NONLINEAR WAVES AND SOLITONS
Notes of the course.

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Contents

1 Introduction

In this course we present analytic techniques to investigate important nonlinear aspects of natural phenomena. Most of the model equations introduced and studied in the first three years of physics courses are linear, like the harmonic oscillator, describing the small oscillations around a stable equilibrium position, the wave equation describing the propagation of a physical signal in a medium under the hypothesis of small (elastic) deformations of the materials, or the propagation of an electromagnetic wave in the vacuum, the heat (diffusion) equation, describing heat transfer phenomena, but also the random walk in probability theory.

There is a practical motivation for the success of linear equations. The terrestrial environment is quite exceptional in the universe: its temperature is very low, and this allows for the coexistence of solid, liquid and gaseous systems, due to the fact that the energetic content of the degrees of freedom of the constituents of matter is very low with respect to the binding energies of these constituents. As a comparison, the stellar environment consists of ionized matter, and the energies of the degrees of freedom of the atomic constituents is extremely high. Therefore the matter in terrestrial environment is almost always near stable equilibrium states, and the solicitations exercised through ordinary means are just perturbations, and the restoring forces are elastic, i.e., linear. In addition, there are important physical theories that are intrinsecally linear, like the Maxwell theory of the electromagnetic field in a vacuum, or the Schrödinger wave mechanics. Therefore the study of linear problems plays a central role in the development of our knowledge of nature.

However there are many phenomena, even in terrestrial environment, that cannot be explained through linear theories. The equations of fluid dynamics are systems of nonlinear partial differential equations (PDEs), and phenomena like the turbulence, wave breaking, and shock waves cannot be explained through linear equations, and problems like the meteorological predictions cannot be solved through linear theories. Electromagnetism in the matter is a nonlinear theory for intense fields, due to polarization and magnetization effcts. Also the Einstein theory of gravitation is nonlinear, and the effects of intense gravitational fields can be understood only using the full nonlinear theory.

It turns out that many of these nonlinear phenomena in nature can be understood perturbatively considering the first nonlinear corrections to the linearized theories, and the aim of this course is twofold.
i) On one hand we introduce, through the multiscale perturbation method, several nonlinear model equations of Mathematical Physics generalizing to a nonlinear context the classical linear equations of Mathematical Physics

known since a long time. We derive these equations in a small field regime (but not so small to be allowed to completely neglect nonlinear corrections), sometimes in the presence of a weak dispersion, or diffusion, or dissipation, then capturing the first nonlinear corrections to the linearized theory.
ii) On the other hand, it turns out that the constructed nonlinear models are not only physically relevant, but they are also special from a mathematical point of view, since they can be integrated through particular mathematical techniques. Therefore the second goal of this course is to introduce these techniques, allowing one to describe analytically the nonlinear dynamics of generic and physically relevant initial data. The fact that physically relevant model PDEs are also special from a mathematical point of view, so special to be often integrable, is not a coincidence, as we shall see later on.

The nonlinear model equations constructed from physics and solved in this course are the following. The Riemann equation

$$u_t + c(u)u_x = 0, \qquad x,t \in \mathbb{R},\ u(x,t) \in \mathbb{R}, \tag{1}$$

quasi-linear hyperbolic generalization of the unidirectional linear wave equation $u_t + cu_x = 0$, that can be solved through the method of characteristics.

The Burgers equation

$$u_t + uu_x = \nu u_{xx},\ \nu > 0, \qquad x,t \in \mathbb{R},\ u(x,t) \in \mathbb{R}, \tag{2}$$

nonlinear generalization of the heat (diffusion) equation $u_t = \nu u_{xx},\ \nu > 0$, that can be integrated using a suitable nonlinear contact transformation of the dependent variable, the Hopf-Cole transformation.

The Korteweg - de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0, \qquad x,t \in \mathbb{R},\ u(x,t) \in \mathbb{R}, \tag{3}$$

nonlinear generalization of the linear dispersive PDE $u_t + u_{xxx} = 0$, describing weakly nonlinear and weakly dispersive wave phenomena, that can be integrated using the inverse scattering (spectral) transform (IST) method or the finite gap method, depending respectively on whether the involved fields are localized or periodic.

The nonlinear Schrödinger (NLS) equations

$$i\psi_t + \psi_{xx} + 2\eta|\psi|^2\psi = 0,\ \eta = \pm 1, \qquad x,t \in \mathbb{R},\ \psi(x,t) \in \mathbb{C}, \tag{4}$$

nonlinear analogues of the linear Schrödinger equation for a free particle: $i\psi_t + \psi_{xx} = 0$, describing the propagation of small amplitude quasi monochromatic waves in a nonlinear dispersive medium, that can be solved through the same techniques used for the KdV equation.

The course is organized as follows.
In §2 we summarize the basic properties of linear dispersive PDEs, and the asymptotic techniques used to investigate the longtime dispersion of a localized wave packet.
In §3 we study through the method of characteristics quasi-linear hyperbolic waves, how they break in space-time, giving rise to the so-called gradient catastrophe, and how they subsequently evolve into multivalued waves.
In §4 we describe how it is possible to avoid wave breaking and multivaluedness i) introducing the so-called shock waves, discontinuous and single valued waves manifestation of small dissipative effects; ii) regularizing the hyperbolic model adding suitable dissipative or dispersive corrections.
In §5 we introduce the multiscale perturbation method for ordinary differential equations (ODEs) and for PDEs, and we use it to derive the above nonlinear model equations of Mathematical Physics from large classes of nonlinear PDEs.
In §6 we solve the KdV and NLS equations through the so-called Inverse Spectral (or Scattering) Transform (IST) method, a nonlinear analogue of the Fourier transform method, and we show how a generic localized initial datum evolves into a train of "solitons", localized solitary waves interacting nonlinearly in an elastic way.
In §7 we use a direct technique, the Darboux transformation, to construct exact nontrivial solutions from elementary solutions of soliton PDEs like the KdV and NLS equations.
In §8 we investigate the mathematical properties and the dynamics of NLS periodic anomalous (rogue) waves, nonlinear waves of anomalously large amplitude with respect to the surrounding waves, appearing apparently from nowhere and disappearing without leaving any trace.

2 Linear Dispersive waves [36, 9, 29]

A linear partial differential equation (PDE) with constant coefficients can be written as

$$\mathcal{P}(\partial_t, \partial_x)u = 0, \quad u = u(x,t), \tag{5}$$

where $u = u(x,t)$ is the unknown field and $\mathcal{P}(a,b)$ is an polynomial function of a and b; here we limit, for simplicity, our considerations to the case of PDEs in $1+1$ dimensions, but the generalization to higher dimensions is obvious. It is dispersive if it admits a monochromatic plane wave as solution:

$$u(x,t) = a \exp\left[i(kx - \omega(k)t)\right] \tag{6}$$

$\forall k \in \mathbb{R}$ and for a certain function $\omega(k)$ such that

- $\omega(k) \in \mathbb{R},\ \forall k \in \mathbb{R}$,
- $\omega''(k) \neq 0$ for almost every $k \in \mathbb{R}$.

k and $\omega(k)$ are called, respectively, the wave number and the dispersion relation. The first condition implies that there is no dissipation or a source term; the second condition avoids that $\omega(k) = ck$ (in this case, the wave would travel rigidly with the constant speed c and the equation would be hyperbolic and not dispersive).

Important examples are

- $iu_t + u_{xx} = 0$ the Schrödinger equation for a free particle, with $\mathcal{P}(\partial_t, \partial_x) = i\partial_t + \partial_x^2$ and $\omega(k) = k^2$;
- $u_t + u_{xxx} = 0$ the linearized Korteweg-de Vries (KdV) equation, with $\mathcal{P}(\partial_t, \partial_x) = \partial_t + \partial_x^3$ and $\omega(k) = -k^3$;
- $u_{tt} - c^2 u_{xx} + \mu^2 u = 0$ the Klein-Gordon equation, with $\mathcal{P}(\partial_t, \partial_x) = \partial_t^2 - c^2\partial_x^2 + \mu^2$ and $\omega^2 = c^2k^2 + \mu^2$.

In general, the PDE $\mathcal{P}(\partial_t, \partial_x)u = 0$ is dispersive if, solving the equation $\mathcal{P}(-i\omega, ik) = 0$ with respect to ω, one obtains one or more solutions $\omega(k)$ satisfying the two properties above.

2.1 Fourier Representation

The linearity of the equation implies that, if (6) is solution $\forall k$, then a linear combination of monochromatic waves with different k's is also solution. If one is interested in localized solutions in space, then one obtains the Fourier transform (FT) representation of the solution. More precisely, the solution of the Cauchy problem for the linear dispersive PDE

$$
\begin{aligned}
&\mathcal{P}(\partial_t, \partial_x)u = 0,\ \ u = u(x,t),\\
&u(x,0) = u_0(x) \text{ given },\\
&u(x,t) \to 0 \text{ as } x \to \pm\infty
\end{aligned}
\tag{7}
$$

is (see Fig. 45)

$$
\begin{aligned}
&u(x,t) = \int\limits_{-\infty}^{\infty} \tfrac{dk}{2\pi} \hat{u}_0(k) e^{i(kx-\omega(k)t)},\\
&\hat{u}_0(k) := \int\limits_{-\infty}^{\infty} u_0(x) e^{-ikx} dx \qquad \text{FT of the initial condition}
\end{aligned}
\tag{8}
$$

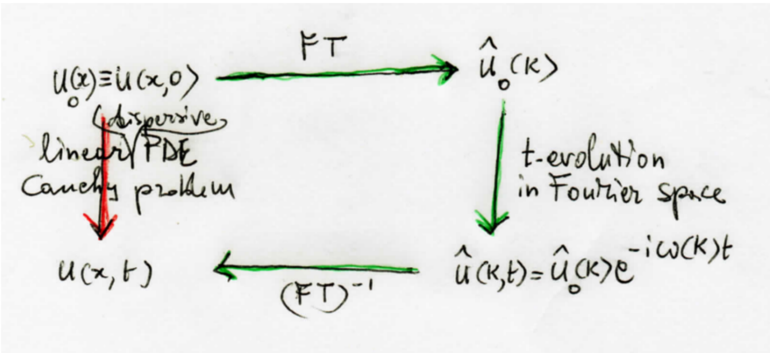


Figure 1: The solution scheme of the Fourier method. We go from the physical space to the Fourier space because there the evolution is simpler, being given by an ODE.

Remarks

1) We observe that, if $f(x) \in \mathbb{R}$, then $\overline{\hat{f}(k)} = \hat{f}(-k)$; and if $\hat{f}(k)$ can be analytically prolongued outside the real axis, then $\overline{\hat{f}(k)} = \hat{f}(-\bar{k})$. In addition, if $u(x,t) \in \mathbb{R}$ in (8), then $\omega(k)$ is an odd function and $\overline{\hat{u}_0(k)} = \hat{u}_0(-k)$. SHOW IT!

2) The solution (8) can also be written in the following suggestive form:

$$
\begin{aligned}
&u(x,t) = \int_{\mathbb{R}} \tfrac{dk}{2\pi} e^{i(kx-\omega(k)t)} \int_{\mathbb{R}} dy e^{-iky} u_0(y)\\
&= \int_{\mathbb{R}} dy u_0(y) \int_{\mathbb{R}} \tfrac{dk}{2\pi} e^{i(k(x-y)-\omega(k)t)}\\
&= \int_{\mathbb{R}} S(x-y,t) u_0(y) dy
\end{aligned}
\tag{9}
$$

(in the second step we used the Fubini theorem), where $S(x,t)$ is the so-called "fundamental solution" of the PDE

$$S(x,t) := \int_{\mathbb{R}} \frac{dk}{2\pi} e^{i[kx-\omega(k)t]}, \tag{10}$$

that particular solution corresponding to a Dirac δ function initial datum: $u(x,0) = \delta(x)$.

3) If $\omega(k) = k^n$, SHOW THAT the fundamental solution is a "similarity solution" of the PDE:

$$S(x,t) = \tfrac{1}{\sqrt[n]{t}} f\left(\tfrac{x}{\sqrt[n]{t}}\right), \quad f(\xi) := \int_{\mathbb{R}} \tfrac{ds}{2\pi} e^{i(s\xi - s^n)} \tag{11}$$

(see Appendix 1 for the definition of similarity solution with examples).

The solution (8) is exact but not expressible, in general, in terms of elementary functions. It is the superposition of infinitely many elementary monochromatic waves. Each wave travels with its phase velocity

$$v_{phase} = \frac{\omega(k)}{k}, \tag{12}$$

different for different k's. It follows that an initially localized wave packet disperses as time goes. As we shall see, it is the group velocity

$$v_{group} = \omega'(k) \tag{13}$$

that plays a relevant role in the dispersion of the wave packet. This can be easily seen in the quasi-monochromatic approximation in which the Fourier transform $\hat{u}_0(k)$ is localized around the wave number k_0, suggesting the change of variables

$$k = k_0 + \epsilon k', \quad 0 < \epsilon \ll 1. \tag{14}$$

Substituting (14) into (8), and expanding in power series we have

$$\begin{aligned} &u(x,t) \sim \int_{\mathbb{R}} \tfrac{d\tilde{k}}{2\pi} \hat{u}_0(k_0 + \epsilon k') e^{i[k_0+\epsilon k')x - \omega(k_0+\epsilon k')t]} \\ &\sim \epsilon e^{i[k_0 x - \omega(k_0)t]} \int_{\mathbb{R}} \tfrac{dk'}{2\pi} \hat{u}_0(k_0 + \epsilon k') e^{ik'[\epsilon(x-\omega'(k_0)t)]} \\ &=: \epsilon A\left(\epsilon\left(x - \omega'(k_0)t\right)\right) e^{i[k_0 x - \omega(k_0)t]}. \end{aligned} \tag{15}$$

Then the monochromatic crest $e^{i[k_0 x - \omega(k_0)t]}$, traveling with the phase velocity $v_{phase} = \frac{\omega(k_0)}{k_0}$, is modulated by the slowly varying amplitude $A\left(\epsilon\left(x - \omega'(k_0)t\right)\right)$, traveling with the group velocity $v_{group} = \omega'(k_0)$. In the dispersion of a free quantum particle, described by the Schrödinger equation $iu_t + u_{xx} = 0$, $\omega(k) = k^2$, and

$$v_{phase} = k_0 < 2k_0 = v_{group}. \tag{16}$$

It follows that the amplitude travels with twice the speed of a crest (ripple). The approximation used in this derivation does not capture some relevant properties of linear dispersive waves. To get them, it is convenient to consider the longtime behavior of these solutions.

2.2 Longtime behavior and stationary phase approximation

Often one is interested in the solution (8) for large x and t, and $x/t = O(1)$; this is consistent, for example, with the quantum mechanical case, where the times of the experimental measures are much larger than the atomic times. In this case it is possible to describe analytically such a dispersion. Let's rewrite (8) as

$$\begin{aligned} & u(x,t) = \int_{-\infty}^{\infty} \tfrac{dk}{2\pi} \hat{u}_0(k) e^{i\phi(k,x/t)t}, \\ & \phi\left(k, \tfrac{x}{t}\right) := k\ \tfrac{x}{t} - \omega(k) \end{aligned} \tag{17}$$

For large t and $x/t = O(1)$, the exponential oscillates very rapidly and the integral goes to zero as t tends to ∞, by the Riemann-Lebesque lemma. To see how it tends to zero, we use the **stationary phase method**.
We first observe that the main contribution comes from the points in which the phase $\phi(k, x/t)$ is stationary with respect to k. Let us suppose that there exists a unique real stationary point $k_0(x/t) \in \mathbb{R}$ such that

$$\begin{aligned} & \tfrac{\partial\phi}{\partial k}\left(k, \tfrac{x}{t}\right)\Big|_{k_0} = \tfrac{\partial\phi}{\partial k}\left(k_0, \tfrac{x}{t}\right) = \tfrac{x}{t} - \omega'(k_0) = 0, \\ & \Rightarrow \quad \omega'(k_0) = \tfrac{x}{t} \quad \Rightarrow \quad k_0 = k_0\left(\tfrac{x}{t}\right). \end{aligned} \tag{18}$$

In a neighborhood of k_0:

$$\begin{aligned} & \hat{u}_0(k) = \hat{u}_0\left(k_0\left(\tfrac{x}{t}\right)\right) + O(k - k_0), \\ & \phi\left(k, \tfrac{x}{t}\right) = \phi\left(k_0, \tfrac{x}{t}\right) + \tfrac{1}{2}\tfrac{\partial^2\phi}{\partial k^2}\left(k_0, \tfrac{x}{t}\right)(k - k_0)^2 \\ & + O(k - k_0)^3, \end{aligned} \tag{19}$$

Therefore

$$
\begin{aligned}
u(x,t) &\sim \int\limits_{k_0-\epsilon}^{k_0+\epsilon} \tfrac{dk}{2\pi}\hat{u}_0(k)e^{i\phi(k,x/t)t} \\
&\sim \tfrac{\hat{u}_0(k_0)}{2\pi}e^{i\phi(k_0,x/t)t}\int\limits_{k_0-\epsilon}^{k_0+\epsilon} e^{\frac{i}{2}\phi''(k_0,x/t)(k-k_0)^2 t}dk \\
&\sim \tfrac{\hat{u}_0(k_0)}{\pi\sqrt{2|\phi''(k_0,x/t)|t}}e^{i\phi(k_0,x/t)t}\int\limits_{-\epsilon\sqrt{\frac{|\phi''(k_0,x/t)|t}{2}}}^{\epsilon\sqrt{\frac{|\phi''(k_0,x/t)|t}{2}}} e^{i\nu\xi^2}d\xi \\
&\sim \tfrac{\hat{u}_0(k_0)}{\pi\sqrt{2|\phi''(k_0,x/t)|t}}e^{i\phi(k_0,x/t)t}\int\limits_{\mathbb{R}} e^{i\nu\xi^2}d\xi \\
&= \tfrac{\hat{u}_0(k_0)}{\sqrt{2\pi|\phi''(k_0,x/t)|t}}e^{i\phi(k_0,x/t)t+i\nu\frac{\pi}{4}},
\end{aligned}
\tag{20}
$$

where

$$
\nu := \operatorname{sign}\left(\frac{\partial^2\phi}{\partial k^2}(k_0,x/t)\right) = -\operatorname{sign}\left(\frac{\partial^2\omega}{\partial k^2}(k_0,x/t)\right). \tag{21}
$$

In the first step we use the fact that the main contribution is in a neighborhood of k_0; in the second step we use the relevant terms of the Taylor expansions; in the third step we change variables: $\xi = \sqrt{\frac{|\phi''(k_0,x/t)|t}{2}}(k-k_0)$; in the fourth step the finite integral is approximated by the integral over the real line, due again to the fast oscillation; in the last step we evaluate the Fresnell integral as:

$$
\int\limits_{\mathbb{R}} e^{i\nu\xi^2}d\xi = \sqrt{\pi}e^{i\nu\frac{\pi}{4}}, \tag{22}
$$

expressing this integral in terms of the gaussian integral $\int_{\mathbb{R}}\exp(-x^2)dx = \sqrt{\pi}$, using the Cauchy theorem. We have qualitatively constructed the leading order term of the asymptotic expansion of the longtime solution. A quantitative proof, together with the estimate of the correction, will be made in (35) and in the Appendix 9.2.4.

Summarizing, for $t \gg 1$ and $x/t = O(1)$:

$$
\begin{aligned}
&u(x,t) \sim \tfrac{1}{\sqrt{t}}\mathcal{A}\left(k_0\right)e^{i\Theta(x,t,k_0)}, \\
&\Theta\left(x,t,k_0\right) := k_0 x - \omega\left(k_0\right)t, \\
&\mathcal{A}\left(k_0\right) := \tfrac{\hat{u}_0(k_0)}{\sqrt{2\pi|\omega''(k_0)|}}\exp\left[-i\operatorname{sign}(\omega''(k_0))\tfrac{\pi}{4}\right], \\
&k_0 = k_0\left(\tfrac{x}{t}\right).
\end{aligned}
\tag{23}
$$

Since k_0 depends on x/t, the leading order solution (23) represents a non uniform wave train, and since k_0 depends on x and t slowly for $t \gg 1$:

$$
\begin{aligned}
\tfrac{1}{k_0}\tfrac{\partial k_0}{\partial x} &= \tfrac{k_0'}{k_0}\tfrac{1}{t} = \tfrac{1}{k_0\omega''(k_0)}\tfrac{1}{t} \ll 1, \\
\tfrac{1}{k_0}\tfrac{\partial k_0}{\partial t} &= -\tfrac{k_0'}{k_0}\tfrac{x}{t^2} = -\tfrac{\omega'(k_0)}{k_0\omega''(k_0)}\tfrac{1}{t} \ll 1,
\end{aligned}
\tag{24}
$$

it represents a **slowly varying (in space-time) wave train**. In (24) we have used the relation $\omega'(k_0) = \frac{x}{t}$ and its consequence $k_0'\omega''(k_0) = 1$, obtained taking the x derivative of it.

In addition,

$$\begin{array}{l} \frac{\partial\Theta}{\partial x} = k_0 + k_0'\frac{x}{t} - \omega'(k_0)k_0' = k_0, \\ \frac{\partial\Theta}{\partial t} = -k_0'\frac{x^2}{t^2} + \omega'(k_0)k_0'\frac{x}{t} - \omega(k_0) = -\omega(k_0). \end{array} \tag{25}$$

Therefore
i) we have the same formulas as for the single monochromatic wave $a\exp(i\theta)$, where $\theta = kx - \omega(k)t$, for which: $\theta_x = k,\ \theta_t = -\omega(k)$. But now wave number and dispersion relation are not constant;
ii) the Schwarz lemma ($\Theta_{xt} = \Theta_{tx}$) implies that **the wave number k_0 propagates with the group velocity $\omega'(k_0)$ according to the Riemann nonlinear hyperbolic PDE** (see §3):

$$\frac{\partial k_0}{\partial t} + \omega'(k_0)\frac{\partial k_0}{\partial x} = 0. \tag{26}$$

An observer traveling with constant speed $v_0 = x/t = \omega'(k_0(x/t))$ sees waves with wave number $k_0(v_0)$ and angular frequency $\omega(k_0(v_0))$; therefore **wave number and angular frequency travel with the group velocity** $\omega'(k_0(v_0))$, but the amplitude of the wave train changes and the crests move (they separate or get closer). If the phase $\Theta(x,t)$ is constant, then

$$\frac{d\Theta}{dt} = \Theta_t + \Theta_x\frac{dx}{dt} = 0 \quad \Rightarrow \quad \frac{dx}{dt} = -\frac{\Theta_t}{\Theta_x} \sim \frac{\omega(k_0(x/t))}{k_0(x/t)} \tag{27}$$

Therefore **an observer moving with a crest travels with the phase velocity**, and wave number and angular frequency vary (the neighboring crests separate or get closer).

As we shall see now, the energy of the wave train (or probability to find the quantum particle, or mass if u is a mass density of something, or power in optics) travels with the group velocity $\omega'(k_0)$.

$$\begin{array}{l} E(t) = \int\limits_{x_1}^{x_2} |u(x,t)|^2 dx \sim \frac{1}{2\pi t}\int\limits_{x_1}^{x_2} \frac{|\hat{u}_0(k_0(x/t))|^2}{|\omega''(k_0(x/t))|}dx \\ = \frac{1}{2\pi}\int\limits_{k_m}^{k_M} |\hat{u}_0(k_0)|^2 dk_0 \end{array} \tag{28}$$

The change of variables used: $x \to k_0(x/t)$ at fixed t is given by $x = \omega'(k_0(x/t))t$, with $dx = \omega''(k_0(x/t))tdk_0$, and with the end points $x_j = \omega'(k_j)t,\ j = 1,2$. In addition: $k_m = min\{k_1,k_2\},\ k_M = max\{k_1,k_2\}$.

Therefore the energy is constant in the interval (x_1, x_2), where x_1, x_2 travel with the group velocity: $x_j = \omega'(k_j)t$. Since $x_2(t) - x_1(t) = |\omega'(k_2) - \omega'(k_1)|t$ grows linearly in t, the energy disperses on an interval growing linearly with time.

Now we apply these results to two important examples, the Schrödinger and the linearized KdV equations.

2.3 The Schrödinger equation

For the Schrödinger equation for a free particle

$$iu_t + u_{xx} = 0, \tag{29}$$

we have $\omega(k) = k^2$; then

$$\begin{array}{l} \frac{x}{t} = \omega'(k_0) = 2k_0, \quad \Rightarrow \quad k_0 = \frac{x}{2t}, \\ \omega''(k) = 2, \end{array} \tag{30}$$

and the group lines are straight lines in the (x, t) plane.

$$\Theta(x,t) = k_0(x/t)x - \omega(k_0(x/t))t = x^2/(4t) = \Theta_0 \tag{31}$$

and the phase lines are parabolas (see Fig. 2).

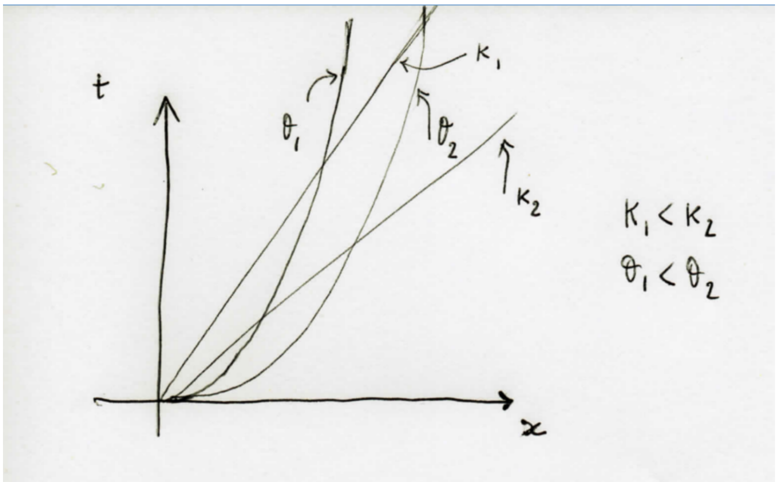


Figure 2: The group straight lines k_1 and k_2 constant, and the parabolic phase lines Θ_1 and Θ_2 constant.

Traveling with the group velocity (with constant speed v), one sees the wave number $k_0(v)$ and frequency $\omega(k_0(v))$. Since $k_0 = const = x/(2t) =$

$\sqrt{\Theta}/\sqrt{t} = 2\Theta/x$, increasing x, the phase increases. In addition, at fixed t, increasing x, k_0 increases (the distance between two crests decreases).
If one travels with a crest (Θ is constant): $\Theta = \Theta_0 = x^2/(4t) = k_0^2 t$. Increasing time, k_0 decreases (neighboring crests separate), and since k_0 increases increasing x, it means that the observer traveling with constant speed overcomes the crests (see Fig. 3).

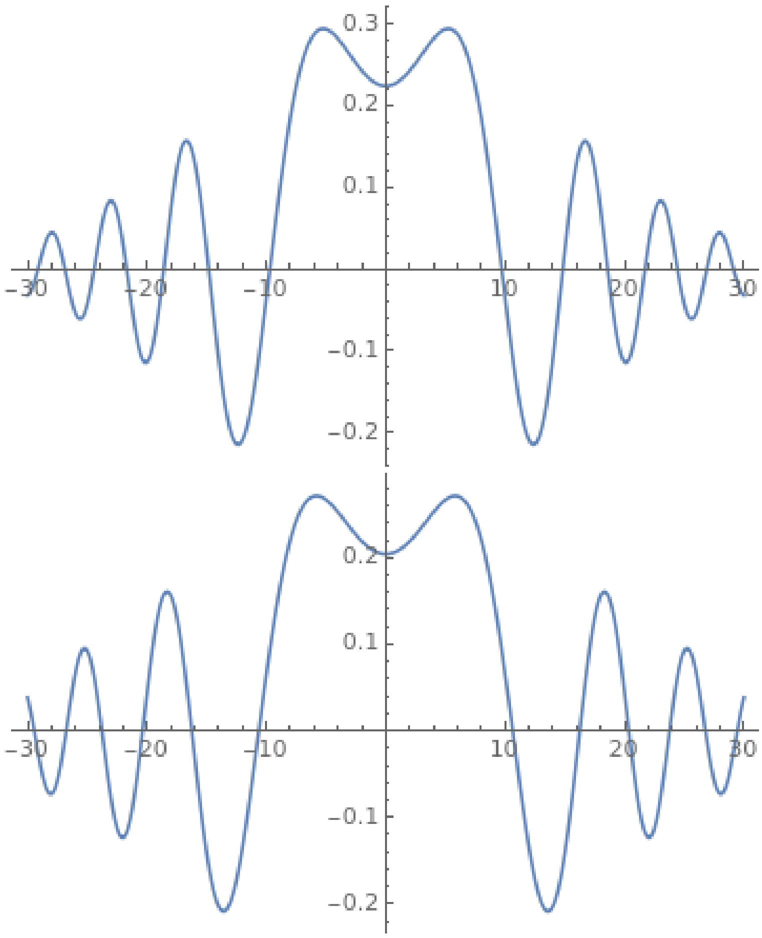


Figure 3: The graphs of the analytic formula (23) describing the asymptotics of the real part of the solution of the linear Schrödinger equation for a gaussian initial condition at $t = 10$ and at $t = 12$, to show the following. i) at fixed t, increasing x the wave-length decreases, since $k_0 = x/(2t)$; ii) as t increases, if one travels with a crest, the distance of the neighboring crests increases (k_0 decreases), and to see the same k_0, one should move to the right (the observer traveling with constant speed overcomes the crests).

At last:

$$u(x,t) \sim \frac{\hat{u}_0\left(\frac{x}{2t}\right)}{\sqrt{4\pi t}} e^{i\left(\frac{x^2}{4t}-\frac{\pi}{4}\right)}, \quad t \gg 1, \ \frac{x}{t} = O(1). \tag{32}$$

We remark that the fundamental similarity solution of (29)

$$S(x,t) = \int_{\mathbb{R}} \frac{dk}{2\pi} e^{i(kx-k^2t)} = \frac{e^{i\left(\frac{x^2}{4t}-\frac{\pi}{4}\right)}}{\sqrt{4\pi t}} \tag{33}$$

(CALCULATE THIS INTEGRAL COMPLETING THE SQUARE AND REDUCING IT TO THE FRESNELL INTEGRAL) is the main ingredient of the longtime solution:

$$\begin{array}{l} u(x,t) \sim U(x,t) := A\left(\frac{x}{2t}\right) S(x,t), \quad t \gg 1, \quad x/t = O(1), \\ A\left(\frac{x}{2t}\right) := \hat{u}_0\left(\frac{x}{2t}\right), \end{array} \tag{34}$$

where $A(\cdot)$ is then an arbitrary localized function (the Fourier transform of an arbitrary localized initial datum).

One can show that the correction to the leading order formula (34) reads:

$$\begin{array}{l} u(x,t) \sim S(x,t)\left[A\left(\frac{x}{2t}\right) + \frac{1}{t}B\left(\frac{x}{2t}\right) + O(t^{-2})\right], \\ B(\xi) = -\frac{i}{4}A''(\xi), \qquad t \gg 1, \qquad x/t = O(1). \end{array} \tag{35}$$

Indeed, observing that the leading order solution (34) solves the PDE up to $O(t^{-2})$ corrections:

$$iU_t + U_{xx} = A\cancel{(iS_t + S_{xx})} - i\cancel{\frac{x}{2t}A'\frac{S}{t}} + \cancel{\frac{1}{t}A'S_x} + A''\frac{S}{(2t)^2} = O(t^{-2}), \tag{36}$$

it follows that (35a) is the correct ansatz; substituting it in the PDE, one obtains (35b). VERIFY IT!

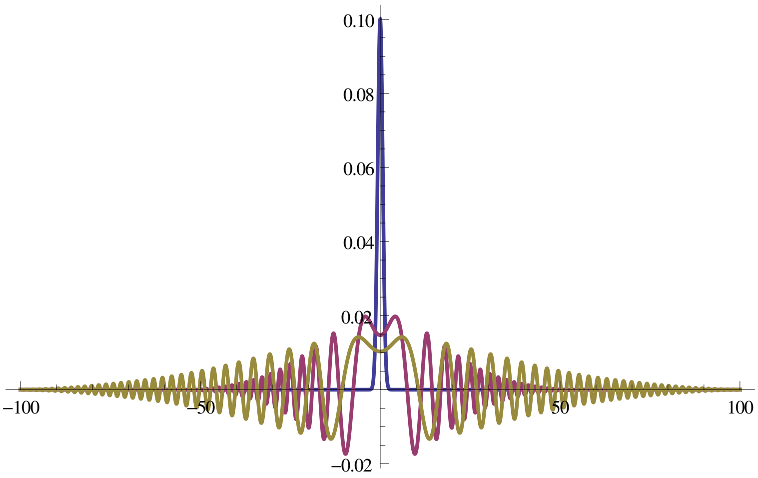


Figure 4: Three snapshots of the numerical evolution, according to NLS, of the real part of u, for a gaussian initial condition (in blue). COMPARE IT with the analytic plots of Fig. 3

2.4 The linearized KdV equation

The fundamental (similarity) solution of the linearized KdV equation

$$u_t + u_{xxx} = 0, \quad u(x,t) \in \mathbb{R} \tag{37}$$

is

$$\begin{aligned} S(x,t) &= \int\limits_{\mathbb{R}} \tfrac{dk}{2\pi} e^{i(kx+k^3t)} = \tfrac{1}{(3t)^{1/3}} \int\limits_{\mathbb{R}} \tfrac{ds}{2\pi} e^{i(\xi s + \frac{s^3}{3})} \\ &=: \tfrac{1}{(3t)^{1/3}} Ai\left(\xi\right), \qquad \xi = \tfrac{x}{(3t)^{1/3}} \end{aligned} \tag{38}$$

(in the second step we used the change of variables $k \to s, \; s = (3t)^{1/3}k$). SHOW that the Airy function $Ai(\xi)$ defined in (38) solves the ODE

$$Ai_{\xi\xi}(\xi) = \xi Ai(\xi) \tag{39}$$

and then plays an important role in Quantum Mechanic, describing the wave function of the stationary Schrödinger equation in the small region around the point in which the potential changes its sign.

Now we look for the longtime behavior of the solution of the Cauchy problem for the linearized KdV equation

$$\begin{aligned} &u_t + u_{xxx} = 0, \quad u(x,t) \in \mathbb{R}, \\ &u(x,0) = u_0(x), \quad u(x,t) \to 0, \; x \to \pm\infty, \end{aligned} \tag{40}$$

whose solution is

$$u(x,t) = \int_{\mathbb{R}} \frac{dk}{2\pi} \hat{u}_0(k) e^{i(kx+k^3t)}. \tag{41}$$

Now $\phi(k, x/t) = kx/t + k^3$, $\phi'(k_0) = x/t + 3k_0^2 = 0$ and there are two stationary points

$$k_0^\pm = \pm\left(-\frac{x}{3t}\right)^{1/2}. \tag{42}$$

If $x/t < 0$ (the observer travels with negative speed), then $k_0^\pm = \pm\sqrt{\left|\frac{x}{3t}\right|} \in \mathbb{R}$ and we can use, as before, the stationary phase method with two stationary points:

$$\begin{aligned}
&\phi(k_0^\pm, x/t) = \mp\left|\tfrac{x}{3t}\right|^{3/2}, \\
&\phi''(k_0^\pm, x/t) = 6k_0^\pm = \pm 2\left|\tfrac{3x}{t}\right|^{1/2} \quad \Rightarrow \\
&u(x,t) \sim \frac{1}{\sqrt{4\pi\left|\frac{3x}{t}\right|^{1/2} t}} \left[\hat{u}_0\left(\sqrt{\left|\tfrac{x}{3t}\right|}\right) e^{-i\left(2\left|\frac{x}{3t}\right|^{3/2} t - \frac{\pi}{4}\right)}\right. \\
&\left. + \hat{u}_0\left(-\sqrt{\left|\tfrac{x}{3t}\right|}\right) e^{i\left(2\left|\frac{x}{3t}\right|^{3/2} t - \frac{\pi}{4}\right)}\right], \; t \gg 1, \; x/t = O(1) < 0.
\end{aligned} \tag{43}$$

Since $u \in \mathbb{R}$ and $\omega(k)$ is odd, then $\hat{u}_0(k) = \overline{\hat{u}_0(-k)}$ for $k \in \mathbb{R}$, and the asymptotics can be rewritten in the simpler form

$$\begin{aligned}
&u(x,t) \sim \frac{\left|\hat{u}_0\left(\sqrt{\left|\frac{x}{3t}\right|^{1/2}}\right)\right|}{\sqrt{\pi t}\left|\frac{3x}{t}\right|^{1/4}} \cos\Theta(x,t), \; t \gg 1, \; x/t = O(1) < 0, \\
&\Theta(x,t) := 2\left|\tfrac{x}{3t}\right|^{3/2} t - \arg\left(\hat{u}_0\left(\sqrt{\left|\tfrac{x}{3t}\right|^{1/2}}\right)\right) - \tfrac{\pi}{4}.
\end{aligned} \tag{44}$$

If $x/t > 0$ (the observer travels with positive speed), then $k_0^\pm = \pm i\sqrt{\left|\frac{x}{3t}\right|} \in i\mathbb{R}$, the stationary phase method cannot be used anymore and must be replaced by the "Steepest descent method", or "saddle point method", illustrated in Appendix 2.

We have

$$u(x,t) = \int_{\mathbb{R}} \frac{dk}{2\pi} \hat{u}_0(k) e^{i(kx+k^3t)} = \int_{\mathbb{R}} \frac{dk}{2\pi} \hat{u}_0(k) e^{f(k,x/t)t}, \tag{45}$$

with

$$\begin{aligned}
&f(k, x/t) = i(kx/t + k^3), \; f'(k, x/t) = i(x/t + 3k^2) = 0 \;\Rightarrow\; k_0^\pm = \pm i\sqrt{\left|\tfrac{x}{3t}\right|}, \\
&f''(k_0^\pm, x/t) = 6ik_0^\pm = \mp 6\sqrt{\left|\tfrac{x}{3t}\right|}, \quad e^{f(k_0^\pm, x/t)t} = e^{\mp 2\left|\frac{x}{3t}\right|^{3/2} t}.
\end{aligned} \tag{46}$$

Since now $k = k_R + ik_I \in \mathbb{C}$, $k_R = \mathrm{Re}\ k$, $k_I = \mathrm{Im}\ k$,

$$
\begin{array}{l}
f(k, x/t) = u(k_R, k_I) + iv(k_R, k_I), \\
f_R = u(k_R, k_I) = k_I^3 - 3k_R^2 k_I - k_I \frac{x}{t}, \\
f_I = v(k_R, k_I) = k_R \frac{x}{t} + k_R^3 - 3k_R k_I^2, \\
v(k_0^{\pm}) = 0, \qquad u(k_0^{\pm}) = \mp 2 \left| \frac{x}{3t} \right|^{3/2}.
\end{array}
\tag{47}
$$

Then the curves of steepest variation, defined by

$$
\begin{array}{l}
v(k_R, k_I) - v(k_0^{\pm}) = v(k_R, k_I) \\
= k_R(k_R^2 - 3k_I^2 + \frac{x}{t}) = 0,
\end{array}
\tag{48}
$$

are the imaginary axis $k_R = 0$ and the hyperbola $k_R^2 - 3k_I^2 + \frac{x}{t} = 0$ of the complex k plane. The conditions of steepest descent

$$
u(k_R, k_I) - u(k_0^{\pm}) = k_I^3 - 3k_R^2 k_I - k_I \tfrac{x}{t} \pm 2 \left| \tfrac{x}{3t} \right|^{3/2} < 0 \tag{49}
$$

select the upper branch of the hyperbola passing through k_0^+ (on which $\varphi_0 = \arg(f''(k_0^+)) = \pi$ and $\theta_0 = 0$), and the semi straight line $k_R = 0$, $k_I < (x/3t)^{1/2}$ passing through k_0^- (on which $\varphi_0 = 0$ and $\theta_0 = \pi/2$) (VERIFY IT).

In addition, for $|k| \gg 1$, then $f(k) \sim ik^3 = i|k|^3(\cos(3\varphi) + i\sin(3\varphi)) = |k|^3(i\cos(3\varphi) - \sin(3\varphi))$, where $k = |k|e^{i\varphi}$. It follows that the integrand goes to 0 exponentially at ∞ when $\sin(3\varphi) > 0$; i.e., when $0 < \varphi < \pi/3$, $2\pi/3 < \varphi < \pi$, and $4\pi/3 < \varphi < 5\pi/3$ (see Fig. 5).

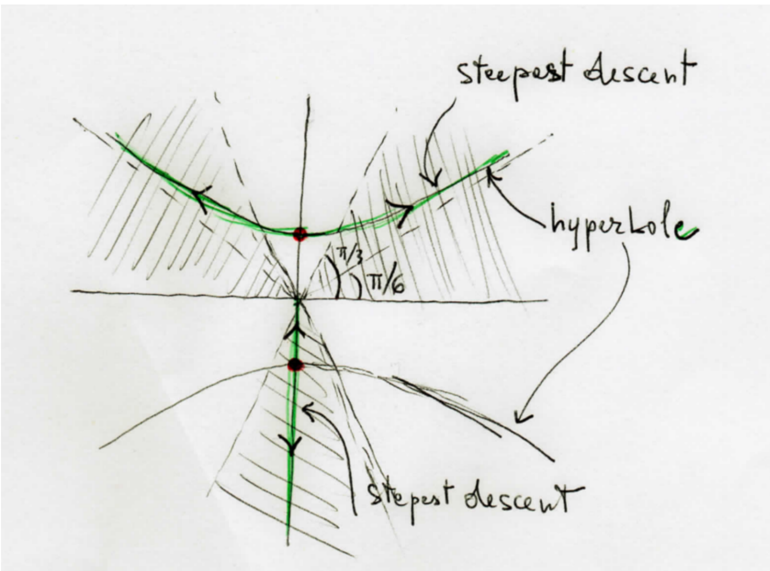


Figure 5: The two steepest descent contours passing through k_0^+ and k_0^- are in green. In the gray regions the exponential converges to zero at ∞

Therefore the only steepest descent contour that can be connected to the initial integration contour $(-\infty, \infty)$ through two arcs at ∞ is the upper branch of the parabola passing through k_0^+, and the direction at k_0^+ is $\theta_0 = 0$. More precisely, the integral over the real line is equal to the integral over the contour $\tilde{\gamma}$ consisting of the union of the infinite arc $(-\infty, \infty\exp(5i\pi/6))$ the upper branch of the hyperbola γ, from $\infty\exp(5i\pi/6)$ to $\infty\exp(i\pi/6)$, and the infinite arc $(\infty\exp(i\pi/6), \infty)$, by the Cauchy theorem, having assumed that $\hat{u}_0(k)$ be analytic inside the closed contour $\mathbb{R} \cup \tilde{\gamma}$. In addition, since the integrals over the infinite arcs are zero, we obtain, using formula (1269) in Appendix 2,

$$
\begin{aligned}
&u(x,t) = \int\limits_{\mathbb{R}} \tfrac{dk}{2\pi} \hat{u}_0(k) e^{i(kx+k^3t)} = \int\limits_{\gamma} \tfrac{dk}{2\pi} \hat{u}_0(k) e^{i(kx+k^3t)} \\
&= \frac{\hat{u}_0\left(i\sqrt{\left|\frac{x}{3t}\right|}\right)}{2\sqrt{\pi t}\left|\frac{3x}{t}\right|^{1/4}} e^{-2\left|\frac{x}{3t}\right|^{3/2} t} \left(1 + O\left(\tfrac{1}{t}\right)\right), \\
&t \gg 1, \quad x/t = O(1) > 0.
\end{aligned} \tag{50}
$$

We observe that the RHS of (50) is real, as it has to be, since $\overline{\hat{u}_0(k)} = \hat{u}_0(-\overline{k})$.

From (44) and (50) we conclude that a localized initial condition evolves into a slowly varying wave train moving left.

