kappa^2)K(\kappa^2)}{E(\kappa^2)-(1-\kappa^2)K(\kappa^2)}, \\
v_3(\vec{r}) = \frac{r_1+r_2+r_3}{3} - \frac{2}{3}(r_3 - r_1)\frac{(1-\kappa^2)K(\kappa^2)}{E(\kappa^2)}, \\
\kappa^2 = \frac{\gamma-\beta}{\gamma-\alpha} = \frac{r_2-r_1}{r_3-r_1}.
\end{array}
\tag{104}
$$

We want to use these results to solve the Cauchy problem

$$
\begin{array}{l}
u_t + uu_x + \varepsilon^2 u_{xxx} = 0, \\
u(x,0) = H(-x)
\end{array}
\tag{105}
$$

corresponding to

$$
r_1(x,0) = 0, \quad r_2(x,0) = H(x), \quad r_3(x,0) = 1.
\tag{106}
$$

Therefore $dr_i(x,0)/dx \geq 0, \ \forall x$, and we have rarefaction waves for the Riemann invariants, with regular behavior. In addition: $r_1(x,0) \leq r_2(x,0) \leq r_3(x,0)$, and

$$
\overline{u(x,0)} = r_1(x,0)+r_2(x,0)-r_3(x,0)+2(r_3(x,0)-r_1(x,0))\frac{E(\kappa^2(\vec{r}))}{K(\kappa^2(\vec{r}))}|_{t=0} = H(-x).
\tag{107}
$$

Verify that the solutions of the above Witham equations are:

$$r_1(x,t) = 0, \quad r_2(x,t) = v_2^{-1}(x/t), \quad r_3(x,t) = 1, \tag{108}$$

where $v_2^{-1}(\cdot)$ is the function inverse of v_2.

Therefore the dispersive shock is described by

$$u(x,t) \sim 1 + r_2(x/t) - 2r_2(x/t) sn^2 \left(\sqrt{\frac{1}{\sqrt{6}} \frac{1}{\varepsilon} \left(x - \frac{1 + r_2(x/t)}{3} t \right), r_2(x/t)} \right), \tag{109}$$

a slowly modulated train of oscillations developing at approximately the breaking time, in a region opening up with a positive velocity 2/3 for the front, and with a negative velocity −1 for the back.

2 La propagazione ondosa in Natura, il metodo multiscala e le equazioni modello [4, 8, 1, 3, 17]

1) Consider the two anharmonic oscillators

$$\begin{array}{l} i)\ \ddot{q} + q - \frac{\epsilon}{6} q^3 = 0, \quad \text{Hamiltonian cubic pendulum}, \quad 0 < \epsilon << 1, \\ ii)\ \ddot{q} + q + \epsilon \dot{q}^3 = 0, \quad \text{with nonlinear friction} \end{array} \tag{110}$$

with the same initial conditions

$$q(0) = 1, \quad \dot{q}(0) = 0. \tag{111}$$

Use the multiscale method to show that

$$\begin{array}{l} i)\ q(t) = \cos\left(t - \frac{1}{16}\varepsilon t\right) + O(\varepsilon), \\ ii)\ q(t) = \left(1 + \frac{3}{4}\varepsilon t\right)^{-1/2} \cos t + O(\varepsilon). \end{array} \tag{112}$$

2) Use the multiscale method to construct the solution

$$q(t) = \frac{a_0 e^{\varepsilon t/2}}{\sqrt{1 + \left(\frac{a_0}{2}\right)^2 (e^{\varepsilon t} - 1)}} \cos(t + \phi_0) + O(\varepsilon) \tag{113}$$

of the Van Der Pol oscillator

$$\ddot{q} + q - \varepsilon(1 - q^2)\dot{q} = 0, \tag{114}$$

and show that

$$q(t) \to 2\cos(t + \phi_0) + O(\varepsilon), \quad t \to \infty, \tag{115}$$

i.e., the solution tends to a limiting cycle (at O(ε): the circle of radius 2).