We remark that both asymptotics (44) and (50) diverge for $x/t \sim 0$. It means that, for $x/t \sim 0$, there must exist another asymptotic region in which the solution is regular and matches well with the left and right asymptotic regions. To construct the asymptotics in this intermediate region, we observe that

$$
\begin{aligned}
&u(x,t) = \tfrac{1}{2\pi} \int_{\mathbb{R}} \hat{u}_0(k) e^{i(kx+k^3t)} dk \\
&= \tfrac{1}{2\pi(3t)^{1/3}} \int_{\mathbb{R}} \hat{u}_0\left(\tfrac{s}{(3t)^{1/3}}\right) e^{i(s\xi + s^3/3)} ds, \\
&\xi := \tfrac{x}{(3t)^{1/3}}.
\end{aligned} \tag{51}
$$

If $t \gg 1$ and $x(3t)^{-1/3} = O(1)$ $(x/t \sim 0)$, the argument of $\hat{u}_0$ is small inside the second integral : $\hat{u}_0(k) = \hat{u}_0(0) + \hat{u}_0'(0)\ k + O(k^2)$, and

$$
\begin{aligned}
&u(x,t) \sim \tfrac{\hat{u}_0(0)}{2\pi(3t)^{1/3}} \int_{\mathbb{R}} e^{i(s\frac{x}{(3t)^{1/3}} + s^3/3)} ds + \tfrac{\hat{u}_0'(0)}{2\pi(3t)^{2/3}} \int_{\mathbb{R}} s e^{i(s\frac{x}{(3t)^{1/3}} + s^3/3)} ds \\
&= \tfrac{\hat{u}_0(0)}{(3t)^{1/3}} Ai\left(\tfrac{x}{(3t)^{1/3}}\right) - \tfrac{i\hat{u}_0'(0)}{(3t)^{2/3}} Ai'\left(\tfrac{x}{(3t)^{1/3}}\right), \quad t \gg 1, \quad \tfrac{x}{(3t)^{1/3}} = O(1).
\end{aligned} \tag{52}
$$

It is possible to show that the asymptotics (52) match perfectly with the leading order terms (44) and (50) of the left and right regions, using the asymptotics of the Airy function (see Appendix 2):

$$
Ai(\xi) \sim \begin{cases} \frac{e^{-\frac{2}{3}\xi^{3/2}}}{2\sqrt{\pi}\xi^{1/4}}, & \xi \gg 1 \\ \frac{\cos(\frac{2}{3}\xi^{3/2} - \pi/4)}{\sqrt{\pi}|\xi|^{1/4}}, & \xi \ll -1. \end{cases} \tag{53}
$$

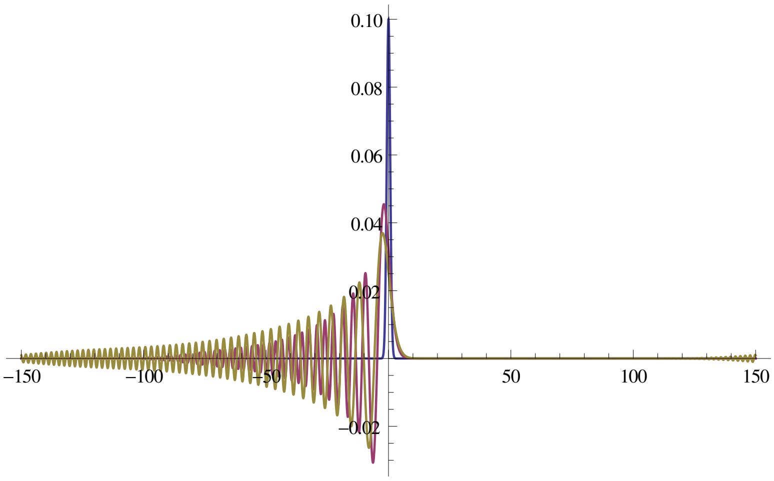


Figure 6: Three snapshots of the numerical evolution, according to the linearized KdV, of a gaussian initial condition show that the non uniform wave train propagates to the left, as shown analytically.

2.5 Exercises

1) Given the Cauchy problem

$$u_t + i\omega(-i\partial_x)u = 0, \quad u(x,0) \text{ given}, \;\; x \in \mathbb{R}, \;\; t \geq 0, \tag{54}$$

where $\omega(k)$ is a polynomial function of k, so that, f.i., if $\omega(k) = k^2$, then $\omega(-i\partial_x) = (-i\partial_x)^2 = -\partial_x^2$,
1. show that the Fourier integral representation of its solution is

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}_0(k)e^{i(kx-\omega(k)t)}dk, \tag{55}$$

where $\hat{u}_0(k)$ is the Fourier transform of the initial condition $u(x,0)$:

$$\hat{u}_0(k) = \int_{\mathbb{R}} e^{-iky}u(y,0)dy. \tag{56}$$

2. Show that (55) can be written as a convolution integral, in the suggestive form:

$$u(x,t) = \int_{\mathbb{R}} S(x-y,t)u(y,0)dy, \tag{57}$$

where $S(x,t)$ is the "fundamental" solution of the PDE, defined as:

$$S(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(kx-\omega(k)t)}dk. \tag{58}$$

3. If $\omega(k) = k^n$, then $S(x,t)$ is the following similarity solution of the PDE:

$$\begin{aligned} S(x,t) &= \tfrac{1}{t^{1/n}} f\left(\tfrac{x}{t^{1/n}}\right), \\ f(\xi) &= \tfrac{1}{2\pi}\int_{\mathbb{R}} e^{i(k\xi - k^n)}dk. \end{aligned} \tag{59}$$

4. Show that, if $u \in \mathbb{R}$, then:

$$\begin{array}{l} \overline{\hat{u}_0(k)} = \hat{u}_0(-k), \quad k \in \mathbb{R} \\ u(x,t) = \frac{1}{\pi} Re \int\limits_0^\infty \hat{u}_0(k) e^{i(kx-\omega(k)t)} dk \end{array} \tag{60}$$

If, in addition, $\hat{u}_0(k)$ can be prolongued outside the real axis, then

$$\overline{\hat{u}_0(k)} = \hat{u}_0(-\bar{k}). \tag{61}$$

(for the second of (60) we have also assumed that $\omega(k)$ is odd: $\omega(-k) = -\omega(k)$)

2) Given the following linear PDEs:

$$\begin{array}{ll} i) \;\; iu_t + u_{xx} = 0, & \text{free particle Schrödinger equation,} \\ ii) \;\; u_t + u_{xxx} = 0, & \text{linearized KdV equation,} \\ iii) \;\; u_{tt} - u_{xx} + u = 0, & \text{Klein - Gordon equation,} \end{array} \tag{62}$$

1. Construct the fundamental similarity solution (1232) (only for i) and ii)).
2. Study the longtime behavior, for $t >> 1$, $x/t = O(1)$, of the solutions of their Cauchy problem using the stationary phase, Laplace, or saddle point methods, depending on the situation, and estimate the error.
3. Study of the relevance of the exact similarity solution in the longtime behavior (only for i) and ii)).

Solution:

i) Free particle Schrödinger equation:

$$\begin{array}{l} S(x,t) = \frac{1}{2\sqrt{\pi t}} e^{i(\frac{x^2}{4t} - \frac{\pi}{4})}, \\ u(x,t) = S(x,t) \left(A(\xi) + \frac{1}{t} B(\xi) + O(t^{-2}) C(\xi) \right), \quad \xi = \frac{x}{2t} = O(1), \quad t >> 1 \\ A(\xi) = \hat{u}_0(\xi), \quad B(\xi) = -\frac{i}{4} A_{\xi\xi} \end{array} \tag{63}$$

ii) Linear KdV. For $x/t > 0$, the lines of constant $v(k)$ are the imaginary axis and the hyperbola ${k_R}^2 - 3{k_I}^2 + x/t = 0$. The steepest descent contour passing through the critical point $i\sqrt{\frac{x}{3t}}$ is the upper branch of the hyperbola, while the steepest descent contour passing through the critical point $-i\sqrt{\frac{x}{3t}}$ is the imaginary axis. The asymptotics is obtained replacing the integration real line by the steepest descent contour passing through $i\sqrt{\frac{x}{3t}}$.

$$\begin{array}{l} S(x,t) = \frac{1}{(3t)^{1/3}} Ai\left(\frac{x}{(3t)^{1/3}}\right), \\ u(x,t) \sim \frac{\hat{u}_0(|x/3t|^{1/2})}{\sqrt{4\pi|3x/t|^{1/2}t}} e^{-i2|x/3t|^{3/2}t + i\pi/4} + \text{c.c.}, \quad \frac{x}{3t} = O(1) < 0, \quad t >> 1, \\ u(x,t) \sim \frac{\hat{u}_0(i|x/3t|^{1/2})}{\sqrt{12\pi|3x/t|^{1/2}t}} e^{-2|x/3t|^{3/2}t}, \quad \frac{x}{3t} = O(1) > 0, \quad t >> 1, \\ u(x,t) \sim \frac{\hat{u}_0(0)}{2\pi(3t)^{1/3}} Ai\left(\frac{x}{(3t)^{1/3}}\right) - \frac{i\hat{u}_0'(0)}{2\pi(3t)^{2/3}} Ai'\left(\frac{x}{(3t)^{1/3}}\right), \quad \frac{x}{(3t)^{1/3}} = O(1), \quad t >> 1, \\ u(x,t) \sim \frac{\hat{u}_0(0)}{2\pi} S(x,t), \quad \frac{x}{(3t)^{1/3}} = O(1), \quad t >> 1, \end{array} \tag{64}$$

where $Ai(\xi)$ is the Airy function

$$Ai(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(k\xi + k^3/3)} dk, \tag{65}$$

solution of the ODE: $f''(\xi) - \xi f(\xi) = 0$.

iii) Klein-Gordon equation. The dispersion relation is two-valued (since the PDE is second order in t):

$$\omega^{\pm}(k) = \pm\sqrt{k^2 + 1}; \tag{66}$$

therefore the phase velocity is greater than the light speed 1, while the group velocity is less than 1:

$$\frac{\omega}{k} = \frac{\sqrt{k^2+1}}{k} > 1, \quad \frac{d\omega}{dk} = \frac{k}{\sqrt{k^2+1}} < 1 \tag{67}$$

The Fourier representation of the real solution reads:

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} B(k) e^{i(kx+\sqrt{k^2+1}t)} dk + \frac{1}{2\pi}\int_{\mathbb{R}} \overline{B(-k)} e^{i(kx-\sqrt{k^2+1}t)} dk, \tag{68}$$

where

$$B(k) = \frac{1}{2}\left(\hat{u}_0(k) - i\frac{\hat{u}_0'(k)}{\sqrt{k^2+1}}\right), \tag{69}$$

where $\hat{u}_0(k)$ and $\hat{u}_0'(k)$ are the Fourier transforms of respectively $u(x,0)$ and $u_t(x,0)$. For $|x/t| < 1$ (inside the light cone) and $t >> 1$ (see Fig. 7):

$$u \sim \frac{1}{\sqrt{2\pi t}}\left(1 - \left(\frac{x}{t}\right)^2\right)^{-3/4} B\left(-\frac{x}{\sqrt{t^2-x^2}}\right) e^{i\sqrt{t^2-x^2}+i\pi/4} + c.c. \tag{70}$$

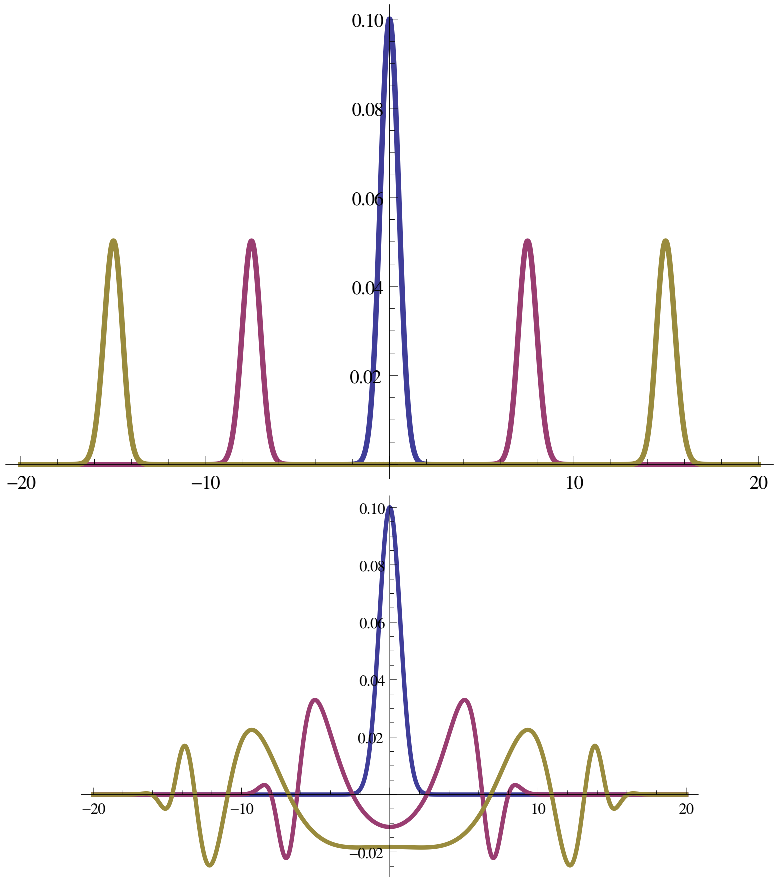


Figure 7: Three time steps ($t = 0$, $t = T/2$, $t = T$) of the numerical evolution of a gaussian initial condition according to the wave equation equation $u_{tt} - c^2 u_{xx} = 0$ (the figure above) and the Klein-Gordon equation (the figure below).

3) Study the longtime behavior, for $t >> 1,\ x/t = O(1)$, of the Fourier integral

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}_0(k) e^{i(kx-\omega(k)t)} dk \tag{71}$$

under the hypothesis that there exists a unique stationary phase point $k_0(x/t) \in \mathbb{R}$, and that $\omega''(k_0) = 0,\ \omega'''(k_0) \neq 0$.

4) Given the linear PDE $\mathcal{P}(\partial_t, \nabla_{\vec{x}})u(\vec{x}, t) = 0$, $\vec{x} \in \mathbb{R}^n,\ t \in \mathbb{R}$ in $(n+1)$ dimensions, with $u \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$,
i) show that the solution of its Cauchy problem:

$$\mathcal{P}(\partial_t, \nabla_{\vec{x}})u(\vec{x}, t) = 0, \quad u(\vec{x}, 0) = u_0(\vec{x}) \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n) \tag{72}$$

is given by the Fourier integral:

$$\begin{array}{l} u(\vec{x}, t) = \int_{\mathbb{R}^n} \hat{u}_0(\vec{k}) e^{i(\vec{k}\cdot\vec{x} - \omega(\vec{k})t)} \frac{d\vec{k}}{(2\pi)^n} \\ \hat{u}_0(\vec{k}) = \int_{\mathbb{R}^n} u_0(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} d\vec{x} \end{array} \tag{73}$$

where $\omega(\vec{k})$ is obtained solving the equation $\mathcal{P}(-i\omega, i\vec{k}) = 0$ wrt ω.
ii) Show that, under the hypothesis that the vector equation for $\vec{k}$

$$\frac{\vec{x}}{t} = \nabla_{\vec{k}}\omega(\vec{k}) \tag{74}$$

admits a unique real solution $\vec{k}_0 = \vec{k}_0(\vec{x}/t) \in \mathbb{R}^n$, the extension of the stationary phase method for multiple integrals gives the following longtime behavior:

$$\begin{array}{l} u \sim \left(\frac{1}{2\pi t}\right)^{n/2} \left(det\left(\frac{\partial^2\omega(\vec{k}_0)}{\partial k_i \partial k_j}\right)\right)^{-1/2} \hat{u}_0(\vec{k}_0) e^{i(\vec{k}_0\cdot\vec{x} - \omega(\vec{k}_0)t + m\frac{\pi}{4})}, \\ m \equiv -\sum\limits_{j=1}^{n} \text{sign}(\lambda_j) \end{array} \tag{75}$$

where $\lambda_j,\ j = 1, .., n$ are the (real) eigenvalues of the symmetric matrix $\left(\frac{\partial^2\omega(\vec{k})}{\partial k_i \partial k_j}\right)\Big|_{\vec{k}_0}$.

5) Let $\Gamma(z)$ be the Euler Γ function:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad Re\ z > 0. \tag{76}$$

i) Show, integrating by parts, that it is the generalization of the factorial:

$$\Gamma(n+1) = n!,\ n \in \mathbb{N}. \tag{77}$$

ii) Use the Laplace method to construct the Stirling formula:

$$n! = n^n e^{-n}\sqrt{2\pi n}\left(1 + O(n^{-1})\right), \quad n >> 1. \tag{78}$$

6) Use the Laplace method to show that, if $f(t)$ has a max of order $n-1$ in $t_0 \in (a, b)$:

$$f(t) = f(t_0) + \frac{f^{(n)}(t_0)}{n!}(t - t_0)^n + O(t - t_0)^{n+1}, \quad f^{(n)}(t_0) < 0, \quad n \text{ even}, \tag{79}$$

then

$$\int_a^b g(t) e^{pf(t)} dt \sim \frac{2\ \Gamma(1/n)}{n} \sqrt[n]{\frac{n!}{p|f^{(n)}(t_0)|}} g(t_0) e^{pf(t_0)}, \tag{80}$$

having also used the formula

$$\int_{\mathbb{R}} e^{-s^n} ds = \frac{2\ \Gamma(1/n)}{n}. \tag{81}$$

7) Obtain the following asymptotics for the Bessel and modifies Bessel functions:

$$\begin{aligned} I_n(x) &= \frac{1}{\pi}\int\limits_0^{\pi} \cos(nt)e^{x\cos t}dt \sim \frac{1}{\sqrt{2\pi x}}e^x, \quad x \gg 1, \\ K_\nu(x) &= \int\limits_0^{\infty} \cosh(\nu t)e^{-x\cosh t}dt \sim \sqrt{\frac{\pi}{2x}}e^{-x}, \quad x \gg 1. \end{aligned} \tag{82}$$

8) Given the Airy function $Ai(x)$, defined by

$$Ai(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(kx-k^3)}dk, \quad x \in \mathbb{R}, \tag{83}$$

i) use the saddle point method to show that

$$\begin{aligned} Ai(x) &= \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}}(1+O(x^{-3/2})), \quad x \gg 1, \\ Ai(x) &= \frac{1}{\pi|x|^{1/4}}\cos\left(\tfrac{2}{3}|x|^{3/2} - \tfrac{\pi}{4}\right)(1+O(x^{-3/2})), \quad x \ll -1. \end{aligned} \tag{84}$$

ii) Use the above asymptotics to show that the longtime behavior of the solutions of the Cauchy problem for the linearized KdV equation in the region $|x|/t^{1/3} = O(1),\ t \gg 1$, matches well with the asymptotics in the left and right regions $|x|/t = O(1),\ t \gg 1,\ x < 0$ and $x > 0$.

9) Given the integral

$$f(x,t) = \int\limits_a^b g(k)e^{i(kx-k^2t)}dk, \quad t \gg 1, \quad x/t = O(1), \tag{85}$$

where we integrate over a contour $\mathcal{C}$ from a to b, inside a domain $\mathcal{D}$ of analyticity of f and g, i) show that the saddle point is $x/2t$ and the steepest descent contour is given by the straight line passing through $x/2t$ and parallel to the line bisecting the second and fourth quadrants. ii) Show that, if $a < x/2t$ and $b > x/2t$, including the cases $a = -\infty$ and $b = \infty$, the asymptotics of $f(x,t)$ are given by the saddle point formula

$$f(x,t) = \frac{1}{\sqrt{4\pi t}}g\left(\frac{x}{2t}\right)e^{i\frac{x^2}{4t}-i\frac{\pi}{4}}(1+O(1/t)), \quad t \gg 1, \quad x/t = O(1). \tag{86}$$

iii) Show that, if $a < x/2t$ and $b \in \mathbb{C}$, with $0 < \arg b < \pi/2$, the leading asymptotics is given, instead, by the integration by parts formula at the end point b:

$$f(x,t) = \frac{g(b)e^{i(bx/t-b^2)t}}{2\pi i(x/t-2b)t}(1+O(1/t)), \quad t \gg 1, \quad x/t = O(1). \tag{87}$$

3 Hyperbolic Waves [36, 12, 23]

This chapter is dedicated to nonlinear hyperbolic PDEs. The prototype example is the Riemann equation

$$u_t + c(u)u_x = 0, \quad u = u(x,t) \in \mathbb{R}, \tag{88}$$

where $c(\cdot)$ is a smooth function of its argument playing the role of a field dependent velocity.

If $c = c_0$ does not depend on u, the equation reduces to the linear first order wave equation $u_t + c_0 u_x = 0$ (also called the "advection equation") describing a rigid propagation of the initial profile with constant speed c_0:

$$u(x,t) = f(x - c_0 t). \tag{89}$$

The simplest and most relevant example of nonlinear equation in the class (88) is the so-called Hopf equation

$$u_t + uu_x = 0, \quad u = u(x,t) \in \mathbb{R}, \tag{90}$$

corresponding to a speed $c(u)$ depending linearly on the field u.

We have already seen that equation (88) is satisfied by the wave number of the slowly varying wave train describing the longtime behavior of a linear dispersive wave (equation (26)).

Equation (88) has also the following physical interpretation. Let $\rho(x,t)$ be the density of some physical quantity (mass, charge, ...) and $q = \rho v$ the corresponding flux per unit of time; then the conservation law in integral form reads

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx = q(x_1,t) - q(x_2,t) \tag{91}$$

Taking the limit $x_2 \to x_1$, and assuming that density and flux be smooth functions, one obtains the continuity equation (SHOW IT)

$$\rho_t + q_x = 0. \tag{92}$$

In addition, if the flux is a smooth function of ρ: $q = Q(\rho)$, then ρ satisfies (88) with

$$c(\rho) = Q'(\rho). \tag{93}$$

3.1 The Method of Characteristics

The integration of equation (88) is obtained via the method of characteristics, through which the integration is reduced to the solution of an algebro-transcendental equation.

Suppose for a moment that we know the solution $u(x,t)$ of the Riemann equation $u_t + c(u)u_x = 0$; then we introduce the ODE

$$\frac{dx}{dt} = c(u(x,t)) \tag{94}$$

defining a one-parameter family of curves $x = x(t,\eta)$ in the (x,t) plane, called "characteristic curves". The parameter is, for instance, the integration constant, or it can be identified with the intersection of the characteristic curve with the x-axis:

$$x = \eta \quad \text{if } t = 0. \tag{95}$$

On each characteristic curve, u does not vary, since

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}c(u) = 0. \tag{96}$$

Therefore the PDE (88) is equivalent to the system of two ODEs

$$\frac{dx}{dt} = c(u), \qquad \frac{du}{dt} = 0 \tag{97}$$

(a great simplification). In addition, since u is constant on the characteristic curve parametrized by η, it will be an arbitrary function of η: $u = A(\eta)$. The first equation becomes $\frac{dx}{dt} = c(A(\eta)) = const$, implying that

$$x = c(A(\eta))t + \eta, \qquad u(x,t) = A(\eta). \tag{98}$$

This system of two algebro-transcendental equations defines the general solution of (88) (general because it depends on the arbitrary function $A(\eta)$).

If, for instance, we are interested in the solution of the Cauchy problem

$$\begin{aligned} &u_t + c(u)u_x = 0, \quad u = u(x,t) \in \mathbb{R}, \\ &u(x,0) = u_0(x), \end{aligned} \tag{99}$$

we remark from the first equation in (98) that, at $t = 0$, $x = \eta$, implying that $u(x,0) = u_0(x) = A(x)$. Then the solution of (99) is given by the system

$$x = c(u_0(\eta))t + \eta, \qquad u(x,t) = u_0(\eta), \tag{100}$$

that can be solved implicitly in the following way.

1) One solves the first equation in (100) with respect to η, obtaining $\eta =$

$\eta(x, t)$.
2) Replacing $\eta = \eta(x, t)$ in the second equation, one obtains the wanted solution

$$u(x, t) = u_0(\eta(x, t)) = u_0(x - c(u)t). \tag{101}$$

Comparing the second formula in (101) with (89) we infer that the initial condition propagates with a velocity $c(u)$ depending on the field u. If, for simplicity, $c(u) = u$, and if the initial condition $u_0(x)$ is a localized positive bump, the max of the profile travels faster than the other parts of the profile, then the profile deforms and, at a certain time, called the breaking time t_b and in certain point x_b the profile becomes vertical. We obtain the so-called "gradient catastrophe" at finite time. For $t > t_b$ the solution becomes multivalued (three-valued in Fig. 8) in a segment $(x_1(t), x_2(t))$ (with $x_1(t_b) = x_2(t_b) = x_b$) growing with time:

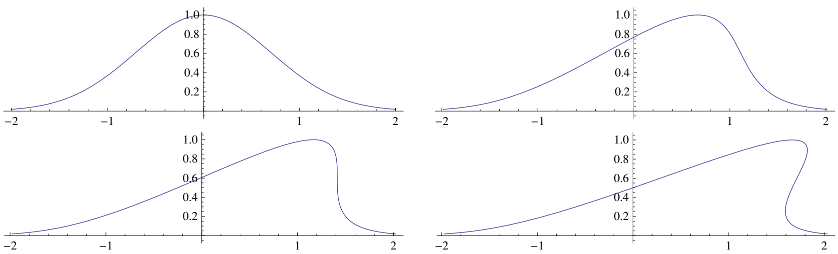


Figure 8: Four snapshots of the evolution of a localized initial condition for $c(u) = u$, obtained through the numerical inversion of the algebro-transcendental solution. 1st shot: the initial condition of gaussian type; 2nd picture: the deformation of the profile for $t < t_b$; 3rd picture: the gradient catastrophe at $t = t_b$; 4th picture: the three-valued profile for $t > t_b$.

A geometric description of the process comes from the study of the slopes of the one-parameter family of characteristics. Each portion of the initial profile $u_0(\eta)$ travels on the characteristic straight line of parameter η. Different portions travel on different characteristics having different slopes $c(u_0(\eta))$, and one expects that different characteristics may intersect in space-time. In a point of intersection (x, t) the solution u has more values, the values carried by the intersecting characteristics (see Fig. 9):

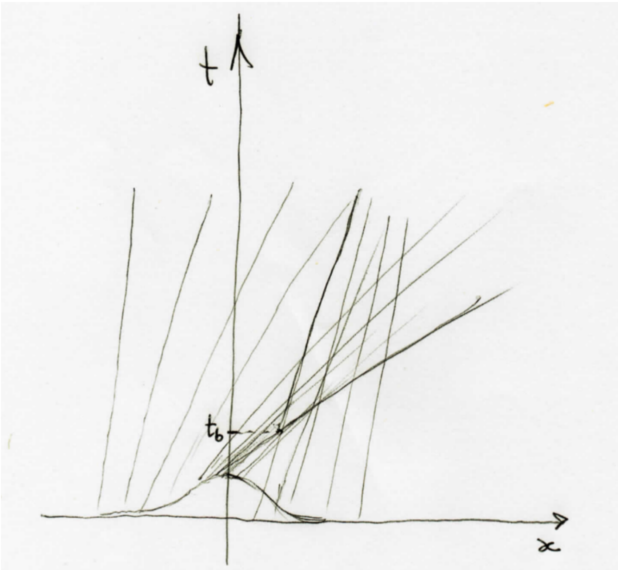


Figure 9: The (handmade) figure shows the characteristics generated by a generic localized bump initial condition for $c(u) = u$ (the characteristics are vertical lines far away from the bump, and they have a maximal slope where the bump has its maximum). For $t \geq t_b$ the characteristics start intersecting. COMPARE this figure with Fig. 8

How to find the breaking time t_b, the time at which the first gradient catastrophe takes place? There are many ways.

1) *Two characteristics intersect.* Suppose the characteristics η and $\eta + \delta\eta$ intersect in the space-time point (x, t):

$$\begin{aligned} &x = \eta + F(\eta)t, \\ &x = \eta + \delta\eta + F(\eta + \delta\eta)t \sim \eta + F(\eta)t \\ &+\delta\eta(1 + F'(\eta)t), \end{aligned} \tag{102}$$

where

$$F(\eta) := c(u_0(\eta)), \tag{103}$$

implying the condition

$$1 + F'(\eta)t = 0, \quad \Rightarrow \quad t = -\frac{1}{F'(\eta)}. \tag{104}$$

Since $t > 0$, then η must be such that

$$F'(\eta) < 0. \tag{105}$$

2) *The slope must be* ∞. Taking the partial derivative with respect to x and t of equations (100):

$$\begin{aligned} &u_t = u_0'(\eta)\eta_t, \quad 0 = F(\eta) + F'(\eta)t\eta_t + \eta_t \\ &= F(\eta) + \eta_t(1 + F'(\eta)t) \;\Rightarrow \\ &\eta_t = -\tfrac{F(\eta)}{1+F'(\eta)t} \quad \Rightarrow \quad u_t = -\tfrac{F(\eta)u_0'(\eta)}{1+F'(\eta)t}. \end{aligned} \tag{106}$$

$$\begin{aligned} &u_x = u_0'(\eta)\eta_x, \quad 1 = F'(\eta)t\eta_x + \eta_x \\ &= \eta_x(1 + F'(\eta)t) \;\Rightarrow \\ &\eta_x = \tfrac{1}{1+F'(\eta)t} \quad \Rightarrow \quad u_x = \tfrac{u_0'(\eta)}{1+F'(\eta)t}. \end{aligned} \tag{107}$$

Therefore u_t and u_x are infinity when (104) holds.
3) *Impossibility to solve the first of equ.s (100) with respect to* η. From the Dini condition, it is not possible to solve the first of equations (100) when its partial derivative with respect to η is zero, and one obtains again (104).

Since t_b is the first time at which one has multivaluedness, then $-F'(\eta)$ must take its maximum value at $\eta = \eta_b$. It follows that the corresponding characteristic parameter η_b is defined by the equations

$$F'(\eta_b) < 0, \quad F''(\eta_b) = 0, \quad F'''(\eta_b) > 0. \tag{108}$$

Known η_b, then

$$t_b = -\frac{1}{F'(\eta_b)}, \quad x_b = \eta_b + F(\eta_b)t_b. \tag{109}$$

If, for instance, $c(u) = u$ ($F(\eta) = u_0(\eta)$) and $u_0(x) = \exp(-x^2)$, VERIFY THAT

$$\eta_b = 1/\sqrt{2}, \quad t_b = \sqrt{\frac{e}{2}}, \quad x_b = \sqrt{2}. \tag{110}$$

In general, the system of algebro-transcendental equations (100) cannot be solved in terms of elementary functions. In §3.2 we shall discuss two relevant cases that can be treated via elementary functions. In §3.3 we will see that, using a regular perturbation theory, it is possible to describe the breaking features of the solution near breaking in terms of elementary functions.

3.2 Compression and Rarefaction Waves

Consider the following Cauchy problem for compression and rarefaction waves of the Hopf equation:

$$u_t + uu_x = 0, \tag{111}$$

$$u_0(x) = \begin{cases} a_1, & x < -l, \\ \frac{a_2-a_1}{2l}x + \frac{a_1+a_2}{2}, & -l < x < l, \\ a_2, & x > l; \end{cases} \tag{112}$$

a rarefaction wave if $a_1 < a_2$; a compression wave if $a_1 > a_2$ (see Fig. 10).

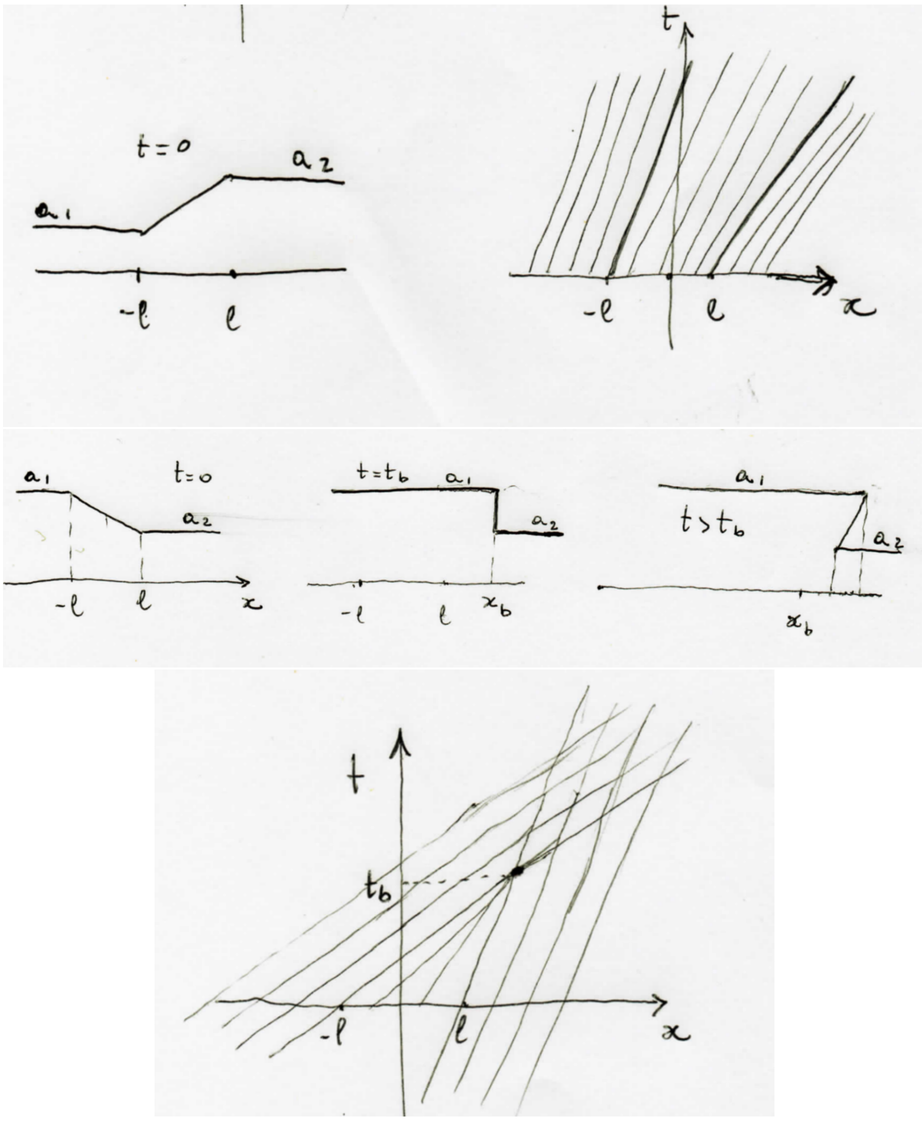


Figure 10: The first figure shows a rarefaction wave and the non intersecting characteristic curves; no breaking in this case. The second picture shows three snapshots of the evolution of a compression wave, before, at, and after t_b, and the third picture the corresponding intersecting characteristic curves.

We recall that the solution is:

$$x = u_0(\eta)t + \eta, \quad u = u_0(\eta) \tag{113}$$

For $\eta < -l$, $u_0(\eta) = a_1$; then $x = a_1 t + \eta$, $\Rightarrow$ $\eta = x - a_1 t < -l$, and $u = a_1$.

Therefore

$$u(x,t) = a_1, \quad x < a_1 t - l. \tag{114}$$

For $|\eta| < l$, $u_0(\eta) = \frac{a_2-a_1}{2l}\eta + \frac{a_1+a_2}{2}$; then

$$\begin{aligned} & x = [\tfrac{a_2-a_1}{2l}\eta + \tfrac{a_1+a_2}{2}]t + \eta, \ \Rightarrow \ \eta = \frac{x-\frac{a_1+a_2}{2}t}{1+\frac{a_2-a_1}{2l}t} \\ \Rightarrow \ & -l < \frac{x-\frac{a_1+a_2}{2}t}{1+\frac{a_2-a_1}{2l}t} < l, \ \Rightarrow \ x < a_2 t + l, \ x > a_1 t - l. \end{aligned} \tag{115}$$

Therefore

$$u = u_0(\eta) = \frac{a_2-a_1}{2l}\frac{x-\frac{a_1+a_2}{2}t}{1+\frac{a_2-a_1}{2l}t} + \frac{a_1+a_2}{2}, \ a_1 t - l < x < a_2 t + l \tag{116}$$

If $\eta > l$, $u_0(\eta) = a_2$; then $x = a_2 t + \eta \ \Rightarrow \ \eta = x - a_2 t > l \ \Rightarrow \ x > a_2 t + l$; therefore

$$u = a_2, \quad x > a_2 t + l. \tag{117}$$

Summarizing, the solution is

$$u(x,t) = \begin{cases} a_1, & x < a_1 t - l \\ \frac{a_2-a_1}{2l}\frac{x-\frac{a_1+a_2}{2}t}{1+\frac{a_2-a_1}{2l}t} + \frac{a_1+a_2}{2}, & a_1 t - l < x < a_2 t + l, \\ a_2, & x > a_2 t + l. \end{cases} \tag{118}$$

For $|\eta| < l$ then

$$t_b = -\frac{1}{u_0'(\eta)} = \frac{2l}{a_1 - a_2}. \tag{119}$$

In the rarefaction case $t_b < 0$; then there is no breaking and the wave is more and more rarefacted as time goes. In the compression case $t_b > 0$, then there is breaking, with

$$\begin{aligned} & |\eta_b| < l, \quad t_b = \tfrac{2l}{a_1-a_2} > 0, \\ & x_b = \left(\tfrac{a_2-a_1}{2l}\eta_b + \tfrac{a_1+a_2}{2}\right)\tfrac{2l}{a_1-a_2} + \eta_b = \tfrac{a_1+a_2}{a_1-a_2}\ l > l. \end{aligned} \tag{120}$$

In the limit $l \to 0$, the initial condition is discontinuous, the compression wave breaks immediately ($t_b = 0$), but the the rarefaction wave case becomes continuous for $t > 0$ and is described by (VERIFY IT):

$$u(x,t) = \begin{cases} a_1, & x < a_1 t \\ \frac{x}{t}, & a_1 t < x < a_2 t, \\ a_2, & x > a_2 t. \end{cases} \tag{121}$$

3.3 Study of the solution near breaking

Consider the evolution of a localized one-dimensional wave according to the Hopf equation

$$\begin{aligned} &u_t + uu_x = 0, \\ &u(x,0) = u_0(x) =: F(x), \quad x \in \mathbb{R}. \end{aligned} \tag{122}$$

We have seen that such evolution is described by the implicit equations

$$u = F(\xi), \quad \xi = x - F(\xi)t, \tag{123}$$

in which one solves the second equation with respect to ξ, obtaining $\xi = \xi(x,t)$, and substitute it into the first, to get the solution $u = F(\xi(x,t))$. The 1-dimensional (movable) Singularity Manifold is:

$$\mathcal{S}(\xi,t) = 1 + F_\xi(\xi)t = 0 \quad \Rightarrow \quad t = -\frac{1}{F_\xi(\xi)}. \tag{124}$$

Since

$$u_x = \frac{F_\xi}{1+tF_\xi}, \tag{125}$$

the wave breaks on the singularity manifold.

We are interested in the first time t_b in which the breaking of the solution occurs, corresponding to the characteristic values ξ_b such that

$$\begin{aligned} &t_b = t(\xi_b) = \text{global min}\{t(\xi)\} > 0 \quad \Rightarrow \\ &F_\xi(\xi_b) < 0, \quad F_{\xi\xi}(\xi_b) = 0, \quad F_{\xi\xi\xi}(\xi_b) > 0, \end{aligned} \tag{126}$$

ξ_b is an inflection point of the initial profile.

At t_b, the wave breaks in the point x_b of the x-axis defined by

$$x_b = F(\xi_b)t_b + \xi_b. \tag{127}$$

To study the solution (123) near breaking we introduce the variables:

$$\begin{aligned} &x = x_b + x', \quad t = t_b + t', \quad \xi = \xi_b + \xi', \\ &|x'|, |t'|, |\xi'| \ll 1. \end{aligned} \tag{128}$$

Subtracing the equations $\xi = x - F(\xi)t$ and $\xi_b = x_b - F(\xi_b)t_b$, expanding in terms of the small parameters x', t', ξ', and using $t_b = -1/F_\xi(\xi_b),\ F_{\xi\xi}(\xi_b) = 0$, we obtain, to leading order, the following cubic equation for ξ':

$$\xi'^3 + b(t')\xi' - \gamma X(x',t') \sim 0, \tag{129}$$

where

$$b(t') = \tfrac{6F_\xi}{t_b F_{\xi\xi\xi}}t', \quad X(x',t') = x' - Ft', \quad \gamma = \tfrac{6}{t_b F_{\xi\xi\xi}} \tag{130}$$

and from now on, unless specified, F and its derivatives are evaluated at ξ_b.

The cubic corresponds to the maximal balance

$$|X| = O(|t'|^{3/2}), \qquad |\xi'| = O(|t'|^{1/2}). \tag{131}$$

The three roots of this cubic are given explicitly by the well-known Cardano (Tartaglia, Ferro) formulas:

$$\begin{aligned} &\xi'_0\,(x', y', t') = (S_+)^{\frac{1}{3}} + (S_-)^{\frac{1}{3}}, \\ &\xi'_\pm\,(x', y', t') = \tfrac{1}{2}\left((S_+)^{\frac{1}{3}} + (S_-)^{\frac{1}{3}}\right) \pm \tfrac{\sqrt{3}}{2} i \left((S_+)^{\frac{1}{3}} - (S_-)^{\frac{1}{3}}\right), \end{aligned} \tag{132}$$

where

$$\begin{aligned} &S_\pm = R \pm \sqrt{\Delta}, \quad \Delta = R^2 + Q^3, \\ &Q(t') = \tfrac{b(t')}{3} = -\tfrac{2}{t_b^2 F_{\xi\xi\xi}} t', \quad R(x', t') = \tfrac{\gamma}{2} X(x', t') = \tfrac{\gamma}{2}(x' - Ft'). \end{aligned} \tag{133}$$

Expanding the SM equation one gets the parabola (see Fig. 11):

$$0 = \mathcal{S}(\xi, t) \sim F_\xi(\xi_b) t' + \frac{F_{\xi\xi\xi}(\xi_b)}{2} t_b \xi'^2 = \frac{t_b F_{\xi\xi\xi}}{2}(\xi'^2 + Q). \tag{134}$$

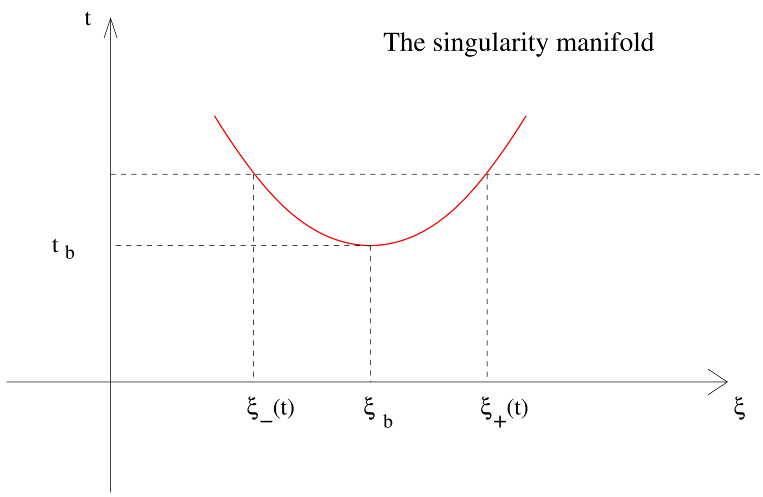


Figure 11: The singularity manifold near breaking.

Before breaking

If $t < t_b$ ($t' < 0$), then $b(t')$, $Q(t')$, Δ and $\mathcal{S}$ are strictly positive, and only the root ξ'_0 is real; correspondingly, the real solution of (122) is single valued:

$$u \sim F(\xi_b + \xi'_0(x', t')) \tag{135}$$

and the negative slope u_x of the profile, finite $\forall x$, reaches its minimum at the inflection point $x_f(t')$:

$$x_f(t') = x_b + F(\xi_b)t' \;\Rightarrow\; X = x - x_f(t')). \tag{136}$$

Since, at $x = x_f(t')$, $X = 0$, then $\xi'_0 = 0$ and

$$u(x_f(t), t) = F(\xi_b), \quad u_x(x_f(t), t) \sim \tfrac{1}{t-t_b}, \quad u_{xx}(x_f(t), t) = 0. \tag{137}$$

Indeed we have the formulas

$$\begin{aligned}
&\xi_x = \tfrac{1}{S} \sim \tfrac{1}{F_\xi t' + \frac{F_{\xi\xi\xi}}{2} t_b \xi'}, \quad \xi_{xx} = -F_{\xi\xi}(\xi)\xi_x^3 t, \quad u_x = F'(\xi)\xi_x \sim \tfrac{1}{t' + \frac{F_{\xi\xi\xi}}{2F_\xi} t_b \xi'}, \\
&u_{xx} = F_{\xi\xi}(\xi)\xi_x^2 + F'(\xi)\xi_{xx} = F_{\xi\xi}(\xi)\xi_x^3 \sim \tfrac{F_{\xi\xi\xi}}{(F_\xi t' + \frac{F_{\xi\xi\xi}}{2} t_b \xi')^3}\xi',
\end{aligned} \tag{138}$$

that evaluated at $\xi' = \xi'_0 = 0$, give (137).

To analyse the solution in a smaller region around the inflection point, we choose

$$|X| = |x - x_f(t')| = O(|t'|^{p+\frac{1}{2}}), \quad p > 1, \tag{139}$$

Then ${\xi'}^3 << b\xi' \sim -\gamma X$ and the solution becomes more explicit:

$$\xi'_0 \sim \frac{\gamma X}{b} = \frac{x' - Ft'}{F_\xi t'} \tag{140}$$

reducing to the exact similarity solution of the Hopf equation:

$$u \sim F(\xi_b + \xi'_0) \sim F + F_\xi \xi'_0 \sim \frac{x - x_b}{t - t_b}, \tag{141}$$

describing the tangent to the profile at the inflection point (see Fig. 12), with

$$u_x \sim (t - t_b)^{-1}. \tag{142}$$

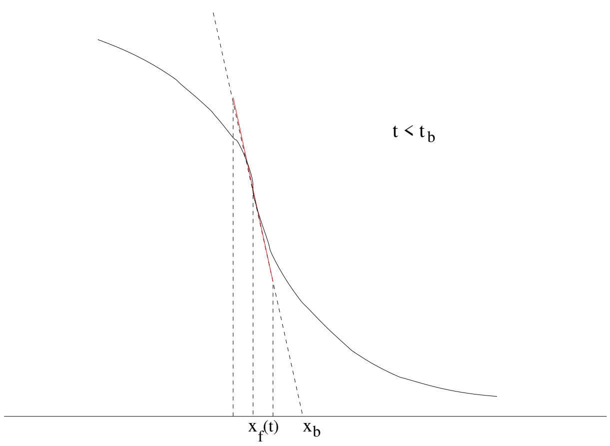


Figure 12: The profile immediately before breaking.

At breaking

In the limit $t \uparrow t_b$,

i) the inflection point reaches the breaking point: $x_f(t) \to x_b$, and the tangent to the inflection point becomes the vertical line $x = x_b$.

ii) the cubic becomes ${\xi'}^3 \sim \gamma X(x', t') = 0$ whose solution reads:

$$\xi' = \sqrt[3]{\gamma(x - x_b)}, \tag{143}$$

and, correspondingly,

$$u \sim F\left(\xi_b + \sqrt[3]{\gamma(x - x_b)}\right), \quad u_x \sim \frac{\sqrt[3]{\gamma}}{3} \frac{F_\xi}{(x - x_b)^{2/3}}, \tag{144}$$

describing the typical vertical inflection at $t = t_b$, in the neighborhood of x_b (see Fig. 13):

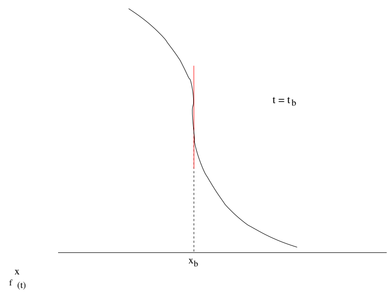


Figure 13: The profile at breaking.

After breaking

For $t > t_b$, the line $t = const,\ t > t_b$ intersects the SM in the two points (see Fig. 11)

$$\xi_\pm(t) - \xi_b = \pm\sqrt{\tfrac{2|F_\xi|}{t_b F_{\xi\xi\xi}}(t - t_b)} = \pm\sqrt{|Q(t')|}. \tag{145}$$

and, correspondingly, $\mathcal{S} \le 0$ for $\xi_-(t) \le \xi' \le \xi_+(t)$. In addition $Q(t') < 0$ and the discriminant Δ can be positive or negative, depending on the space-time regions we consider. When $\Delta < 0$ the cubic has three real roots and we have multivaluedness:

$$\Delta = R^2 + Q^3 = \frac{\gamma^2}{4}(x' - Ft')^2 - \frac{8}{t_b^6 F_{\xi\xi\xi}^3} {t'}^3 \le 0. \tag{146}$$

It follows that the multivaluedness region is given by

$$\begin{aligned} &\Delta \le 0 \ \Leftrightarrow \ x^-(t) \le x \le x^+(t), \\ &x^\pm(t') = x_b + F(\xi_b)(t - t_b) \pm \tfrac{2\sqrt{2}}{3t_b^2\sqrt{F_{\xi\xi\xi}}}(t - t_b)^{3/2}. \end{aligned} \tag{147}$$

We conclude that, for $t > t_b$, the solution is three-valued for $x \in [x^-(t'), x^+(t')]$:

$$u_0(x,t) = F(\xi_b + \xi'_0(x',t')), \quad u_\pm(x,t) = F(\xi_b + \xi'_\pm(x',t')). \tag{148}$$

and single valued outside. At the end points $x^\pm(t')$ of the interval $\Delta = 0$, and two of the three solutions coincide (see Fig. 14).

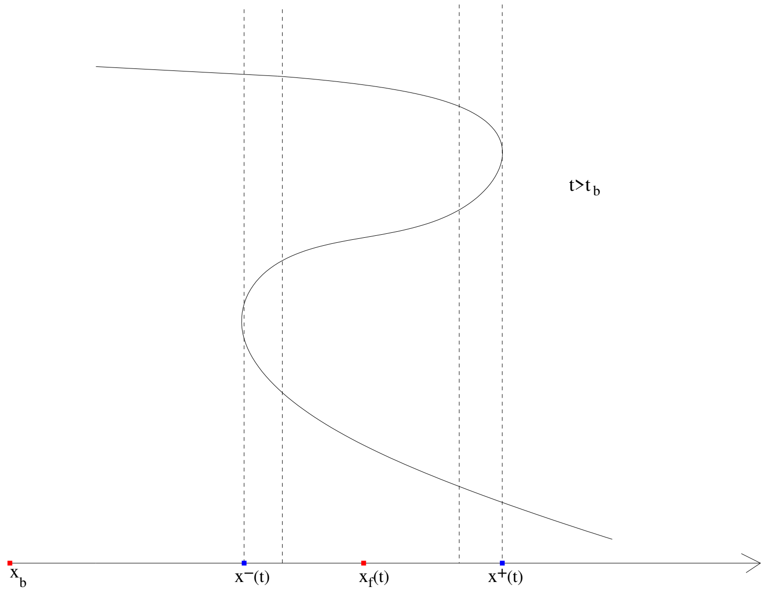


Figure 14: The profile immediately after breaking, and the interval $[x^-(t'), x^+(t')]$ in which the solution is three valued ($\Delta \leq 0$).

The movable singularity manifold presents several universal features. Correspondingly, also the solution of the Cauchy problem for the Hopf equation presents universality features near the singularity manifold.

3.4 Geometric Meaning of the scalar hyperbolic PDE in arbitrary dimensions

Now we consider a scalar quasi-linear PDE in $M \geq 2$ dimensions

$$\sum_{j=1}^{M} P_j(\underline{x}, u) u_{x_j} = Q(\underline{x}, u), \quad \underline{x} = (x_1, \ldots, x_M), \tag{149}$$

or, in vector form:

$$\vec{P}(\underline{x}, u) \cdot \nabla_{\underline{x}} u = Q(\underline{x}, u). \tag{150}$$

If

$$\varphi(\underline{x}, u) = c = const \tag{151}$$

defines implicitly a solution of (149), and if $\partial\varphi/\partial u \neq 0$, then we can in principle solve (151) with respect to u, obtaining

$$u = u(\underline{x}, c). \tag{152}$$

This solution defines the M-dimensional "integral hypersurface S of (149) in the space $(\underline{x}, u) = \mathbb{R}^{M+1}$.

Since

$$D_{x_i}\varphi = \frac{\partial\varphi}{\partial x_i} + \frac{\partial\varphi}{\partial u} u_{x_i} = 0, \quad i = 1, \ldots, M, \tag{153}$$

in vector form:

$$\nabla_{\underline{x}}\varphi + \frac{\partial\varphi}{\partial u} \nabla_{\underline{x}}\, u = \underline{0}, \tag{154}$$

we have that $\nabla_{\underline{x}}\varphi \parallel \nabla_{\underline{x}} u$.

If $\underline{V} = (\vec{P}, Q)$, then equation (150) (with (154)) becomes

$$\underline{V} \cdot \nabla_{(\underline{x},u)}\varphi = 0. \tag{155}$$

Since $\nabla_{(\underline{x},u)}\varphi$ is normal to S, it follows that $\underline{V}$ is tangent to the integral hypersurface S at the point $\underline{r} = (\underline{x}, u) \in S$, defining a direction on S. Moving along that direction, one constructs a "characteristic curve", always tangent to $\underline{V}$ (see Fig. 15).

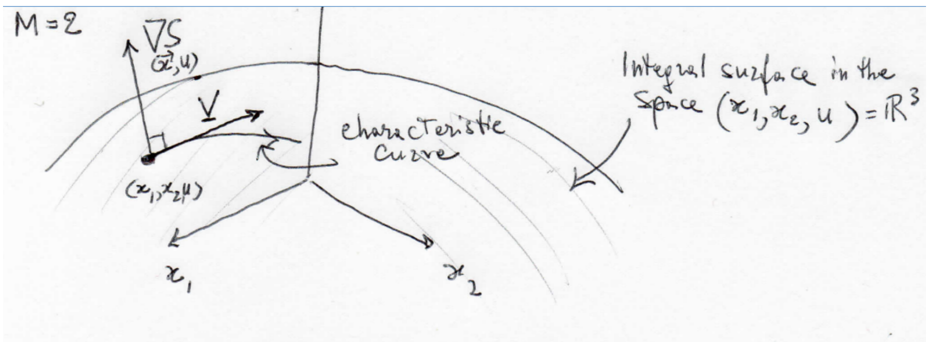


Figure 15: The geometric meaning of the hyperbolic PDE

If s is the arc length parameter of the characteristic curve, then $d\underline{r}/ds$ is tangent to S and parallel to $\underline{V} = (\vec{P}, Q)$; then $d\underline{r}/ds \parallel \underline{V}$. Therefore

$$\begin{aligned} \tfrac{dx_j}{ds} &= \tfrac{1}{\mu}P_j, \quad j = 1, \ldots, M, \\ \tfrac{du}{ds} &= \tfrac{1}{\mu}Q, \end{aligned} \tag{156}$$

Therefore the PDE (149) is equivalent to the following system of M independent ODEs

$$\frac{dx_j}{P_j(\underline{x}, u)} = \frac{du}{Q(\underline{x}, u)} \left(= \frac{ds}{\mu} \right), \quad j = 1, \ldots, M, \tag{157}$$

in the philosophy of the characteristics method. It is easy to verify that these equations reduce to the characteristic equations (97) if $M = 2$.

Examples. Find the general solution of the following equations.
1. $xu_x + yu_y = u$.

Equations (157) become

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}, \tag{158}$$

implying

$$\begin{cases} \ln y = \ln x + c \\ \ln u = \ln x + d \end{cases} \Rightarrow \begin{cases} y = c_1 x \\ u = c_2 x \end{cases} \tag{159}$$

One can write $c_2 = f(c_1)$, where $f(\cdot)$ is an arbitrary function; then

$$u = x\ f\left(\frac{y}{x}\right) = y\ g\left(\frac{y}{x}\right) \tag{160}$$

is the general solution (with $f(\xi) = \xi g(\xi)$).

2. $u_t + uu_x = 1$.
Equations (157) become

$$dt = \frac{dx}{u} = du \ \Rightarrow \ \frac{dx}{dt} = u, \ \frac{du}{dt} = 1, \tag{161}$$

The second equation implies $u = t + b$, and the first equation becomes $dx/dt = t + b$, implying

$$\begin{array}{l} x = \frac{t^2}{2} + bt + c = \frac{t^2}{2} + (u - t)t + c = ut - \frac{t^2}{2} + c, \\ u = t + b = t + f(c). \end{array} \tag{162}$$

Then

$$u = t + f\left(x - ut + \frac{t^2}{2}\right) \tag{163}$$

is the general solution, where $f(\cdot)$ is an arbitrary function.

3.5 Hyperbolic System of PDEs

Consider the system of N quasi linear 1st order PDEs:

$$u_{it} + \sum_{j=1}^{N} C_{ij}(\vec{u}) u_{j_x} + h_i(\vec{u}) = 0, \quad i = 1, \ldots, N \tag{164}$$

that can be written in vector form as follows

$$\vec{u}_t + C(\vec{u})\vec{u}_x + \vec{h}(\vec{u}) = \vec{0}, \tag{165}$$

$$\vec{u}(x,t) = \begin{pmatrix} u_1(x,t) \\ \vdots \\ u_N(x,t) \end{pmatrix}, \ \vec{h}(\vec{u}) = \begin{pmatrix} h_1(\vec{u}) \\ \vdots \\ h_N(\vec{u}) \end{pmatrix} \in \mathbb{R}^N. \tag{166}$$

where $C(\vec{u})$ is the $N \times N$ matrix of components $C_{ij}(\vec{u})$.

Construct eigenvalues $c(\vec{u})$ and left (row) eigenvectors $\underline{L}(\vec{u})$ of the matrix $C(\vec{u})$, and suppose that there are N independent eigenvectors:

$$\begin{aligned} &\underline{L}^{(k)}(\vec{u})C(\vec{u}) = c_k(\vec{u})\underline{L}^{(k)}(\vec{u}), \quad k = 1, \ldots, N, \\ &\underline{L}^{(k)} = (L_1^{(k)}, \ldots, L_N^{(k)}). \end{aligned} \tag{167}$$

If one takes the (real) scalar product of these left eigenvectors times the vector equation (165), using (167), one obtains N scalar equations:

$$\begin{aligned} &\underline{L}^{(k)}(\vec{u}) \cdot \left(\vec{u}_t + C(\vec{u})\vec{u}_x + \vec{h}(\vec{u})\right) \\ &= \underline{L}^{(k)}(\vec{u}) \cdot \left(\vec{u}_t + c_k(\vec{u})\vec{u}_x + \vec{h}(\vec{u})\right) = 0, \quad k = 1, \ldots, N \end{aligned} \tag{168}$$

that can be written in the characteristic form:

$$\underline{L}^{(k)}(\vec{u}) \cdot \left(\frac{d_k \vec{u}}{dt} + \vec{h}(\vec{u})\right) = 0, \quad k = 1, \ldots, N \tag{169}$$

on the characteristics defined by

$$\frac{d_k x}{dt} = c_k(\vec{u}), \quad k = 1, \ldots, N. \tag{170}$$

Therefore, for each k, we have a system of two ODEs defined on the characteristic $c_k(\vec{u})$. In components:

$$\sum_{i=1}^{N} L_i^{(k)} \left(\frac{du_i}{dt} + h_i\right) = 0 \quad \text{on} \quad \frac{d_k x}{dt} = c_k(\vec{u}). \tag{171}$$

We are finally ready to give the definition of hyperbolic system of PDEs.
Definition. The system of N quasi linear PDEs (165) is hyperbolic if the eigenvalue equation (167) has only real eigenvalues (not necessarily distinct) and N independent left (row) eigenvectors.

Remarks.
1) Since $\vec{u}$, $\vec{h}$, and matrix $C(\vec{u})$ are real, if the eigenvalues are real, then also the eigenvectors are real.
2) If $C(\vec{u})$ is a symmetric matrix, then the system is hyperbolic.
3) The existence of N independent eigenvectors implies that the N informations associated with the N real fields $u_i's$ propagate on the N characteristics associated with the N real eigenvalues. Some of the eigenvalues may coincide, corresponding to the situation in which some of the informations may travel on the same characteristic curve.
Example: The equations of the gas dynamics.

$$\begin{array}{l} \rho_t + u\rho_x + \rho u_x = 0, \\ u_t + uu_x + \frac{1}{\rho}\, p_x = 0, \\ S_t + uS_x = 0, \\ p = p(\rho, S), \quad \text{equation of state of the gas,} \end{array} \tag{172}$$

where ρ is the density, u is the gas velocity, p is the pressure and S is the entropy.

It might be convenient to eliminate ρ using the equation of state. From

$$p_t = \frac{\partial p}{\partial \rho}\rho_t + \frac{\partial p}{\partial S}S_t, \quad p_x = \frac{\partial p}{\partial \rho}\rho_x + \frac{\partial p}{\partial S}S_x, \tag{173}$$

it follows that

$$\rho_t = \frac{1}{a^2}\left(p_t - \frac{\partial p}{\partial S}S_t\right), \ \rho_x = \frac{1}{a^2}\left(p_x - \frac{\partial p}{\partial S}S_x\right), \tag{174}$$

where

$$a^2 = \frac{\partial p}{\partial \rho} > 0. \tag{175}$$

Substituting (174) in (172), on obtains the new form of gas equation:

$$\begin{array}{l} p_t + up_x + \rho a^2 u_x = 0, \\ u_t + uu_x + \frac{1}{\rho}\, p_x = 0, \\ S_t + uS_x = 0, \end{array} \tag{176}$$

that can be written in the form (165) with:

$$\vec{u} = \begin{pmatrix} p \\ u \\ S \end{pmatrix}, \quad C(\vec{u}) = \begin{pmatrix} u & \rho a^2 & 0 \\ 1/\rho & u & 0 \\ 0 & 0 & u \end{pmatrix}, \quad \vec{h} = \vec{0}. \tag{177}$$

VERIFY THAT the three eigenvalues of C are

$$c_0 = u, \quad c_\pm = u \pm a \tag{178}$$

and the corresponding independent left eigenvectors are

$$\underline{L}_0 = (0, 0, 1), \quad \underline{L}_\pm = (1, \pm a\rho, 0). \tag{179}$$

We conclude that the gas dynamics equations are hyperbolic, and can be written in the following characteristic form:

$$\begin{array}{l} \frac{dS}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = u, \\ \frac{dp}{dt} \pm \rho a \frac{du}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = u \pm a \end{array} \tag{180}$$

We infer that the entropy travels on the characteristics generated by the gas velocity, and that $a = \sqrt{\partial p / \partial \rho} > 0$ has the meaning of velocity of the sound.

Exercise. (DO IT) Solve the gas dynamics equations near the constant equilibrium state: $\rho = \rho_0, \ p = p_0, \ u = 0, \ S = S_0$; namely in the case:

$$\begin{array}{l} \rho = \rho_0 + \epsilon \rho_1, \quad p = p_0 + \epsilon p_1, \\ u = \epsilon u_1, \quad S = S_0 + \epsilon S_1, \quad 0 < \epsilon \ll 1. \end{array} \tag{181}$$

SHOW that (180) becomes

$$\begin{array}{l} \frac{dS_1}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} \sim 0, \\ \frac{d(p_1 \pm \rho_0 a_0 u_1)}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = \pm a_0, \end{array} \tag{182}$$

where

$$a_0 = \sqrt{\frac{\partial p}{\partial \rho}(\rho_0, S_0)} \tag{183}$$

is the constant sound speed of the linearized theory, implying that

$$\begin{array}{l} p_1 \pm \rho_0 a_0 u_1 = f_\pm(\eta_\pm) \quad \text{on} \quad x = \pm a_0 t + \eta_\pm, \\ S_1 = g(x), \end{array} \tag{184}$$

where $f_\pm, g$ are arbitrary functions to be fixed through the initial data. At last we obtain the general solution:

$$\begin{array}{l} p - p_0 = \frac{\epsilon}{2} \left[f_+(x - a_0 t) + f_-(x + a_0 t) \right] + O(\epsilon^2), \\ u = \frac{\epsilon}{2\rho_0 a_0} \left[f_+(x - a_0 t) - f_-(x + a_0 t) \right] + O(\epsilon^2), \\ S = S_0 + \epsilon g(x) + O(\epsilon^2). \end{array}$$

3.6 Riemann invariants

The obtained characteristic form (169),(170)

$$\begin{array}{l} \underline{L}^{(k)}(\vec{u}) \cdot \left(\frac{d_k \vec{u}}{dt} + \vec{h}(\vec{u})\right) = 0, \quad k = 1, \ldots, N, \\ \frac{d_k x}{dt} = c_k(\vec{u}), \quad k = 1, \ldots, N. \end{array} \tag{185}$$

of the hyperbolic system is very complicated, since it couples all the fields. Is it possible to simplify further the equation? In particular, is it possible to find a change of variables $\vec{u} \to \vec{r}(\vec{u}) = (r_1(\vec{u}), \ldots, r_N(\vec{u}))^T$ through which the dynamics is decoupled in the form

$$\begin{array}{l} \frac{dr_k}{dt} + \tilde{f}_k(\vec{r}) = 0, \quad k = 1, \ldots, N, \\ \frac{dx}{dt} = c_k(\vec{u}) = \tilde{c}_k(\vec{r}) \end{array} \tag{186}$$

If so, we have found the Riemann variables $\vec{r}$, the **Riemann invariants** if $\tilde{f}_k = 0, \ k = 1, \ldots, N$.

Comparing (185) and (186) it follows that the Riemann variables can be found if the differentials $\underline{L}^{(k)} \cdot d\vec{u}, \ k = 1, \ldots, N$ are exact, i.e, if there exist $2N$ functions $r_k(\vec{u}), \ \lambda_k(\vec{u}), \ k = 1, \ldots, N$ such that:

$$\sum_{i=1}^{N} L_i^{(k)}(\vec{u}) du_i = \lambda_k dr_k \ (= \lambda_k \sum_{i=1}^{N} \frac{\partial r_k(\vec{u})}{\partial u_i} du_i), \tag{187}$$

implying the following N^2 equations for the $2N$ unknowns $r_k, \lambda_k, \ k = 1, \ldots, N$

$$L_i^{(k)}(\vec{u}) = \lambda_k(\vec{u}) \frac{\partial r_k(\vec{u})}{\partial u_i}, \quad i, k = 1, \ldots, N. \tag{188}$$

If $N = 2$, we have 4 linear PDEs for the 4 unknowns $r_k, \lambda_k, \ k = 1, \ldots, 2$, that can be manipulated to the form

$$\begin{array}{l} \left(\frac{L_1^{(k)}(\underline{u})}{\lambda_k}\right)_{u_2} = \left(\frac{L_2^{(k)}(\underline{u})}{\lambda_k}\right)_{u_1}, \quad k = 1, 2, \\ L_1^{(k)}(\underline{u}) \frac{\partial r_k}{\partial u_2} = L_2^{(k)}(\underline{u}) \frac{\partial r_k}{\partial u_1}, \quad k = 1, 2, \end{array} \tag{189}$$

and the Riemann variables can always be constructed in principle; if $N > 2$, $N^2 > 2N$ and we have an overdetermined system of N^2 equations for the $2N$ unknowns $r_k, \lambda_k, \ k = 1, \ldots, N$. The construction of these unknowns is not possible, in general, unless the $L_i^{(k)}$'s are suitably constrained.

Example: the isentropic gas. Let us consider, as a basic illustrative example, the case of the gas dynamics equations under the hypothesis of constant entropy $S = S_0$ (the isentropic gas). In this case we have only the two variables p, u and the construction of the Riemann invariants is possible. We recall that the characteristic form of the equations is

$$\tfrac{dp}{dt} \pm \rho a \tfrac{du}{dt} = 0 \quad \text{on} \quad \tfrac{dx}{dt} = u \pm a. \tag{190}$$

It is more convenient to use here the density ρ; since $p = p(\rho, S_0)$, then $dp/dt = a^2 d\rho/dt$, and equations (190) become

$$\frac{a(\rho)}{\rho}\frac{d\rho}{dt} \pm \frac{du}{dt} = 0, \quad \text{on} \quad \frac{dx}{dt} = u \pm a. \tag{191}$$

The $N^2 = 4$ equations (188) for the $2N = 4$ unknowns $r_\pm, \lambda_\pm$ read

$$\begin{aligned} \tfrac{a(\rho)}{\rho} &= \lambda_\pm \tfrac{\partial r_\pm}{\partial \rho}, \\ \pm 1 &= \lambda_\pm \tfrac{\partial r_\pm}{\partial u}, \end{aligned} \tag{192}$$

defining the Riemann invariants

$$r_\pm = \int^\rho \tfrac{a(\rho')}{\rho'} d\rho' \pm u, \tag{193}$$

with $\lambda_\pm = \pm 1$ (VERIFY IT). We conclude that the equations for an isoentropic gas decouple in the form

$$\frac{dr_\pm}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = u \pm a. \tag{194}$$

If the gas is polytropic:

$$p = \kappa \rho^\gamma \tag{195}$$

(in the case of an adiabatic transformation: $\gamma = c_p/c_v > 1$, where c_p and c_v are respectively the constant pressure and constant volume specific heats), then $a^2 = \partial p/\partial \rho = \kappa\gamma\rho^{\gamma-1}$ and (VERIFY IT)

$$r_\pm = \frac{2\sqrt{\kappa\gamma}}{\gamma - 1}\rho^{\frac{\gamma-1}{2}} \pm u = \frac{2}{\gamma - 1} a \pm u. \tag{196}$$

3.7 Exercises

1) Show that the following linear PDE for the field $\rho(x, t)$:

$$\rho_t + c(x,t)\rho_x + a(x,t)\rho = b(x,t) \tag{197}$$

is equivalent to the system of two ODEs:

$$\begin{aligned} \tfrac{d\rho}{dt} &+ a(x,t)\rho = b(x,t), \\ \tfrac{dx}{dt} &= c(x,t). \end{aligned} \tag{198}$$

2) Find the general solution of the following linear PDEs:

$$\begin{array}{l} u_t + t^2 u_x + xu = 0, \quad \left(u = F(x - t^3/3)e^{-(t^4/12 + t(x - t^3/3))}\right), \\ i\gamma u_t + yu_x - xu_y = 0, \quad (...), \\ yu_x - xu_y = 0, \quad \left(u = F(x^2 + y^2)\right), \\ yu_x + xu_y = 0, \quad \left(u = F(x^2 - y^2)\right), \\ xu_x + yu_y = 0, \quad (u = F(y/x)), \\ xu_x - yu_y = 0, \quad (u = F(xy)), \\ xu_x + yu_y = x^2, \quad \left(u = x^2/2 + F(y/x)\right), \\ xu_x + yu_y = u, \quad (u = xF(y/x)), \\ xu_x + yu_y + zu_z = 0, \quad (u = F(y/x, z/x)), \\ g_y u_x - g_x u_y = 0, \quad g(x, y) \text{ given}, \quad (u = F(g(x, y))) \end{array} \tag{199}$$

3) Find the general solution of the following quasi-linear PDEs:

$$\begin{array}{l} i) \;\; u_t + c(u)u_x = 0, \quad u = F(x - c(u)t), \\ ii) \;\; u_t + c(u)u_x = 1, \\ c(u) = u \quad \Rightarrow \quad u = t + F(x - ut + t^2/2), \\ c(u) = u^2 \quad \Rightarrow \quad u = t + F(x - u^2 t + ut^2 - t^3/3) \end{array} \tag{200}$$

4) Given the two Cauchy problems for the Hopf equation:

$$\begin{array}{l} u_t + uu_x = 0, \quad u = u(x, t), \quad x \in \mathbb{R}, \; t \geq 0, \\ i) \;\; u(x, 0) = e^{-x^2}, \\ ii) \;\; u(x, 0) = (x^2 + 1)^{-1}, \end{array} \tag{201}$$

i) draw the 1-parameter family of characteristic curves; ii) find the first characteristic parameter η_b and the first breaking point (x_b, t_b).
A. i) $\eta_b = 1/\sqrt{2}, \; t_b = \sqrt{e/2}, \; x_b = \sqrt{2}$. ii) $\eta_b = 1/\sqrt{3}, \; t_b = 8\sqrt{3}/9, \; x_b = \sqrt{3}$.

5) Compression and rarefaction waves.
Consider the Cauchy problem:

$$\begin{array}{l} u_t + uu_x = 0, \quad u = u(x, t), \quad x \in \mathbb{R}, \; t \geq 0, \\ u(x, 0) = a_1 H(-l - x) + a_2 H(x - l) + H(l^2 - x^2)\left(\frac{a_1 + a_2}{2} - \frac{a_1 - a_2}{2} x\right), \end{array} \tag{202}$$

in the two cases

$$\begin{array}{l} i) \;\; a_1 > a_2 > 0, \quad \text{compression wave}, \\ ii) \;\; a_2 > a_1 > 0 \quad \text{rarefaction wave}. \end{array} \tag{203}$$

Solve it explicitely, draw the characteristic curves and show that they describe respectively a compression and a rarefaction wave. Indicate if there is wave breaking and, if so, find η_b and (x_b, t_b).
A. For the compression wave:

$$u(x, t) = \begin{cases} a_1, & x < a_1 t - l, \\ -\frac{a_1 - a_2}{2l} \frac{x - \frac{a_2 + a_1}{2} t}{1 - \frac{a_1 - a_2}{2l} t} + \frac{a_2 + a_1}{2}, & -l + a_1 t < x < l + a_2 t, \\ a_2, & x > l + a_2 t. \end{cases} \tag{204}$$

There is wave breaking:

$$t_b = \frac{2l}{a_1 - a_2}, \quad x_b = \frac{a_1 + a_2}{a_1 - a_2} l, \quad |\eta_b| < 1 \tag{205}$$

6) Consider the Cauchy problem

$$\begin{array}{l} u_t + uu_x = 0, \\ u(x, 0) = f(x), \end{array} \tag{206}$$

where f describes a single localized bump, and study analytically the behavior of the solution near breaking (immediately before, at, and immediately after breaking). See section §2.3.

7) More on rarefaction waves.
i) Show that the solution of the Cauchy problem

$$u_t + uu_x = 0, \quad u(x,0) = a_1 H(-x) + a_2 H(x), \quad a_1 < a_2 \tag{207}$$

is given by

$$u = \begin{cases} a_1, & x < a_1 t, \\ x/t, & a_1 t < x < a_2 t, \\ a_2, & x > a_2 t \end{cases} \tag{208}$$

Hint. Observe that this Cauchy problem can be viewd as the $l \to 0$ limit of that of the previous problem. But there are other ways of doing it ...
ii) Show that the solution of the Cauchy problem

$$u_t + c(u)u_x = 0, \quad u(x,0) = a_1 H(-x) + a_2 H(x), \quad a_1 < a_2 \tag{209}$$

is given by

$$u = \begin{cases} a_1, & x < c(a_1)t, \\ c^{-1}(x/t), & c(a_1)t < x < c(a_2)t, \\ a_2, & x > c(a_2)t \end{cases} \tag{210}$$

where $c^{-1}(\xi)$ is the inverse of function $c(u)$.

8) Given the following system of PDEs, establish if they are hyperbolic and, if so, write them in characteristic form.

i) The wave equation $u_{tt} - c^2 u_{xx} = 0$.
ii) The Klein - Gordon equation $u_{tt} - c^2 u_{xx} + u = 0$.
iii) The system

$$\begin{array}{l} u_t + c(u,v)u_x = 0, \\ v_t + c(u,v)v_x = u \end{array} \tag{211}$$

iv) The system

$$\begin{array}{l} u_t + c(u)u_x = 0, \\ v_t + c(u)v_x + c'(u)vu_x = 0 \end{array} \tag{212}$$

v) The gas dynamics equations

$$\begin{array}{l} \rho_t + u\rho_x + \rho u_x = 0, \\ u_t + uu_x + \frac{p_x}{\rho} = 0, \\ S_t + uS_x = 0, \end{array} \tag{213}$$

where $p = p(\rho, S)$.
R. i)

$$\begin{array}{l} \frac{d}{dt}(w - cv) = 0, \quad \frac{dx}{dt} = c, \quad \Rightarrow \quad w - cv = A(x - ct), \\ \frac{d}{dt}(w + cv) = 0, \quad \frac{dx}{dt} = -c, \quad \Rightarrow \quad w + cv = B(x + ct), \\ v \equiv u_x, \ w \equiv u_t \end{array} \tag{214}$$

implying the well-known result $u = f(x - ct) + g(x + ct)$, with

$$f'(\cdot) = -\frac{1}{2c}A(\cdot), \quad g'(\cdot) = \frac{1}{2c}B(\cdot). \tag{215}$$

ii)

$$\begin{array}{l} \varphi_t - c\varphi_x + u = 0, \\ u_t + cu_x - \varphi = 0, \\ \varphi \equiv u_t + cu_x. \end{array} \tag{216}$$

iii) it is already in characteristic form, with the single characteristic $dx/dt = c(u,v)$ and two different characteristic forms (two different eigenvectors $(1,0)$ and $(0,1)$).
iv) The first equation is in characteristic form for the single field u; the second one cannot be put in

characteristic form; therefore the system is not hyperbolic. Nevertheless it can be solved solving first the first equation, hyperbolic, on the characteristic $dx/dt = c(u)$, and then solving the second one on that characteristic (do it!).
v) Rewrite (213) in the form

$$\begin{array}{l} p_t + up_x + \rho a^2 u_x = 0, \\ u_t + uu_x + \frac{p_x}{\rho} = 0, \\ S_t + uS_x = 0, \end{array} \tag{217}$$

where $a^2(\rho) = \partial p/\partial\rho > 0$, obtaining the following eigenvalues and eigenvectors:

$$\begin{array}{ll} c_0 = u \text{ (gas speed)}, & \underline{L}_0 = (0,0,1), \\ c_\pm = u \pm a \text{ (sound speeds)}, & \underline{L}_\pm = (1, \pm a\rho, 0). \end{array} \tag{218}$$

Therefore the system in characteristic form reads:

$$\begin{array}{l} \frac{dp}{dt} \pm \rho a \frac{du}{dt} = 0, \quad \frac{dx}{dt} = u \pm a, \\ \\ \frac{dS}{dt}, \quad \frac{dx}{dt} = u. \end{array} \tag{219}$$

Verify that, in the linear limit in which we study small perturbations of the constant solution:

$$\begin{array}{l} \rho = \rho_0 + \epsilon\rho_1(x,t) + O(\epsilon^2), \quad p = p_0 + \epsilon p_1(x,t) + O(\epsilon^2), \\ u = \epsilon u_1(x,t) + O(\epsilon^2), \quad S = S_0 + \epsilon S_1(x,t) + O(\epsilon^2), \end{array} \tag{220}$$

we obtain

$$\begin{array}{l} p = p_0 + \epsilon[f_-(x - a_0 t) + f_+(x + a_0 t)] + O(\epsilon^2), \\ u = \frac{\epsilon}{a_0\rho_0}[f_-(x - a_0 t) - f_+(x + a_0 t)] + O(\epsilon^2), \\ S = S_0 + \epsilon g(x) + O(\epsilon^2), \end{array} \tag{221}$$

where $a_0 = \sqrt{\partial p(\rho_0, S_0)/\partial\rho}$, and the functions $f_\pm$ and g are arbitrary.

9) Show that i) the Riemann invariants of the wave equation $u_{tt} - c^2 u_{xx} = 0$, $c > 0$ are given by $r_\pm = w \mp cv$, where $v = u_x$ and $w = u_t$, so that the PDE is written as the system of ODEs in characteristic form:

$$\frac{dr_\pm}{dt} = 0, \quad \frac{dx}{dt} = \pm c. \tag{222}$$

ii) The Riemann invariants of the gas dynamics equations (213) (under the constant entropy S hypothesis) are given by

$$r_\pm = \int^\rho \frac{a(\rho')}{\rho'} d\rho' \pm u, \tag{223}$$

where $a^2(\rho) = p'(\rho) > 0$, so that the system (219) decouples as follows:

$$\frac{dr_\pm}{dt} = 0, \quad \frac{dx}{dt} = u \pm a(\rho). \tag{224}$$

Show that, for an adiabatic process ($p = \kappa\rho^\gamma$),

$$\begin{array}{l} a^2 = \kappa\gamma\rho^{\gamma-1}, \\ r_\pm = \frac{2\sqrt{\kappa\gamma}}{\gamma-1}\rho^{\frac{\gamma-1}{2}} \pm u = \frac{2a}{\gamma-1} \pm u. \end{array} \tag{225}$$

4 Regularization of hyperbolic waves [36]

If a wave breaking of the profile with a subsequent multivaluedness is acceptable when we study the amplitude of a water wave near the shore, it is not acceptable if one studies other wave phenomena, like the sound propagation. In this case the field is a density, or a pressure, and it cannot have more than one value in a space-time point. It means that the model equation we used is not adequate to describe the phenomenon near breaking. There are two ways to deal with the problem.

1) **Regularization of the solution**. One introduces "weak solutions", substituting the continuous multivalued solution with a discontinuous BUT single valued solution (the so-called "shock wave").

2) **Regularization of the model**. One improves the model equation, introducing suitable corrective terms. If the model is the Riemann equation, then the corrective terms are usually u_{xx}, in the presence of dissipation or diffusion, and u_{xxx} in the presence of dispersion.

4.1 Regularization of the solution and shock waves

We begin with the first way, going back to the physical motivation of the Riemann equation (88), the conservation law in integral form

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)dx = q(x_1,t) - q(x_2,t) \tag{226}$$

for the density u and the flux q.

If there is a discontinuity of u at $x = S(t)$ (the shock trajectory in space-time) then (226) must be rewritten as

$$\begin{aligned} q(x_1,t) - q(x_2,t) &= \tfrac{d}{dt}\left(\int_{x_1}^{S(t)} + \int_{S(t)}^{x_2}\right) u(x,t)dx \\ &= \left[u(S^-,t) - u(S^+,t)\right]\dot{S}(t) + \left(\int_{x_1}^{S(t)} + \int_{S(t)}^{x_2}\right) u_t(x,t)dx, \end{aligned}$$

where $u(S^-,t)$ and $u(S^+,t)$ are respectively the values of $u(x,t)$ for $x \uparrow S(t)$ and $x \downarrow S(t)$. Taking the limits $x_1 \to S^-$ and $x_2 \to S^+$, we obtain (VERIFY IT) the Rankine-Hugoniot law, or "shock condition"

$$\dot{S}(t) = \frac{q(S^-,t) - q(S^+,t)}{u(S^-,t) - u(S^+,t)} \tag{227}$$

equivalent to the conservation law in the case of a shock.

Since the multivalued continuous curve, obeying the equation $u_t + c(u)u_x = 0$, and the discontinuous curve both satisfy the conservation law, it follows that **the shock front cuts away lobi of equal area** of the multivalued profile.

It is important to remark that, cutting away the two lobi one loses a part of the information contained in the initial condition. Therefore in this process the entropy increases and the shock condition describes how (see Fig. 16).

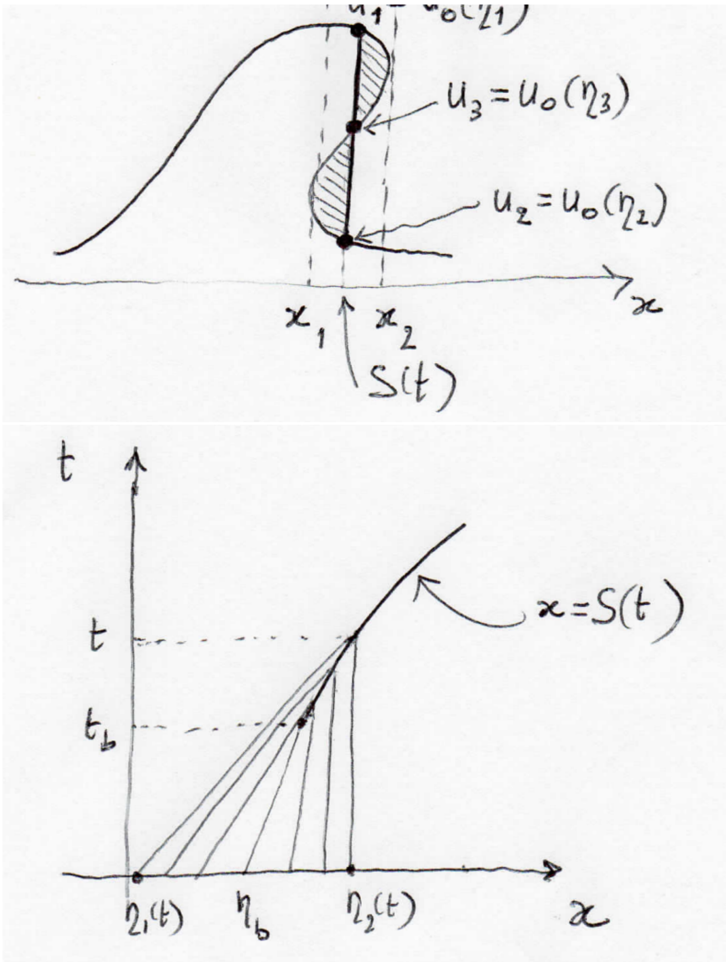


Figure 16: For $t > t_b$ the solution is 3-valued in the interval (x_1, x_2), with values $u_j = u_0(\eta_j)$, $j = 1, 2, 3$ and η_j, $j = 1, 2, 3$ are the 3 characteristics meeting at that time. The shock front $S(t) \in (x_1, x_2)$ cuts away lobi of equal area. Correspondingly, the characteristic curves meets on the shock line trajectory but do not intersect anymore.

The unknowns $S(t), \eta_j(t),\ j = 1, 2, 3$ are constructed, in principle, for $t > t_b$, through the following equations

$$S(t) = \eta_j + F(\eta_j)t, \quad j = 1, 2, \quad F(\eta) := c(u_0(\eta)), \tag{228}$$

$$\dot{S}(t) = \frac{Q(u_0(\eta_1)) - Q(u_0(\eta_2))}{u_0(\eta_1) - u_0(\eta_2)}, \tag{229}$$

with the initial conditions

$$S(t_b) = x_b = F(\eta_b)t_b + \eta_b, \quad \eta_1(t_b) = \eta_2(t_b) = \eta_b, \tag{230}$$

where

$$q = Q(u), \qquad Q'(u) = c(u). \tag{231}$$

Equations (228) mean that the characteristics $\eta_1(t), \eta_2(t)$ intersect at $t > t_b$ in the point $S(t)$; equation (229) is just the shock condition (227). For $t > t_b$, the inversion of (228) with respect to η gives the 3 solutions $\eta_1(t, S), \eta_2(t, S), \eta_3(t, S)$ satisfying the initial condition $\eta_j(t_b, x_b) = \eta_b,\ j = 1, 2, 3$. Substituting $\eta_1(t, S), \eta_2(t, S)$ into (229), this equation becomes a first order ODE for S, that can be uniquely solved, in principle, assigning the initial condition $S(t_b) = x_b$.

In the simplest case of the Hopf equation $u_t + uu_x = 0$,

$$Q(u) = u^2/2 \tag{232}$$

and the translation in time is proportional to u; it follows that the vertical discontinuity shock is mapped backward at $t = 0$ in a segment intersecting the initial profile in the points $\eta_1 < \eta_3 < \eta_2$. The property of equal area lobi is preserved in this backward mapping (see Fig. 17)

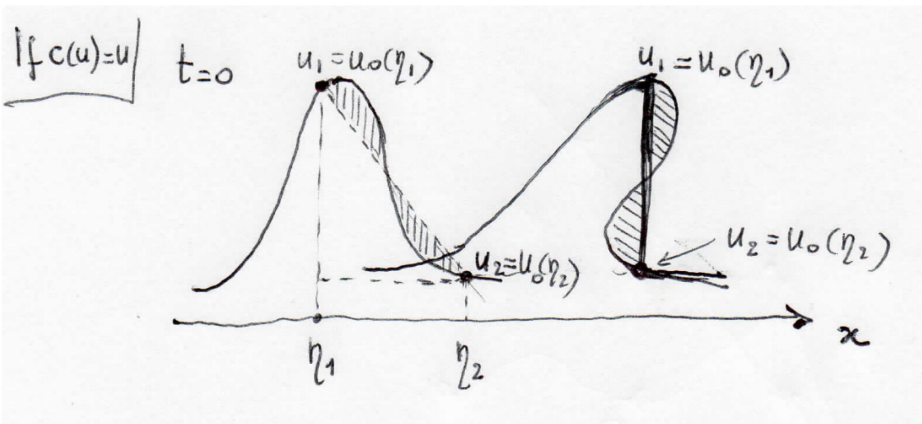


Figure 17: The backword mapping for the Hopf equation.

and can be written more easily on the initial profile. Indeed the area below the initial profile equals the area of the rectangular trapeze of vertices η_1, η_2, u_2, u_1 (see the initial profile in Fig. 17), and reads:

$$\begin{aligned}&\int_{\eta_1}^{\eta_2} u_0(\eta)d\eta = u_0(\eta_2)(\eta_2-\eta_1) + \tfrac{(u_0(\eta_1)-u_0(\eta_2))(\eta_2-\eta_1)}{2}\\ &= \tfrac{1}{2}(\eta_2-\eta_1)(u_0(\eta_1)+u_0(\eta_2)).\end{aligned} \tag{233}$$

This formula can be obtained also from equations (228)-(230) as follows. First, equation (229) becomes

$$\dot{S}(t) = \frac{u_0(\eta_1)) + u_0(\eta_2)}{2}. \tag{234}$$

Then we subtract (228) for $j=1$ to the same equation for $j=2$, obtaining

$$t = \frac{\eta_1-\eta_2}{u_0(\eta_2)-u_0(\eta_1)}; \tag{235}$$

then we take the t-derivative of (228) for $j = 1, 2$, we add the obtained equations and use (235) to get:

$$\dot{S} = \frac{1}{2}\left(\left[u_0'(\eta_1)\dot{\eta}_1 + u_0'(\eta_2)\dot{\eta}_2\right] \frac{\eta_1-\eta_2}{u_0(\eta_2)-u_0(\eta_1)} + (\dot{\eta}_1+\dot{\eta}_2) + (u_0(\eta_2)+u_0(\eta_1)) \right). \tag{236}$$

Comparing this equation with (234) we end up with

$$\begin{aligned}0 &= \tfrac{1}{2}\left[u_0'(\eta_1)\dot{\eta}_1 + u_0'(\eta_2)\dot{\eta}_2\right](\eta_1-\eta_2) + \tfrac{1}{2}(\dot{\eta}_1+\dot{\eta}_2)(u_0(\eta_2)-u_0(\eta_1)\\ &= u_0'(\eta_2)\dot{\eta}_2 - u_0'(\eta_1)\dot{\eta}_1 - \tfrac{1}{2}\tfrac{d}{dt}\left((u_0(\eta_2)+u_0(\eta_1))(\eta_2-\eta_1)\right)\\ &= \tfrac{d}{dt}\left(\int_{\eta_1}^{\eta_2} u_0(\eta)d\eta - \tfrac{1}{2}(u_0(\eta_2)+u_0(\eta_1))(\eta_2-\eta_1)\right).\end{aligned} \tag{237}$$

It follows that the quantity $\int_{\eta_1}^{\eta_2} u_0(\eta)d\eta - \frac{1}{2}(u_0(\eta_2)+u_0(\eta_1))(\eta_2-\eta_1)$ does not depend on time, and since it is zero at $t = t_b$ (see (230)), it is zero (formula (233)).