3) Derive the Hopf equation $u_t + uu_x = 0$ from the Riemann equation $u_t + c(u)u_x = 0$ using multiscale expansions, in the weakly nonlinear regime.

4) Derive the Burgers equation $u_t + uu_x = \nu u_{xx}$ from the following class $u_t + c(u)u_x = (D(u)u_x)_x, \ D(u) > 0$ of PDEs, in the weakly nonlinear regime, using multiscale expansions.

5) Derive the KdV equation $u_t + uu_x + u_{xxx} = 0$ from the following class $u_t + c(u)u_x + K_1(u)[K_2(u)(K_3(u)u_x)_x]_x = 0$ of nonlinear dispersive PDEs, using multiscale expansions.

6) Derive the nonlinear Schrödinger equation from the Sine Gordon equation $u_{tt} - c^2 u_{xx} + \mu^2 \sin u = 0$ (or, more in general, from a large class of nonlinear dispersive PDEs), using multiscale expansions.

7) Derive the dKP(3,1) equation $(u_t + uu_x)_x + u_{yy} + u_{zz} = 0$ from the equations of Acoustics, under the hypothesis of i) weak nonlinearity and ii) quasi one-dimensionality.

8) i) Derive the equations of surface water waves from the Euler equations, linearize them under a small field hypothesis, and show their dispersive nature, with the dispersion relation

$$\omega^2(k) = gk \tanh(h\ k), \tag{116}$$

where g is the acceleration of gravity and h is the depth of the fluid.

9) i) Derive the KdV equation (see [1, 3]) in the context of surface water waves in $(1 + 1)$ dimensions, under the hypothesis of ia) small amplitudes and iib) shallow water ($kh \ll 1$, where k is the wave number and h is the depth of the fluid). ii) Derive the KP equation (see [2, 3]) in the context of surface water waves in $(2+1)$ dimensions, under the hypothesis of iia) small amplitudes, iib) shallow water, and iic) quasi one-dimensionality. Show that, neglecting dispersion, one obtains the dKP(2,1) equation.
iii) Derive (see [3]) the NLS equation in the context of surface water waves in (1+1) dimensions, under the hypothesis of iiia) small amplitude ($a << \lambda$) and iiib) quasi monocromatic waves in sufficiently deep water ($kh \gg 1$). iv)

Derive its multidimensional generalization in the context of surface water waves in $(2+1)$ dimensions.

10) Derive (see [21]) the NLS equation in the framework of Langmuir waves in a plasma, described by the system of equations:

$$n_t + (nv)_x = 0, \quad v_t + vv_x = \phi_x - n_x/n, \quad \phi_{xx} = n - 1,$$

with boundary onditions $n \to 1, \ v \to 0, \ \phi \to 0$ as $|x| \to \infty$, where n is the electron density, v is the electron velocity and ϕ is the electrostatic potential in dimensionless variables, expanding the fields around the equilibrium solution:

$$n = 1 + \varepsilon n_1 + \varepsilon^2 n_2 + O(\varepsilon^3), \quad v = \varepsilon v_1 + O(\varepsilon^2), \quad \phi = \varepsilon\phi_1 + O(\varepsilon^2).$$

11) Derive (see [8]) the NLS equation in nonlinear optics, for a homogeneous and isotropic dielectric.

3 La teoria dei solitoni

1) Analyticity projectors. Show that the operators

$$P^{\pm} f(\lambda) := \pm\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(\lambda')}{\lambda' - (\lambda \pm i\varepsilon)} d\lambda. \tag{117}$$

are analyticity projectors on the real line; i.e., they map a Holder function $f(\lambda), \ \lambda \in \mathbb{R}$ decaying at ∞ sufficiently fast into functions analytic in the upper and lower halves of the complex λ plane respectively. ii) Show, in particular, that

$$(P^+)^2 = P^+, \ (P^-)^2 = P^-, \ P^+P^- = P^-P^+ = 0, \ P^+ + P^- = 1. \tag{118}$$