4.1.1 Explicit Example: the compression wave

We deal with the Hopf equation $u_t + uu_x = 0$ whose initial condition is given by the compression wave (112), with $a_1 > a_2$. Without regularization the solution breaks as we have seen in §2.2 (see Fig. 18)

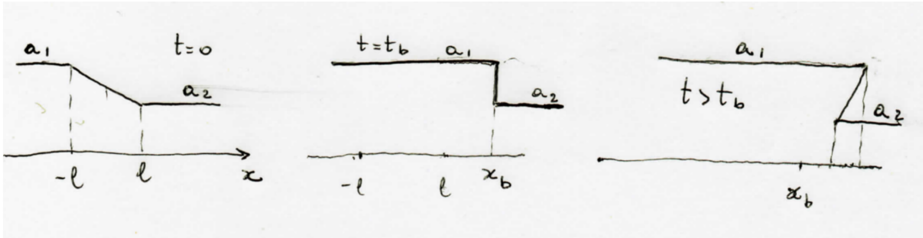


Figure 18: The breaking of a compression wave.

Since $Q'(u) = c(u) = u$, then $Q = u^2/2$+const, and the shock equations read, for $t > t_b$:

$$\begin{array}{l} S(t) = \eta_j + u_0(\eta_j)t, \quad j = 1, 2, \\ \dot{S}(t) = \left(\frac{u_0^2(\eta_1) - u_0^2(\eta_2)}{2[u_0(\eta_1) - u_0(\eta_2)]} = \right) \frac{1}{2}[u_0(\eta_1) + u_0(\eta_2)], \\ S(t_b) = x_b, \quad \eta_j(t_b) = \eta_b, \ j = 1, 2. \end{array} \tag{238}$$

As we have seen, the discontinuity of the initial condition implies that all the characteristics $|\eta| \leq l$ meet at the breaking point

$$(x_b, t_b) = \left(\frac{a_1 + a_2}{a_1 - a_2} l, \frac{2l}{a_1 - a_2} \right); \tag{239}$$

then we do not have a unique η_b, and the above condition $\eta_1(t_b) = \eta_2(t_b) = \eta_b$ may not be valid.

Equations (238) become

$$\begin{array}{l} S(t) = \eta_1 + a_1 t, \quad j = 1, 2, \\ S(t) = \eta_2 + a_2 t, \quad j = 1, 2, \\ \dot{S}(t) = \frac{a_1 + a_2}{2}, \end{array} \tag{240}$$

and we evaluate the first two equations (240) at $t = t_b$, using $S(t_b) = x_b = \frac{a_1+a_2}{a_1-a_2} l$, obtaining

$$\eta_1(t_b) + \eta_2(t_b) = 0, \quad \eta_2(t_b) - \eta_1(t_b) = 2l \quad \Rightarrow \quad \eta_2(t_b) = l, \ \eta_1(t_b) = -l. \tag{241}$$

Now we take the difference and the sum of the first two equations:

$$\begin{array}{l} t = -\frac{\eta_1 - \eta_2}{a_1 - a_2}, \\ S(t) = \frac{a_1 + a_2}{2} t + \frac{\eta_1 + \eta_2}{2}, \end{array} \tag{242}$$

and the derivative of the second equation (242):

$$\dot{S} = \frac{a_1 + a_2}{2} + \frac{\dot{\eta}_1 + \dot{\eta}_2}{2}. \tag{243}$$

Comparing (243) and the third of (240) we infer that

$$\dot{\eta}_1 + \dot{\eta}_2 = 0 \quad \Rightarrow \quad \eta_1(t) + \eta_2(t) = const, \tag{244}$$

and evaluating the last equation at $t = t_b$, using (241), we infer that

$$\eta_1(t) + \eta_2(t) = 0 \quad \Rightarrow \quad S(t) = \frac{a_1 + a_2}{2} t. \tag{245}$$

Summarizing we have the following shock evolution

$$\begin{array}{l} \eta_1(t) = -\frac{a_1 - a_2}{2} t, \quad \eta_2(t) = -\eta_1(t) = \frac{a_1 - a_2}{2} t, \\ S(t) = \frac{a_1 + a_2}{2} t, \quad t > t_b, \end{array} \tag{246}$$

indicating that the multivalued region of the compression wave in Fig.18 is replaced by a single valued shock traveling with a speed that is the average of the amplitudes in front and behind the shock (see Fig. 19).

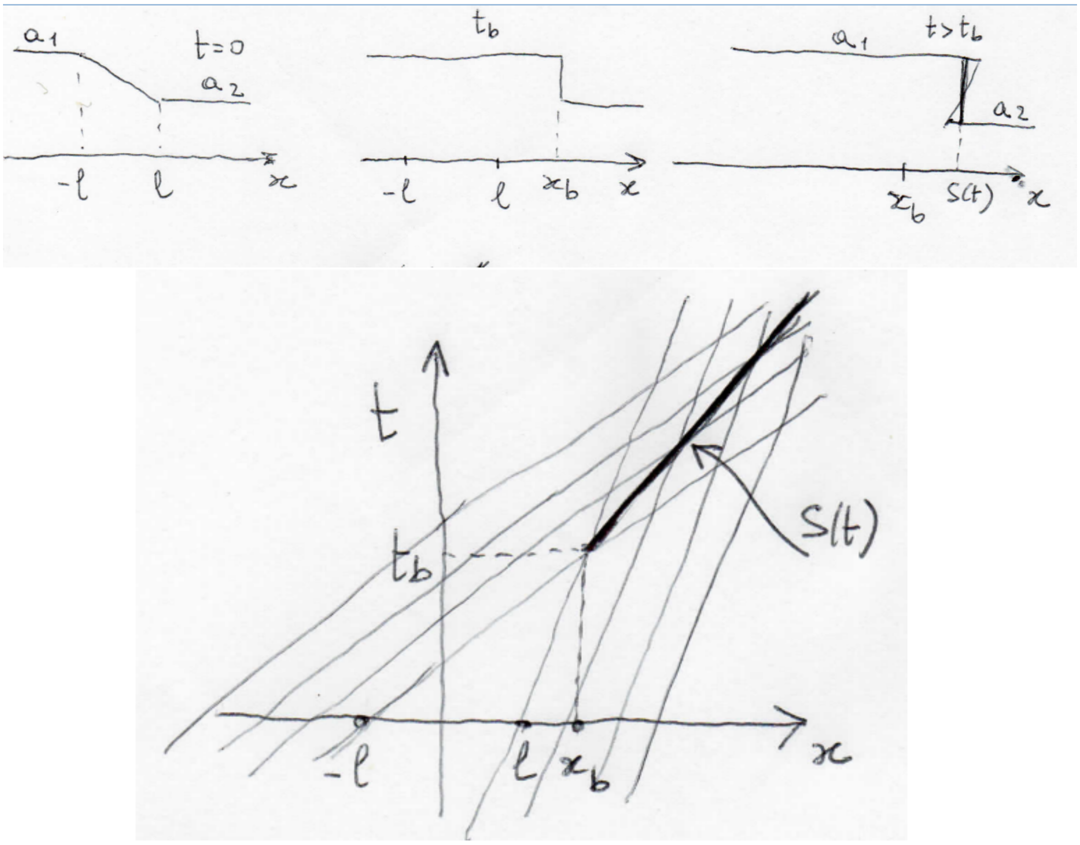


Figure 19: The regularization of a compression wave.

4.2 Important application: the piston problem

As an application of the gas dynamics equations, that can be written in the hyperbolic form

$$\begin{aligned} &\tfrac{dS}{dt} = 0, \quad \text{on} \quad \tfrac{dx}{dt} = u, \\ &\tfrac{dp}{dt} \pm \rho a \tfrac{du}{dt} = 0, \quad \text{on} \quad \tfrac{dx}{dt} = u \pm a, \end{aligned} \tag{247}$$

we consider the piston problem described in the following Fig. 20

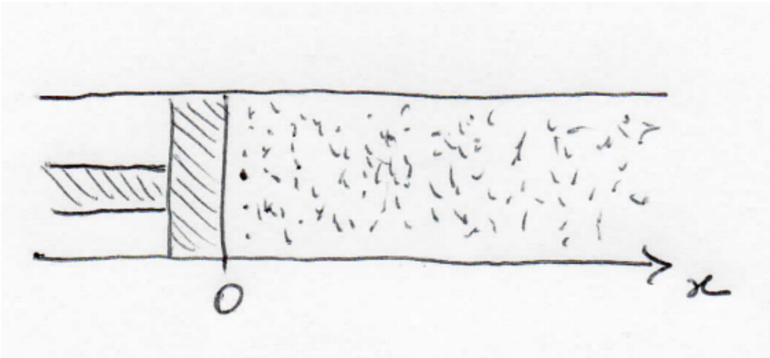


Figure 20: The piston problem

At $t = 0$ the gas is at rest in a uniform state:

$$u = 0, \quad \rho = \rho_0, \quad S = S_0, \quad a^2 = a_0^2 = \frac{\partial p}{\partial \rho}(\rho_0, S_0). \tag{248}$$

The piston moves in the x direction on the given trajectory

$$x = X(t) \tag{249}$$

and the gas follows it accordingly. The gas particle paths are the characteristic curves $dx/dt = u$, and the piston trajectory is itself a particle path trajectory, since the gas follows the piston. Therefore the particle trajectories are similar to that of the piston. It follows that they all originate from the x axis (see Fig. 21).

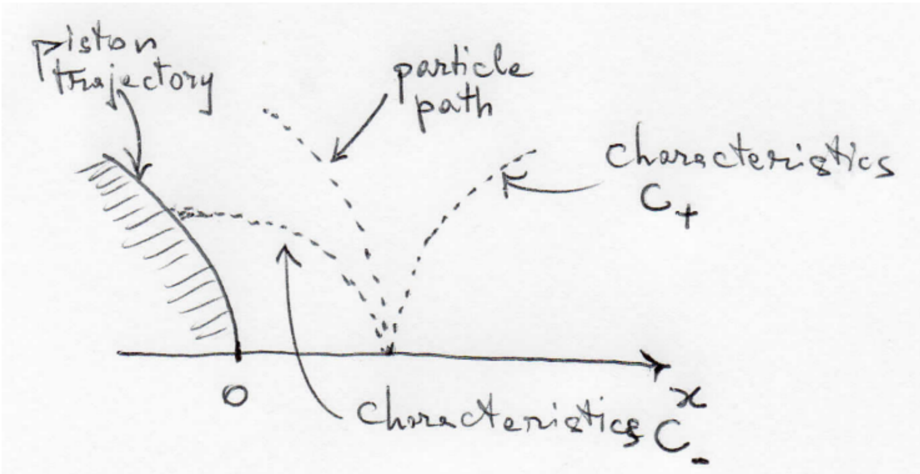


Figure 21: The three characteristics curves in the piston problem.

Since $dS/dt = 0$ on the particle paths $dx/dt = u$, all originating from the x axis, it follows that $S = S_0$ on each path $\Rightarrow S = S_0$ everywhere in the part of the (x, t) plane accessible to the gas. Then the flow is isentropic and the gas equations (247) simplify in terms of the Riemann invariants $r_\pm$

$$r_\pm = \frac{2a}{\gamma - 1} \pm u = const \quad \text{on} \quad \frac{dx}{dt} = u \pm a \tag{250}$$

for a polytropic gas.

Consider the characteristics C_- such that $dx/dt = u - a$. Since $a > 0$, $dx/dt = u - a < u$, where u is the slope of the gas particles. Then also the characterisctics C_- start from the x axis (they cannot start from the piston) (see Fig. 21). Consequently

$$r_- = \frac{2a}{\gamma - 1} - u = \left(\frac{2a}{\gamma - 1} - u\right)\bigg|_{t=0} = \frac{2a_0}{\gamma - 1} \tag{251}$$

on all the characteristics C_- covering all the accessible space-time. It follows that

$$r_- = \frac{2a}{\gamma - 1} - u = \frac{2a_0}{\gamma - 1} \tag{252}$$

everywhere, not only on its characteristics, implying, in particular, that a can be expressed in terms of u

$$a = a_0 + \frac{\gamma - 1}{2} u \tag{253}$$

everywhere.

Consider now the characteristics C_+ such that $dx/dt = u + a$. On them

$$r_+ = \frac{2a}{\gamma - 1} + u = const. \tag{254}$$

The slopes $u+a$ of C_+ is greater than the slope of the particle path, implying that we have to distinguish two cases: C_+ curves originating from the x axis and C_+ curves originating from the piston.
i) If C_+ originates from the x axis, we have, as before:

$$r_+ = \frac{2a}{\gamma - 1} + u = \left(\frac{2a}{\gamma - 1} + u\right)\bigg|_{t=0} = \frac{2a_0}{\gamma - 1}. \tag{255}$$

Comparing (255) and (251) we infer that

$$u = 0, \quad a = a_0, \quad \text{on} \quad \frac{dx}{dt} = a + u = a_0 \;\Rightarrow\; \text{on } x = a_0 t + \eta. \tag{256}$$

Therefore the characteristics C_+ originating from the x-axis are the straight lines $x = a_0 t + \eta,\ \eta \geq 0$, and on them u, a are constants: $u = 0,\ a = a_0$.

ii) if C_+ originates from the piston, we first rewrite the characteristic equations for r_+ using (253):

$$\begin{array}{l} r_+ = \frac{2a}{\gamma-1} + u = \frac{2a_0}{\gamma-1} + 2u = const \quad \text{on} \\ \frac{dx}{dt} = a + u = a_0 + \frac{\gamma+1}{2} u. \end{array} \tag{257}$$

Therefore u is constant on its characteristics (that depends only on u). It follows that the C_+ characteristics originating from the pistons are also straight lines:

$$x = \left(a_0 + \frac{\gamma + 1}{2} u\right) t + \zeta, \tag{258}$$

where ζ is the characteristic parameter enumerating the family of characteristics emanating from the piston. The characteristic ζ meets the piston at $t = \tau$:

$$u = \dot{X}(\tau) \;\text{ at }\; x = X(\tau). \tag{259}$$

Then $u = \dot{X}(\tau)$ on C_+, and at $t = \tau$:

$$X(\tau) = (a_0 + \frac{\gamma + 1}{2}\dot{X}(\tau))\tau + \zeta \tag{260}$$

Eliminating ζ from the equations (258) and (260) we obtain the equation of the characteristic curve meeting the piston at $t = \tau$:

$$x = X(\tau) + [a_0 + \frac{\gamma + 1}{2}\dot{X}(\tau)](t - \tau), \tag{261}$$

on which

$$u = \dot{X}(\tau), \quad a = a_0 + \frac{\gamma - 1}{2}\dot{X}(\tau), \quad S = S_0. \tag{262}$$

The equations (261) and (262) provide the solution in the space-time region covered by the characteristics origination from the piston in the usual way: one solves equation (261) with respect to τ, obtaining $\tau = \tau(x, t; a_0, \gamma)$, and substitutes it into equations (262) to finally get u and a (see Fig. 22).

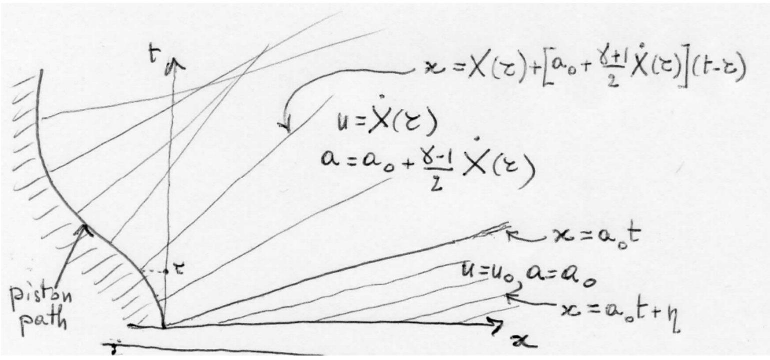


Figure 22: The C_+ characteristics and the solution in space-time. In the first part of its trajectory $X(\tau)$, the piston moves backward with increasing speed: $\dot{X}(\tau) < 0$, $\ddot{X}(\tau) < 0$; the slope decreases, u decreases and we have a rarefaction wave with no breaking. In the second part of the trajectory the piston moves backward with decreasing speed: $\dot{X}(\tau) < 0$, $\ddot{X}(\tau) > 0$; the slope increases and we have breaking.

The piston problem can be solved explicitly when the piston moves with constant speed V:

$$X(t) = Vt. \tag{263}$$

We recall that

$$S = S_0, \quad a = a_0 + \frac{\gamma - 1}{2}u \ \text{ everywhere}, \tag{264}$$

on the characteristics C_+ originating from the x-axis

$$u = 0, \quad a = a_0, \quad \text{on} \ \ x = a_0 t + \eta, \tag{265}$$

and on the characteristics originating from the piston we have, from (261) and (262), the solution

$$\begin{aligned} &x = V\tau + [a_0 + \tfrac{\gamma+1}{2}V](t - \tau), \\ &u = V, \quad a = a_0 + \tfrac{\gamma-1}{2}V. \end{aligned} \tag{266}$$

The characteristics C_+ originating from the piston have slope $a_0 + \frac{\gamma+1}{2}V$. We distinguish two cases.

i) If $V < 0$ this constant slope is less than the slope a_0 of the characteristics C_+ originating from the x-axis; therefore they never intersect (see Fig. 23). There is a fan of characteristics originating from the origin $(0,0)$ of the space-time

$$x = \left(a_0 + \frac{\gamma+1}{2}u\right)t, \quad -|V| \le u \le 0. \tag{267}$$

Inside the fan, the gas velocity is obtained solving (267) with respect to u, obtaining

$$u = \frac{2a_0}{\gamma+1}\left(\frac{x}{a_0 t} - 1\right) \Rightarrow a = \frac{2a_0}{\gamma+1} + \frac{\gamma-1}{\gamma+1}\frac{x}{t}, \tag{268}$$

see Fig. 23:

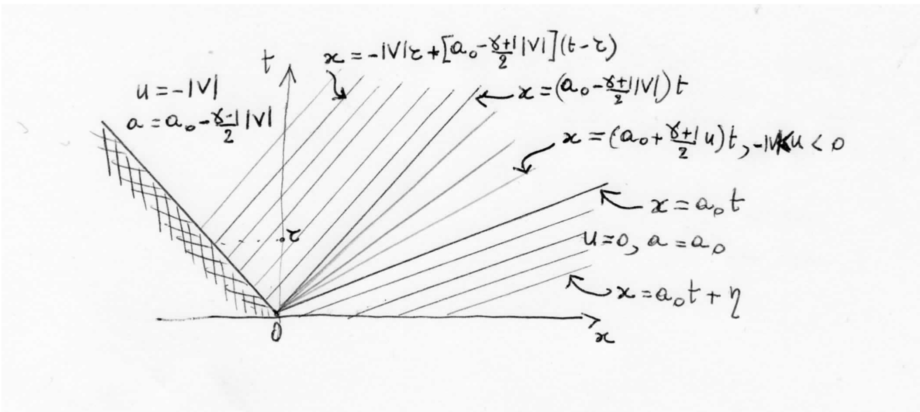


Figure 23: The characteristics curves when the piston moves with a negative constant speed (a rarefaction wave).

Summarizing:

$$u = \begin{cases} -|V|, & -|V|t \le x \le (a_0 - \frac{\gamma+1}{2}|V|)t, \\ \frac{2a_0}{\gamma+1}\left(\frac{x}{a_0 t} - 1\right), & (a_0 - \frac{\gamma+1}{2}|V|)t \le x \le a_0 t, \\ 0, & x \ge a_0 t. \end{cases} \tag{269}$$

$$a = \begin{cases} a_0 - \frac{\gamma-1}{2}|V|, & -|V|t \le x \le (a_0 - \frac{\gamma+1}{2}|V|)t, \\ \frac{2a_0}{\gamma+1} + \frac{\gamma-1}{\gamma+1}\frac{x}{t}, & (a_0 - \frac{\gamma+1}{2}|V|)t \le x \le a_0 t, \\ a_0, & x \ge a_0 t. \end{cases} \tag{270}$$

If $V > 0$, the slope of the characteristics C_+ originating from the piston is greater than the slope a_0 of the characteristics C_+ originating from the x-axis; therefore they intersect immediately at $(x_b, t_b) = (0,0)$ (see Fig. 24).

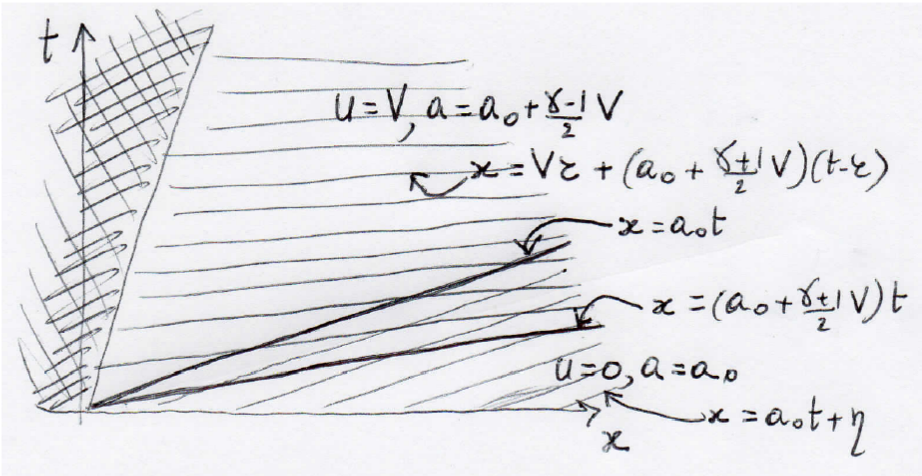


Figure 24: The characteristics curves when the piston moves with a positive constant speed (a compression wave).

Summarizing:

$$u = \begin{cases} V, & Vt < x < a_0 t, \\ V, 0, & a_0 t < x < \left(a_0 + \frac{\gamma+1}{2}V\right) t, \\ 0, & x > \left(a_0 + \frac{\gamma+1}{2}V\right) t. \end{cases} \tag{271}$$

$$a = \begin{cases} a_0 + \frac{\gamma-1}{2}V, & Vt < x < a_0 t, \\ a_0, a_0 + \frac{\gamma-1}{2}V, & a_0 t < x < \left(a_0 + \frac{\gamma+1}{2}V\right) t, \\ a_0, & x > \left(a_0 + \frac{\gamma+1}{2}V\right) t. \end{cases} \tag{272}$$

At last we remark that, known u and a, one obtains ρ and p from

$$p = p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \quad \rho = \rho_0 \left(\frac{a}{a_0}\right)^{\frac{2}{\gamma-1}}. \tag{273}$$

The first equation is for a polytropic gas; the second equation comes from

$$a^2 = \frac{\partial p}{\partial \rho} = \frac{\gamma p_0}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = a_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}. \tag{274}$$

4.2.1 Regularization and shock in the piston compression problem

.

We have established, see (261), that the equation of the characteristics originating from the piston are

$$x = X(\tau) + c(u)(t-\tau), \quad u = \dot{X}(\tau), \tag{275}$$

where

$$c(u) = a_0 + \frac{\gamma + 1}{2} u. \tag{276}$$

This velocity fields corresponds to the flux

$$Q(u) = a_0 u + \frac{\gamma + 1}{4} u^2 + const \tag{277}$$

and leads to the shock equation

$$\dot{S}(t) = \frac{Q(u_1) - Q(u_2)}{u_1 - u_2} = a_0 + \frac{\gamma + 1}{4}(u_1 + u_2), \tag{278}$$

where u_1 and u_2 are respectively the velocities behind and in front of the shock:

$$u_1 = \dot{X}(\tau), \quad u_2 = 0; \tag{279}$$

therefore

$$\dot{S}(t) = a_0 + \frac{\gamma + 1}{4} \dot{X}(\tau) \tag{280}$$

with, at $x = S(t)$:

$$S(t) = X(\tau) + \left(a_0 + \frac{\gamma + 1}{2} \dot{X}(\tau) \right) (t - \tau). \tag{281}$$

Again we solve (281) with respect to τ, obtaining $\tau = \tau(S, t)$, and we substitute it in (280) to get the first order ODE defining $S(t)$, with $S(0) = 0$ and $\tau\big|_{t=0} = 0$ ($t_b = x_b = 0$ in the compression problem), and the problem is implicitly solved.

These formulas can be simplified remarking that the sound speed is $a_0 \sim$ 343 m/s (at $T \sim 20$ Celsius degrees) while the piston speed is in general much smaller. If, f.i., $V \sim 10$ cm/s, then $V/a_0 \sim 3 \cdot 10^{-4}$, the slope of the trajectory of the piston is much smaller than the slope of the parallel characteristics C_+ originating from the x-axis, and

$$\dot{X}(\tau) \ll a_0, \quad X(\tau) \ll a_0 \tau. \tag{282}$$

Then

$$S(t) \sim \left(a_0 + \frac{\gamma + 1}{2} \dot{X}(\tau) \right) t - a_0 \tau, \tag{283}$$

implying

$$\dot{S} \sim a_0 + \frac{\gamma + 1}{2} \dot{X}(\tau) + \left(-a_0 + \frac{\gamma + 1}{2} \ddot{X}(\tau) t \right) \frac{d\tau}{dt}. \tag{284}$$

Eliminating $\dot{S}$ from (280) and (284) and dividing by a_0, one obtains (VERIFY IT)

$$\frac{\gamma+1}{4a_0}\dot{X}(\tau)+\frac{\gamma+1}{2a_0}\ddot{X}(\tau)t\frac{d\tau}{dt}\sim\frac{d\tau}{dt} \tag{285}$$

Multiplying by $\dot{X}(\tau)$:

$$\begin{aligned}&\tfrac{\gamma+1}{4a_0}(\dot{X}(\tau))^2+\tfrac{\gamma+1}{2a_0}\ddot{X}(\tau)\dot{X}(\tau)t\tfrac{d\tau}{dt}=\tfrac{d}{dt}\left(\tfrac{\gamma+1}{4a_0}(\dot{X}(\tau))^2t\right)\\&\sim\dot{X}(\tau)\tfrac{d\tau}{dt}=\tfrac{d}{dt}\int\limits_0^\tau\dot{X}(\tau')d\tau',\end{aligned} \tag{286}$$

and integrating with respect to t, we obtain

$$\frac{\gamma+1}{4a_0}(\dot{X}(\tau))^2t\sim\int\limits_0^\tau\dot{X}(\tau')d\tau', \tag{287}$$

where we have set the integration constant to be 0, since, from (283), at $t=0$ $S=0$ and $\tau\big|_{t=0}=0$.

With equations (283) and (287) the shock problem is reduced to the solution of the algebro-transcendental equation (287) with respect to τ, to get $\tau=\tau(t)$, and its substitution in (283) gives the shock trajectory without solving the usual nonlinear ODE.

If $\dot{X}(0)>0$ we have an initial compression, the shock starts at the origin: $S(0)=0$, $\tau\big|_{t=0}=0$, and equation (287) simplifies further, for $0<t\ll 1$:

$$\frac{\gamma+1}{4a_0}\dot{X}(0)t\sim\tau,\qquad 0<t\ll 1. \tag{288}$$

Then (283) becomes

$$\begin{aligned}S(t)&\sim\left(a_0+\tfrac{\gamma+1}{2}\dot{X}(0)\right)t-a_0\tfrac{\gamma+1}{4a_0}\dot{X}(0)t\\&=\left(a_0+\tfrac{\gamma+1}{4}\dot{X}(0)\right)t\end{aligned} \tag{289}$$

and the shock starts with speed $a_0+\frac{\gamma+1}{4}\dot{X}(0)$.

If the piston moves with constant speed $V>0$, everything simplifies. The C_+ characteristics $x=a_0t+\eta$ originating from the x axis and those $x=V\tau+(a_0+\frac{\gamma+1}{2}V)(t-\tau)$ originating from the piston meet on the shock front $x=S(t)=(a_0+\frac{\gamma+1}{2}V)t$, and the solution reads (VERIFY IT) (see Fig. 25)

$$\begin{pmatrix}u\\a\end{pmatrix}=\begin{cases}\begin{pmatrix}V\\a_0+\frac{\gamma-1}{2}V\end{pmatrix}, & Vt<x<\left(a_0+\frac{\gamma+1}{4}V\right)t,\\ \begin{pmatrix}0\\a_0\end{pmatrix}, & x>\left(a_0+\frac{\gamma+1}{4}V\right)t.\end{cases} \tag{290}$$

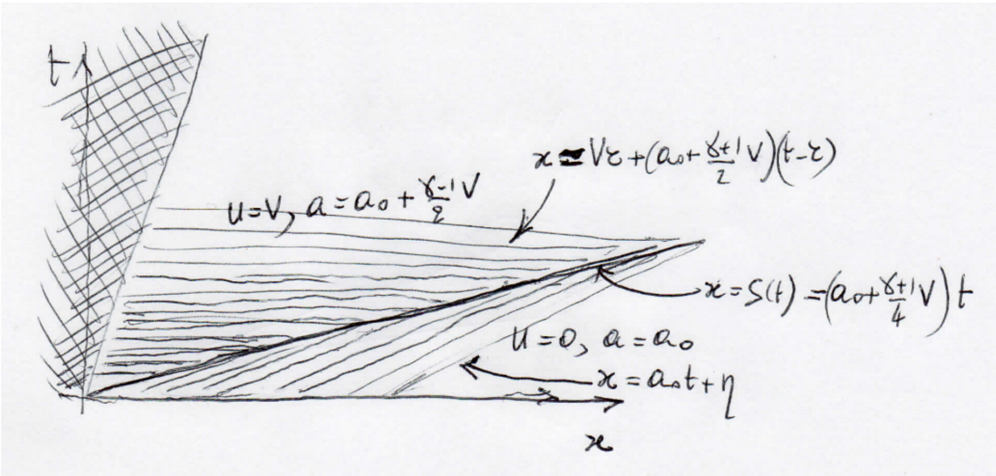


Figure 25: The regularized characteristics curves when the piston moves with a positive constant speed (a regularized compression wave).

4.3 Dissipative regularization of hyperbolic equations and the Burgers equation [36]

As we have already said, the regularization of hyperbolic systems based on physical considerations is the best (but, in general, more difficult) way to deal with the problem of the gradient catastrophe.

In this section we deal with the "dissipative regularization" of the Riemann equation (88), introducing a small dissipative term:

$$u_t + c(u)u_x = \nu u_{xx}, \quad 0 < \nu \ll 1. \tag{291}$$

This equation implies conservation of the mass:

$$\begin{array}{l} \frac{d}{dt}\int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} u_t(x,t)dx \\ = \int_{\mathbb{R}}(-c(u)u_x + \nu u_{xx})dx \\ = [-Q(u) + \nu u_x]_{-\infty}^{\infty} = 0, \quad Q'(u) = c(u), \end{array} \tag{292}$$

but a loss of energy. Indeed, multiply (291) by u and integrate:

$$\frac{1}{2}(u^2)_t = -(\tilde{Q}(u))_x + \nu u u_{xx}, \quad \tilde{Q}'(u) = uc(u). \tag{293}$$

$$\begin{array}{l} \frac{d}{dt}\int_{\mathbb{R}} \frac{u^2(x,t)}{2}dx = \int_{\mathbb{R}} \left(\frac{u^2}{2}\right)_t dx \\ = \int_{\mathbb{R}}(-uc(u)u_x + \nu u u_{xx})dx \\ = \left[-\tilde{Q}(u)\right]_{-\infty}^{\infty} + \nu \int_{\mathbb{R}} u u_{xx}dx = -\nu \int_{\mathbb{R}} u_x^2 dx < 0. \end{array} \tag{294}$$

In both cases we have assumed that $u(x,t), u_x(x,t), Q(u), \tilde{Q}(u) \to 0$ as $x \to \pm\infty$.

The introduction of the small dissipation introduces a vector field contrasting the steepening of the nonlinear term (see Fig. 26)

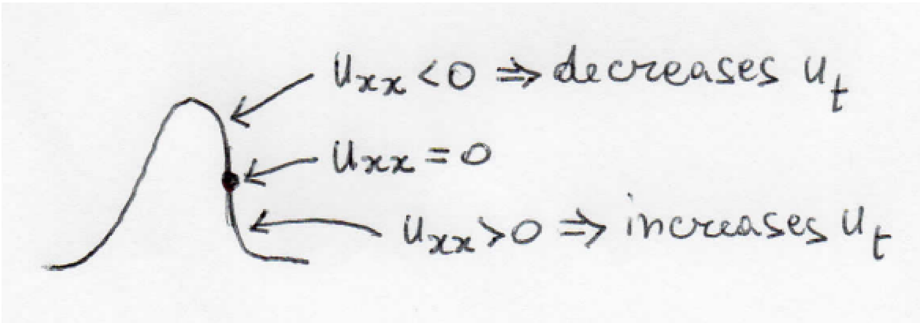


Figure 26: The vector field νu_{xx} contrasts the steepening effect of the vector field $-uu_x$.

4.3.1 The Burgers equation and its Cauchy problem

From now on we limit our considerations to the simplest among the nonlinear equations (291), the so-called "Burgers equation"

$$u_t + uu_x = \nu u_{xx}, \quad \nu > 0. \tag{295}$$

It turns out that also the Burgers equation can be integrated, through a method based on the fact that the equation is the compatibility (integrability) condition for the following system of linear equations for an auxiliary field $\varphi(x,t)$:

$$\begin{array}{l} \varphi_x = -\frac{1}{2\nu} u\varphi, \quad () \\ \varphi_t = \nu \varphi_{xx}. \end{array} \tag{296}$$

The first equation is the so-called "Hopf-Cole transformation", the second equation is the famous "heat (or diffusion) equation".

To show it, we first establish that (VERIFY IT)

$$\begin{array}{l} \varphi_{xx} = -\frac{1}{2\nu}\left(u_x - \frac{1}{2\nu}u^2\right)\varphi, \\ \varphi_{xxx} = -\frac{1}{2\nu}\left(u_{xx} - \frac{3}{2\nu}uu_x + \frac{1}{(2\nu)^2}u^3\right)\varphi, \\ \varphi_{tx} = -\frac{1}{2\nu}[u_t - \frac{1}{2}uu_x + \frac{1}{4\nu}u^3]\varphi, \\ \varphi_{xt} = \nu\varphi_{xxx} = -\frac{1}{2\nu}[\nu u_{xx} - \frac{3}{2}uu_x + \frac{1}{4\nu}u^3]\varphi \end{array} \tag{297}$$

and the Schwarz lemma $\varphi_{xt} = \varphi_{tx}$ is satisfied if and only if u satisfies the Burgers equation (295). Sometimes one preferes the potential form of the

Burgers equation

$$\phi_t - \frac{1}{2}\phi_x^2 = \nu\phi_{xx} \tag{298}$$

linearized to the heat equation $\varphi_t = \nu\varphi_{xx}$ via the point transformation $\phi = -2\nu\log\varphi$.

The integrability scheme (296) can be used to solve the Cauchy problem for the Burgers equation

$$\begin{array}{l} u_t + uu_x = \nu u_{xx}, \quad u = u(x,t;\nu) \in \mathbb{R}, \\ u(x,0;\nu) = u_0(x). \end{array} \tag{299}$$

through the following steps.
1. We first go from $u(x,0)$ to $\varphi(x,0)$ solving the first ODE in (296);
2. Given $\varphi(x,0)$, we solve the heat equation in (296) constructing $\varphi(x,t)$;
3. One constructs the solution $u(x,t) = 2\nu\varphi_x(x,t)/\varphi(x,t)$ again from the first equation (296).
It is important to remark that the step 2 itself requires the use of the Fourier transform method scheme of solution (see Fig. 27).

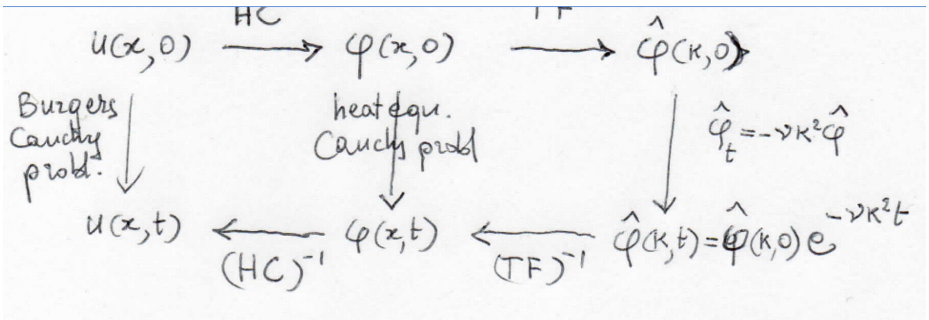


Figure 27: Integration scheme for the Cauchy problem of the Burgers equation.

Let us construct the solution of the Cauchy problem (299) of the Burgers equation using this integration scheme. We first construct the solution of the Cauchy problem for the heat equation

$$\begin{array}{l} \varphi_t = \nu\varphi_{xx}, \\ \varphi(x,0) = \varphi_0(x). \end{array} \tag{300}$$

using the Fourier method. The solution

$$\varphi(x,t) = \frac{1}{2\sqrt{\pi\nu t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4\nu t}}\varphi(y,0)dy, \tag{301}$$

is obtained as follows:

$$\begin{aligned}
&\varphi(x,t) = \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ikx-\nu k^2 t}\hat{\varphi}(k,0) = \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ikx-\nu k^2 t} \int_{\mathbb{R}} dy e^{-iky}\varphi(y,0) \\
&= \int_{\mathbb{R}} dy \varphi(y,0) \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ik(x-y)-\nu k^2 t} = \int_{\mathbb{R}} \frac{dy}{2\pi}\varphi(y,0) e^{-\frac{(x-y)^2}{4\nu t}} \int_{\mathbb{R}} dk e^{-\nu t\left(k-i\frac{x-y}{2\nu t}\right)^2} \\
&= \frac{1}{\sqrt{\nu t}} \int_{\mathbb{R}} \frac{dy}{2\pi}\varphi(y,0) e^{-\frac{(x-y)^2}{4\nu t}} \int_{\mathbb{R}} e^{-s^2} ds = \frac{1}{2\sqrt{\pi\nu t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4\nu t}} \varphi(y,0) dy.
\end{aligned} \tag{302}$$

In the 3rd step we exchange, as usual, the two integrals using Fubini; in the 4th step we complete the square in the second integral; in the 5th step we use the identity

$$\int_{\mathbb{R}} e^{-(x-ia)^2} dx = \int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}, \quad a \in \mathbb{R}, \tag{303}$$

coming from the application of the Cauchy theorem to the integral $\oint e^{-z^2} dz$, where the closed contour is the rectangle $(-R, R, R-ia, -R-ia)$.

The solution is exact but, in general, not expressible in terms of elementary functions. Let us consider the following two interesting examples, in which non smooth initial conditions evolve for $t > 0$ into smooth functions.
1. If $\varphi(x,0) = \delta(x-x_0)$, then

$$\varphi(x,t) = \frac{1}{2\sqrt{\pi\nu t}} e^{-\frac{(x-x_0)^2}{4\nu t}}. \tag{304}$$

A Dirac δ initial condition evolves into a gaussian whose amplitude decays as $1/\sqrt{t}$, and whose variance grows as t (see the left Figure 28).
2. If $\varphi(x,0) = H(x-x_0)$, where $H(x)$ is the Heaviside step function ($H(x) = 1$ for $x > 0$, and $H(x) = 0$ for $x < 0$), then

$$\begin{aligned}
&\varphi(x,t) = \int_{x_0}^{\infty} e^{-\frac{(x-y)^2}{4\nu t}} dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x-x_0}{2\sqrt{\nu t}}} e^{-y^2} dy \\
&= \frac{1}{2}\left(1 + \mathrm{Erf}\left(\frac{x-x_0}{\sqrt{4\nu t}}\right)\right),
\end{aligned} \tag{305}$$

where $\mathrm{Erf}(x)$ is the error function

$$\mathrm{Erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy, \tag{306}$$

with $\mathrm{Erf}(-x) = -\mathrm{Erf}(x)$, $\mathrm{Erf}(0) = 0,\ \mathrm{Erf}(\infty) = 1$ (see the right Figure 28).

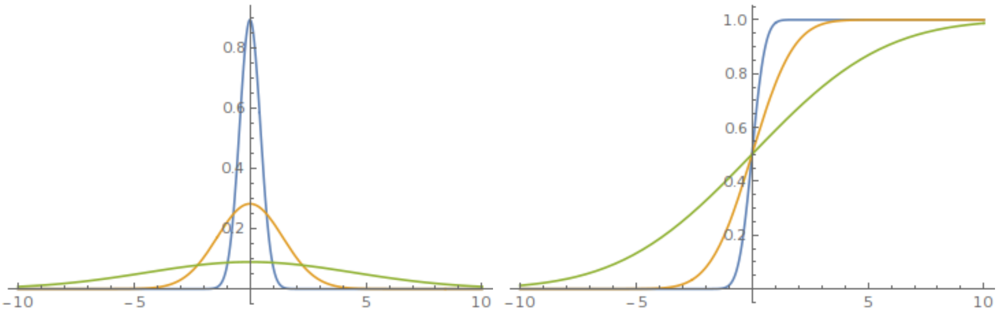


Figure 28: Three snapshots of the evolution of the initial conditions $\varphi(x,0) = \delta(x)$ (left) and $\varphi(x,0) = H(x)$ (right) at $t = 0.1,\ 1.0,\ 10$, according to the Burgers equation for $\nu = 1$.

Solving the first of equations (296) for $\varphi(x,0)$, with $u = u(x,0)$, one obtains $\varphi(x,0) = A\exp\left(-\frac{1}{2\nu}\int\limits_0^x u(y,0)dy\right)$. Therefore

$$\begin{aligned} \varphi(x,t) &= \tfrac{1}{2\sqrt{\pi\nu t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4\nu t}}\varphi(y,0)dy \\ &= \tfrac{A}{2\sqrt{\pi\nu t}}\int_{\mathbb{R}} e^{-\frac{1}{2\nu}G(x,y,t)}dy, \end{aligned} \tag{307}$$

where

$$G(x,y,t) := \int\limits_0^y u(y',0)dy' + \frac{(x-y)^2}{2t}. \tag{308}$$

Using again the first of equations (296) with $u = u(x,t)$ one obtains the solution of the Cauchy problem (299):

$$u(x,t) = -2\nu\frac{\varphi_x(x,t)}{\varphi(x,t)} = \frac{\int_{\mathbb{R}} \frac{x-y}{t} e^{-\frac{1}{2\nu}G(x,y,t)}dy}{\int_{\mathbb{R}} e^{-\frac{1}{2\nu}G(x,y,t)}dy}. \tag{309}$$

If $\nu > 0$ the solution $u(x,t)$ is continuous and single valued $\forall x \in \mathbb{R},\ t > 0$, since the solutions of the heat equation are smooth and single valued $\forall x \in \mathbb{R},\ t > 0$.

Although the Burgers equation reduces to the Hopf equation when $\nu \to 0$, it is not automatic that the solution of the Cauchy problem for the Burgers equation tend to the solution of the Cauchy problem of the Hopf equation, for the same initial condition, when $\nu \to 0$. Now we shall show that it tends to the single valued discontinuous solution (the shock solution).

To do so, we first remark that, if $u(x,0)$ is localized, G behaves asymptotically like a parabola: $G(x,\eta,t) \sim \eta^2/(2t)$ as $\eta \to \pm\infty$, and has one or more local minima for η finite, depending on x and t (see Fig. 29).

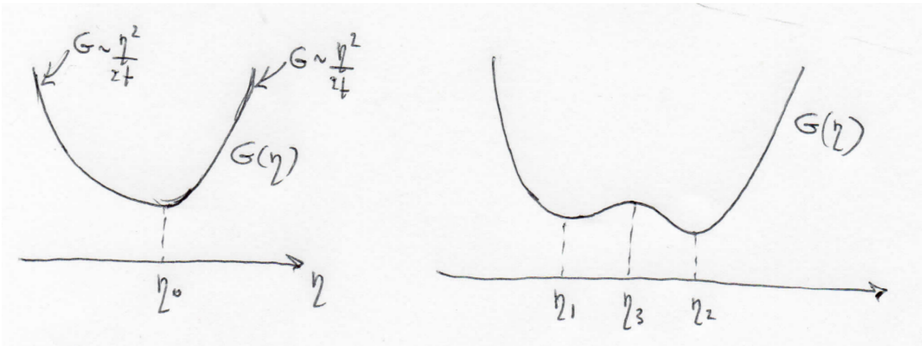


Figure 29: Local minima of $G(\eta)$, varying x and t.

Then $-G(x,\eta,t)$ has local maxima, and since $1/(2\nu) \gg 1$, one can apply the Laplace method (see Appendix 2) to evaluate the leading order term of the solution (309).

Suppose that we are in a space-time region in which there exists only one max at η_0 such that

$$\frac{\partial G(x,\eta,t)}{\partial \eta}\Big|_{\eta_0} = 0 \ \Rightarrow\ \eta_0 = \eta_0(x,t). \tag{310}$$

Then

$$\frac{\partial G(x,\eta,t)}{\partial \eta}\Big|_{\eta_0} = u_0(\eta_0) - \frac{x-\eta_0}{t} \ \Rightarrow\ x = \eta_0 + u_0(\eta_0)t \tag{311}$$

and

$$u(x,t) \sim \frac{x-\eta_0}{t} = u_0(\eta_0), \quad 0 < \nu \ll 1. \tag{312}$$

Therefore, **in the space-time region in which there exists only one stationary point η_0, the solution of the Cauchy problem (299) tends to the well-known solution**

$$u(x,t) = u_0(\eta_0), \quad x = \eta_0 + u_0(\eta_0)t \tag{313}$$

of the Cauchy problem for the Hopf equation, when $\nu \to 0$, corresponding to the same initial condition.