2) Given a Holder function $f(\lambda)$ for $\lambda \in \mathbb{R}$ decaying at ∞ sufficiently fast, a polynomial $P(\lambda)$, a set of complex numbers $\{k_j^+, \ R_j^+, \ j = 1, \dots, N^+,$ $k_j^-, \ R_j^-, \ j = 1, \dots, N^-\}$, where Im $k_j^+ > 0$ and Im $k_j^- < 0$, show that the unique solution of the Riemann problem

$$\psi^+(\lambda) - \psi^-(\lambda) = f(\lambda), \quad \lambda \in \mathbb{R} \tag{119}$$

where $\psi^+(\lambda)$ is analytic in the upper half of the complex λ plane, except for the simple poles k_j^+'s with residues R_j^+'s, $\psi^-(\lambda)$ is analytic in the lower half of the complex λ plane, except for the simple poles k_j^-'s with residues R_j^-'s, and $\psi^\pm(\lambda) \to P(\lambda),\ |\lambda| >> 1$, is

$$\psi^\pm(\lambda) = P(\lambda) + \sum_{j=1}^{N^+} \frac{R_j^+}{\lambda - k_j^+} + \sum_{j=1}^{N^-} \frac{R_j^-}{\lambda - k_j^-} \pm P^\pm f(\lambda), \tag{120}$$

where $P^\pm$ are the analyticity projectors defined in (117).

3) Let $u(x) = -A\delta(x - x_0),\ A \in \mathbb{R}$, be the potential of the Schrödinger equation $[-\partial_x^2 + u(x)]\psi = k^2\psi$. Evaluate explicitly: i) the eigenfunctions of the continuous spectrum and the coefficients $a(k), b(k), R(k), T(k)$; ii) the discrete spectrum p_j, the corresponding eigenfunctions and the norming constants b_j. Show that the existence of discrete spectrum depends on the sign of A.

4) Assume $u(x) = O(\varepsilon),\ \varepsilon << 1$, and construct the first two terms of the ε - expansion of the eigenfunctions and of the spectral data.

5) *Scattering problem.* Study the scattering problem described by the Schrödinger equation

$$-\psi''(x,k) + u(x)\psi(x,k) = k^2\psi(x,k), \quad x \in \mathbb{R}, \quad k > 0,$$

where $\psi(x,k)$, the eigenfunction of the continuous spectrum of the Schrödinger operator $-d^2/dx^2 + V(x)$, represents the wave function of a particle beam scattered by the localized potential $u(x)$ e $E = k^2 > 0$ is the energy of the beam (the continuous spectrum $\sigma_c = \{E > 0\}$), with the following boundary conditions:

$$\psi(x,k) \sim\ R(k)e^{-ikx} + e^{ikx},\ x \sim -\infty; \qquad \psi(x,k) \sim\ T(k)e^{ikx},\ x \sim \infty$$

describing an incoming beam of particles of wave number k and intensity 1, partially reflected and transmitted through the potential ($R(k)$ e $T(k)$ are respectively the reflection and transmission coefficients).
i) Observe that the function $\phi(x,k) = \psi(x,k)/T(k)$ satisfies a simpler scattering problem:

$$\phi''(x,k) + k^2\phi(x,k) = u(x)\phi(x,k), \quad x \in \mathbb{R},, \quad k > 0$$

$$\phi(x,k) \sim \frac{R(k)}{T(k)}e^{-ikx} + \frac{e^{ikx}}{T(k)},\ x \sim -\infty; \quad \phi(x,k) \sim\ e^{ikx},\ x \sim \infty$$

and use the advanced Green function of the operator d^2/dx^2+k^2 to rewrite such a problem as a Volterra integral equation [5], obtaining:

$$\phi(x,k)=e^{ikx}-\int_x^{\infty}dy\frac{\sin k(x-y)}{k}u(y)\phi(y,k)$$

and the following integral representations for the reflection and transmission coefficients:

$$\frac{1}{T(k)}=1-\int_{\mathbb{R}}dk\frac{e^{-iky}}{2ik}u(y)\phi(y,k),\quad \frac{R(k)}{T(k)}=\int_{\mathbb{R}}dk\frac{e^{iky}}{2ik}u(y)\phi(y,k).$$