We remark, from (311) and (312), that the stationary point equation for G corresponds to the equation defining the characteristics, and the stationary point $\eta_0(x,t)$ is the characteristic parameter. Since multivaluedness and shocks appear when three or more characteristics meet, this corresponds to the case in which there exist more stationary points. From Fig. 29 the simplest case after that of one stationary point is the case of three stationary points $\eta_j,\ j=1,2,3$ and two maxima η_1,η_2:

$$\left.\frac{\partial G(x,\eta,t)}{\partial \eta}\right|_{\eta_j}=0,\ \Rightarrow\ x=\eta_j+u_0(\eta_j)t,\ j=1,2,3. \tag{314}$$

Then the Laplace method gives, for $0<\nu\ll 1$:

$$u(x,t)\sim \frac{\frac{x-\eta_1}{t}(G''(\eta_1))^{-1/2}e^{-\frac{G(\eta_1)}{2\nu}}+\frac{x-\eta_2}{t}(G''(\eta_2))^{-1/2}e^{-\frac{G(\eta_2)}{2\nu}}}{(G''(\eta_1))^{-1/2}e^{-\frac{G(\eta_1)}{2\nu}}+(G''(\eta_2))^{-1/2}e^{-\frac{G(\eta_2)}{2\nu}}}. \tag{315}$$

If (x,t) are such that $G(\eta_1)<G(\eta_2)$, then $e^{-\frac{G(\eta_1)}{2\nu}}\gg e^{-\frac{G(\eta_2)}{2\nu}}$, and (315) reduces to

$$u(x,t)\sim\frac{x-\eta_1}{t}=u_0(\eta_1). \tag{316}$$

If (x,t) are such that $G(\eta_1)>G(\eta_2)$, then $e^{-\frac{G(\eta_1)}{2\nu}}\ll e^{-\frac{G(\eta_2)}{2\nu}}$, and (315) reduces to

$$u(x,t)\sim\frac{x-\eta_2}{t}=u_0(\eta_2). \tag{317}$$

If (x,t) are such that $G(\eta_1)=G(\eta_2)$, we have

$$\int_0^{\eta_1}u(y,0)dy+\frac{(x-\eta_1)^2}{2t}=\int_0^{\eta_2}u(y,0)dy+\frac{(x-\eta_2)^2}{2t}, \tag{318}$$

implying the shock condition (233):

$$\begin{aligned}\int_{\eta_1}^{\eta_2}u(y,0)dy&=\tfrac{(x-\eta_1)^2}{2t}-\tfrac{(x-\eta_2)^2}{2t}=-\tfrac{t}{2}[u_0^2(\eta_2)-u_0^2(\eta_2)]\\&=\tfrac{\eta_2-\eta_1}{2[u_0(\eta_2)-u_0(\eta_1)]}[u_0^2(\eta_2)-u_0^2(\eta_2)]=(\eta_2-\eta_1)\tfrac{u_0(\eta_1)+u_0(\eta_1)}{2}.\end{aligned} \tag{319}$$

In the 3rd step, we have used the fact that $t=\frac{\eta_2-\eta_1}{u_0(\eta_2)-u_0(\eta_1)}$, coming from subtracting equations $x=\eta_j+u_0(\eta_j)t,\ j=1,2$ in (314).

We have established that **in the space-time region in which there exists three stationary point $\eta_j,\ j=1,2,3$, the solution of the Cauchy problem (299) for the Burgers equation tends, when $\nu\to 0$, to the shock wave solution of the Cauchy problem for the Hopf equation, corresponding to the same initial condition**. Therefore the shock solution of the Hopf equation describes the evolution, according to the Burgers equation, in the presence of a small dissipation (see Fig. 30).

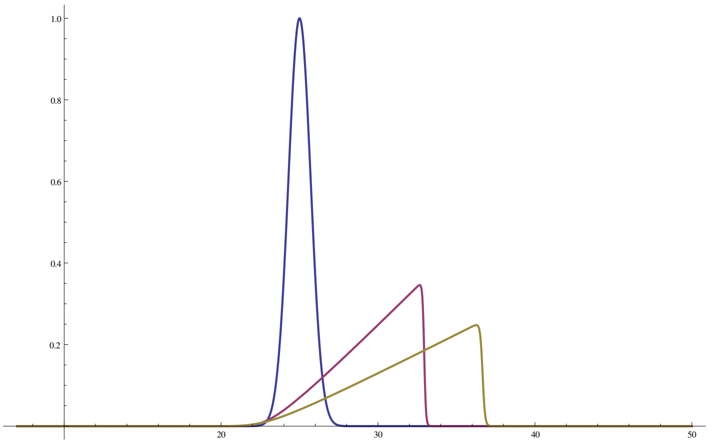


Figure 30: Three snapshots of the numerical dynamics of an initial gaussian profile according to the Burgers equation with a small dissipation, and the formation of a smooth shock.

4.3.2 The shock structure

Since in the Burgers equation $u_t = -uu_x + \nu u_{xx}$ the vector fields $-uu_x$ and νu_{xx} play the opposite role of steepening and smoothening out of the profile, respectively, one expects the existence of a steady state, suggested also by the shock solution. Therefore we look for a solution of the Burgers equation in the form

$$u = U(X), \quad X = x - vt; \tag{320}$$

then U satisfies the ODE $-vU' + UU' = \nu U''$. Integrating it once we have the first order ODE

$$\nu U' = U^2/2 - vU + A, \tag{321}$$

where A is an arbitrary constant, that can be integrated once more by quadratures:

$$\frac{X}{\nu} = \int^U \frac{dy}{y^2/2 - vy + A}. \tag{322}$$

Looking for a solution such that

$$U \to U_2, \ \text{as } x \to \infty; \quad U \to U_1, \ \text{as } x \to -\infty, \tag{323}$$

then equation (321) implies:

$$\frac{U_1^2}{2} - vU_1 + A = \frac{U_2^2}{2} - vU_2 + A = 0 \tag{324}$$

fixing the constants v, A in the following way:

$$v = \frac{U_1 + U_2}{2}, \quad A = \frac{U_1 U_2}{2}. \tag{325}$$

We remark that the velocity of the profile coincides with the velocity of the shock wave of the Hopf equation.

Since, from (324), U_1 and U_2 are the roots of the denominator of the integrand in (322), we have

$$\tfrac{X}{2\nu} = \int^U \tfrac{dy}{(y-U_1)(y-U_2)} = \tfrac{1}{U_1-U_2} \ln \tfrac{U_1-U}{U-U_2} \tag{326}$$

and

$$u = \frac{U_1 + U_2 e^{\frac{U_1-U_2}{2\nu}X}}{1 + e^{\frac{U_1-U_2}{2\nu}X}}. \tag{327}$$

This formula describes a compression wave moving rigidly with velocity $v = \frac{U_1+U_2}{2}$, with shock strength (the relative jump) $\frac{U_1-U_2}{U_1}$ and shock thickness $\frac{2\nu}{U_1-U_2}$. The shock thickness tends to zero when the dissipation parameter $\nu \to 0$ and leads to a discontinuous shock (see Fig. 31).

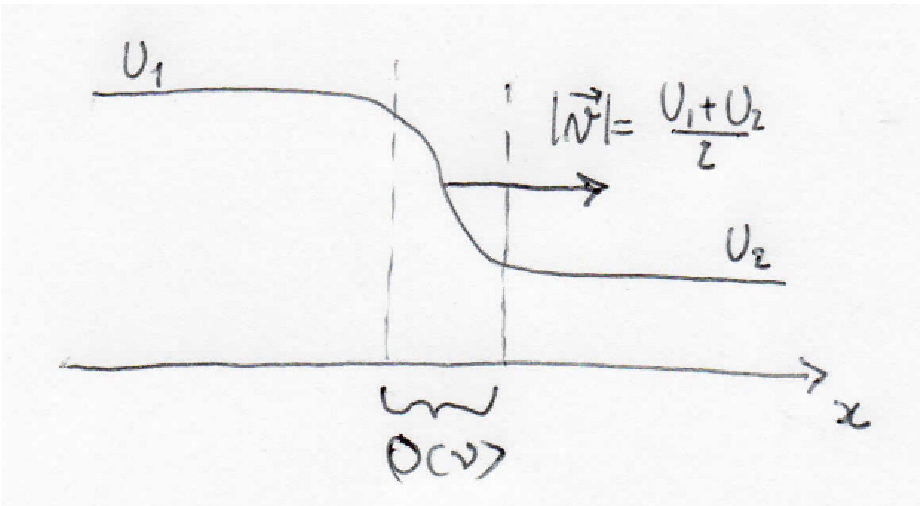


Figure 31: The shock structure in the case of a small dissipation.

4.3.3 Conclusions

Let us summarize what we learned from the previous considerations.

1) If the initial condition of the Burgers Cauchy problem (299) when $0 < \nu \ll 1$ is a smooth localized bump, initially

$$\|\nu u_{xx}\| \ll \|u_t\|, \|uu_x\| = O(1) \tag{328}$$

and the dynamics is ruled by the Hopf equation. Then the wave deforms and, when $|t - t_b| \ll 1$, $t < t_b$, the profile has a fast variation. In the neighborhood of (x_b, t_b) the space-time dependence is rule by the "fast variables"

$$\tilde{x} = \frac{x}{\nu}, \ \tilde{t} = \frac{t}{\nu} = O(1), \tag{329}$$

and the dynamics is now ruled by the Burgers equation

$$u_{\tilde{t}} + uu_{\tilde{x}} = u_{\tilde{x}\tilde{x}} \tag{330}$$

in which there is no small parameter anymore; the solution (327) describes the shape of the profile around the shock.
2. If the initial condition is given by a smooth compression wave, the Hopf equation rules the dynamics until we are in the neighborhood of (x_b, t_b), when the dynamics is ruled by (330) and the solution near the shock is described by (327).
3. If the initial condition is given by a smooth rarefaction wave, the Hopf equation rules again the dynamics, but now the solution becomes more and more rarefacted, and the Burgers equation does not play any role.

4.4 Dispersive regularization of hyperbolic equations and the KdV equation

In the KdV equation $u_t + uu_x + \epsilon^2 u_{xxx} = 0, \ u = u(x,t) \in \mathbb{R}$, dispersion and nonlinearity have opposite effect, and one expect that a rigidly traveling wave solution can exists. Therefore we look for a solution in the form

$$u = U(\Theta), \quad \Theta := x - ct - x_0, \tag{331}$$

obtaining the following ODE for U:

$$-cU_\Theta + UU_\Theta + \epsilon^2 U_{\Theta\Theta\Theta} = 0. \tag{332}$$

Integrating once we obtain:

$$\epsilon^2 U_{\Theta\Theta} + \frac{1}{2}U^2 - cU = \frac{E_1}{6}; \tag{333}$$

multiplying by U_Θ and integrating one more time we obtain the first order ODE

$$\epsilon^2 \frac{U_\Theta^2}{2} + \frac{U^3}{6} - \frac{c}{2}U^2 = \frac{E_1}{6}U + \frac{E_2}{6}, \tag{334}$$

where E_1 and E_2 are the two constants of integration. It can be rewritten as

$$\begin{aligned} &\epsilon^2 U_\Theta^2 = \tfrac{1}{3}P(U), \\ &P(U) := -U^3 + 3cU^2 + E_1 U + E_2 = -(U-\alpha)(U-\beta)(U-\gamma). \end{aligned} \tag{335}$$

Since $U_\Theta^2 > 0$, then also $P(U) > 0$; therefore the real constants E_1, E_2 are chosen to have three real roots, and we order them as $\alpha \leq \beta \leq \gamma$.

Figure

From the figure it follows that the region in which $U \in \mathbb{R}$ and bounded is $\beta \leq U \leq \gamma$.

The relation among the coefficients of the polynomial and its roots are

$$\begin{aligned} &c = \tfrac{\alpha+\beta+\gamma}{3}, \\ &E_1 = -(\alpha\beta + \beta\gamma + \alpha\gamma), \\ &E_2 = \alpha\beta\gamma. \end{aligned} \tag{336}$$

From the figure it follows that the region in which $U \in \mathbb{R}$ and bounded is $\beta \leq U \leq \gamma$.

The ODE (335) becomes

$$\frac{dU}{d\Theta} = \frac{1}{\sqrt{3}\epsilon}\sqrt{P(U)} \tag{337}$$

that is integrated by quadrature

$$\frac{1}{\sqrt{3}\epsilon}\Theta = \int\limits_{\gamma}^{U} \frac{dU'}{\sqrt{P(U')}} \tag{338}$$

Since the polynomial has degree 3, the solution can be expressed in terms of elliptic functions.

Looking for the solution in the form

$$U = \gamma - (\gamma - \beta)\sin^2\left(\varphi\left(\frac{\Theta}{\epsilon}\right)\right) \tag{339}$$

one can show that

$$P(U) = (\gamma - \alpha)(\gamma - \beta)^2\left(1 - \kappa^2\sin^2(\varphi)\right)\sin^2(\varphi)\cos^2(\varphi), \tag{340}$$

where

$$\kappa = \frac{\gamma - \beta}{\gamma - \alpha}. \tag{341}$$

Therefore......................

and show that the solution can be written in terms of the elliptic sine as:

$$\begin{aligned} &U = \gamma - (\gamma - \beta)sn^2\left(\sqrt{\tfrac{\gamma-\alpha}{12}}\tfrac{\zeta}{\epsilon}, \kappa\right), \\ &\kappa = \sqrt{\tfrac{\gamma-\beta}{\gamma-\alpha}}. \end{aligned} \tag{342}$$

11) Show that, if $\beta \to \alpha$ (the case of two coinciding roots), then $\kappa \to 1$, and

$$sn(u, \kappa) \to \tanh u. \tag{343}$$

Consequently, the travelling wave solution of KdV reduces to

$$U = \gamma - (\gamma - \alpha)\tanh^2\left(\sqrt{\frac{\gamma - \alpha}{12}}\frac{\zeta}{\epsilon}\right) = \alpha + \frac{\gamma - \alpha}{\cosh^2\left(\sqrt{\frac{\gamma-\alpha}{12}}\frac{\zeta}{\epsilon}\right)} \tag{344}$$

If, in addition, $\alpha = 0$, then the travelling wave solution reduces to the so-called 1-soliton solution of KdV

$$U = \frac{3c}{\cosh^2\left(\frac{\sqrt{c}}{2}\frac{x-ct-x_0}{\epsilon}\right)}, \tag{345}$$

an exponentially localized travelling wave whose velocity is proportional to the amplitude and inversely proportional to the $\sqrt{width}$.

4.5 Exercices

1) Regularize the compression wave of problem 5) of section 2.1.2

2) What happens if we look for discontinuous solutions of $u_t + uu_x = 0$ in the form $u = H(s(t) - x)u^-(x,t) + H(x - s(t))u^+(x,t)$, where $H(x)$ is the Heaviside step function and $u^{\pm}(x,t)$ are smooth functions?

3) Consider the Cauchy problem

$$\begin{aligned} &u_t + uu_x = 0, \\ &u(x,0) = f(x), \end{aligned} \tag{346}$$

where $f(x)$ describes a single bump, and study the behavior of the regularized (shock) solution near breaking.
A. See section 4 of Appunti 1.

4) Given the Cauchy problem

$$\begin{aligned} &u_t + c(u)u_x = 0, \quad c(u) = Q'(u), \\ &u(x,0) = f(x), \end{aligned} \tag{347}$$

where $f(x)$ describes a single bump,
i) construct the shock condition

$$\dot{s} = \frac{Q(u_2) - Q(u_1)}{u_2 - u_1} \tag{348}$$

and show that it is equivalent of placing the vertical shock to cut equal area lobi of the three valued solution.
ii) Show that, if $c(u) = u,\ Q(u) = u^2/2$, the shock equations involving $s(t), \eta_1(t), \eta_2(t)$ can be reformulated as cutting equal area lobi on the initial profile:

$$\int_{\eta_1}^{\eta_2} f(\eta)d\eta = \frac{1}{2}(\eta_1 - \eta_2)(f(\eta_1) + f(\eta_2)) \tag{349}$$

5) Given the Burgers equation $u_t + uu_x = \nu u_{xx},\ \nu > 0$,
i) Show that, for localized solutions in $\mathbb{R}$, $dM/dt = 0$ and $dE/dt < 0$, where M is the mass and E is the energy:

$$M = \int_{\mathbb{R}} u(x,t)dx, \quad E = \int_{\mathbb{R}} u^2(x,t)dx. \tag{350}$$

ii) find its traveling wave solution satisfying the boundary conditions $u(x,t) \to u_\pm,\ x \to \pm\infty$, where $u_\pm$ are constants, and discuss the shock structure.
iii) Find its similarity solutions.

6) Show that the solution of the Cauchy problem for the Burgers equation $u_t + uu_x = \nu u_{xx}$ with initial condition $u(x,0) = f(x)$ is given by

$$u(x,t) = \frac{\int_{\mathbb{R}} \frac{x-\eta}{t} e^{-\frac{G(x,\eta,t)}{2\nu}} d\eta}{\int_{\mathbb{R}} e^{-\frac{G(x,\eta,t)}{2\nu}} d\eta} \tag{351}$$

where

$$G(x,\eta,t) = \int\limits_0^\eta f(\eta')d\eta' + \frac{(x-\eta)^2}{2t} \tag{352}$$

7) Consider the Cauchy problem for the Burgers equation $u_t + uu_x = \nu u_{xx}$ with Gaussian initial condition $u(x,0) = f(x) = e^{-x^2}$, and let $\eta_b = 1/\sqrt{2} \sim 0.71,\ x_b = \sqrt{2} \sim 1.41,\ t_b = \sqrt{e/2} \sim 1.16$ be the breaking parameters of the Hopf equation $u_t + uu_x = 0$ corresponding to the above Gaussian initial condition (see a previous excercise).
7a) Study the function

$$G(x,\eta,t) = \int\limits_0^\eta f(\eta')d\eta' + \frac{(x-\eta)^2}{2t} \tag{353}$$

as function of the variable η, with $x \in \mathbb{R},\ t > 0$ parameters in the following way. i) Show that, for $\eta \to \pm\infty$, $G(x,\eta,t)$ behaves as a parabola: $G \sim \eta^2/2t$. ii) Show that, for $0 < t < t_b$, $G(x,\eta,t)$ possesses just one extremal point, a global minimum η_0. iii) Show that, for $t > t_b$, there is a finite interval $x \in (x_-, x_+)$ in which $G(x,\eta,t)$ possesses three extremal points $\eta_2 < \eta_0 < \eta_1$ such that η_1, η_2 are local minima and η_0 is a local maximum. iv) Show that: if $x \in (x_-, x_+)$ and is close to x_-, the global minimum is η_2; if it is close to x_+, the global minimum is η_1; there is an intermediate value of $x \in (x_-, x_+)$ for which η_1, η_2 give the same value of G: $G(x,\eta_1,t) = G(x,\eta_2,t)$ and are then global minima. v) Show that, if $x \notin (x_-, x_+)$, then there is only one extremal point, a global minimum η_0. vi) Make plottings of all the above cases (see Fig. 32).
7b) Use the above results to investigate the solution (351) of the Cauchy problem for the Burgers equation $u_t + uu_x = \nu u_{xx}$ with Gaussian initial condition $u(x,0) = f(x) = e^{-x^2}$, when $0 < \nu \ll 1$ (small dissipation), showing that such solution tends, for $\nu \to 0$, to the shock solution of the Hopf equation, for the same initial condition.

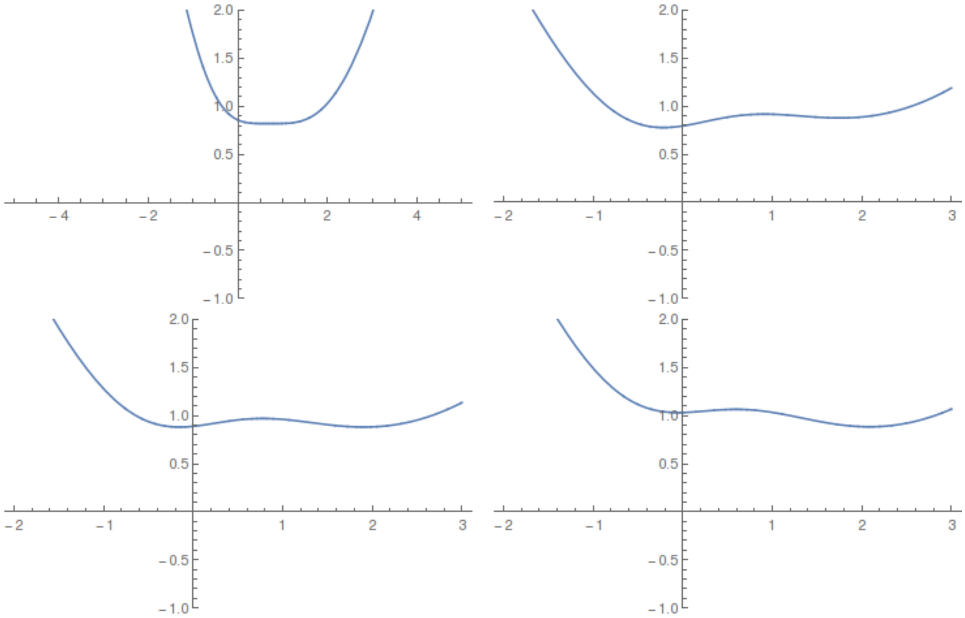


Figure 32: Plots of the function $G(x, \eta, t)$ vaying η, for the Gaussian initial condition $f(\eta) = e^{-\eta^2}$, and for the following choices of (x, t): (x_b, t_b), $(x_b + 0.440, t_b + 1)$, $(x_b + 0.547, t_b + 1)$, $(x_b + 0.700, t_b + 1)$. We remark that, at (x_b, t_b), $G(x, \eta, t)$ has the global minimum at the triple point $\eta = \eta_b$; at $t = t_b + 1$, varying x in a suitable interval, the global minimum changes: if $x = x_b + 0.440$, the global minimum is for $\eta = \eta_2 < 0 < \eta_1$; if $x \sim x_b + 0.547$, the first η_2 and third η_1 local minima give rise to approximately the same value of $G = 0.8807$ and are global minima; if $x = x_b + 0.700$, the global minimum is for $\eta = \eta_1$.

8) Multidimensional generalization of the Burgers equation. Consider the natural multidimensional generalization of the Burgers equation, a pressureless Navier-Stokes equation:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = \nu \nabla^2 \mathbf{u}, \quad \mathbf{u}(\mathbf{x}, t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}. \tag{354}$$

i) Show that it admits the irrotational reduction $\mathbf{u} = -\nabla \phi$ to

$$\phi_t = \frac{1}{2} |\nabla \phi|^2 + \nu \nabla^2 \phi; \tag{355}$$

ii) show that also this equation can be linearized to the heat equation

$$\varphi_t = \nu \nabla^2 \varphi \tag{356}$$

via the Hopf-Cole transformation $\phi = -2\nu \log \varphi$.
iii) Conclude that the solution of the Cauchy problem

$$\begin{array}{l} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = \nu \nabla^2 \mathbf{u}, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) = \nabla \phi_0(\mathbf{x}), \quad \phi_0(\mathbf{x}) \text{ assigned} \end{array} \tag{357}$$

is given by ....

5 Multiscale Expansions and Model Equations of the Nonlinear Mathematical Physics

We have already seen dynamics described by different space-time scales like in the case of the slowly varying wave trains of the linear dispersive PDEs, or in the case of the hyperbolic dynamics in the presence of a small dissipation. In this chapter we introduce the pertubation theory corresponding to the "multiscale expansions method" and we derive several important examples of "model equations of the nonlinear mathematical physics". We begin with the simpler case of the ODEs.

5.1 Multiscale expansions for ODEs

The first example we consider is the celebrated equation for the simple pendulum

$$\ddot{u} + \sin u = 0, \quad u = u(t) \in \mathbb{R} \tag{358}$$

whose general solution is described by special functions called "elliptic functions". We ignore it and look for the solution under the hypothesis of "small amplitudes":

$$u = \delta q, \quad 0 < \delta \ll 1. \tag{359}$$

Then we expand in power series, keeping only the cubic nonlinearity:

$$\ddot{q} + q = \frac{\epsilon}{6} q^3 + O(\epsilon^2), \quad \epsilon = \delta^2, \tag{360}$$

and, to fix the problem, we choose some initial conditions

$$q(0) = 1, \quad \dot{q}(0) = 0. \tag{361}$$

Since equation (360) depends on the small parameter ϵ, it is natural (but, as we shall see, wrong) to look for the solution in the form of a power series in ϵ of the type:

$$q(t) = \sum_{n \geq 0} \epsilon^n q_n(t), \tag{362}$$

implying, from (361), that

$$q_0(0) = 1; \quad q_n(0) = 0, \ n \geq 1; \ \dot{q}_n(0) = 0, \ n \geq 0. \tag{363}$$

In order to keep the expansion ordered (the n^{th} term must be much bigger than the next term), we require that the expansion be asymptotic

$$\frac{\epsilon \|q_{n+1}\|_\infty}{\|q_n\|_\infty} \ll 1, \quad \|f\|_\infty := \sup_{t \geq 0} |f(t)|. \tag{364}$$

Equation (360) becomes

$$\ddot{q}_0 + q_0 + \epsilon\left(\ddot{q}_1 + q_1 - \frac{1}{6}q_0^3\right) + O(\epsilon^3). \tag{365}$$

At $O(1)$ we see that q_0 satisfies the harmonic oscillator equation $\ddot{q}_0 + q_0 = 0$, whose general solution reads

$$q_0(t) = Ae^{it} + \bar{A}e^{-it}, \tag{366}$$

where A is an arbitrary constant complex amplitude, fixed by (361) to be $A = 1/2$.

At $O(\epsilon)$ we obtain a forced harmonic oscillator equation for q_1:

$$\ddot{q}_1 + q_1 = \frac{1}{6}q_0^3 = \frac{1}{6}\left(A^3e^{3it} + 3A^2\bar{A}e^{it} + c.c.\right). \tag{367}$$

This equation expresses two important effects of the nonlinearity.
i) Although the initial condition excites the first harmonic only, the cubic term excites the third harmonic at $O(\epsilon)$.
ii) The terms $e^{\pm it}$ in the forcing are solutions of the harmonic oscillator equation; therefore they are resonant terms and the solution grows linearly in time:

$$\begin{array}{l} q_1(t) = t\alpha e^{it} + \beta e^{3it} + \gamma e^{it} + c.c., \\ \alpha = \frac{1}{4i}A^2\bar{A}, \quad \beta = -\frac{1}{48}A^3, \quad \gamma \in \mathbb{C} \text{ arbitrary}. \end{array} \tag{368}$$

It follows that

$$\epsilon q_1(t) = O(\epsilon t) = O(1), \text{ if } \ t = O(\epsilon^{-1}) \tag{369}$$

and the series is no more asymptotic. In addition the solution cannot diverge in t, since equation (367) is a hamiltonian ODE:

$$\begin{array}{l} \ddot{q}_1 = -V'(q_1), \quad V(q_1) = \frac{q_1^2}{2} - \frac{\epsilon}{24}q_1^4, \\ E = \frac{\dot{q}_1^2}{2} + V(q_1) = E(0) \\ = \left(\frac{\dot{q}_1^2}{2} + \frac{q_1^2}{2} - \frac{\epsilon}{24}q_1^4\right)\Big|_{t=0} \quad \text{finite}. \end{array} \tag{370}$$

We conclude that the ansatz (362) is wrong, and since the expansion ceases to be asymptotic for $t = O(\epsilon^{-1})$, it suggests to look for the following richer ansatz

$$q(t) = \sum_{n\geq 0}\epsilon^n Q_n(t, t_1), \quad t_1 = \epsilon t. \tag{371}$$

Then

$$\begin{array}{l} \frac{d}{dt} = \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial t_1}, \\ \frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\epsilon\frac{\partial}{\partial t}\frac{\partial}{\partial t_1} + \epsilon^2\frac{\partial^2}{\partial t_1^2}, \end{array} \tag{372}$$

and

$$Q_0(0,0) = 1;\; Q_n(0,0) = 0,\; n \geq 1;\; \dot{Q}_n(0,0) = 0,\; n \geq 0. \tag{373}$$

We repeat the previous analysis with this new ansatz in (360), obtaining

$$(\partial_t^2 + 2\epsilon\partial_t\partial_{t_1})(Q_0 + \epsilon Q_1) + Q_0 + \epsilon Q_1 - \frac{\epsilon}{6}Q_0^3 + O(\epsilon^2) = 0. \tag{374}$$

At $O(1)$ we have again the harmonic oscillator:

$$Q_{0tt} + Q_0 = 0 \tag{375}$$

with solution

$$Q_0(t, t_1) = A(t_1)e^{it} + \overline{A(t_1)}e^{-it}, \tag{376}$$

where now A is a function of t_1 to be determined. At $O(\epsilon)$ we have the forced equation

$$\begin{array}{l} Q_{1tt} + Q_1 = -2\partial_t\partial_{t_1}Q_0 + \frac{Q_0^3}{6} \\ = \frac{A^3}{6}e^{3it} - 2i\left(A_{t_1} - \frac{1}{4i}A^2\bar{A}\right)e^{it} + c.c. \end{array} \tag{377}$$

Again we have a secular term, but now we can **choose the dependence of A on the slow variable t_1 to eliminate it**, imposing that the coefficient of the resonant forcing e^{it} be zero:

$$\frac{dA}{dt_1} = \frac{1}{4i}A^2\bar{A}. \tag{378}$$

This equation describes how the complex amplitude of the harmonic oscillation is slowly varying, due to the nonlinearity.

Using the polar representation of A: $A = r\exp(i\theta)$, we obtain the two real equations (VERIFY IT)

$$r_{t_1} = 0, \quad \theta_{t_1} = -\frac{r^2}{4}, \tag{379}$$

implying the solution

$$r = const = r(0) := r_0, \quad \theta = -\frac{r_0^2}{4}t_1 + \theta_0. \tag{380}$$

Then the solution reads

$$q = r_0 e^{i\left(t - \frac{r_0^2}{4}t_1 + \theta_0\right)} + O(\epsilon) = r_0 e^{i(\omega t + \theta_0)} + O(\epsilon), \tag{381}$$

where

$$\omega = 1 - \epsilon\frac{r_0^2}{4} \tag{382}$$

indicates that the angula frequency decreases, due to the non linearity; another important fact is its quadratic dependence on the amplitude, another typical nonlinear effect.

Having eliminated the secular forcing in (377), the solution is bounded (VERIFY IT):

$$Q_1 = -\frac{1}{432}e^{3i(t-\frac{t_1}{6})} + \tilde{A}(t_1)e^{it} + c.c., \tag{383}$$

where $\tilde{A}$ is an arbitrary function of t_1.

If we want to impose the initial conditions we get (VERIFY IT)

$$r_0 = \frac{1}{2}, \quad \theta_0 = 0. \tag{384}$$

We conclude remarking that, if we pushed the analysis at $O(\epsilon^2)$ we would have seen again secularities, and to eliminate them it would be necessary to introduce also the slowly varying variable $t_2 = \epsilon^2 t$; and so on. Therefore the correct ansatz for the solution is

$$q(t) = \sum_{n\geq 0} \epsilon^n Q_n(t, \underline{t}), \quad \underline{t} = (t_1, \dots, t_n, \dots), \quad t_n = \epsilon^n t. \tag{385}$$

Other examples of nonlinear ODEs treated using multiscale expansions are presented among the exercises of this chapter.

The natural generalization of the method to the case of nonlinear PDEs is discussed in the remaining part of this chapter.

5.2 Weakly nonlinear quasi monochromatic waves in nonlinear dispersive PDEs and the nonlinear Schrödinger equation

Here we apply the multiscale method to nonlinear dispersive PDEs, i.e., to nonlinear PDEs that are dispersive in the linear approximation. One can show that, under the hypothesis of

- small amplitudes,
- quasi - monochromatic waves,

the complex amplitude of the monochromatic wave is modulated by suitable slow space-time variables, and this modulation is described by the nonlinear Schrödinger (NLS) equations

$$iu_t + u_{xx} + 2\eta|u|^2u = 0, \quad u = u(x,t), \quad \eta = \pm 1. \tag{386}$$

As we have done it in the previous section, we prefer to illustrate the method on an example, and we choose the natural generalization of the pendulum equation (358) to a PDE, the so-called Sine-Gordon (SG) equation

$$u_{tt} - c^2 u_{xx} + \mu^2 \sin u = 0, \tag{387}$$

an integrable nonlinear generalization of the Klein-Gordon equation relevant in differential geometry. A physical derivation of it comes from the Scott model, consisting of a chain of equal masses pendulums hanging by an elastic thread, with first neighbors interactions due to torsion forces described by the equations

$$m\ddot{\theta}_n = -\mu^2 \sin\theta + \gamma(\theta_{n+1} - \theta_n) - \gamma(\theta_n - \theta_{n-1}), \tag{388}$$

where γ is the torsion coefficient (see Fig. 33)

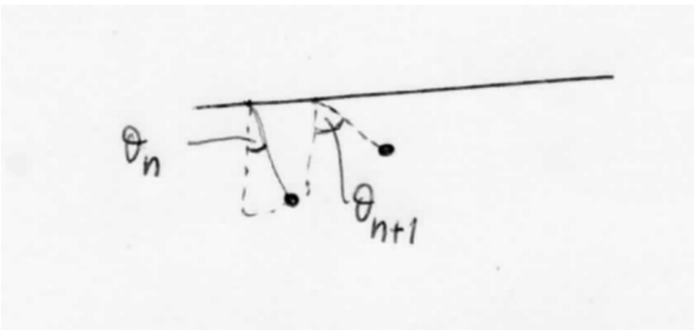


Figure 33: The Scott model of a sequence of pendulums connected by an elastic string

In the case of a large number of pendulums at small distance $\delta \ll 1$, we take the continuous limits $n\delta \sim x$:

$$\begin{aligned} &\theta_n(t) \sim u(x,t), \quad \theta_{n\pm1}(t) \sim u(x \pm \delta, t) \\ &= u(x,t) \pm \delta u_x(x,t) + \tfrac{\delta^2}{2} u_{xx}(x,t) + O\left(\delta^3\right), \end{aligned} \tag{389}$$

obtaining equation (387).

As we have already mentioned, the linear limit of (387) is the Klein-Gordon (KG) equation

$$\mathcal{L}u := u_{tt} - c^2 u_{xx} + \mu^2 u = 0, \tag{390}$$

with two dispersion relations

$$\omega_{\pm} = \pm\sqrt{\mu^2 + c^2 k^2} \tag{391}$$

(from now on we choose $\omega(k) = \omega_+(k)$). Due to the small amplitude hypothesis, we expand the equation (387) in powers:

$$\mathcal{L}u = u_{tt} - c^2 u_{xx} + \mu^2 u = \frac{\mu^2}{6} u^3 + O(u^5) \tag{392}$$

and we look for a solution in the form

$$u = \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + O(\epsilon^4). \tag{393}$$

At $O(\epsilon)$ we have the KG equation

$$\mathcal{L}u_1 = 0. \tag{394}$$

The second hypothesis imposes to choose as starting point of our expansion the monochromatic wave solution of (394):

$$u_1 = Ae^{i\theta(x,t)} + c.c., \quad \theta(x,t) = kx - \omega(k)t. \tag{395}$$

At $O(\epsilon^2)$ we also have $\mathcal{L}u_2 = 0$, and we choose without loss of generality $u_2 = 0$.
At $O(\epsilon^3)$ the nonlinear terms come into play:

$$\mathcal{L}u_3 = \frac{\mu^2}{6} u_1^3 = \frac{\mu^2}{6} \left(A^3 e^{3i\theta} + 3A|A|^2 e^{i\theta} + c.c. \right). \tag{396}$$

As in the ODE example of the previous section, the term $A^3 e^{3i\theta}$ indicates that, due to the nonlinearity, the energy, initially concentrated on the first harmonics, spreads on higher harmonics, and the term $3A|A|^2 e^{i\theta}$ is a resonant forcing that cannot be eliminated, destroying the asymptotic character of the expansion when $t = O(\epsilon^{-1})$. Motivated by the considerations made in the ODE case, we assume that the amplitude A depend on the slow variables

$$x_1 = \epsilon x, \quad t_n = \epsilon^n t, \quad n \in \mathbb{N}^+. \tag{397}$$

Then

$$\begin{array}{l} \partial_t \to \partial_t + \epsilon \partial_{t_1} + \epsilon^2 \partial_{t_2} + O(\epsilon^3), \\ \partial_t^2 \to \partial_t^2 + 2\epsilon \partial_t \partial_{t_1} + \epsilon^2 \left(2\partial_t \partial_{t_2} + \partial_{t_1}^2 \right) + O(\epsilon^3), \\ \partial_x \to \partial_x + \epsilon \partial_{x_1}, \\ \partial_x^2 \to \partial_x^2 + 2\epsilon \partial_x \partial_{x_1} + \epsilon^2 \partial_{x_1}^2. \end{array} \tag{398}$$

At $O(\epsilon)$ we have again $\mathcal{L}u_1 = 0$, and

$$\begin{array}{l} u_1 = A(x_1, \underline{t}) e^{i\theta} + c.c., \\ \underline{t} = (t_1, t_2, \dots, t_n, \dots) \end{array} \tag{399}$$

with the complex amplitude modulated by the slowly varying space-time variables in a way to be prescribed.

At $O(\epsilon^2)$ we have

$$\begin{array}{l} \mathcal{L}u_2 = -2\left(\partial_t\partial_{t_1} - c^2\partial_t\partial_{t_1}\right)u_1 = \\ 2i\omega(k)\left(A_{t_1} + \frac{c^2k}{\omega(k)}A_{x_1}\right)e^{i\theta} + c.c. \end{array} \tag{400}$$

Since $\exp(\pm i\theta)$ are secular unacceptable forcings, their coefficients must be zero; therefore

$$A_{t_1} + \frac{c^2k}{\omega(k)}A_{x_1} = A_{t_1} + \omega'(k)A_{x_1} = 0, \tag{401}$$

implying

$$\begin{array}{l} A = A(x_1 - \omega'(k)t_1, t_2, \dots), \\ u_2 = A_2(x_1, \underline{t})e^{i\theta} + c.c. \end{array} \tag{402}$$

At $O(\epsilon^3)$ we have

$$\begin{array}{l} \mathcal{L}u_3 = -2\left(\partial_t\partial_{t_1} - c^2\partial_x\partial_{x_1}\right)u_2 - \left(2\partial_t\partial_{t_2} + \partial_{t_1}^2 - c^2\partial_{x_1}^2\right)u_1 + \frac{\mu^2}{6}u_1^3 \\ = 2i\omega\left[\left(\partial_{t_1} + \omega'(k)\partial_{x_1}\right)A_2\right]e^{i\theta} \\ +2\omega\left[\left(i\partial_{t_2} + \frac{c^2-\omega'^2}{2\omega}\partial_{x_1}^2\right)A + \frac{\mu^2}{4\omega}A|A|^2\right]e^{i\theta} + \frac{\mu^2}{6}A_1^3e^{3i\theta} + c.c. \end{array} \tag{403}$$

Since $\exp(\pm i\theta)$ are secular unacceptable forcings, and

$$\omega''(k) = \frac{c^2 - \omega'^2}{\omega} = \frac{c^2\mu^2}{\omega^3(k)}, \tag{404}$$

we must impose

$$A_{2t_1} + \omega'(k)A_{2x_1} = 0 \quad \Rightarrow \quad A_2 = A_2(x_1 - \omega'(k)t_1, t_2, \dots), \tag{405}$$

and

$$iA_{t_2} + \frac{\omega''(k)}{2}A_{x_1x_1} + \frac{\mu^2}{4\omega(k)}A|A|^2 = 0 \tag{406}$$

Therefore the elimination of all secularities at this order implies that the amplitude A depends on t_2 through the nonlinear Schrödinger (NLS) equation (406).

Summarizing, for real nonlinear PDEs whose linear approximation is dispersive with dispersion relation $\omega(k)$, looking for small amplitude quasi-monochromatic wave solutions:

$$u(x,t) = \epsilon A\exp(i(kx - \omega(k)t)) + O(\epsilon^2) + c.c., \tag{407}$$

the amplitude A is modulated by the slowly varying variables (397) in the following way:

$$A = A(\xi, t_2, \dots), \quad \xi = \epsilon(x - \omega'(k)t), \tag{408}$$

$$iA_{t_2} + \tfrac{\omega''(k)}{2} A_{\xi\xi} + b(k)A|A|^2 = 0. \tag{409}$$

Therefore, with respect to the slow space-time variables x_1, t_1 there is a rigid translation with the group velocity; with respect to the slower time t_2 the amplitude evolves according to the NLS equation (409). The dependence on the slower times t_n, $n \geq 3$ can be fixed, in principle, eliminating the secularities at higher order.

We remark that the model equation NLS depends on the original nonlinear dispersive PDE from 1) the dispersion relation $\omega(k)$ of its linearized theory, and 2) from the coefficient $b(k)$ containing informations also on the nonlinear terms ($b(k) = \mu^2/(4\omega(k))$ in our example).

We also remark that equation (409) can be written in the suggestive form of a time dependent Schrödinger equation

$$iA_{t_2} + \frac{\omega''(k)}{2} A_{\xi\xi} - V(\xi, t_2)A = 0, \tag{410}$$

with the "self-induced" potential

$$V(\xi, t_2) = -b(k)|A(\xi, t_2)|^2. \tag{411}$$

To fix the ideas, suppose that $\omega''(k) > 0\ \forall k$. We have two cases.
1) If $b(k) > 0\ \forall k$, $V < 0$, and we have a potential well, the attractive case with bound states. In nonlinear optics we have a focalization of the energy in suitable regions of space-time.
2) If $b(k) < 0\ \forall k$, $V > 0$ and we have a potential barrier, with scattering states; in nonlinear optics we have defocusing effects.
Therefore, depending on the relative sign of the dispersive and nonlinear terms, we have two completely different dynamics. If the dispersive and nonlinear terms have the same sign, we have the "focusing NLS" equation; if the dispersive and nonlinear terms have opposite sign, we have the "defocusing NLS" equation.

In the example we considered: $\omega''(k) > 0\ \forall k$ (see (404)) and $b(k) = \mu^2/(4\omega(k)) > 0\ \forall k$; then we have obtained the focusing NLS equation. It is also possible that the signs of $\omega''(k)$ and $b(k)$ change varying $k \in \mathbb{R}$; in this case the same NLS equation is focusing or defocusing depending on the value of the wave number k.

We close this section remarking that the slow variables of the multiscale analysis could be generated also through a linear analysis and the dispersion relation.

The two hypothesis of small amplitude and quasi monochromatic wave imply that, to leading order, the solution is described by its Fourier transform approximation

$$u(x,t) \sim \int_{\mathbb{R}} \frac{dk}{2\pi} \hat{u}_0(k) e^{i(kx-\omega(k)t)} \tag{412}$$

in which $\hat{u}_0(k)$ is concentrated around a specific wave number k_0, suggesting the change of variables

$$k = k_0 + \epsilon \tilde{k}, \quad 0 < \epsilon \ll 1, \tag{413}$$

and, consequently,

$$\theta(x,t) = kx - \omega(k)t = (k_0 x - \omega(k_0)t) + \tilde{k}\,(x_1 - \omega'(k_0)t_1) - \frac{\omega''(k_0)}{2}\tilde{k}^2 t_2, \tag{414}$$

and the slow variables x_1, t_1, t_2 are the ones defined in (397).

$$\xi = x_1 - \omega'(k_0)t_1. \tag{415}$$

Substituting (413),(414) into (412), we have

$$\begin{array}{l} u(x,t) \sim \epsilon \int_{\mathbb{R}} \frac{d\tilde{k}}{2\pi} \hat{u}_0(k_0 + \epsilon\tilde{k}) e^{i[k_0+\epsilon\tilde{k})x - \omega(k_0+\epsilon\tilde{k})t]} \\ \sim \epsilon e^{i[k_0 x - \omega(k_0)t]} \int_{\mathbb{R}} \frac{d\tilde{k}}{2\pi} \hat{u}_0(k_0 + \epsilon\tilde{k}) e^{i[\tilde{k}\xi - i\frac{\omega''(k_0)}{2}\tilde{k}^2 t_2]} \\ = \epsilon A(\xi, t_2) e^{i[k_0 x - \omega(k_0)t]}, \\ A(\xi, t_2) := \int_{\mathbb{R}} \frac{d\tilde{k}}{2\pi} \hat{u}_0(k_0 + \epsilon\tilde{k}) e^{i[\tilde{k}\xi - i\frac{\omega''(k_0)}{2}\tilde{k}^2 t_2]}, \\ \xi = x_1 - \omega'(k_0)t_1. \end{array} \tag{416}$$

Then the slowly varying amplitude satisfies the PDEs

$$A_{t_1} + \omega'(k)A_{x_1} = 0, \qquad iA_{t_2} + \tfrac{\omega''(k)}{2} A_{\xi\xi} = 0; \tag{417}$$

the first one is (401), and the second one is the linearized version of (406).

This argument allows one to construct the linear part of the nonlinear PDE describing weakly nonlinear quasi-monochromatic waves in 1+1 dimensions; the nonlinear part is generically described by the self-induced cubic potential (411), where the coefficient $b(k)$ is obtained from the multiscale expansion when $t = O(\epsilon^{-2})$.

This simple approach, based on the expansion of the dispersion relation around a wave number k_0, can be generalized in a straightforward way to higher dimensions. For instance, in $d+1$ dimensions, expanding around the wave vector $\mathbf{k}_0 = (k_{10}, k_{20}, \dots, k_{d0})$: $\mathbf{k}_0 + \epsilon\tilde{\mathbf{k}}$, $\tilde{\mathbf{k}} = (k_1, k_2, \dots, k_d)$, one obtains:

$$\begin{array}{l} \theta(\mathbf{x},t) = \left(\mathbf{k}_0 + \epsilon\tilde{\mathbf{k}}\right) \cdot \mathbf{x} - \omega\left(\mathbf{k}_0 + \epsilon\tilde{\mathbf{k}}\right) t = \left[\mathbf{k}_0 \cdot \mathbf{x} - \omega\left(\mathbf{k}_0\right) t\right] \\ + \tilde{\mathbf{k}} \cdot \left(\mathbf{x_1} - \nabla_{\mathbf{k}}\omega\left(\mathbf{k}_0\right)\ t_1\right) - \frac{1}{2} \sum\limits_{i,j=1}^{d} \frac{\partial^2 \omega(\mathbf{k}_0)}{\partial k_i \partial k_j}\ t_2 + O(\epsilon^2), \end{array} \tag{418}$$

where

$$\mathbf{x_1} = (x_1, x_2, \ldots, x_d),\ \mathbf{x_1} = \epsilon \mathbf{x},\ t_j = \epsilon^j t,\ j = 1, 2 \tag{419}$$

are slow space-time variables. Correspondingly, the slowly varying amplitude A satisfies the linear PDEs:

$$\left(\partial_{t_1} + \nabla_{\mathbf{k}} \omega\left(\mathbf{k}_0\right) \cdot \nabla_{\mathbf{x_1}}\right) A = \mathbf{0}, \tag{420}$$

$$iA_{t_2} + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 \omega}{\partial k_i \partial k_j}\Big|_{\mathbf{k}=\mathbf{k}_0} A_{x_i x_j} = 0. \tag{421}$$

As before, the nonlinear part is generically described by the self-induced cubic potential (411), where the coefficient b is obtained from the multiscale expansion at order $t = O(\epsilon^{-2})$. Therefore, taking care of equations (419),(421), the following NLS equation in $2+1$ dimensions

$$\begin{array}{l} iu_{t_2} + \frac{1}{2} \sum\limits_{i,j=1}^{d} \frac{\partial^2 \omega(\mathbf{k}_0)}{\partial k_i \partial k_j} A_{\xi_i \xi_j} + b(\mathbf{k}_0)|A|^2 A = 0, \\ A = A(\xi, t_2), \quad \xi_j = x_j - \frac{\partial \omega(\mathbf{k}_0)}{\partial k_j}\, t_1. \end{array} \tag{422}$$

plays an important role as model equation in the description of weakly nonlinear, quasi-monochromatic waves in $d+1$ dimensions.

As an application, for the dispersion relation of the Euler equations in the deep water regime ($d = 2$):

$$\omega(\mathbf{k}) = \sqrt{g}\left(k_1^2 + k_2^2\right)^{1/4}, \tag{423}$$

and assuming without loss of generality that the monochromatic wave travels along the $x_1 = x$ direction: $\mathbf{k}_0 = (k_0, 0)$, then

$$\tfrac{\partial^2 \omega}{\partial k_1^2}(\mathbf{k}_0) = -\tfrac{C_g}{2|k_0|} < 0, \quad \tfrac{\partial^2 \omega}{\partial k_1 k_2}(\mathbf{k}_0) = 0, \quad \tfrac{\partial^2 \omega}{\partial k_2^2}(\mathbf{k}_0) = \tfrac{C_g}{|k_0|} > 0, \tag{424}$$

where $C_g = \frac{1}{2}\sqrt{\frac{g}{|k_0|}}$ is the group velocity. Then equation (424) reduces the NLS equation:

$$iA_{t_2} - \frac{C_g}{4|k_0|}\left(\partial_{x_1}^2 - 2\partial_{x_2}^2\right) - \frac{2k_0^4}{\omega}|A|^2 A = 0, \tag{425}$$

whose dimensionless form reads

$$iu_\tau + u_{XX} - u_{YY} + |u|^2 u = 0. \tag{426}$$

This equation, called the hyperbolic NLS equation in $2+1$ dimensions, is focusing in the wave propagation direction x, and defocusing in the transversal direction y.

5.3 Weakly nonlinear hyperbolic PDEs and the Hopf model

Now we apply the multiscale method to the Riemann class of hyperbolic equations

$$u_t + c(u)u_x = 0, \quad u = u(x,t) \tag{427}$$

under the hypothesis of small perturbation around the constant solution u_0. Therefore we look for solutions in the naive form

$$u(x,t) = u_0 + \epsilon u_1(x,t) + \epsilon^2 u_2(x,t) + O(\epsilon^3). \tag{428}$$

Then the equation becomes

$$\begin{aligned} &\epsilon u_{1t} + \epsilon^2 u_{2t} + \epsilon^3 u_{3t} + O(\epsilon^4) + [c_0 + \epsilon c_0'(u_1 + \epsilon u_2) \\ &+\epsilon^2 \tfrac{c_0''}{2} u_1^2](\epsilon u_{1x} + \epsilon^2 u_{2x} + \epsilon^3 u_{3x}) = 0, \end{aligned} \tag{429}$$

where

$$c_0 = c(u_0), \quad c_0' = c'(u_0), \quad c_0'' = c''(u_0). \tag{430}$$

At $O(\epsilon)$ we have the advection equation

$$u_{1t} + c_0 u_{1x} = 0, \quad \Rightarrow \quad u_1 = f(x - c_0 t), \quad f \text{ arbitrary.} \tag{431}$$

At $O(\epsilon^2)$:

$$u_{2t} + c_0 u_{2x} = -c_0' u_1 u_{1x} := g(x - c_0 t). \tag{432}$$

Since $g(x - c_0 t)$ is a secular forcing, we have the usual linear growth:

$$u_2 = t g(x - c_0 t) + h(x - c_0 t), \quad h \text{ arbitrary,} \tag{433}$$

and the expansion (428) ceases to be asymptotic when $t = O(1/\epsilon)$ unless we impose that $u_1 u_{1x} = f f_x = 0$, trivializing the solution. In addition, the theory of hyperbolic waves is not compatible with this linear time growth of the amplitude.

The way to fix the problem is by now clear: we have to allow the solution to depend on slower time variables:

$$u_n = u_n(x, t, t_1, t_2, \dots), \quad t_n = \epsilon^n t. \tag{434}$$

Then

$$\partial_t \to \partial_t + \epsilon \partial_{t_1} + \epsilon^2 \partial_{t_2} + \dots \tag{435}$$

and we start again the analysis. At $O(\epsilon)$ and $O(\epsilon^2)$:

$$u_{1t} + c_0 u_{1x} = 0, \quad \Rightarrow \quad u_1 = f(x - c_0 t, t_1, t_2, \dots), \; f \text{ arbitrary.} \tag{436}$$

$$u_{2t} + c_0 u_{2x} = -(u_{1t_1} + c_0' u_1 u_{1x}) =: g(x - c_0 t, t_1, \dots). \tag{437}$$

Again the forcing is secular and, to avoid the linear growth, we must impose that the main perturbation satisfy the well-known Hopf equation

$$u_{1t_1} + c_0' u_1 u_{1x} = 0. \tag{438}$$

Therefore **the Hopf equation is the model equation for the class of Riemann equations in the weakly nonlinear regime**.

We remark that, in the non generic case in which $c_0' = 0,\ c_0'' \neq 0$, then $u_{1t_1} = 0$, and one has to go to the next order to establish that the model equation is the cubic Riemann equation

$$u_{1t_2} + \frac{c_0''}{2} u_1^2 u_{1x} = 0. \tag{439}$$

5.4 Weakly nonlinear and weakly dissipative PDEs and the Burgers equation

Consider the following class of PDEs

$$u_t + c(u) u_x = \left(D(u) u_x\right)_x, \quad D(u) > 0, \quad u = u(x,t) \in \mathbb{R}, \tag{440}$$

depending on two smooth arbitrary functions $c(u), D(u)$. It is easy to verify that the RHS is a dissipative term, showing that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2 dx = -\int_{\mathbb{R}} D(u) u_x^2 dx < 0, \tag{441}$$

if u is localized.

Again we consider small perturbations of the constant solution u_0:

$$u(x,t) = u_0 + \epsilon u_1(x,t) + \epsilon^2 u_2(x,t) + O(\epsilon^3), \tag{442}$$

leading, at $O(\epsilon)$, to the equation

$$\begin{aligned} & u_{1t} + c_0 u_{1x} = D_0 u_{1xx}, \\ & c_0 = c(u_0), \quad D_0 = D(u_0), \end{aligned} \tag{443}$$

reducing to the heat equation $u_{1\tau} = D_0 u_{1\xi\xi}$ via the change of variables

$$\xi = x - c_0 t, \quad \tau = t. \tag{444}$$

Consider now two different situations.

1. **Small dissipation**: $D_0 = \epsilon\nu > 0,\ \nu = O(1)$.

Then, at $O(\epsilon)$ and $O(\epsilon^2)$:

$$u_{1t} + c_0 u_{1x} = 0,\ \Rightarrow\ u_1 = u_1(x - c_0 t), \tag{445}$$

$$u_{2t} + c_0 u_{2x} = -\left(c_0' u_1 u_{1x} - \nu u_{1xx}\right) = f(x - c_0 t), \tag{446}$$

where $c_0' = c'(u_0)$. We obtain the usual secular forcing that cannot be eliminated without trivializing the expansion. Therefore we introduce slow times

$$u_j = u_j(x, t, t_1, \dots),\quad t_n = \epsilon^n t,\quad j \geq 1, \tag{447}$$

obtaining now, at $O(\epsilon)$ and $O(\epsilon^2)$:

$$u_{1t} + c_0 u_{1x} = 0,\quad u_1 = u_1(x - c_0 t, t_1, \dots), \tag{448}$$

$$u_{2t} + c_0 u_{2x} = -\left(u_{1t_1} + c_0' u_1 u_{1x} - \nu u_{1xx}\right) = f(x - c_0 t, t_1, \dots). \tag{449}$$

Eliminating the secular forcing is equivalent to fixing the dependence of u_1 on the slower time t_1 through the Burgers equation

$$\begin{aligned} &u_1 = u_1(x - c_0 t, t_1, \dots), \\ &u_{1t_1} + c_0' u_1 u_{1x} - \nu u_{1xx} = 0. \end{aligned} \tag{450}$$

Therefore **the Burgers equation describes weakly nonlinear and weakly dissipative waves**.

2. **Weakly nonlinear long waves in dissipative media**.

We approach this problem as we did with the derivation of the slow variables of NLS in the second part of §4.2. Let us start with the heat equation (443) and observe that $u = \exp(i\theta),\ \theta = kx - W(k)t$ is solution of it if

$$W(k) = c_0 k - iD_0 k^2 t. \tag{451}$$

If we look for long waves (small wave numbers) in dissipative media, we assume that

$$k = \epsilon^\gamma \kappa,\ \gamma > 0,\quad D_0 = O(1), \tag{452}$$

implying that

$$\begin{aligned} &\theta = kx - W(k)t = \epsilon^\gamma \kappa(x - c_0 t) + iD_0 \epsilon^{2\gamma} \kappa^2 t \\ &= \kappa(x_1 - c_0 t_1) + iD_0 \kappa^2 t_2, \end{aligned} \tag{453}$$

generating the slow variables

$$x_1 = \epsilon^\gamma x, \quad t_1 = \epsilon^\gamma t, \quad t_2 = \epsilon^{2\gamma} t, \tag{454}$$

and the partial derivatives

$$\partial_x \to \epsilon^\gamma \partial_{x_1}, \quad \partial_t \to \epsilon^\gamma \partial_{t_1} + \epsilon^{2\gamma} \partial_{t_2}. \tag{455}$$

As in the NLS case, using the linearized theory (coming from the weak non-linearity) + the physical hypothesis (long waves), we have generated the proper slow variables of the problem and the correct ansatz

$$u \sim u_0 + \epsilon u_1(x_1, t_1, t_2) + O(\epsilon^2). \tag{456}$$

At $O(\epsilon^{1+\gamma})$:

$$u_{1t_1} + c_0 u_{1x_1} = 0, \quad u_1 = u_1(x_1 - c_0 t_1, t_2), \tag{457}$$

At higher order, we have contributions of, in principle, different order:

$$u_{2t_1}, u_{2x_1}, u_1 u_{1x_1} = O(\epsilon^{2+\gamma}), \quad u_{1t_2}, u_{1x_1x_1} = O(\epsilon^{1+2\gamma}). \tag{458}$$

Using the principle of "maximal balance", stating that nature favors situations in which the maximal number of terms balance at a certain order, we observe that this maximal balance is achieved when

$$2 + \gamma = 1 + 2\gamma \quad \Rightarrow \quad \gamma = 1. \tag{459}$$

Therefore the next order is $O(\epsilon^3)$, with

$$u_{2t_1} + c_0 u_{2x_1} = -\left(u_{1t_2} + c_0' u_1 u_{1x_1} - D_0 u_{1x_1x_1}\right) = g(x_1 - c_0 t_1, t_2). \tag{460}$$

The usual secular forcing is eliminated imposing that u_1 evolves with respect to t_2 according to the Burgers equation

$$u_{1t_2} + c_0' u_1 u_{1x_1} = D_0 u_{1x_1x_1}. \tag{461}$$

Therefore **the Burgers equation describes also weakly nonlinear long waves in dissipative media**.

5.5 Weakly nonlinear and weakly dispersive PDEs and the Korteweg - de Vries (KdV) equation

Consider the following class of nonlinear dispersive PDEs

$$u_t + c(u) u_x + K_1(u) \left[K_2(u) \left(K_3(u) u_x\right)_x\right]_x = 0, \tag{462}$$

that can be rewritten in the form

$$\begin{array}{l} u_t + c(u)u_x + F(u)u_{xxx} + G(u)u_x u_{xx} + H(u)u_x^3 = 0, \\ F(u) = K_1(u)K_2(u)K_3(u), \\ G(u) = K_1(u)[3K_2(u)K_3'(u) + K_2'(u)K_3(u)], \\ H(u) = K_1(u)K_2'(u)K_3'(u). \end{array} \tag{463}$$

The small amplitude hypothesis implies the usual ansatz

$$u = u_0 + \epsilon u_1 + \dots, \quad u_0 = \text{const}, \tag{464}$$

and the linearized equation, at $O(\epsilon)$, is the linear dispersive PDE

$$u_{1t} + c_0 u_{1x} + F_0 u_{xxx} = 0, \quad F_0 = F(u_0). \tag{465}$$

with dispersive relation

$$\omega(k) = c_0 k - F_0 k^3. \tag{466}$$

We observe that, in the long wave regime (small wave numbers)

$$\omega(k) \sim c_0 k = O(k), \quad v_f = \omega(k)/k \sim c_0 = O(1), \quad |k| \ll 1. \tag{467}$$

Therefore we are in the so-called "weakly dispersive regime".

Since we are interested in long waves, as before we write $k = \epsilon^\gamma \kappa$, $\gamma > 0$, and

$$\theta = kx - \omega(k)t = \epsilon^\gamma \kappa(x - c_0 t) + F_0 \epsilon^{3\gamma} \kappa^3 t = \kappa(x_1 - c_0 t_1) + F_0 \kappa^3 t_3, \tag{468}$$

with the introduction of the slow variables

$$x_1 = \epsilon^\gamma x, \quad t_1 = \epsilon^\gamma t, \quad t_3 = \epsilon^{3\gamma} t \tag{469}$$

implying

$$\partial_x \to \epsilon^\gamma \partial_{x_1}, \qquad \partial_t \to \epsilon^\gamma \partial_{t_1} + \epsilon^{3\gamma} \partial_{t_3}. \tag{470}$$

Assuming

$$u_j = u_j(x_1, t_1, t_3), \quad j \geq 1, \tag{471}$$

we have, at the leading $O(\epsilon^{1+\gamma})$:

$$u_{1t_1} + c_0 u_{1x_1} = 0, \quad u_1 = u_1(x_1 - c_0 t_1, t_3). \tag{472}$$

As before, at higher order we have contributions of different order:

$$u_{2t_1}, u_{2x_1}, u_1 u_{1x_1} = O(\epsilon^{2+\gamma}), \quad u_{1t_3}, u_{1x_1x_1x_1} = O(\epsilon^{1+3\gamma}), \tag{473}$$

and the maximal balance principle gives now

$$2+\gamma = 1+3\gamma \;\Rightarrow\; \gamma = \frac{1}{2}. \tag{474}$$

Therefore the next order is $O(\epsilon^{5/2})$, and the equation reads

$$u_{2t_1} + c_0 u_{2x_1} = -\left(u_{1t_3} + c_0' u_1 u_{1x_1} + F_0 u_{1x_1x_1x_1}\right) = g(x_1 - c_0 t_1, t_3) \tag{475}$$

Again the resonant forcing can be eliminated if the dependence of the field on t_3 is described by the KdV equation

$$u_{1t_3} + c_0' u_1 u_{1x_1} + F_0 u_{1x_1x_1x_1} = 0. \tag{476}$$

Therefore the **KdV equation is a model equation in the description of weakly nonlinear and weakly dispersive (long) waves**.

It would be possible to show that, if $c_0' = 0$ and $c_0'' \neq 0$, the model equation would be the so-called modified KdV equation

$$u_{1t} + \frac{c_0''}{2} u_1^2 u_{1x_1} + F_0 u_{1x_1x_1x_1} = 0. \tag{477}$$

5.6 Water wave equations and their NLS and KdV limits

Consider a small volume of fluid subjected to volume forces $\vec{F} = (F_1, F_2, F_3)$ (f.i., gravity) and pressure forces, with acceleration $\vec{a} = (a_1, a_2, a_3)$ (see Fig. 34).

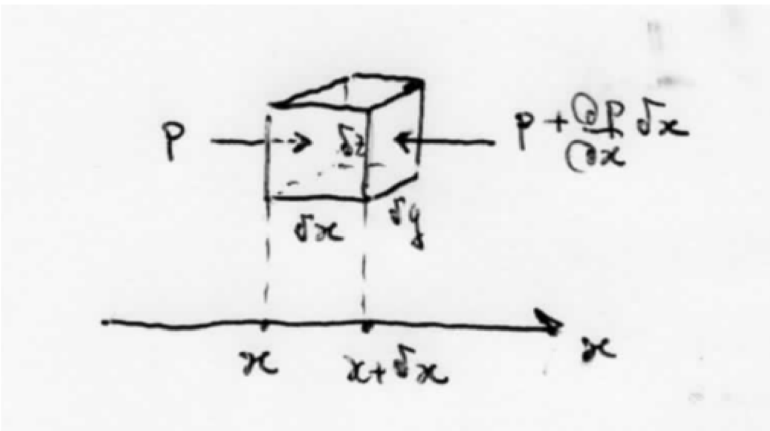


Figure 34: Small volume $\delta x \delta y \delta z$ of fluid subjected to volume and pressure forces.

The Newton equation in the x direction reads:

$$\rho a_1 \delta x \delta y \delta z = \left[p(x) - \left(p(x) + \frac{\partial p}{\partial x} \delta x \right) \right] \delta y \delta z + F_1 \delta x \delta y \delta z, \tag{478}$$

where $\delta x \delta y \delta z$ is the small volume and ρ is the density of the fluid. In vector form:

$$\rho \vec{a} = \vec{F} - \bigtriangledown p. \tag{479}$$

Let $\vec{x}(t)$ be the trajectory of the small volume of fluid; its velocity is $\vec{v} = \vec{v}(\vec{x}(t), t)$, and the acceleration is

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{v}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{v}}{\partial z}\frac{dz}{dt} = \vec{v}_t + (\vec{v} \cdot \bigtriangledown)\, \vec{v}. \tag{480}$$

Therefore the Newton law (479) becomes the Euler equations

$$\vec{v}_t + (\vec{v} \cdot \bigtriangledown)\, \vec{v} = \frac{\vec{F} - \bigtriangledown p}{\rho}. \tag{481}$$

If $\vec{F}$ is the weight force: $\vec{F} = - \bigtriangledown U(z),\ U(z) = \rho g z$:

$$\vec{v}_t + (\vec{v} \cdot \bigtriangledown)\, \vec{v} = -\frac{\bigtriangledown (U + p)}{\rho}. \tag{482}$$

If there is dissipation with coefficient $\nu > 0$, one obtains the Navier-Stokes equations

$$\vec{v}_t + (\vec{v} \cdot \bigtriangledown)\, \vec{v} = -\frac{\bigtriangledown (U + p)}{\rho} + \nu \bigtriangleup \vec{v}. \tag{483}$$

In addition, the conservation of mass

$$\frac{d}{dt} \int_V \rho dV + \int_{\partial V} \rho \vec{v} \cdot \hat{n} d\sigma = 0 \tag{484}$$

implies, via the Gauss theorem, the continuity equation

$$\rho_t + \bigtriangledown \cdot (\rho \vec{v}) = 0. \tag{485}$$

The last equation (5 equations for the 5 unknowns $\vec{v}, p, \rho$) is the equation of state:

$$p = p(\rho, S). \tag{486}$$

In hydrodynamics, $\rho = \rho_0$ constant, and the continuity equation becomes

$$\bigtriangledown \cdot \vec{v} = 0. \tag{487}$$

Applying the operator $\bigtriangledown\wedge$ to the Euler equations (482) and using the identity

$$(\vec{v}\cdot\bigtriangledown)\,\vec{v}=\frac{1}{2}\bigtriangledown|\vec{v}|^2-\vec{v}\wedge(\bigtriangledown\wedge\vec{v}), \tag{488}$$

one obtains the vorticity equation

$$\vec{\omega}_t-\bigtriangledown\wedge(v\wedge\vec{\omega})=0, \tag{489}$$

for the vorticity

$$\vec{\omega}:=\bigtriangledown\wedge\vec{v}. \tag{490}$$

This equation implies that, if $\vec{\omega}$ is initially zero, it remains zero during the evolution. Therefore $\vec{\omega}=\vec{0}$ is a constraint compatible with the evolution that can be imposed to the equations, obtaining an "irrotational flow", with two consequences:
i) the existence of a potential ϕ such that

$$\vec{v}=\bigtriangledown\phi. \tag{491}$$

ii) the simplification of the Euler equations (due to (488))

$$\vec{v}_t+\frac{1}{2}\bigtriangledown|\vec{v}|^2=-\frac{\bigtriangledown(U+p)}{\rho_0}. \tag{492}$$

Writing this equation in terms of ϕ:

$$(\bigtriangledown\phi)_t+\frac{1}{2}\bigtriangledown|\bigtriangledown\phi|^2+\bigtriangledown\left(\frac{U+p}{\rho_0}\right)=0, \tag{493}$$

integrating it:

$$\phi_t+\frac{1}{2}|\bigtriangledown\phi|^2+\frac{U+p-p_0}{\rho_0}=f(t), \tag{494}$$

and observing that ϕ is defined up to a function of t, we get rid of $f(t)$, obtaining the Bernoulli

$$\phi_t+\frac{1}{2}|\bigtriangledown\phi|^2+\frac{U+p-p_0}{\rho_0}=0,\quad U=g\rho_0 z, \tag{495}$$

together with the Laplace equation

$$\triangle\phi=0, \tag{496}$$

consequence of (487).

Surface water waves For surface water waves on a flat bottom (see Fig. 35):

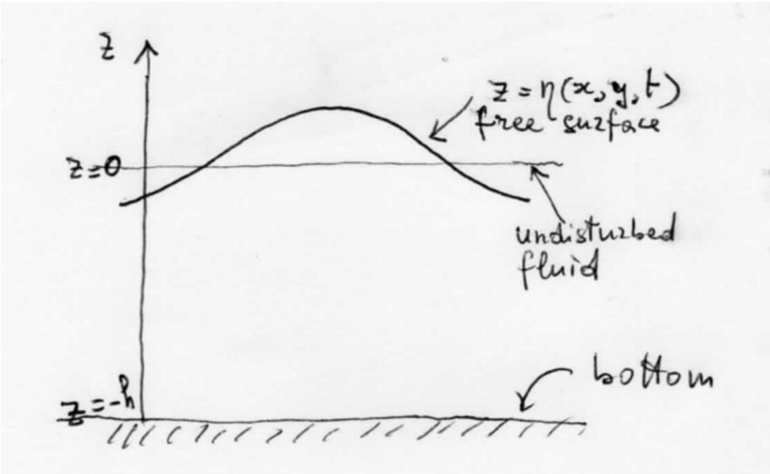


Figure 35: A surface water waves on a flat bottom.

one has to add the boundary conditions on the bottom:

$$\phi_z = 0, \quad z = -h, \tag{497}$$

(the water cannot penetrate through the bottom) and the corresponding boundary conditions on the free surface of separation between water and air, defined by the equation

$$F(\vec{x}, t) = z - \zeta(x, y, t) = 0. \tag{498}$$

Since water cannot mix with air, water particles on the free surface at t remain on the free surface at later times:

$$\begin{array}{l} F(\vec{x}_p(t), t) = 0, \\ F(\vec{x}_p(t) + \delta\vec{x}_p, t + dt) = \nabla_{\vec{x}} F \cdot \delta\vec{x}_p + F_t dt = 0. \end{array} \tag{499}$$

Since $\delta\vec{x}_p/dt$ is the particle velocity, we infer that the "matter derivative" of the free surface equation is zero:

$$\frac{DF}{dt} = F_t + \vec{v} \cdot \nabla_{\vec{x}} F = F_t + \nabla_{\vec{x}}\phi \cdot \nabla_{\vec{x}} F = 0, \tag{500}$$

and, using (498), one finally obtains

$$\phi_z = \zeta_t + \phi_x \zeta_x + \phi_y \zeta_y, \qquad z = \zeta(x, y, t). \tag{501}$$

We remark that also (497) can be written as $DF/dt = 0$, for the bottom surface $F = z + h = 0$.

Summarizing, the surface water wave equations are:

$$
\begin{array}{ll}
\triangle\phi = 0, & -h \le z \le \zeta(x,y,t), \\
\phi_z = \zeta_t + \phi_x\zeta_x + \phi_y\zeta_y, & z = \zeta(x,y,t), \\
\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\zeta = 0, & z = \zeta(x,y,t), \\
\phi_z = 0, & z = -h.
\end{array}
\tag{502}
$$

In the small amplitude case $|\phi|, |\zeta| \ll 1$ with their derivatives, equation (502) reduce to

$$
\begin{array}{ll}
\triangle\phi = 0, & -h \le z \le \zeta(x,y,t) \sim 0, \\
\phi_z - \zeta_t \sim 0, & z = \zeta(x,y,t) \sim 0, \\
\phi_t + g\zeta \sim 0, & z = \zeta(x,y,t) \sim 0, \\
\phi_z = 0, & z = -h.
\end{array}
\tag{503}
$$

Applying ∂_t to the third equation and using the second one, one obtains

$$
\phi_{tt} + g\phi_z = 0, \quad z \sim 0 \tag{504}
$$

Looking for a solution using separation of variables:

$$
\phi = \phi_0 + R(z)e^{i(k_1x+k_2y-\omega t)} + c.c., \quad \zeta = \zeta_0(k)e^{i(k_1x+k_2y-\omega t)} + c.c., \tag{505}
$$

the first and fourth of equations (503) imply that

$$
\begin{array}{l}
R''(z) = (k_1^2 + k_2^2)R(z), \quad -h \le z \le 0, \\
R'(-h) = 0,
\end{array}
\tag{506}
$$

whose solution is

$$
R(z) = C\cosh(k(z+h)), \quad k = \sqrt{k_1^2 + k_2^2}, \quad C = const. \tag{507}
$$

Replacing (505) in (504) one obtains

$$
-\omega^2 R(0) + gR'(0) = 0 \tag{508}
$$

and using (507), we finally obtain $\omega^2\cosh(kh) = gk\sinh(kh)$, implying the well-known dispersion relation of surface water waves

$$
\omega^2 = gk\tanh(hk), \quad k = \sqrt{k_1^2 + k_2^2}. \tag{509}
$$

At last, from the 2nd equation, we get

$$\zeta = iC\frac{\omega(k)}{g}\cosh(hk)e^{i(k_1x+k_2y-\omega(k)t)} + c.c. \tag{510}$$

We distinguish two basic regimes.

1. Long waves (shallow water waves): $h/\lambda \ll 1, \;\Rightarrow\; hk \ll 1 \;\Rightarrow\; \omega \sim \sqrt{gh}\sqrt{k_1^2+k_2^2}$.

2. Short waves (deep water waves): $h/\lambda \gg 1, \;\Rightarrow\; hk \gg 1 \;\Rightarrow\; \omega \sim \sqrt{gk}$.

An important example of long wave (shallow water wave) is the Tsunami. Indeed, for it:

$$\lambda \sim 100\ Km, \quad \zeta \sim 1\ m. \tag{511}$$

If $h \sim 6000\ m$ (off of Polynesia), then $h/\lambda \sim 0.06 \ll 1$ and $v \sim \sqrt{gh} \sim 870\ Km/h$, the speed of an airliner.

The wind waves are instead examples of short waves (deep water waves); indeed

$$\lambda \sim 100\ m, \quad \zeta \sim 2\ m, \tag{512}$$

and, if $h \sim 6000\ m$, $h/\lambda \sim 60 \gg 1$.

The KdV and KP water wave regimes. As we have seen in this chapter, from the dispersion relation of the linearized theory it is possible to extract important informations concerning the multiscale expansion, establishing the proper slow variables to use, and the linearized form of the model equations to be obtained. Here we apply this approach on the water wave theory in the case of surface waves.

The start with the dispersion relation

$$\omega^2(k) = gk\tanh(hk), \quad k = \sqrt{k_1^2+k_2^2}. \tag{513}$$

We first observe that, in the case of long waves (shallow water theory), $hk \ll 1$, and (513) is expanded in the form

$$\begin{aligned}\omega^2(k) &= gk\left(kh - \tfrac{1}{3}(kh)^3 + O(kh)^5\right)\\ &= ghk^2\left(1 - \tfrac{1}{3}(kh)^2 + O(kh)^4\right),\end{aligned} \tag{514}$$

implying that

$$\omega(k) = \sqrt{gh}k\left(1 - \frac{1}{6}(hk)^2 + O(kh)^4\right). \tag{515}$$

One dimensional waves. In the case of one dimensional waves ($k_2 = 0$), this dispersion relation corresponds to the linear PDE

$$\eta_t + \sqrt{gh}\left(\eta_x + \frac{h^2}{6}\eta_{xxx}\right) = 0. \tag{516}$$

It turns out that, if one takes account of the quadratic nonlinearity of the water wave equations, in the long wave approximation, one should add the nonlinear KdV term $\frac{3\sqrt{gh}}{2h}\eta\eta_x$, obtaining the KdV equation

$$\eta_t + \sqrt{gh}\left(\eta_x + \frac{h^2}{6}\eta_{xxx} + \frac{3}{2h}\eta\eta_x\right) = 0 \tag{517}$$

for the amplitude η of the surface wave.

Observe that, near the shore (h small) the dispersive term is small with respect to the nonlinear term, and the equation reduces to the Hopf equation, describing wave breaking.

If we wanted to play the game of the slow variables, we should proceed as we did in the previous sections. In the long wave regime: $k = \epsilon^\gamma \kappa,\ \gamma > 0$; then $\omega \sim \sqrt{gh}(\epsilon^\gamma\kappa - \frac{h^2}{6}\epsilon^{3\gamma}\kappa^3)$, and

$$\begin{aligned} \theta &= kx - \omega(k)t = \epsilon^\gamma \kappa x - \sqrt{gh}(\epsilon^\gamma\kappa - \tfrac{h^2}{6}\epsilon^{3\gamma}\kappa^3)t \\ &= \kappa(x_1 - \sqrt{gh}t_1) + \tfrac{\sqrt{gh}h^2}{6}\kappa^3 t_3 = \kappa\xi + \tfrac{\sqrt{gh}h^2}{6}\kappa^3 t_3, \end{aligned} \tag{518}$$

for the slow variables

$$x_1 = \epsilon^\gamma x,\ t_1 = \epsilon^\gamma t,\ t_3 = \epsilon^{3\gamma} t,\ \xi = x_1 - \sqrt{gh}t_1. \tag{519}$$

corresponding to the linear KdV

$$\begin{aligned} &\eta = \eta(\xi, t_3), \quad \xi = x_1 - \sqrt{gh}t_1, \\ &\eta_{t_3} + \sqrt{gh}\tfrac{h^2}{6}\eta_{\xi\xi\xi} = 0. \end{aligned} \tag{520}$$

If one takes account of the quadratic nonlinearity, one would obtain, in the proper multiscale expansion, $\gamma = 1$, and the KdV equation

$$\eta_{t_3} + \sqrt{gh}\left(\frac{h^2}{6}\eta_{\xi\xi\xi} + \frac{3}{2h}\eta\eta_\xi\right) = 0 \tag{521}$$

as the condition of elimination of the usual secularity.

Quasi one dimensional waves. In the case of long and quasi one dimensional waves, the wave length in the y direction is much larger than the wave length in the x direction: $\lambda_2 \gg \lambda_1 \gg 1 \ \Rightarrow\ 1 \gg k_1 \gg k_2$. Then

$$k = \sqrt{k_1^2 + k_2^2} = k_1\sqrt{1 + \frac{k_2^2}{k_1^2}} \sim k_1\left(1 + \frac{1}{2}\frac{k_2^2}{k_1^2}\right). \tag{522}$$

and the dispersion relation (515) is expanded in the following way:

$$\begin{aligned} \omega &\sim \sqrt{gh}k\left(1-\tfrac{h^2}{6}k^2\right) \\ &\sim \sqrt{gh}k_1\left(1+\tfrac{1}{2}\tfrac{k_2^2}{k_1^2}\right)\left(1-\tfrac{h^2}{6}(k_1^2+k_2^2)\right) \\ &\sim \sqrt{gh}\left(k_1+\tfrac{1}{2}\tfrac{k_2^2}{k_1}-\tfrac{h^2}{6}k_1^3\right), \end{aligned} \tag{523}$$

corresponding to the linear PDE in 2+1 dimensions

$$\eta_t+\sqrt{gh}\left(\eta_x+\frac{h^2}{6}\eta_{xxx}+\frac{1}{2}\partial_x^{-1}\eta_{yy}\right)=0. \tag{524}$$

If one takes account of the quadratic nonlinearity of the water wave equations in this long wave regime, one obtains the celebrated Kadomtsev-Petviashvili (KP) equation

$$\eta_t+\sqrt{gh}\left(\eta_x+\frac{h^2}{6}\eta_{xxx}+\frac{3}{2h}\eta\eta_x+\frac{1}{2}\partial_x^{-1}\eta_{yy}\right)=0 \tag{525}$$

for the amplitude η of the surface wave. Again, near the shore, h is small, and the KP equation reduces to the dispersionless KP (dKP) equation

$$\eta_t+\sqrt{gh}\left(\eta_x+\frac{3}{2h}\eta\eta_x+\frac{1}{2}\partial_x^{-1}\eta_{yy}\right)=0. \tag{526}$$

Slow variables can be introduced as follows:

$$k_1=\epsilon^p\kappa_1,\quad k_2=\epsilon^{p+q}\kappa_2,\quad p,q>0. \tag{527}$$

Then

$$\begin{aligned} \theta &= k_1x+k_2y-\omega(k_1,k_2)t \\ &= \epsilon^p\kappa_1x+\epsilon^{p+q}\kappa_2y-\sqrt{gh}\Big[\epsilon^p\kappa_1+\tfrac{\epsilon^{p+2q}}{2}\tfrac{k_2^2}{k_1}-\tfrac{h^2}{6}\epsilon^{3p}\kappa_1^3\Big]t \end{aligned} \tag{528}$$

The maximal balance principle imposes that

$$3p=p+2q \quad\Rightarrow\quad q=p, \tag{529}$$

and we obtain

$$\begin{aligned} \theta &= \kappa_1(x_1-\sqrt{gh}t_1)+\kappa_2y_2-\sqrt{gh}\Big[\tfrac{1}{2}\tfrac{k_2^2}{k_1}-\tfrac{h^2}{6}\kappa_1^3\Big]t_3 \\ &= \kappa_1\xi+\kappa_2y_2-W(\kappa_1,\kappa_2)t_3 \end{aligned} \tag{530}$$

with the slow variables

$$x_1=\epsilon^px,\ t_1=\epsilon^pt,\ y_2=\epsilon^{2p}y,\ t_3=\epsilon^{3p}t,\ \xi=(x_1-\sqrt{gh}t_1), \tag{531}$$

and $W(\kappa_1, \kappa_2)$ is the new dispersion relation

$$W(\kappa_1, \kappa_2) = \frac{\sqrt{gh}}{2}\left(\frac{k_2^2}{k_1} - \frac{h^2}{3}\kappa_1^3\right). \tag{532}$$

The dispersion relation in (530) corresponds to the linearized KP equation in the slow variables:

$$\eta_{t_3} + \sqrt{gh}\left(\frac{h^2}{6}\eta_{\xi\xi\xi} + \frac{1}{2}\partial_\xi^{-1}\eta_{y_2 y_2}\right) = 0. \tag{533}$$

If one takes account of the quadratic nonlinearity, one would obtain, in the proper multiscale expansion, the KP equation

$$\eta_{t_3} + \sqrt{gh}\left(\frac{h^2}{6}\eta_{\xi\xi\xi} + \frac{3}{2h}\eta\eta_\xi + \frac{1}{2}\partial_\xi^{-1}\eta_{y_2 y_2}\right) = 0 \tag{534}$$

as the condition of elimination of the usual secularity.

We remark that, if we break the maximal balance principle in this way

$$p > q, \tag{535}$$

the dispersion in the x direction is negligeable, and the KP equation reduces to the dKP equation:

$$\eta_{t_3} + \sqrt{gh}\left(\frac{3}{2h}\eta\eta_\xi + \frac{1}{2}\partial_\xi^{-1}\eta_{y_2 y_2}\right) = 0. \tag{536}$$

The NLS deep water regime. We consider small amplitude surface waves in one dimension, in the deep water regime $kh \gg 1$, then

$$\begin{array}{l} \omega(k) = \sqrt{gk},\ k \in \mathbb{R}, \\ R(z) = C\cosh(k(z+h)) \sim \tilde{C}e^{|k|z}. \end{array} \tag{537}$$

If we are interested in small amplitude quasi monochromatic waves, we know from (505,(507),(510) that the solutions read as follows

$$\begin{array}{l} \phi = \epsilon\Phi(x_1, z_1, t_1, t_2, ..)e^{i\theta+|k|z} + c.c. + O(\epsilon^2), \\ \zeta = \epsilon A(x_1, 0, t_1, t_2, ..)e^{i\theta} + c.c. + O(\epsilon^2),\ A = i\frac{\omega}{g}\Phi \end{array} \tag{538}$$

where the slow variables are

$$x_1 = \epsilon x,\ z_1 = \epsilon z,\ t_j = \epsilon^j t,\ j \geq 1, \tag{539}$$

and

$$\theta = kx - \sqrt{gk}t. \tag{540}$$

From the above considerations we expact the following result The above multiscale expansion leads to the following equations

$$\begin{array}{l} A_{t_1} + \omega'(k) A_{x_1} = 0, \quad A = A(\xi, t_2, \dots), \\ iA_{t_2} + \frac{\omega''(k)}{2} A_{\xi\xi} + \beta(k)|A|^2 A = 0, \end{array} \tag{541}$$

where

$$\begin{array}{l} \xi = x_1 - \omega'(k) t_1, \\ \omega(k) = \sqrt{g\ k}, \ \Rightarrow \ \omega'(k) = \frac{\omega(k)}{2k}, \ \omega''(k) = -\frac{\omega(k)}{4k^2} < 0, \end{array} \tag{542}$$

and where the coefficient $\beta(k)$ is obtained from multiscale analysis; since A is dimensionally a length, an elementary dimensional analysis suggests for β the form $\beta(k) = \beta_0 k^2 \omega(k)$, where β_0 is a adimensional constant. Multiscale analysis confirms this simple argument and fixes $\beta_0 = 1/2$. Therefore we obtain the following equation [39]

$$\begin{array}{l} iA_{t_2} + \frac{\omega''(k)}{2} A_{\xi\xi} - \frac{\omega(k)k^2}{2} |A|^2 A \\ = iA_{t_2} - \frac{\omega(k)}{8k^2} A_{\xi\xi} - \frac{\omega(k)k^2}{2} |A|^2 A = 0, \\ A = A(x_1 - \frac{\omega}{2k} t_1, t_2). \end{array} \tag{543}$$

Since the sign of the dispersive and nonlinear terms are both negative, small amplitude, quasi monochromatic surface waves in deep water are described by the focusing NLS equation.

Verify that the change of variables to the canonical dimensionless form $iu_t + u_{xx} + 2|u|^2 u = 0$ is

$$x = \sqrt{2} k^2 a_0 (x_1 - \frac{\omega}{2k} t_1), \quad t = -\frac{\omega a_0^2 k^2}{4} t_2, \quad u = A/a_0, \tag{544}$$

where a_0 is a characteristic elevation.

To study the more realistic $2+1$ dimensional deep water regime:

$$\omega = \sqrt{g\kappa}, \quad \kappa = \sqrt{k_1^2 + k_2^2}, \tag{545}$$

we begin with equation (424) for $d = 2$

$$\begin{array}{l} iu_{t_2} + \frac{1}{2} \sum\limits_{i,j=1}^{2} \frac{\partial^2 \omega(\mathbf{k}_0)}{\partial k_i \partial k_j} A_{\xi_i \xi_j} + b(\mathbf{k}_0)|A|^2 A = 0, \\ A = A(\vec{\xi}, t_2), \quad \xi_j = x_j - \frac{\partial \omega(\mathbf{k}_0)}{\partial k_j}\ t_1, \end{array} \tag{546}$$

and we assume without loss of generality that the monochromatic wave of the linearized theory travel along the $x_1 = x$ direction: $\mathbf{k}_0 = (k_0, 0)$. Then one obtains

$$\omega_{k_1 k_1}|_{\mathbf{k}_0} = -\frac{\sqrt{g}}{4|k_0|}, \quad \omega_{k_1 k_2}|_{\mathbf{k}_0} = 0, \quad \omega_{k_2 k_2}|_{\mathbf{k}_0} = \frac{\sqrt{g}}{2|k_0|}, \tag{547}$$

and equation (546) becomes

$$\begin{array}{l} iA_{t_2} - \frac{\omega(k)}{8k^2}\left(A_{\xi\xi} - 2A_{x_2x_2}\right) - \frac{\omega(k)k^2}{2}|A|^2A = 0, \\ \xi = x_1 - \omega'(k_0)t_1, \end{array} \tag{548}$$

, in a suitable adimensional form,

$$iu_t + u_{xx} - u_{yy} + 2|u|^2u = 0, \quad u = u(x,y,t) \in \mathbb{C}. \tag{549}$$

This equation, often called the hyperbolic NLS equation in $2+1$ dimensions, is focusing in the wave propagation direction x, and defocusing in the transversal direction y.