Such an integral equation, equivalent to the Schrödinger differential equation + boundary conditions, is the most convenient formulation of the problem to extract informations.

ii) Use the method of successive approximations to study the properties of ϕ in the following way.

a) Rerwrite the integral equation for the unknown $f(x,k)=\phi(x,k)e^{-ikx}$, such that $f\sim 1,\ x\to\infty$:

$$f(x,k)=1+\int_x^{\infty}\frac{e^{2ik(y-x)}-1}{2ik}u(y)f(y,k)dy$$

and look for the solution as a Neumann series:

$$f(x,k)=\sum_{i=0}^{\infty}h_i(x,k),\quad h_0=1, \tag{121}$$

obtaining the recursion relation:

$$h_{j+1}(x,k)=\int_x^{\infty}\frac{e^{2ik(y-x)}-1}{2ik}u(y)h_j(y,k)dy,\quad j\geq 0. \tag{122}$$

b) From the inequality: $|e^{2ik(y-x)}-1|/|2ik|\leq 1/|k|$, valid for Im $k\geq 0$, $k\neq 0$, show that

$$|h_{j+1}(x,k)|\leq\frac{1}{|k|}\int_x^{\infty}|u(y)||h_j(y,k)|dy, \tag{123}$$

and then that:

$$\begin{aligned}&|h_n(x,k)|\leq\frac{1}{n!}\left(\frac{A(x)}{|k|}\right)^n\leq\frac{1}{n!}\left(\frac{A(-\infty)}{|k|}\right)^n,\\&A(x):=\int_x^{\infty}|V(y)|dy.\end{aligned} \tag{124}$$

Therefore the Neumann series representing the solution is absolutely and uniformely convergent for Im $k \geq 0,\ k \neq 0$, if $u(x) \in L_1(\mathbb{R})$. Under these conditions, the solution exists unique, and it is analytic in the upper half of the complex k plane. Analogously one can prove that $1/T(k)$ is analytic in the upper half of the complex k plane. Under more stringent conditions on u, one could show, in a similar manner, that the eigenfunction is also continuous on the real k axes, where the physics takes place.
c) Let $k_j,\ j=1,..,N$ be the zeroes of the function $1/T(k)$ in the upper half of the complex k plane (the poles of the transmission coefficient). Then, since $\lambda_j = E_j = k_j^2 \in \mathbb{R}$, it follows that a) k_j is purely imaginary: $k_j = ip_j,\ p_j > 0,\ j=1,..,N$, b) the functions $\phi(x,k_j),\ j=1,..,N$ are exponentially localized:

$$\phi_j(x) := \phi(x,k_j) = O(e^{-p_j|x|}), \quad |x| \to \infty, \quad j=1,..N$$

and then they are eigenfunctions of the Schrödinger operator in $L_2(\mathbb{R})$:

$$-\phi_j''(x) + u(x)\phi_j(x) = -p_j^2\phi_j(x), \quad x \in \mathcal{R}$$

corresponding to negative eigenvalues $\lambda_j = E_j = -p_j^2 < 0$ of the energy (the discrete spectrum: $\sigma_p = \{-p_j^2\}_1^N$). Summarizing: $\sigma = \sigma_p \cup \sigma_c = \{-p_j^2\}_1^N \cup \mathbb{R}^+$.
d) Show that the set of $\lambda_j = -p_j^2,\ j=1,..,N$ is bounded from below.
Hint. Take the scalar product of the eigenfunction ϕ_j, normalized to 1, with the Schrödinger equation, obtaining:

$$\lambda_j - (\phi_j, u\phi_j) = (\phi_j', \phi_j') \geq 0 \quad \Rightarrow \quad |\lambda_j| \leq -(\phi_j, V\phi_j) \leq |(\phi_j, u\phi_j)| \leq ||u||_\infty.$$