5.7 Nonlinear optics and NLS [33]

For a non magnetic medium ($\vec{M} = \vec{0}$), in the absence of external charges and currents, the electric field $\vec{E}$ and the polarization $\vec{P}$ are connected by the equations

$$\begin{array}{l} \bigtriangledown \cdot (\epsilon_0\vec{E} + \vec{P}) = 0, \\ \bigtriangledown^2\vec{E} - \bigtriangledown(\bigtriangledown \cdot \vec{E}) = \mu_0\frac{\partial^2}{\partial t^2}(\epsilon_0\vec{E} + \vec{P}). \end{array} \tag{550}$$

We assume that $\vec{P}$ be connected to $\vec{E}$ through the convolution integral

$$\vec{P}(\vec{r},t) = \epsilon_0\int_{-\infty}^{t}\chi(\vec{r},t-t')\vec{E}(\vec{r},t')dt' \tag{551}$$

where χ is the electric susceptibility. We remark that the electric susceptibility $\chi(\vec{r},t)$ is 0 for $t<0$ by causality (the effect, the polarization, cannot preceed the cause, the electric field). The convolution product becomes a simple product in Fourier space

$$\vec{P}_\omega(\vec{r}) = \epsilon_0\chi_\omega(\vec{r})\vec{E}_\omega(\vec{r}), \tag{552}$$

where $\vec{P}_\omega(\vec{r}), \chi_\omega(\vec{r}), \vec{E}_\omega(\vec{r})$ are the Fourier transforms of $\vec{P}(\vec{r},t), \chi(\vec{r},t), \vec{E}(\vec{r},t)$ with respect to t:

$$f_\omega = \int_{\mathbb{R}} e^{-i\omega t}f(t)dt, \quad f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}f_\omega d\omega. \tag{553}$$

In Fourier space, equations (550) read:

$$\begin{array}{l} \bigtriangledown \cdot (\epsilon_0\vec{E}_\omega + \vec{P}_\omega) = 0, \\ \bigtriangledown^2\vec{E}_\omega - \bigtriangledown(\bigtriangledown \cdot \vec{E}_\omega) + \mu_0\omega^2\left(\epsilon_0\vec{E}_\omega + \vec{P}_\omega\right) = 0. \end{array} \tag{554}$$

Using (552) and the first of (554), we obtain

$$\begin{aligned}&0 = \bigtriangledown \cdot (\epsilon_0 \vec{E}_\omega + \vec{P}_\omega) = \bigtriangledown \cdot (\epsilon \vec{E}_\omega)\\&= (\bigtriangledown \epsilon) \cdot \vec{E}_\omega + \epsilon \bigtriangledown \cdot \vec{E}_\omega,\end{aligned} \tag{555}$$

where

$$\epsilon(\vec{r}, \omega) := \epsilon_0 (1 + \chi_\omega(\vec{r})) \tag{556}$$

is called dielectric constant (when it is constant), implying

$$\bigtriangledown \cdot \vec{E}_\omega = -\vec{E}_\omega \cdot \frac{\bigtriangledown \epsilon}{\epsilon}. \tag{557}$$

Substituting (557) into the second of (554), we obtain

$$\bigtriangledown^2 \vec{E}_\omega + \bigtriangledown \left(\vec{E}_\omega \cdot \frac{\bigtriangledown \epsilon}{\epsilon} \right) + \frac{\omega^2}{c^2} \frac{\epsilon}{\epsilon_0} \vec{E}_\omega = \vec{0}. \tag{558}$$

Introducing the refraction index $n(\vec{r}, \omega)$ as follows

$$n^2(\vec{r}, \omega) := 1 + \chi_\omega(\vec{r}), \tag{559}$$

equation (558) becomes

$$\bigtriangledown^2 \vec{E}_\omega + 2 \bigtriangledown \left(\vec{E}_\omega \cdot \bigtriangledown \ln n \right) + \frac{\omega^2}{c^2} n^2 \vec{E}_\omega = \vec{0}, \tag{560}$$

and assuming a slow dependence of n on the space variables, this equation reduces to the Helmholtz equation

$$\bigtriangledown^2 \vec{E}_\omega + \frac{\omega^2}{c^2} n^2(\vec{r}, \omega) \vec{E}_\omega = \vec{0}. \tag{561}$$

Assuming

$$n(\vec{r}, \omega) = n_0(\omega) + \delta n(\vec{r}, \omega), \quad \left| \frac{\delta n}{n_0} \right| \ll 1 \quad \Rightarrow \; n^2 \sim {n_0}^2 + 2 n_0 \delta n, \tag{562}$$

equation (561) becomes

$$\begin{aligned}&\bigtriangledown^2 \vec{E}_\omega + \left(k_0^2 + 2 k_0^2 \tfrac{\delta n(\vec{r}, \omega)}{n_0} \right) \vec{E}_\omega = \vec{0},\\&k_0 := \tfrac{\omega}{c} n_0\end{aligned} \tag{563}$$

The last assumption is paraxiality in the direction of propagation z:

$$\vec{E}_\omega(\vec{r}) = \vec{A}(\vec{r}_\perp, z) e^{i k_0 z} \tag{564}$$

where $\vec{A}(\vec{r}_\perp, z)$ depends on the transversal variables $\vec{r}_\perp = (x, y)$, and slowly on the propagation variable z (slow modulation of a monochromatic wave in the direction z):

$$\partial_z^2 \vec{E}_\omega \sim (2ik_0\vec{A}_z - k_0^2\vec{A})e^{ik_0 z}, \tag{565}$$

we finally end up with equation

$$\begin{array}{l} i\vec{A}_z + \frac{1}{2k_0}\bigtriangledown_\perp^2 \vec{A} + k_0\frac{\delta n(\vec{r},\omega)}{n_0(\omega)}\vec{A} = \vec{0}, \\ \bigtriangledown_\perp^2 = \partial_x^2 + \partial_y^2. \end{array} \tag{566}$$

At last, if $\delta n = \delta n(I)$, where I is the light intensity $I = |\vec{A}|^2$, we obtain a general nonlinear Schrödinger equation

$$\begin{array}{l} i\vec{A}_z + \frac{1}{2k_0}\bigtriangledown_\perp^2 \vec{A} + \varphi(|\vec{A}|^2)\vec{A} = \vec{0}, \\ \varphi(|\vec{A}|^2) := k_0\frac{\delta n(|\vec{A}|^2)}{n_0(\omega)}. \end{array} \tag{567}$$

In the small fields limit:

$$\frac{\delta n}{n_0} \sim a|\vec{A}|^2, \tag{568}$$

it reduces to the cubic elliptic NLS equation in 2+1 dimension:

$$i\vec{A}_z + \frac{1}{2k_0}\bigtriangledown_\perp^2 \vec{A} + ak_0|\vec{A}|^2\vec{A} = \vec{0}. \tag{569}$$

In the large field limit the refraction index δn saturates and $\varphi(|\vec{A}|^2) \sim const.$

5.8 Exercices

1) Consider the two anharmonic oscillators

$$\begin{array}{l} \ddot{q} + q - \frac{\epsilon}{6}q^3 = 0, \quad \text{Hamiltonian cubic pendulum}, \quad 0 < \epsilon << 1, \\ \ddot{q} + q + \epsilon\dot{q}^3 = 0, \quad \text{with nonlinear friction} \end{array} \tag{570}$$

with the same initial conditions

$$q(0) = 1, \quad \dot{q}(0) = 0. \tag{571}$$

Use the multiscale method to show that, respectively:

$$\begin{array}{l} q(t) = \cos\left(t - \frac{1}{16}\epsilon t\right) + O(\epsilon), \\ q(t) = \left(1 + \frac{3}{4}\epsilon t\right)^{-1/2}\cos t + O(\epsilon) \end{array} \tag{572}$$

2) Use the multiscale method to construct the solution

$$q(t) = \frac{a_0 e^{\epsilon t/2}}{\sqrt{1 + \left(\frac{a_0}{2}\right)^2 (e^{\epsilon t} - 1)}}\cos(t + \phi_0) + O(\epsilon) \tag{573}$$

of the Van Der Pol oscillator

$$\ddot{q} + q - \epsilon(1 - q^2)\dot{q} = 0, \tag{574}$$

and show that

$$q(t) \to 2\cos(t+\phi_0) + O(\epsilon), \quad t \to \infty, \tag{575}$$

i.e., the solution tends to a limiting cycle (at O(ϵ): the circle of radius 2).

3) Derive the Hopf equation $u_t + uu_x = 0$ from the Riemann equation $u_t + c(u)u_x = 0$ using multiscale expansions, in the weakly nonlinear regime.

4) Derive the Burgers equation $u_t+uu_x = \nu u_{xx}$ from the following class $u_t+c(u)u_x = (D(u)u_x)_x,\ D(u) > 0$ of PDEs, in the weakly nonlinear regime, using multiscale expansions.

5) Derive the KdV equation $u_t+uu_x+u_{xxx} = 0$ from the following class $u_t+c(u)u_x+K_1(u)[K_2(u)(K_3(u)u_x)_x]_x = 0$ of nonlinear dispersive PDEs, using multiscale expansions.

6) Derive the NLS equation from the Sine Gordon equation $u_{tt} - c^2u_{xx} + \mu^2 \sin u = 0$ (or, more in general, from a large class of nonlinear dispersive PDEs), using multiscale expansions.

7) Derive the NLS equation from the KdV equation $u_t + uu_x + u_{xxx} = 0$ using multiscale expansions.

8) Derive the dKP(3,1) equation $(u_t + uu_x)_x + u_{yy} + u_{zz} = 0$ from the equations of Acoustics, under the hypothesis of i) weak nonlinearity and ii) quasi one-dimensionality.

9) i) Derive the equations of surface water waves from the Euler equations, linearize them under a small field hypothesis, and show their dispersive nature, with the dispersion relation

$$\omega^2(k) = gk\tanh(h\ k), \tag{576}$$

where g is the acceleration of gravity and h is the depth of the fluid.

10) i) Derive the KdV equation (see [1, 3]) in the context of surface water waves in $(1+1)$ dimensions, under the hypothesis of ia) small amplitudes and iib) shallow water ($kh \ll 1$, where k is the wave number and h is the depth of the fluid). ii) Derive the KP equation (see [2, 3]) in the context of surface water waves in $(2+1)$ dimensions, under the hypothesis of iia) small amplitudes, iib) shallow water, and iic) quasi one-dimensionality. Show that, neglecting dispersion, one obtains the dKP(2,1) equation.
iii) Derive (see [3]) the NLS equation in the context of surface water waves in $(1+1)$ dimensions, under the hypothesis of iiia) small amplitude ($a << \lambda$) and iiib) quasi monocromatic waves in sufficiently deep water ($kh \gg 1$). iv) Derive its multidimensional generalization in the context of surface water waves in $(2+1)$ dimensions.

11) Derive (see [**?**]) the NLS equation in the framework of Langmuir waves in a plasma, described by the system of equations:

$$n_t + (nv)_x = 0, \quad v_t + vv_x = \phi_x - n_x/n, \quad \phi_{xx} = n - 1,$$

with boundary onditions $n \to 1,\ v \to 0,\ \phi \to 0$ as $|x| \to \infty$, where n is the electron density, v is the electron velocity and ϕ is the electrostatic potential in dimensionless variables, expanding the fields around the equilibrium solution:

$$n = 1 + \epsilon n_1 + \epsilon^2 n_2 + O(\epsilon^3), \quad v = \epsilon v_1 + O(\epsilon^2), \quad \phi = \epsilon\phi_1 + O(\epsilon^2).$$

12) Derive (see [14]) the NLS equation in nonlinear optics, for a homogeneous and isotropic dielectric.

6 Soliton Equations and the IST Method

6.1 The KdV example

The KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \quad u = u(x,t) \in \mathbb{R}, \tag{577}$$

(we have rescaled the independent and dependent variables to reach this convenient form) can be written as the integrability (compatibility) condition of the following pair (the so-called "Lax pair") of linear differential equations for the auxiliary field ψ:

$$\begin{aligned} &L\psi = E\psi, \quad \psi = \psi(x,t,E), \\ &\psi_t = M\psi, \end{aligned} \tag{578}$$

where L, M are the following differential operators

$$\begin{aligned} &L := -\partial_x^2 + u(x,t), \\ &M := -4\partial_x^3 + 6u\partial_x + 3u_x + c(k) = -4\partial_x^3 + 3(u\partial_x + \partial_x u) + c(k), \end{aligned} \tag{579}$$

and $c(k)$ is an arbitrary scalar such that $c_x = 0$.

We remark that the operator L is the Schrödinger operator and the first equation in (578) is the time independent Schrödinger equation with energy E and eigenfunction ψ. We have the following results:

$$E_t = 0 \quad \Leftrightarrow \quad L_t + [L, M] = 0 \quad \Leftrightarrow \quad u_t - 6uu_x + u_{xxx} = 0, \tag{580}$$

implying that the KdV dynamics is "isospectral": $u_t - 6uu_x + u_{xxx} = 0 \ \Leftrightarrow \ E_t = 0$; i.e., the spectrum of the Schrödinger operator is a constant of motion iff the potential u of the Schrödinger operator evolves according to the KdV equation. Let's prove the first equivalence in (580):

$$\begin{aligned} &(L\psi)_t = L_t\psi + L\psi_t = (L_t + LM)\psi; \\ &(E\psi)_t = E_t\psi + E\psi_t = E_t\psi + EM\psi \\ &= E_t\psi + M(E\psi) = (E_t + ML)\psi \end{aligned} \tag{581}$$

implying

$$(L_t + [L, M])\,\psi = E_t\psi. \tag{582}$$

If $L_t + [L, M] = 0$, then $E_t = 0$; if $E_t = 0$, then $(L_t + [L, M])\psi = 0$ for every eigenfunction ψ, implying $L_t + [L, M] = 0$.

The second equivalence is left as an exercise to the reader.

We also observe that, taking the x-derivative of the Schrödinger equation: $\psi_{xxx} = u_x\psi + (u - E)\psi_x$, the second of equations (578) can be rewritten as

$$\psi_t = (c(k) - u_x)\psi + (4E + 2u)\psi_x, \tag{583}$$

expressing ψ_t in term of ψ and ψ_x only.

Our goal is to use the above Lax pair to solve the Cauchy problem for the KdV equation on the line, for localized u's

$$\begin{aligned} &u_t - 6uu_x + u_{xxx} = 0, \quad u = u(x,t), \\ &u(x,0) = u_0(x), \\ &u(x,t) \to 0, \quad x \to \pm\infty, \end{aligned} \tag{584}$$

through the Inverse Scattering (or Spectral) Transform (IST) method, summarized in the following scheme (see Fig. 36)

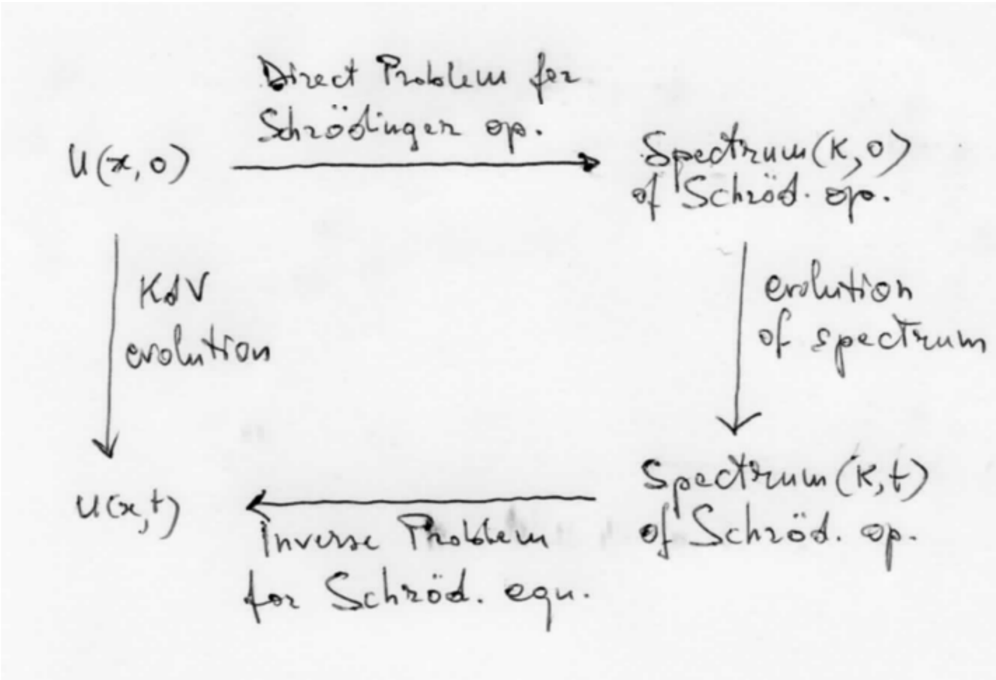


Figure 36: The IST scheme for KdV

6.1.1 Direct problem

It is convenient to write the energy E in terms of the momentun (wave number) k:

$$E = k^2. \tag{585}$$

If $E > 0$, then $k \in \mathbb{R}$ and since $u \to 0,\ x \to \pm\infty$, for large values of $|x|$, ψ oscillates like $\exp(\pm ikx)$. It follows that $\psi \notin L^2(\mathbb{R})$ and the spectrum in continuous ($\psi(x, k)$ does not describes quantum particles, but diffusion states). In addition, for a given energy E, there are two independent solutions corresponding to $\pm k$ (the continuous spectrum is doubly degenerate).

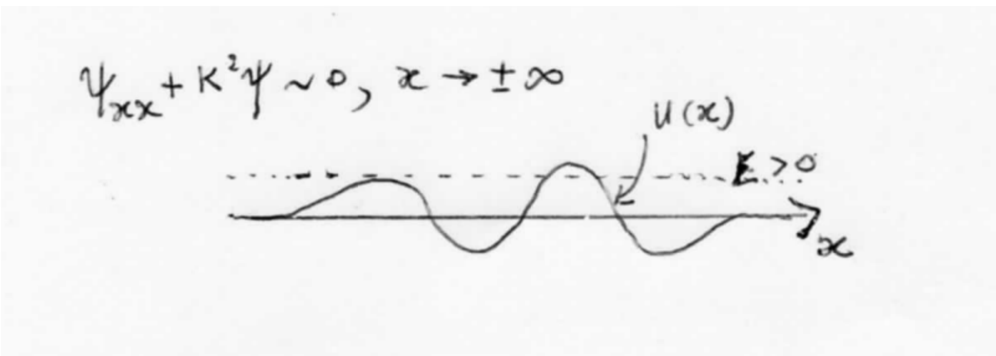


Figure 37: If $E > 0$, the eigenfunctions of a localized potential oscillate at $x \sim \pm\infty$.

We introduce the so-called Jost eigenfunctions

$$\left.\begin{array}{l}\psi_1(x,k)\sim e^{-ikx},\\ \psi_2(x,k)\sim e^{ikx},\end{array}\right\}\quad x\sim+\infty, \tag{586}$$

$$\left.\begin{array}{l}\varphi_1(x,k)\sim e^{-ikx},\\ \varphi_2(x,k)\sim e^{ikx},\end{array}\right\}\quad x\sim-\infty. \tag{587}$$

and $\{\psi_1(x,k),\psi_2(x,k)\}$ and $\{\varphi_1(x,k),\varphi_2(x,k)\}$ are both good basis in the space of solutions of the Schrödinger equation.

In addition there are two symmetries
1. since $E=k^2$, $k\to -k$ is a symmetry: *if $\psi(x,k)$ is a solution of the Schrödinger equation, also $\psi(x,-k)$ is a solution.* It follows that

$$\psi_2(x,k)=\psi_1(x,-k),\quad \varphi_2(x,k)=\varphi_1(x,-k), \tag{588}$$

since $\psi_2(x,k),\psi_1(x,-k)$ are solutions of the Schrödinger equation with the same asymptotic behavior, they coincide.
2. since u is real, and $k\in\mathbb{R}$, if $\psi(x,k)$ is solution then $\overline{\psi(x,k)}$. It follows that

$$\psi_2(x,k)=\overline{\psi_1(x,k)},\quad \varphi_2(x,k)=\overline{\varphi_1(x,k)}. \tag{589}$$

Since the Schrödinger equation is real and $\psi_2(x,k),\overline{\psi_1(x,k)}$ are solutions of the Schrödinger equation with the same asymptotic behavior, they coincide.
3. As a generalization of the symmetry 2, if the eigenfunction can be analytically prolonged off the real k-axis, then, if $\psi(x,k)$ is solution, then $\overline{\psi(x,\bar{k})}$ is also solution.

We can express the four eigenfunctions in terms of only two eigenfunctions: $\psi(x,k),\ \varphi(x,k)$:

$$\begin{array}{l}\varphi_1(x,k)=\varphi(x,k),\quad \varphi_2(x,k)=\varphi(x,-k)=\overline{\varphi(x,k)},\\ \psi_1(x,k)=\psi(x,k),\quad \psi_2(x,k)=\psi(x,-k)=\overline{\psi(x,k)}.\end{array} \tag{590}$$

Writing the φ eigenfunctions in terms of the basis $\{\psi,\bar{\psi}\}$ we obtain the "scattering equation"

$$\varphi(x,k)=a(k)\psi(x,k)+b(k)\overline{\psi(x,k)},\quad k\in\mathbb{R}. \tag{591}$$

In matrix form:

$$\begin{array}{c}\left(\begin{array}{c}\varphi(x,k)\\ \overline{\varphi(x,k)}\end{array}\right)=S(k)\left(\begin{array}{c}\psi(x,k)\\ \overline{\psi(x,k)}\end{array}\right),\\ S(k)=\left(\begin{array}{cc}a(k) & b(k)\\ \overline{b(k)} & \overline{a(k)}\end{array}\right),\end{array} \tag{592}$$

where $S(k)$ is the so called scattering matrix.

Using the Wronskian theorem stating that the Wronskian $W(f_1,f_2):=f_1f_{2x}-f_{1x}f_2$ of two solutions of the Schrödinger equation is x independent, it is convenient to evaluate the Wronskian of the Jost eigenfunctions at ∞:

$$\begin{array}{l}W(\varphi,\bar{\varphi})=\varphi\bar{\varphi}_x-\varphi_x\bar{\varphi}\Big|_{x\sim-\infty}=e^{-ikx}ike^{ikx}-(-ik)e^{-ikx}e^{ikx}=2ik,\\ W(\psi,\bar{\psi})=\psi\bar{\psi}_x-\psi_x\bar{\psi}\Big|_{x\sim\infty}=2ik.\end{array} \tag{593}$$

In addition:

$$\begin{array}{l}2ik=W(\varphi,\bar{\varphi})=W(a\psi+b\bar{\psi},\bar{b}\psi+\bar{a}\bar{\psi})=(a\psi+b\bar{\psi})(\bar{b}\psi_x+\bar{a}\bar{\psi}_x)\\ -(a\psi_x+b\bar{\psi}_x)(\bar{b}\psi+\bar{a}\bar{\psi})\Big|_{x\sim\infty}=(|a|^2-|b|^2)W(\psi,\bar{\psi})=2ik(|a|^2-|b|^2),\end{array} \tag{594}$$

from which

$$|a(k)|^2-|b(k)|^2=1\quad(\det S(k)=1),\quad k\in\mathbb{R}. \tag{595}$$

The coefficients of the scattering matrix are related to the well-know reflection and transmission coefficients as follows

$$\begin{array}{ll} \frac{b(k)}{a(k)} = R(k) & \text{reflection coefficient,} \\ \frac{1}{a(k)} = T(k) & \text{transmission coefficient.} \end{array} \tag{596}$$

Indeed

$$\frac{\varphi(x,k)}{a(k)} = \psi(x,k) + \frac{b(k)}{a(k)}\overline{\psi(x,k)}, \quad k \in \mathbb{R}, \tag{597}$$

$$\frac{\varphi(x,k)}{a(k)} \sim \begin{cases} e^{-ikx} + R(k)e^{ikx}, & x \sim \infty, \\ T(k)e^{-ikx}, & x \sim -\infty \end{cases} \tag{598}$$

(see Fig. 38)

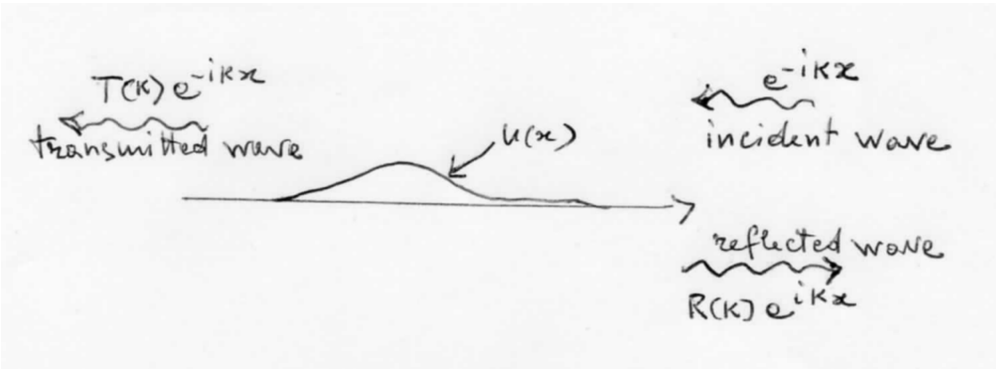


Figure 38: Reflected and transmitted waves by a localized potential.

implying that the scattering is unitary:

$$|T(k)|^2 + |R(k)|^2 = 1 \tag{599}$$

(the probability that an incident particle is transmitted plus the probability that an incident particle is reflected is equal to 1).

Analiticity properties in k. To study the analyticity properties of the eigenfunctions in k, it is necessary to convert the Schrödinger equation + boundary conditions defining them into a single integral equation.

To do so, we use the following result:
If $G(x-y,k)$ is a Green's function of the harmonic oscillator:

$$\left(\frac{d^2}{dx^2} + k^2\right) G(x-y,k) = \delta(x-y), \tag{600}$$

then the solution $\psi(x,k)$ of the following integral equation

$$\psi(x,k) = \psi_0(x,k) + \int_{\mathbb{R}} G(x-y,k)u(y)\psi(y,k)dy \tag{601}$$

(where $\psi_0(x,k)$ is any solution of the harmonic oscillator equation above) is a solution of the Schrödinger equation

$$\left(\frac{d^2}{dx^2} - u(x) + k^2\right) \psi(x,k) = 0. \tag{602}$$

(VERIFY IT applying the harmonic oscillator operator to the integral equation).

We start with the Fourier integral representation of the Green's function

$$G(x,k) = \int_{\mathbb{R}} \frac{dp}{2\pi} \hat{G}(p,k) e^{ipx} \tag{603}$$

implying that

$$\begin{aligned} &\left(\tfrac{d^2}{dx^2} + k^2\right) G(x,k) = \textstyle\int_{\mathbb{R}} \frac{dp}{2\pi} (k^2 - p^2) \hat{G}(p,k) e^{ipx} \\ &= \delta(x) = \textstyle\int_{\mathbb{R}} \frac{dp}{2\pi} e^{ipx} \quad \Rightarrow \quad \hat{G}(p,k) = (k^2 - p^2)^{-1}. \end{aligned} \tag{604}$$

The obtained Green's function

$$G(x,k) = -\int_{\mathbb{R}} \frac{dp}{2\pi} \frac{e^{ikx}}{p^2 - k^2}, \quad k \in \mathbb{R}, \tag{605}$$

has two polar singularities on the real axis that must be avoided to have a finite integral. Let $G^+(x,k)$ and $G^-(x,k)$ the two Green's functions obtained respectively passing below and above both singularities. Then a simple exercice in contour integration gives

$$G^{\pm}(x,k) = \pm H(\pm x) \frac{\sin(kx)}{k}. \tag{606}$$

G^+ is the so-called retarded (causal) Green's function, and G^- is the so-called advanced (anti-causal) Green's function. But since there is no time here, they are both acceptable.

It is easy to verify that the Jost eigenfunctions φ and ψ are constructed using respectively G^+ and G^-, and satisfy the following integral equations

$$\begin{aligned} \varphi(x,k) &= e^{-ikx} + \int_{-\infty}^{x} \tfrac{\sin k(x-y)}{k} u(y) \varphi(y,k) dy, \\ \psi(x,k) &= e^{-ikx} - \int_{x}^{\infty} \tfrac{\sin k(x-y)}{k} u(y) \psi(y,k) dy. \end{aligned} \tag{607}$$

To study the analyticity properties, it is convenient to introduce the functions

$$\mu^+(x,k) := \varphi(x,k) e^{ikx}, \quad \mu^-(x,k) := \psi(x,k) e^{ikx}, \tag{608}$$

satisfying the integral equations

$$\begin{aligned} \mu^+(x,k) &= 1 + \int_{-\infty}^{x} \tfrac{e^{2ik(x-y)}-1}{2ik} u(y) \mu^+(y,k) dy, \\ \mu^-(x,k) &= 1 - \int_{x}^{\infty} \tfrac{e^{2ik(x-y)}-1}{2ik} u(y) \mu^-(y,k) dy. \end{aligned} \tag{609}$$

We look for the solution of the integral equation for μ^+ as a Neumann series

$$\mu^+(x,k) = 1 + \sum_{k \geq 1} \mu_j(x,k), \tag{610}$$

obtaining the iteration formula

$$\begin{aligned} &\mu_{j+1}(x,k) = \int_{-\infty}^{x} K(x-y,k) u(y) \mu_j(y,k), \ j \geq 0, \\ &\mu_0 = 1, \quad K(x,k) := \tfrac{e^{2ikx}-1}{2ik}. \end{aligned} \tag{611}$$

We first observe that, if $\mathrm{Im}k > 0$, $\mu_1(x,k) = \int_{-\infty}^{x} K(x-y,k) u(y) dy$ is analytic in the upper half k-plane, therefore $\mu_2(x,k) = \int_{-\infty}^{x} K(x-y,k) u(y) \mu_1(y,k) dy$ is also analytic in the upper half k-plane; and so on:

all terms of the Neumann series are analytic for Im$k > 0$. To extend the analyticity property to the sum $\mu^+(x,k)$ of the Neumann series (610) it is necessary to prove its uniform convergence.

We first observe that

$$|K(x-y,k)| \leq \frac{e^{-2Imk(x-y)}+1}{2|k|} \leq \frac{1}{|k|}; \tag{612}$$

therefore

$$\begin{aligned} &|\mu_{j+1}(x,k)| \leq \int\limits_{-\infty}^{x} |K(x-y,k)||u(y)||\mu_j(y,k)|dy \\ &\leq \tfrac{1}{|k|} \int\limits_{-\infty}^{x} |u(y)||\mu_j(y,k)|dy. \end{aligned} \tag{613}$$

Define

$$A(x) := \int\limits_{-\infty}^{x} |u(y)|dy, \tag{614}$$

then

$$\begin{aligned} &|\mu_1(x,k)| \leq \tfrac{1}{|k|}A(x), \\ &|\mu_2(x,k)| \leq \tfrac{1}{|k|} \int\limits_{-\infty}^{x} |u(y)||\mu_1(y,k)|dy \\ &\leq \tfrac{1}{|k|^2} \int\limits_{-\infty}^{x} |u(y)|A(y)dy = \tfrac{1}{|k|^2} \int\limits_{-\infty}^{x} \left(\tfrac{A^2(y)}{2}\right)_y dy \\ &= \tfrac{1}{2}\left(\tfrac{A(x)}{|k|}\right)^2. \end{aligned} \tag{615}$$

By induction one can show that

$$|\mu_n(x,k)| \leq \frac{1}{n!}\left(\frac{A(x)}{|k|}\right)^n \leq \frac{1}{n!}\left(\frac{\|u\|_1}{|k|}\right)^n, \tag{616}$$

and we conclude that

$$\begin{aligned} &|\mu^+(x,k)| \leq 1 + \sum_{n\geq 1} |\mu_n(x,k)| \leq \sum_{n\geq 0} \tfrac{1}{n!}\left(\tfrac{\|u\|_1}{|k|}\right)^n \\ &= \exp\left(\tfrac{\|u\|_1}{|k|}\right), \end{aligned} \tag{617}$$

proving the total convergence of the Neumann series. The total convergence of (610) for Im$k \geq 0$, $k \neq 0$, and $u \in L^1(\mathbb{R})$ implies that $\mu^+(x,k)$ is analytic for Im$k > 0$. Under these conditions, the eigenfunction $\varphi(x,k)$ exists unique, and is analytic for Im$k > 0$. Following the same reasoning, one shows that $\mu^-(x,k)$ (and $\psi(x,k)$) are analytic for Im$k < 0$.

It is possible to control the singular point $k = 0$ restricting a bit the properties of u. For $k = 0$ the integral equation (609) for μ^+ becomes

$$f(x) = 1 + \int\limits_{-\infty}^{x} (x-y)u(y)f(y)dy, \quad f(x) := \mu^+(x,0). \tag{618}$$

Looking again for a solution in the form of Neumann series

$$f(x) = 1 + \sum_{n\geq 1} f_n(x) \tag{619}$$

we obtain the recursion

$$f_{j+1}(x) = \int\limits_{-\infty}^{x} (x-y)u(y)f_j(y)dy, \quad j \geq 0, \tag{620}$$

and the inequality

$$|f_{j+1}(x)| \leq \int\limits_{-\infty}^{x} (x-y)|u(y)||f_j(y)|dy, \quad j \geq 0. \tag{621}$$

Therefore

$$|f_1(x)| \leq M(x), \tag{622}$$

where

$$M(x) := \int\limits_{-\infty}^{x} (x-y)|u(y)|dy. \tag{623}$$

Analogously:

$$\begin{aligned} |f_2(x)| &\leq \int\limits_{-\infty}^{x} (x-y)|u(y)||f_1(y)|dy \\ &\leq \int\limits_{-\infty}^{x} dy(x-y)|u(y)| \int\limits_{-\infty}^{y} dy'(y-y')|u(y')| \\ &\leq \int\limits_{-\infty}^{x} dy(x-y)|u(y)| \int\limits_{-\infty}^{y} dy'(x-y')|u(y')| \\ &= \frac{1}{2}\int\limits_{-\infty}^{x} \left(\left(\int\limits_{-\infty}^{y} (x-y')|u(y')|dy'\right)^2\right)_y dy \\ &= \frac{1}{2}M^2(x), \end{aligned} \tag{624}$$

and one can show by induction that

$$|h_j(x)| \leq \frac{1}{j!}M^j(x). \tag{625}$$

Therefore

$$|f(x)| \leq 1 + \sum_{j\geq 1}|f_j(x)| \leq \sum_{j\geq 0}\frac{M^j(x)}{j!} = \exp\left(M(x)\right), \tag{626}$$

and the convergence is uniform in any compact of the interval $(-\infty, a]$, $\forall a \in \mathbb{R}$. It follows that the Jost eigenfunction is well defined at $k = 0$ if $u(x) \in L^1_{loc}(\mathbb{R})$ and goes to 0 at $x \to \pm\infty$ faster than $1/x^2$; equivalently, if

$$u \in L^1_1(\mathbb{R}) \quad \Leftrightarrow \quad \int_{\mathbb{R}}(1+|x|)|u(x)|dx < \infty. \tag{627}$$

The analyticity properties of the scattering coefficients can be found from the Wronskians:

$$\begin{aligned} W(\varphi,\bar{\psi}) &= W(a\psi + b\bar{\psi},\bar{\psi}) = a(k)W(\psi,\bar{\psi}) = 2ik\ a(k), \\ W(\psi,\varphi) &= W(\psi, a\psi + b\bar{\psi}) = b(k)W(\psi,\bar{\psi}) = 2ik\ b(k). \end{aligned} \tag{628}$$

Another useful representation of $a(k), b(k)$ comes from comparing the scattering equation (591) and the first of the integral equations (607) at $x \sim \infty$:

$$\begin{aligned} &\varphi(x,k) \sim a(k)e^{-ikx} + b(k)e^{ikx}, \\ &\varphi(x,k) \sim e^{-ikx} + \int_{\mathbb{R}} \tfrac{e^{ik(x-y)} - e^{-ik(x-y)}}{2ik} u(y)\varphi(y,k)dy \\ &= e^{-ikx}\left(1 - \tfrac{1}{2ik}\int_{\mathbb{R}} u(y)\mu^+(y,k)dy\right) + e^{ikx}\tfrac{1}{2ik}\int_{\mathbb{R}} u(y)\mu^+(y,k)e^{-2ikx}dy, \end{aligned} \tag{629}$$

obtaining

$$\begin{aligned} a(k) &= 1 - \tfrac{1}{2ik}\int_{\mathbb{R}} u(y)\mu^+(y,k)dy, \\ b(k) &= \tfrac{1}{2ik}\int_{\mathbb{R}} u(y)\mu^+(y,k)e^{-2ikx}dy. \end{aligned} \tag{630}$$

Since $\varphi(x,k), \overline{\psi(x,k)}$ are analytic for $\mathrm{Im}k > 0$, it follows from (628) that $a(k)$ is also analytic for $\mathrm{Im}k > 0$, with asymptotics (from the first of (630))

$$a(k) = 1 - \frac{1}{2ik}\int_{\mathbb{R}} u(y)dy + O(k^{-2}), \quad |k| \gg 1. \tag{631}$$

Since $\psi(x,k), \varphi(x,k)$ have analyticity in opposite half planes, it follows from (629) that $b(k)$ does not have analyticity properties, in general, outside the real axis.

At last, for $|k| \gg 1$,

$$\begin{array}{l} \mu^+(x,k) = 1 + \int_{-\infty}^{x} \frac{e^{2ik(x-y)}-1}{2ik} u(y)\mu^+(y,k)dy \\ = 1 - \frac{1}{2ik} \int_{-\infty}^{x} u(y)dy + O(k^{-2}), \\ \mu^-(x,k) = 1 - \int_{x}^{\infty} \frac{e^{2ik(x-y)}-1}{2ik} u(y)\mu^-(y,k)dy \\ = 1 + \frac{1}{2ik} \int_{x}^{\infty} u(y)dy + O(k^{-2}), \end{array} \tag{632}$$

implying that

$$\begin{array}{l} \mu^{\pm}(x,k) = 1 + O(k^{-1}), \quad |k| \gg 1, \\ u(x) = -2i\partial_x \lim_{|k|\to\infty} [k(\mu^-(x,k) - 1)]. \end{array} \tag{633}$$

Dividing the scattering equation (591) by $a(k)$:

$$\frac{\varphi(x,k)}{a(k)} = \psi(x,k) + R(k)\overline{\psi(x,k)}, \quad k \in \mathbb{R}, \tag{634}$$

we obtain a form of the scattering equation

$$\frac{\mu^+(x,k)}{a(k)} = \mu^-(x,k) + R(k)e^{2ikx}\overline{\mu^-(x,k)}, \tag{635}$$

separating functions analytic in different half planes: $\mu^+(x,k)/a(k)$ analytic for $\mathrm{Im}k > 0$, apart from the zeroes of $a(k)$; $\mu^-(x,k)$ analytic for $\mathrm{Im}k < 0$, and $R(k)e^{-2ikx}\overline{\mu^-(x,k)}$ with no analiticity properties for $\mathrm{Im}k \neq 0$.

Discrete spectrum. Normalizable eigenfunctions, cannot oscillate at $\pm\infty$. It follows, from the Schrödinger equation $\psi_{xx} = (u - E)\psi$, that $u - E > 0$ at $x \sim \pm\infty$. Since $u \to 0$ as $x \to \pm\infty$, then the energy must be negative: $E < 0$, implying that $k = ip \in i\mathbb{R}$. Then $E = -p^2$, and choosing $p > 0$, the eigenfunctions behave at ∞ as follows

$$\psi \sim \left\{ \begin{array}{ll} ce^{px}, & x \sim -\infty, \\ be^{-px}, & x \sim \infty, \end{array} \right. \tag{636}$$

Usually they are normalized choosing $c = 1$:

$$\psi \sim \left\{ \begin{array}{ll} e^{px}, & x \sim -\infty, \\ be^{-px}, & x \sim \infty, \end{array} \right. \tag{637}$$

Therefore $\psi \in L^2(\mathbb{R})$, and since the the Schrödinger equation is areal equation with real boundary condition at $-\infty$, then the normalization coefficient is real too: $b \in \mathbb{R}$.

From qualitative considerations, for $E = -p^2 < 0,\ p > 0$, we have, in general, solutions of the form

$$\psi \sim \left\{ \begin{array}{ll} \alpha(E)e^{px} + \beta(E)e^{-px}, & x \sim -\infty, \\ \gamma(E)e^{px} + \delta(E)e^{-px}, & x \sim \infty, \end{array} \right. \tag{638}$$

To have a normalizable solution, we choose $\beta(E) = 0$, and, optionally, $\alpha(E) = 1$. Since I cannot fix more than two integration constants, $\gamma(E)$ and $\delta(E)$ are in general different from zero, and the solution blows up exponentially at $+\infty$: $\psi \sim \gamma(E)\exp(px) \to \infty$, at $x \to \infty$.

Consider a classical potential well $u(x) < 0$, and let $x_1 < x_2$ be the two points at which $u(x_j) = E,\ j = 1,2$ (the inversion points). Then the Schrödinger equation $\psi_{xx} = (u - E)\psi$ implies that, for $x \in (x_1, x_2)$, the concavity of ψ is towards the interior and ψ oscillates; if $x \notin (x_1, x_2)$, the concavity of ψ is towards the exterior and ψ decays or blows exponentially. If E is just above the minimum of the well, the oscillation is not enough to avoid the divergence at $+\infty$. Increasing E, $\gamma(E)$ decreases and ψ diverges more slowly until we reach a value of E, say, E_1, for which $\gamma(E_1) = 0$ and we have the fundamental state. Since $\gamma(E \neq 0)$ in a neighborhood of E_1 excluding E_1, the spectrum is discrete. Increasing more E, the curvature ψ_{xx}/ψ increases and the number of oscillations increases, until we reach the value E_2 for which $\gamma(E_2) = 0$ again, and we have the first excited case. And so on (see Fig. 39). The discrete spectrum $\{E_1, E_2, \ldots, E_n\}$ is inside the interval $(u_{min}, 0)$, with

$$E_1 < E_2 < \cdots < E_n < 0, \quad p_1 > p_2 > \cdots > p_n > 0. \tag{639}$$

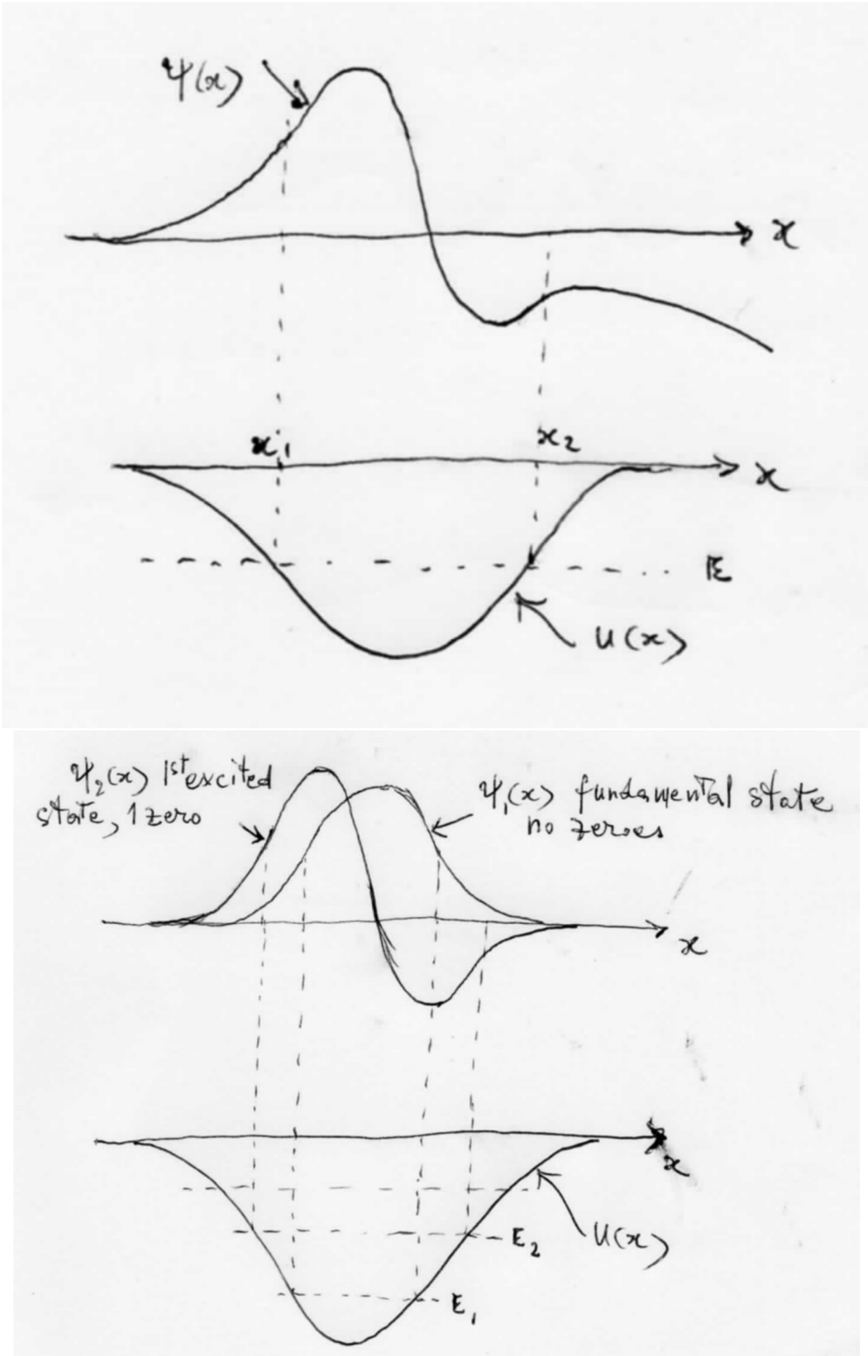


Figure 39:

Relation between the discrete spectum and the zeroes of $a(k)$**.** Let k_0 be a zero of $a(k)$. Since a cannot be zero on the real axis, due to the unitary relation (595), and since $a(k)$ is analytic for $\mathrm{Im}k > 0$, then $\mathrm{Im}k_0 > 0$. In addition, from (628), it follows that $W(\varphi(x, k_0), \bar{\psi}(x, k_0)) = 0$, implying that $\varphi(x, k_0)$

and $\overline{\psi(x,\bar{k}_0)}$ are linearly dependent:

$$\varphi(x,k_0)=b_0\overline{\psi(x,\overline{k_0})}=c\psi(x,-k_0). \tag{640}$$

It follows that

$$\varphi(x,k_0)\sim\left\{\begin{array}{ll} e^{-ik_0x}, & x\sim-\infty,\\ b_0e^{ik_0x}, & x\sim\infty. \end{array}\right. \tag{641}$$

Therefore k_0 belongs to the discrete spectrum: $E_0=k_0^2$. Since the Schrödinger operator is hermitian, then $E_0\in\mathbb{R}$; then $k_0=ip_0$, and Im$k_0>0$ implies that $p_0>0$. In addition

$$\varphi_0(x):=\varphi(x,ip_0)\sim e^{p_0x},\quad x\sim-\infty. \tag{642}$$

All this is valid for all zeroes $k_n=ip_n,\ p_n>0$ of a:

$$\varphi(x,ip_n)=b_n\overline{\psi(x,\overline{ip_n})}=b_n\psi(x,-ip_n), \tag{643}$$

$$\varphi_n(x):=\varphi(x,ip_n)\sim\left\{\begin{array}{ll} e^{p_nx}, & x\sim-\infty,\\ b_ne^{-p_nx}, & x\sim\infty, \end{array}\right. \tag{644}$$

In addition, since the Schrödinger equation is real, and $\varphi_n(x)$ is real at $-\infty$, it follows that $\varphi_n(x)\in\mathbb{R}$; consequently $b_n\in\mathbb{R}$.

Its is easy to convince one self that these zeroes are finite: they are zeroes of an analytic function for Im$k>0$, therefore they are isolated for Im$k>0$, and they can eventually cluster only at the boundary: on the real line and at ∞. They cannot cluster on the real line due to the unitary condition $|a(k)|^2=1+|b(k)|^2>0$, for $k\in\mathbb{R}$. They cannot cluster at ∞, since there $a(k)\sim 1$.