e) Show that, if $u(x) = u_0\delta(x-x_0)$, the integral equation admits the solution

$$\phi(x,k) = e^{ikx} - u_0 H(x_0 - x)\frac{\sin k(x-x_0)}{k}e^{ikx_0}.$$

Then:

$$\phi(x,k) = \frac{2ik - u_0}{2ik}e^{ikx} + \frac{u_0 e^{2ikx_0}}{2ik}e^{-ikx}, \quad x < x_0$$

$$T(k) = \frac{2ik}{2ik - u_0}, \qquad R(k) = \frac{u_0 e^{2ikx_0}}{2ik - u_0}.$$

Found $\phi(x,k)$, at last reconstruct $\psi(x,k) = \frac{2ik}{2ik-u_0}\phi(x,k)$.
f) Verify that the solution we found for $k \in \mathbb{R}$, if extended outside the real k axis, diverges always at + or - infinity, unless $k = -iu_0/2 \in i\mathbb{R}^+$. Therefore,

if the potential is positive ($u_0 > 0$), no eigenfunctions exist in $L_2(\mathbb{R})$; if, instead, the potential is negative, then there exists one and only one $L_2(\mathbb{R})$ eigenfunction $\psi_1(x) := \phi(x, i|u_0|/2) \in L_2(\mathbb{R})$:

$$\psi_1(x) = H(x - x_0)e^{-\frac{|u_0|}{2}x} + H(x_0 - x)e^{\frac{|u_0|}{2}x}$$

corresponding to the negative energy $E_1 = k_1^2 = -u_0^2/4$, and describing a bound state (a localized quantum particle): $\sigma_p = \{E_1\}$.
g) If $u(x) = \epsilon v(x),\ \epsilon << 1$, show that:

$$\phi(x,k) = e^{ikx} - \epsilon \int\limits_x^\infty dy \frac{\sin k(x-y)}{k} v(y) e^{iky} + O(\epsilon^2),$$

$$T(k) = 1 + \frac{\epsilon}{2ik}\int_{\mathcal{R}} dx v(x) + O(\epsilon^2), \quad R(k) = \frac{\epsilon}{2ik}\int_{\mathcal{R}} dx v(x) e^{-2ikx} + O(\epsilon^2)$$

6) Using the above strategy, study the scattering problem

$$\phi''(x,k) + k^2\phi(x,k) = u(x)\phi(x,k), \ \ x \in \mathcal{R}, \qquad \phi(x,k) \sim \ e^{-ikx}, \ x \sim -\infty$$

showing that, in this case, it is convenient to use the retarded Green function of the operator $d^2/dx^2 + k^2$.

7) Let $\varphi(x,k)$ and $\psi(x,k)$ be the Jost eigenfunctions of the Schrödinger operator satisfying the boundary conditions:

$$\varphi(x,k) \sim e^{-ikx}, \ \ x \to -\infty, \qquad \psi(x,k) \sim e^{-ikx}, \ \ x \to \infty \tag{125}$$

i) Write the integral equations satisfied by them; ii) show that $\varphi(x,k)e^{ikx}$ and $\psi(x,k)e^{ikx}$ are analytic respectively in the upper and lower halves of the k plane; iii) show that

$$-2i\frac{d}{dx}[k(\psi(x,k)e^{ikx} - 1)] \to u(x), \ \ |k| \gg 1. \tag{126}$$

8) Let k_0 be a zero of $a(k) = 1/T(k)$, where $T(k)$ is the transmission coefficient of the Schrödinger equation. i) Show that k_0 belongs to the discrete spectrum (therefore $k_0 = ip, \ p > 0$) and, correspondingly, that $\varphi(x,k_0) \in L^2(\mathbb{R})$, with the asymptotics

$$\varphi(x,k_0) \sim e^{px}, \ \ x \sim -\infty, \quad \varphi(x,k_0) \sim be^{-px}, \ \ x \sim \infty \tag{127}$$

where $b \in \mathbb{R}$.
ii) Show that the zeroes $k_0 = ip$ of $a(k)$ are simple, and that $iba'(ip) > 0$.
A. For i), use the Wronskian relation $W(\varphi, \bar{\psi}) = 2ika(k)$ to infer that $\varphi(x, k_0) = b\overline{\psi(x, k_0)} = b\psi(x, -k_0)$.

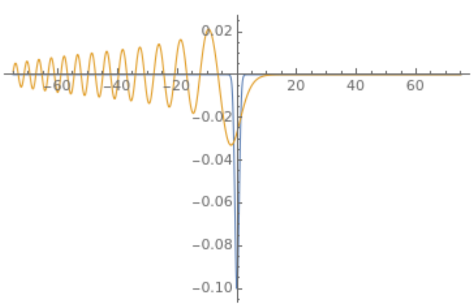
9) *Inverse Problem.* Using the analyticity properties of $\varphi(x, k), \psi(x, k), a(k)$, together with their asymptotics for large k, i) rewrite the scattering equation

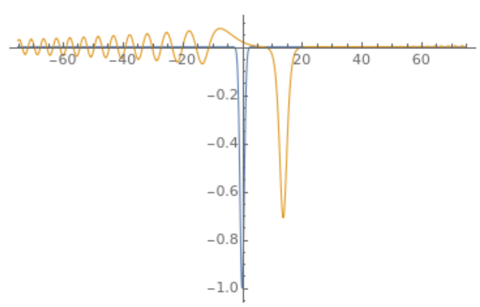
$$\varphi(x, k) = a(k)\psi(x, k) + b(k)\psi(x, -k), \quad k \in \mathbb{R} \tag{128}$$

for the Schrödinger operator as a linear Riemann - Hilbert problem on the real k axis, for a given set of scattering data. ii) Express the solution of such a linear RH problem in terms of integral equations for the eigenfunctions, and iii) reconstruct the potential u(x) in terms of the scattering data.

10) t - *evolution of the scattering data.* Obtain the t evolution of the scattering data if u evolves according to KdV.

11) Consider the Cauchy problem on the line for the KdV equation $u_t + u_{xxx} - 6uu_x = 0$, with the initial condition $u(x, 0) = -b\exp(-x^2)$. Show (numerically) that, i) if $b = 0.1$, the dynamics is described by a pure nonlinear dispersive waves (travelling with negative group velocity); if $b = 1$, by a nonlinear dispersive waves (traveling with negative group velocity) + one soliton, travelling with positive speed; if $b = 4$, by a nonlinear dispersive waves (traveling with negative group velocity) + two solitons, traveling with positive speeds (see the figures below). Interpret these numerical experiments in the light of the IST for KdV.





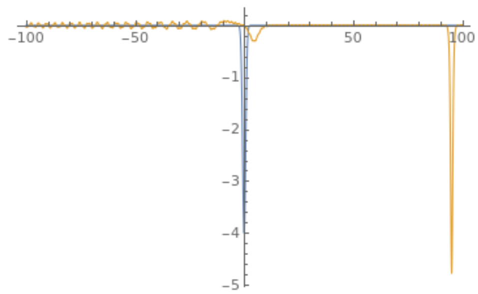


Figure 1 for $b = 0.1$: the area of the well is not large enough to support bound states $\Rightarrow$ the solution evolves into nonlinear dispersive waves; Figure 2 for $b = 1$: the area of the well is large enough to support one bound state $\Rightarrow$ the solution evolves into a one soliton + nonlinear dispersive waves; Figure 3 for $b = 4$: the area of the well is large enough to support two bound states $\Rightarrow$ the solution evolves into two solitons + nonlinear dispersive waves.

11) Construct the 2-soliton solution of KdV and study the interaction of the two solitons.

4 La teoria delle onde anomale in natura

Il materiale di questo capitolo è raccolto negli appunti sul sito del docente

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