It is also possible to prove that they are simple zeroes. We start with the Schrödinger equation for $\varphi(x,k)$:

$$\left(L-k^2\right)\varphi(x,k)=0,\quad L:=-\frac{d^2}{dx^2}+u(x),\quad \text{Im }k>0, \tag{645}$$

and we differentiate it with respect to k and evaluate the result at $k=ip_n$:

$$\partial_k\left(\left(L-k^2\right)\varphi(x,k)\right)\Big|_{k=ip_n}=(L+p_n^2)\varphi_k(x,ip_n)-2ip_n\varphi_n(x)=0. \tag{646}$$

Applying the operator

$$\int_{\mathbb{R}}dx\ \varphi_n(x)\cdot \tag{647}$$

to the above equation one obtains

$$\int_{\mathbb{R}}dx\varphi_n(x)(L+p_n^2)\varphi_k(x,ip_n)=2ip_n\int_{\mathbb{R}}\varphi_n^2(x)dx. \tag{648}$$

Integrating by parts twice the left hand side:

$$\begin{array}{l} -[\varphi_n(x)\varphi_k'(x,ip_n)-\varphi_n'(x)\varphi_k(x,ip_n)]_{-\infty}^{\infty}\\ +\int_{\mathbb{R}}dx\left((L+p_n^2)\varphi_n(x)\right)\varphi_k(x,ip_n)=2ip_n\int_{\mathbb{R}}\varphi_n^2(x)dx \end{array} \tag{649}$$

and using $(L+p_n^2)\varphi_n(x)=0$, one obtains

$$\left[\varphi_n'(x)\varphi_k(x,ip_n)-\varphi_n(x)\varphi_k'(x,ip_n)\right]_{-\infty}^{\infty}=2ip_n\int_{\mathbb{R}}\varphi_n^2(x)dx. \tag{650}$$

To evaluate the LHS we use the asymptotics of $\varphi(x,k)$:

$$\varphi(x,k)\sim\left\{\begin{array}{ll} e^{-ikx}, & x\sim-\infty,\\ a(k)e^{-ikx}+b(k)e^{ikx}, & x\sim\infty, \end{array}\right. \tag{651}$$

from which one obtains

$$\varphi_k(x,k) \sim \begin{cases} -ixe^{-ikx}, & x \sim -\infty, \\ (a'(k) - ixa(k))e^{-ikx} + (b'(k) + ixb(k))e^{ikx}, & x \sim \infty, \end{cases} \tag{652}$$

and

$$\varphi_k(x, ip_n) \sim \begin{cases} -ixe^{p_n x}, & x \sim -\infty, \\ a'(ip_n)e^{p_n x}, & x \sim \infty. \end{cases} \tag{653}$$

Substituting these asymptotics in the LHS of (650) we obtain

$$-2p_n b_n a'(ip_n) = 2ip_n \int_{\mathbb{R}} \varphi_n^2(x)dx \tag{654}$$

implying

$$ib_n a'(ip_n) = \int_{\mathbb{R}} \varphi_n^2(x)dx > 0. \tag{655}$$

Since the RHS is positive, it follows that ip_n is a simple zero of $a(k)$. In addition, since b_n is real, it follows that $ia'(ip_n)$ is also real, with the same sign of b_n.

6.1.2 Inverse problem

In the inverse problem we reconstruct the potential $u(x)$ from a suitable set of spectral data, and we make essential use of the analyticity properties of the eigenfunctions. Therefore we begin with the introduction of the "analyticity projectors" $P^{\pm}$ defined by

$$P^{\pm} f(k) := \pm \frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \frac{f(k')}{k' - (k \pm i\epsilon)} dk', \qquad k \in \mathbb{R}. \tag{656}$$

They map a Holder function [1] $f(k)$, $k \in \mathbb{R}$ decaying at ∞ sufficiently fast into functions analytic in the upper and lower halves of the complex k plane respectively, and satisfy the projection properties

$$P^+P^- = P^-P^+ = 0, \quad {P^+}^2 = P^+, \quad {P^-}^2 = P^-, \quad P^+ + P^- = 1. \tag{657}$$

$f^{\pm}(k) := P^{\pm} f(k)$ are indeed analytic in the upper and lower halves of the complex k plane respectively. For $k + i\epsilon$ and $\text{Im}k \geq 0$, $f^+(k)$ is well defined and analytic for $\text{Im}k > 0$; analogously, for $k - i\epsilon$ and $\text{Im}k \leq 0$, $f^-(k)$ is well defined and analytic for $\text{Im}k < 0$. In addition:

$$P^{\pm} f(k) \sim \mp \frac{1}{2i\pi k} \int_{\mathbb{R}} f(k')dk', \quad |k| \gg 1. \tag{658}$$

They also satisfy the following Plemelj-Sokhotsky formulas:

$$P^{\pm} f(k) = \pm \frac{1}{2\pi i} P \int_{\mathbb{R}} \frac{f(k')}{k' - k} dk' + \frac{1}{2} f(k). \tag{659}$$

To show it quickly, we also assume that $f(k)$ be analytic in a very thin horizontal strip including the real axis. Then

$$\begin{aligned} P^{\pm} f(k) &= \pm \tfrac{1}{2\pi i} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \tfrac{f(k')}{k' - (k \pm i\epsilon)} dk' = \pm \tfrac{1}{2\pi i} \lim_{\epsilon \to 0^+} \int_{\gamma^{\pm}} \tfrac{f(k')}{k' - k} dk' \\ &= \pm P \tfrac{1}{2\pi i} \int_{\mathbb{R}} \tfrac{f(k')}{k' - k} dk' + \tfrac{1}{2} f(k), \end{aligned} \tag{660}$$

where the contours $\gamma^{\pm}$ are shown in Fig. 44.

---

[1] A function $f(k)$ in Holder in $[a, b]$ if there exist $c > 0$ and $0 < \mu < 1$ such that $|f(k_1) - f(k_2)| < c|k_1 - k_2|^{\mu}$ $\forall k_1, k_2 \in [a, b]$.

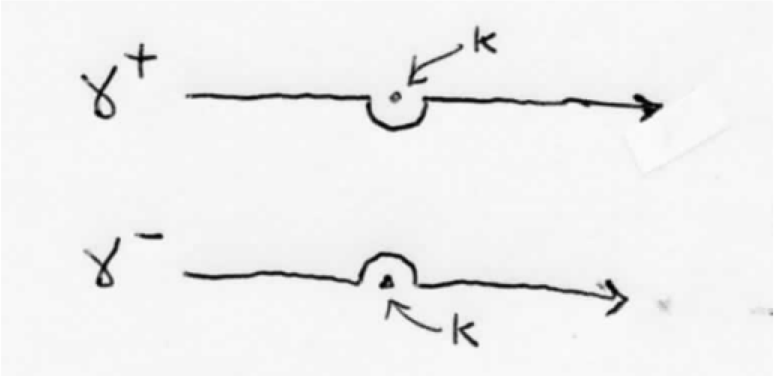


Figure 40:

Moreover the properties (765) are satisfied; f.i.:

$$P^+P^-f(k) = P^+f^-(k) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f^-(k')}{k'-(k+i\epsilon)}dk = 0, \tag{661}$$

closing the contour downstairs, and using the analyticity properties of $f^-(k)$ and the Cauchy theorem;

$$P^{+2}f(k) = P^+f^+(k) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f^+(k')}{k'-(k+i\epsilon)}dk = f^+(k) = P^+f(k), \tag{662}$$

closing the contour upstairs, and using the analyticity properties of $f^+(k)$ and the residue theorem. At last the condition $P^+ + P^- = 1$ follows directly from (768).

After these preliminaries, we begin with the scattering equation (635)

$$\frac{\mu^+(x,k)}{a(k)} = \mu^-(x,k) + R(k)e^{2ikx}\mu^-(x,-k), \quad k \in \mathbb{R}, \tag{663}$$

separating functions analytic in different half planes: $\mu^+(x,k)/a(k)$ analytic for Im$k > 0$, apart from the simple zeroes of $a(k)$; $\mu^-(x,k)$ analytic for Im$k < 0$, and $R(k)e^{-2ikx}\mu^-(x,-k)$ with no analiticity properties for Im$k \neq 0$.

Since $\frac{\mu^+(x,k)}{a(k)}$ is analytic for Im$k > 0$, apart from a finite number N of simple poles in $k_n = ip_n$, whose residues are

$$\begin{aligned}&\tfrac{\mu^+(x,ip_n)}{a'(ip_n)} = \tfrac{\varphi(x,ip_n)e^{-p_nx}}{a'(ip_n)} = \tfrac{b_n}{a'(ip_n)}\psi(x,-ip_n)e^{-p_nx}\\ &= \tfrac{b_n}{a'(ip_n)}\mu^-(x,-ip_n)e^{-2p_nx},\end{aligned} \tag{664}$$

it can be written as

$$\frac{\mu^+(x,k)}{a(k)} = h^+(x,k) + \sum_{n=1}^{N}\frac{b_n}{a'(ip_n)}\frac{\mu^-(x,-ip_n)e^{-2p_nx}}{k-ip_n}, \tag{665}$$

where $h^+(x,k)$ is analytic for Im$k > 0$. Therefore the scattering equation becomes

$$\begin{aligned}&h^+(x,k) + \sum_{n=1}^{N}\tfrac{b_n}{a'(ip_n)}\tfrac{\mu^-(x,-ip_n)e^{-2p_nx}}{k-ip_n} = \mu^-(x,k)\\ &+\left(P^+ + P^-\right)R(k)e^{2ikx}\mu^-(x,-k), \quad k \in \mathbb{R},\end{aligned} \tag{666}$$

having also used the last of properties (765).

Now we separate the functions analytic in the upper half plane from those analytic in the lower half plane:

$$\begin{aligned}&h^+(x,k) - P^+\left(R(k)e^{-2ikx}\mu^-(x,-k)\right) = \mu^-(x,k)\\ &-\sum_{n=1}^{N}\tfrac{b_n}{a'(ip_n)}\tfrac{\mu^-(x,-ip_n)e^{-2p_nx}}{k-ip_n} + P^-\left(R(k)e^{2ikx}\mu^-(x,-k)\right), \quad k \in \mathbb{R}.\end{aligned} \tag{667}$$

On the real axis, we have a function analytic in the upper half plane which is equal to a function analytic in the lower half plane. Therefore, using the Riemann analytic continuation through the boundary (real axis) between two domains (the upper and lower half k planes), the function on the LHS is the analytic continuation of the function on the RHS in the upper half k plane, and viceversa. In this way once defines an analytic function of k in the whole complex plane (an entire function). Since, due to the asymptotics (766),(633), the RHS tends to 1 at $k \sim \infty$, then the entire function is identically 1, by the first Liouville theorem. Therefore:

$$\mu^-(x,k) = 1 + i\sum_{m=1}^{N} \beta_m \frac{\mu^-(x,-ip_m)e^{-2p_m x}}{k-ip_m} + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{R(k')e^{2ik'x}\mu^-(x,-k')}{k'-(k-i\epsilon)}dk', \quad k \in \mathbb{R}, \tag{668}$$

where

$$\beta_m := \frac{b_m}{ia'(ip_m)} > 0, \quad m = 1,\dots,N. \tag{669}$$

This integral equation can be evaluated at $k = -ip_n,\ n = 1,\dots,N$, obtaining:

$$\mu^-(x,-ip_n) = 1 - \sum_{m=1}^{N} \beta_m \frac{\mu^-(x,-ip_m)e^{-2p_m x}}{p_m+p_n} + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{R(k')e^{2ik'x}\mu^-(x,-k')}{k'+ip_n}dk', \ n = 1,\dots,N. \tag{670}$$

Known the spectral data $\mathcal{S} = \{R(k), p_n, \beta_n, n = 1,\dots,N\}$, equations (668) and (670) are a closed system of $N+1$ linear equations for the $N+1$ unknowns $\mu^-(x,k),\ \mu^-(x,-ip_n),\ n = 1,\dots,N$. Once the solution $\mu^-(x,k)$ is constructed from it, the potential u is then reconstructed from (633):

$$u(x) = \partial_x\left(2\sum_{m=1}^{N}\beta_m\mu^-(x,-ip_m)e^{-2p_m x} + \frac{1}{\pi}\int_{\mathbb{R}} R(k)\mu^-(x,-k)e^{2ikx}dk\right). \tag{671}$$

This is the solution of the inverse problem.

6.1.3 Linear time evolution of the spectral data

We recall that the Jost eigenfunction φ evolves according to the equation

$$\varphi_t = (c(k) - u_x)\varphi + (4k^2 + 2u)\varphi_x, \tag{672}$$

with

$$\varphi(x,k) \sim \begin{cases} e^{-ikx}, & x \sim -\infty, \\ a(k)e^{-ikx} + b(k)e^{ikx}, & x \sim \infty. \end{cases} \tag{673}$$

Evaluating (672) at $x \sim -\infty$ one obtains

$$0 = [c(k) + 4k^2(-ik)]e^{-ikx} \quad \Rightarrow \quad c(k) = 4ik^3. \tag{674}$$

Evaluating (672) at $x \sim \infty$:

$$a_t e^{-ikx} + b_t e^{ikx} = 4ik^3(a(k)e^{-ikx} + b(k)e^{ikx}) + 4ik^3(-a(k)e^{-ikx} + b(k)e^{ikx}), \tag{675}$$

implying the following elementary time evolution of the scattering coefficients

$$a_t = 0, \quad b_t = 8ik^3 b \quad \Rightarrow \quad a(k,t) = a(k,0), \ b(k,t) = b(k,0)e^{8ik^3 t} \tag{676}$$

and of the reflection coefficient

$$R(k,t) = R(k,0)e^{8ik^3 t}. \tag{677}$$

From $a(k,t) = a(k,0)$ it follows that its zeroes are constant of motion:

$$p_n(t) = p_n(0). \tag{678}$$

Furthermore for the eigenfunction $\varphi_n(x) = \varphi(x, ip_n)$:

$$\varphi_{n\,t} \sim 4p_n^3 \varphi_n - 4p_n^2 \varphi_{n\,x}, \quad \varphi_n \sim b_n e^{-p_n x}, \quad x \sim \infty, \tag{679}$$

implying that $b_{n\,t} = 8p_n^3 b_n \;\Rightarrow\; b_n(t) = b_n(0)e^{8p_n^3 t}$. From (669) we conclude that

$$\beta_n(t) = \beta_n(0)e^{8p_n^3 t}. \tag{680}$$

Summarizing, the evolution of the spectral data is explicit:

$$\mathcal{S}(t) = \{R(k,t) = R(k,0)e^{8ik^3 t},\; p_n(t) = p_n(0),\; \beta_n(t) = \beta_n(0)e^{8p_n^3 t},\; n = 1, \dots, N\}, \tag{681}$$

and the IST is completed (see Fig. 41).

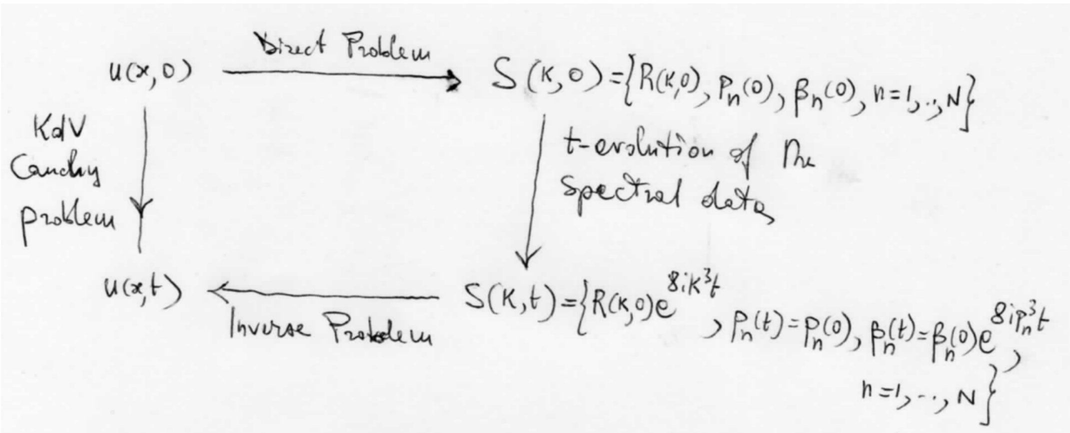


Figure 41:

6.1.4 Pure continuous spectrum

If the potential $u(x)$ does not support bound states, f.i., if $u > 0$, or if u is too small to support bound states, then the inverse problem reduces to

$$\begin{array}{l} \mu^-(x,t,k) = 1 + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{R(k',0)e^{2ik'x+8ik^3 t}\mu^-(x,t,-k')}{k'-(k-i\epsilon)}dk', \quad k \in \mathbb{R}, \\ u(x,t) = \frac{1}{\pi}\partial_x\left(\int_{\mathbb{R}} R(k,0)\mu^-(x,t,-k)e^{2ikx+8ik^3 t}dk\right). \end{array} \tag{682}$$

Therefore the KdV evolution, if the initial condition does not support a discrete spectrum, describes a nonlinear dispersive wave decaying as $1/\sqrt{t}$ for $t \gg 1$.

If $|u| \ll 1$, then $\mu^- \sim 1$ and

$$\begin{array}{l} u(x,t) \sim \frac{1}{\pi}\int_{\mathbb{R}} 2ikR(k,0)e^{i(2kx+8k^3 t)}dk = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}(\lambda)e^{i(\lambda x+\lambda^3 t)}d\lambda, \\ \hat{u}(\lambda) = i\lambda R(\lambda/2), \quad \lambda = 2k. \end{array} \tag{683}$$

Then **the spectral trasform reduces to the Fourier transform in the small field limit**.

6.1.5 Pure discrete spectum and solitons

If the potential $u(x)$ is reflectionless; i.e., if $R(k) = 0$, then the integral equations of the inverse problem reduce to

$$\begin{array}{l} \mu^-(x,t,k) = 1 + i\sum\limits_{m=1}^{N} \frac{\beta_m(0)e^{-2p_m x + 8p_m^3 t}}{k+ip_m}\mu_m(x,t), \quad k \in \mathbb{R}, \\ \mu_n(x,t) + \sum\limits_{m=1}^{N} \frac{\beta_m(0)e^{-2p_m x+8p_m^3 t}}{p_n+p_m}\mu_m(x,t) = 1, \\ \mu_m(x,t) := \mu^-(x,t,-ip_m), \end{array} \tag{684}$$

and

$$u(x) = 2\partial_x\left(\sum_{m=1}^{N}\beta_m(0)e^{-2p_m x+8p_m^3 t}\mu_m(x,t)\right). \tag{685}$$

Since $\beta_n > 0$, it is convenient to use the following notation:

$$\begin{array}{l} \frac{\beta_m(0)}{2p_m}e^{-2p_m x+8p_m^3 t} = e^{-2p_m X_m}, \\ X_m := x - 4p_m^2 t - \gamma_m, \\ \gamma_m := \frac{1}{2p_m}\ln\left(\frac{\beta_m(0)}{2p_m}\right), \end{array} \tag{686}$$

and equations (684), (685) become

$$\begin{array}{l} \mu^-(x,t,k) = 1 + i\sum\limits_{m=1}^{N}\frac{2p_m e^{-2p_m X_m}}{k+ip_m}\mu_m(x,t), \quad k \in \mathbb{R}, \\ \mu_n(x,t) + \sum\limits_{m=1}^{N}\frac{2p_m e^{-2p_m X_m}}{p_n+p_m}\mu_m(x,t) = 1, \\ u(x,t) = 4\partial_x\left(\sum\limits_{m=1}^{N} p_m e^{-2p_m X_m}\mu_m(x,t)\right). \end{array} \tag{687}$$

The second of equations (687) is the linear algebraic system of N equations solving the inverse problem. It is rewritten in the form

$$\left(1+e^{-2p_n X_n}\right)\mu_n(x,t) + \sum_{\substack{m=1 \\ m\neq n}}^{N}\frac{2p_m e^{-2p_m X_m}}{p_n+p_m}\mu_m(x,t) = 1. \tag{688}$$

6.1.6 The 1-soliton case

If $N = 1$, the algebraic system (688) reduces to a single equation

$$\left(1+e^{-2p_1 X_1}\right)\mu_1(x,t) = 1 \quad \Rightarrow \quad \mu_1(x,t) = (1+e^{-2p_1 X_1})^{-1} \tag{689}$$

and one obtains the celebrated 1-soliton solution of KdV:

$$\begin{array}{l} u(x,t) = 4p_1\partial_x\left(e^{-2p_1 X_1}\mu_1(x,t)\right) = 4p_1\partial_x\left(\frac{e^{-2p_1 X_1}}{1+e^{-2p_1 X_1}}\right) \\ = -2\partial_x^2\ln\left(1+e^{-2p_1 X_1}\right) = -\frac{2p_1^2}{\cosh^2(p_1 X_1)} = -\frac{2p_1^2}{\cosh^2(p_1(x-4p_1^2 t-\gamma_1))}. \end{array} \tag{690}$$

It is an exponentially localized solution, coming from the exact balance between the dispersion u_{xxx} and the steepening nonlinearity $-6uu_x$, whose velocity and amplitude are directly proportional to p_1^2, and the localization is directly proportional to p_1.

6.1.7 The 2-soliton case

If $N=2$ the algebraic system (688) becomes

$$\begin{array}{l} \left(1+e^{-2p_1X_1}\right)\mu_1+\frac{2p_2}{p_1+p_2}e^{-2p_2X_2}\mu_2=1, \\ \frac{2p_1}{p_1+p_2}e^{-2p_1X_1}\mu_1+\left(1+e^{-2p_2X_2}\right)\mu_2=1 \end{array} \tag{691}$$

with determinant

$$\Delta_2=1+e^{-2p_1X_1}+e^{-2p_2X_2}+\left(\frac{p_1-p_2}{p_1+p_2}\right)^2 e^{-2p_1X_1-2p_2X_2}. \tag{692}$$

Using the Cramer's rule we solve the system:

$$\mu_1(x,t)=\frac{1+\frac{p_1-p_2}{p_1+p_2}e^{-2p_2X_2}}{\Delta_2},\quad \mu_2(x,t)=\frac{1+\frac{p_2-p_1}{p_1+p_2}e^{-2p_1X_1}}{\Delta_2}, \tag{693}$$

and we obtain the 2-soliton solution, describing the nonlinear interaction of two solitons:

$$\begin{array}{l} u(x,t)=2\partial_x\left(\frac{2p_1\left(1+\frac{p_1-p_2}{p_1+p_2}e^{-2p_2X_2}\right)e^{-2p_1X_1}+2p_2\left(1+\frac{p_2-p_1}{p_1+p_2}e^{-2p_1X_1}\right)e^{-2p_2X_2}}{\Delta_2}\right) \\ =2\partial_x\left(\frac{2p_1e^{-2p_1X_1}+2p_2e^{-2p_2X_2}+2\frac{(p_1-p_2)^2}{p_1+p_2}e^{-2p_1X_1-2p_2X_2}}{\Delta_2}\right) \\ =-2\partial_x^2\ln\Delta_2. \end{array} \tag{694}$$

It is possible to prove that the N-soliton solution of KdV, describing the nonlinear interaction of N solitons, can be written in the compact form

$$u(x,t)=-2\partial_x^2\ln\Delta_N, \tag{695}$$

where Δ_N is the determinant of the algebraic system (688).

6.1.8 Elastic soliton interaction and phase shift

Here we use the 2-soliton solution (694) to describe the interaction of two solitons corresponding to the eigenvalues $p_1>p_2>0$. First we travel with soliton 1, i.e: $X_1=x-4p_1^2t-\gamma_1=O(1)$, or $x=4p_1^2t+const$. Consequently:

$$\begin{array}{l} -2p_2X_2=-2p_2(4p_1^2t+const-4p_2^2t-\gamma_2) \\ =-2p_2[4(p_1^2-p_2^2)t+c']\sim\left\{\begin{array}{ll} +\infty, & t\sim-\infty, \\ -\infty, & t\sim+\infty, \end{array}\right. \end{array} \tag{696}$$

It follows that, at $t\sim-\infty$:

$$\begin{array}{l} \Delta_2\sim e^{-2p_2X_2}\left(1+\left(\frac{p_1-p_2}{p_1+p_2}\right)^2e^{-2p_1X_1}\right)=e^{-2p_2X_2}\left(1+e^{-2p_1X_1^-}\right), \\ X_1^-:=x-4p_1^2t-\gamma_1+\delta_{12},\quad \delta_{12}:=\frac{1}{p_1}\ln\left(\frac{p_1+p_2}{p_1-p_2}\right)>0, \end{array} \tag{697}$$

and

$$\begin{array}{l} u(x,t)=-2\partial_x^2\ln\Delta_2\sim-2\partial_x^2\ln\left(e^{-2p_2X_2}\left(1+e^{-2p_1X_1^-}\right)\right) \\ =-2\partial_x^2\left[-2p_2X_2+\ln\left(1+e^{-2p_1X_1^-}\right)\right]=-2\partial_x^2\ln\left(1+e^{-2p_1X_1^-}\right) \\ =-\frac{2p_1^2}{\cosh^2(p_1(x-4p_1^2t-\gamma_1+\delta_{12}))},\quad t\sim-\infty. \end{array} \tag{698}$$

At $t\sim+\infty$, $\Delta_2\sim1+e^{-2p_1X_1}$, then

$$u(x,t)\sim-2\partial_x^2\ln\left(1+e^{-2p_1X_1}\right)=-2\frac{p_1^2}{\cosh^2(p_1(x-4p_1^2t-\gamma_1))},\ t\sim\infty. \tag{699}$$

Comparing the position $x_1(t)$ of the soliton 1 at $-\infty$ and at $+\infty$.

$$x_1(t) \sim \begin{cases} 4p_1^2 t + \gamma_1 - \delta_{12}, & t \sim -\infty, \\ 4p_1^2 t + \gamma_1, & t \sim +\infty, \end{cases} \tag{700}$$

we infer that, due to the nonlinear elastic interaction with the slower soliton 2, the faster soliton 1 experiences a shift forward, given by

$$\Delta x_1 = x_1(\infty) - x_1(-\infty) = \delta_{12} = \frac{1}{p_1} \ln \left(\frac{p_1 + p_2}{p_1 - p_2} \right) > 0. \tag{701}$$

Analogously, moving with soliton 2, it is possible to show that the slower soliton 2 experiences a shift backward, given by

$$\Delta x_2 = x_2(\infty) - x_2(-\infty) = \delta_{21} = -\frac{1}{p_2} \ln \left(\frac{p_1 + p_2}{p_1 - p_2} \right) < 0, \tag{702}$$

so that the total momentum is conserved (see Fig. 42).

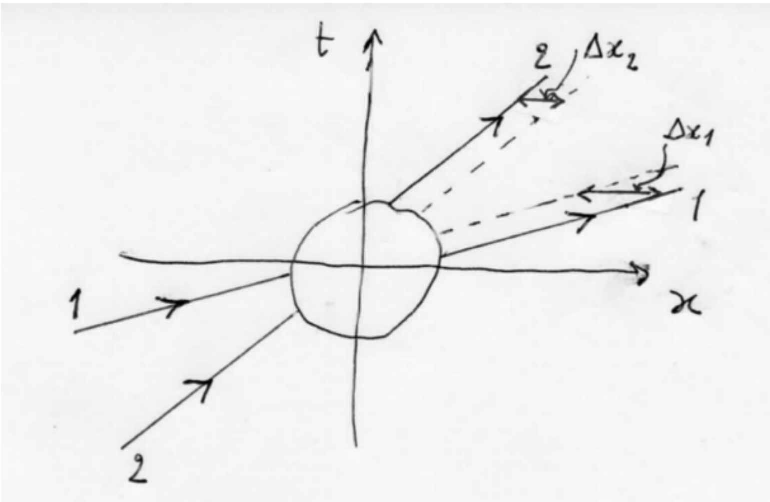


Figure 42:

In the general case of the interaction of N solitons, it is possible to show that the n^{th} soliton experiences the following phase shift

$$\Delta x_n = \frac{1}{p_n} \left(\prod_{j=n+1}^{N} \ln \left| \frac{p_n + p_j}{p_n - p_j} \right| - \prod_{j=1}^{n-1} \ln \left| \frac{p_n + p_j}{p_n - p_j} \right| \right) \tag{703}$$

indicating also that the interaction is pairwise.

6.2 The NLS example

The Cauchy problem for localized initial data, for the focusing and defocusing NLS equations:

$$\begin{array}{l} iu_t + u_{xx} + 2\sigma |u|^2 u = 0, \quad \sigma = \pm 1, \\ u(x,0) = u_0(x), \quad u(x,t) \to 0 \text{ for } x \to \pm\infty, \\ \sigma = 1 \text{ focusing NLS}, \quad \sigma = -1 \text{ defocusing NLS}, \end{array} \tag{704}$$

has been solved by Zakharov and Shabat in [43], using the Inverse Spectral Transform (IST) method, originally discovered by Gardner, Green, Kruskal and Miura in [16] to solve the Cauchy problem for the Korteweg - de Vries (KdV) equation. See also the following books [42, 3, 10, 2], in which several aspects of the method have been discussed in detail.

Lax pair. The NLS equations are the compatibility condition of the following Zakharov-Shabat Lax pair:

$$\underline{\psi}_x(\lambda,x,t) = X(\lambda,x,t)\underline{\psi}(\lambda,x,t), \tag{705}$$

$$\underline{\psi}_t(\lambda,x,t) = T(\lambda,x,t)\underline{\psi}(\lambda,x,t), \tag{706}$$

where

$$\underline{\psi}(\lambda,x,t) = \begin{pmatrix}\psi_1(\lambda,x,t)\\ \psi_2(\lambda,x,t)\end{pmatrix},$$
$$X(\lambda,x,t) = \begin{pmatrix} -i\lambda & iu(x,t) \\ & \\ i\sigma\overline{u(x,t)} & i\lambda \end{pmatrix} = -i\lambda\sigma_3 + iU(x,t),$$

$$T(\lambda,x,t) = \begin{pmatrix} -2i\lambda^2 + i\sigma u(x,t)\overline{u(x,t)} & 2i\lambda u(x,t) - u_x(x,t) \\ 2i\sigma\lambda\overline{u(x,t)} + \sigma\overline{u_x(x,t)} & 2i\lambda^2 - i\sigma u(x,t)\overline{u(x,t)} \end{pmatrix} \tag{707}$$

$$= -i(2\lambda^2 - \sigma|u|^2)\sigma_3 + 2i\lambda U - \sigma_3 U_x,$$

$$\sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad U = \begin{pmatrix}0 & u(x,t)\\ \sigma\bar{u}(x,t) & 0\end{pmatrix}.$$

Indeed the integrability condition $\underline{\psi}_{xt} = \underline{\psi}_{tx}$ is equivalent to the matrix equation

$$X_t - T_x + [X,T] = 0, \tag{708}$$

and the isospectral condition $\lambda_t = 0$, and $\lambda_x = 0$ imply the NLS equations (704) or, in matrix form:

$$iU_t + \sigma_3\left(U_{xx} + 2U^3\right) = 0. \tag{709}$$

$$\begin{aligned} &X_t = iU_t, \quad T_x = i\sigma(|u|^2)_x\sigma_3 + 2i\lambda U_x - \sigma_3 U_{xx}, \\ &[X,T] = 2\lambda^2[\sigma_3,U] + i\lambda[\sigma_3,\sigma_3 U_x] + (2\lambda^2 - \sigma|u|^2)[U,\sigma_3] - i[U,\sigma_3 U] \\ &= 2i\lambda U_x + \sigma|u|^2[\sigma_3,U] + i\sigma(|u|^2)_x\sigma_3, \end{aligned} \tag{710}$$

from which (709) follows.

It might be convenient to write the spectral problem (705) as the eigenvalue problem (multiplying (705) from the left by $i\sigma_3$):

$$\mathcal{L}\underline{\psi} = \lambda\underline{\psi}, \qquad \mathcal{L} := \sigma_3\left(i\partial_x + U(x)\right) = \begin{pmatrix} i\partial_x & u(x) \\ -\sigma\overline{u(x)} & -i\partial_x \end{pmatrix}. \tag{711}$$

Reality symmetry. If $\underline{\psi}(x,t,\lambda) = \begin{pmatrix} \psi_1(x,t,\lambda) \\ \psi_2(x,t,\lambda) \end{pmatrix}$ is a solution of the Lax pair (705),(706),(707), then $\underline{\tilde{\psi}}(x,t,\lambda) = \begin{pmatrix} -\sigma\overline{\psi_2(x,t,\bar{\lambda})} \\ \overline{\psi_1(x,t,\bar{\lambda})} \end{pmatrix}$ is also a solution of (705),(706),(707). Proof is left as exercise.

6.2.1 Direct Problem

IST for rapidly decaying potentials. If $u(x,t) \to 0$ as $|x| \to \infty$, then

$$\underline{\psi}_x \sim -i\lambda\sigma_3\underline{\psi} \quad \Rightarrow \quad \underline{\psi} \sim \begin{pmatrix} c_1^{\pm} e^{-i\lambda x} \\ c_2^{\pm} e^{i\lambda x} \end{pmatrix}, \quad x \to \pm\infty, \quad \lambda \in \mathbb{R}, \tag{712}$$

for the arbitrary constants $c_1^{\pm}, c_2^{\pm}$. Therefore we introduce the Jost solutions

$$\underline{\phi}^{(1)} = \begin{pmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{pmatrix}, \ \underline{\phi}^{(2)} = \begin{pmatrix} \phi_1^{(2)} \\ \phi_2^{(2)} \end{pmatrix}, \ \underline{\psi}^{(1)} = \begin{pmatrix} \psi_1^{(1)} \\ \psi_2^{(1)} \end{pmatrix}, \ \underline{\psi}^{(2)} = \begin{pmatrix} \psi_1^{(2)} \\ \psi_2^{(2)} \end{pmatrix} \tag{713}$$

satisfying the following boundary conditions:

$$\begin{aligned} \underline{\phi}^{(1)} &\sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda x}, \quad \underline{\phi}^{(2)} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\lambda x}, \quad x \sim -\infty, \\ \underline{\psi}^{(1)} &\sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda x}, \quad \underline{\psi}^{(2)} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\lambda x}, \quad x \sim \infty. \end{aligned} \tag{714}$$

Both pairs $\{\underline{\phi}^{(1)}, \underline{\phi}^{(2)}\}$ and $\{\underline{\psi}^{(1)}, \underline{\psi}^{(2)}\}$ are good bases in the space of solutions of the above Lax pair for decaying potentials, and one can write, for instance, $\underline{\phi}^{(1)}$ and $\underline{\phi}^{(2)}$ in terms of the basis $\{\underline{\psi}^{(1)}, \underline{\psi}^{(2)}\}$ as follows:

$$\begin{aligned} \underline{\phi}^{(1)} &= S_{11}(\lambda)\underline{\psi}^{(1)} + S_{21}(\lambda)\underline{\psi}^{(2)}, \quad \lambda \in \mathbb{R}, \\ \underline{\phi}^{(2)} &= S_{12}(\lambda)\underline{\psi}^{(1)} + S_{22}(\lambda)\underline{\psi}^{(2)}, \quad \lambda \in \mathbb{R}, \end{aligned} \tag{715}$$

or, in matrix form:

$$\begin{pmatrix} \phi_1^{(1)} & \phi_1^{(2)} \\ \phi_2^{(1)} & \phi_2^{(2)} \end{pmatrix} = \begin{pmatrix} \psi_1^{(1)} & \psi_1^{(2)} \\ \psi_2^{(1)} & \psi_2^{(2)} \end{pmatrix} \begin{pmatrix} S_{11}(\lambda) & S_{12}(\lambda) \\ S_{21}(\lambda) & S_{22}(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{R}. \tag{716}$$

This is the so-called "scattering equation", and $S(\lambda) = \begin{pmatrix} S_{11}(\lambda) & S_{12}(\lambda) \\ S_{21}(\lambda) & S_{22}(\lambda) \end{pmatrix}$ the scattering matrix. The reason for this terminology comes from observing that

$$\underline{\phi}^{(1)} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda x}, \quad x \sim -\infty, \quad \underline{\phi}^{(1)} \sim \begin{pmatrix} S_{11}(\lambda)e^{-i\lambda x} \\ S_{21}(\lambda)e^{i\lambda x} \end{pmatrix}, \quad x \sim \infty, \tag{717}$$

implying that

$$\begin{aligned} &\tfrac{1}{S_{11}(\lambda)}\underline{\phi}^{(1)} \sim \begin{pmatrix} T(\lambda)e^{-i\lambda x} \\ 0 \end{pmatrix}, \quad x \sim -\infty, \\ &\tfrac{1}{S_{11}(\lambda)}\underline{\phi}^{(1)} \sim \begin{pmatrix} e^{-i\lambda x} \\ R(\lambda)e^{i\lambda x} \end{pmatrix}, \quad x \sim \infty, \end{aligned} \tag{718}$$

where

$$T(\lambda) = \frac{1}{S_{11}(\lambda)}, \quad R(\lambda) = \frac{S_{21}(\lambda)}{S_{11}(\lambda)} \tag{719}$$

are respectively the transmission and the reflection coefficients (see Figure 43)

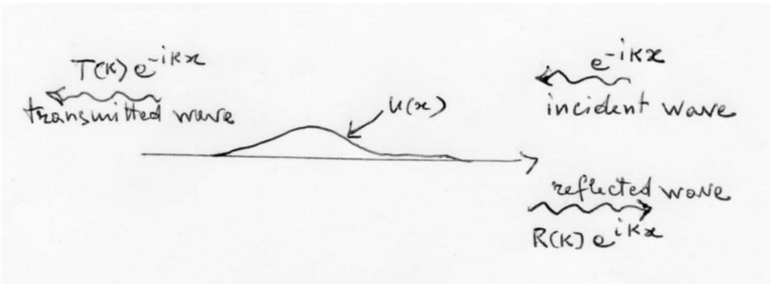


Figure 43: Reflected and transmitted waves by a localized potential.

The Wronskian theorem. If Ψ is a fundamental matrix solution (i.e., $\det \Psi \neq 0$) of the Lax pair (705),(706),(707), then, from the Jacobi (Abel) theorem: $(\det \Psi)_x = (\det \Psi)\mathrm{tr}\,(\Psi_x \Psi^{-1})$, it follows that

$$(\det \Psi)_x = (\det \Psi)_t = 0, \tag{720}$$

since $\mathrm{tr}\,(\Psi_x \Psi^{-1}) = \mathrm{tr}X = 0$ and $\mathrm{tr}\,(\Psi_t \Psi^{-1}) = \mathrm{tr}T = 0$.
Applying this property to the fundamental matrices constructed from the above Jost solutions, we obtain the following formulas:

$$\begin{aligned} &\det\left(\underline{\phi}^{(1)}, \underline{\phi}^{(2)}\right) = \det\left(\underline{\psi}^{(1)}, \underline{\psi}^{(2)}\right) = 1, \\ &\det\left(\underline{\phi}^{(j)}, \underline{\phi}^{(j)}\right) = \det\left(\underline{\psi}^{(j)}, \underline{\psi}^{(j)}\right) = 0, \quad j = 1, 2, \end{aligned} \tag{721}$$

$$\det\left(\underline{\phi}^{(1)},\underline{\psi}^{(2)}\right)=S_{11}(\lambda),\quad \det\left(\underline{\phi}^{(2)},\underline{\psi}^{(2)}\right)=S_{12}(\lambda),$$
$$\det\left(\underline{\psi}^{(1)},\underline{\phi}^{(1)}\right)=S_{21}(\lambda),\quad \det\left(\underline{\psi}^{(1)},\underline{\phi}^{(2)}\right)=S_{22}(\lambda). \tag{722}$$

Since (720) holds, one evaluates the determinant asymptotically, where the eigenfunctions are simpler. For instance, working with $\underline{\phi}^{(j)}, j=1,2$, we do it at $x\sim-\infty$: $\det\left(\underline{\phi}^{(1)},\underline{\psi}^{(2)}\right)=\phi_1^{(1)}\phi_2^{(2)}-\phi_2^{(1)}\phi_1^{(2)}=e^{-i\lambda x}e^{i\lambda x}=1$.

In addition, since $\det\left(\underline{\phi}^{(1)},\underline{\phi}^{(2)}\right)=\det\left(\underline{\psi}^{(1)},\underline{\psi}^{(2)}\right)=1$, it follows from (716) that we have the unimodularity condition:

$$\det S(\lambda)=S_{11}(\lambda)S_{22}(\lambda)-S_{12}(\lambda)S_{21}(\lambda)=1. \tag{723}$$

From the above reality symmetry, it follows that

$$\underline{\phi}^{(2)}(x,t,\lambda)=\begin{pmatrix}-\sigma\overline{\phi_2^{(1)}(x,t,\bar{\lambda})}\\ \overline{\phi_1^{(1)}(x,t,\bar{\lambda})}\end{pmatrix},\quad \underline{\psi}^{(1)}(x,t,\lambda)=\begin{pmatrix}\overline{\psi_2^{(2)}(x,t,\bar{\lambda})}\\ -\sigma\overline{\psi_1^{(2)}(x,t,\bar{\lambda})}\end{pmatrix}, \tag{724}$$

and, consequently,

$$S_{22}(\lambda)=\overline{S_{11}(\bar{\lambda})},\quad S_{12}(\lambda)=-\sigma\overline{S_{21}(\bar{\lambda})}. \tag{725}$$

Then the scattering equations (716) can be rewritten as

$$\begin{pmatrix}\phi_1^{(1)} & \phi_1^{(2)}\\ \phi_2^{(1)} & \phi_2^{(1)}\end{pmatrix}=\begin{pmatrix}\psi_1^{(1)} & \psi_1^{(2)}\\ \psi_2^{(1)} & \psi_2^{(1)}\end{pmatrix}\begin{pmatrix}S_{11}(\lambda) & -\sigma\overline{S_{21}(\lambda)}\\ S_{21}(\lambda) & \overline{S_{11}(\lambda)}\end{pmatrix},\ \lambda\in\mathbb{R}. \tag{726}$$

and the unimodularity condition (723) becomes

$$|S_{11}(\lambda)|^2+\sigma|S_{21}(\lambda)|^2=1,\quad \lambda\in\mathbb{R}. \tag{727}$$

Let us concentrate now on the vector solution $\underline{\phi}^{(1)}(x,\lambda)$. The spectral problem (705) together with the boundary condition (752) are equivalent to the following Volterra integral equation

$$\mu_1^+(x,\lambda)=1+i\int_{-\infty}^{x}u(y)\mu_2^+(y,\lambda)dy,$$
$$\mu_2^+(x,\lambda)=i\sigma\int_{-\infty}^{x}\bar{u}(y)\mu_1^+(y,\lambda)e^{2i\lambda(x-y)}dy, \tag{728}$$

where

$$\underline{\mu}^+=\begin{pmatrix}\mu_1^+\\ \mu_2^+\end{pmatrix}=\underline{\phi}^{(1)}e^{i\lambda x}\sim\begin{pmatrix}1\\0\end{pmatrix},\ x\sim-\infty. \tag{729}$$

Prove it as an exercise.

Replacing the second equation into the first, one obtains the following integral equation for the field μ_1^+:

$$\mu_1^+(x,\lambda) = 1 - \sigma \int\limits_{-\infty}^{x} dy\ u(y) \int\limits_{-\infty}^{y} dz\ \bar{u}(z) e^{2i\lambda(y-z)} \mu_1^+(z,\lambda). \tag{730}$$

We observe that the integral equation (730) is well defined also for $Im\lambda > 0$ (since $y - z > 0$, $e^{2i\lambda(y-z)}$ decays exponentially at ∞ if $Im\lambda > 0$). To prove the existence of the solution of (730) we expand μ_1^+ in the Neumann series

$$\mu_1^+(x,\lambda) = \sum_{j\geq 0} \mu_j(x,\lambda), \quad \mu_0(x,\lambda) = 1, \tag{731}$$

implying the recursion relation

$$\mu_{j+1}(x,\lambda) = -\sigma \int\limits_{-\infty}^{x} dy\ u(y) \int\limits_{-\infty}^{y} dz\ \bar{u}(z) e^{2i\lambda(y-z)} \mu_j(z,\lambda), \quad j \geq 0. \tag{732}$$

Consequently we have the inequality:

$$|\mu_{j+1}(x,\lambda)| \leq \int\limits_{-\infty}^{x} dy\ |u(y)| \int\limits_{-\infty}^{y} dz |u(z)||\mu_j(z,\lambda)|, \quad j \geq 0, \quad \mathrm{Im}\lambda \geq 0, \tag{733}$$

since $|e^{2i\lambda(y-z)}| \leq 1$ for $Im\lambda \geq 0$.

For $j = 0$:

$$\begin{aligned} &|\mu_1(x,\lambda)| \leq \int\limits_{-\infty}^{x} dy|u(y)| \int\limits_{-\infty}^{y} dz|u(z)| = \tfrac{1}{2} \int\limits_{-\infty}^{x} dy \left(\left(\int\limits_{-\infty}^{y} dz|u(z)| \right)^2 \right)_y \\ &= \tfrac{1}{2} U^2(x) \leq \tfrac{1}{2} \|u\|_1^2, \\ &U(x) := \int\limits_{-\infty}^{x} dy|u(y)|, \qquad \|u\|_1 := \textstyle\int_{\mathbb{R}} |u(y)|dy. \end{aligned} \tag{734}$$

For $j = 1$:

$$\begin{aligned} |\mu_2(x,\lambda)| &\leq \int\limits_{-\infty}^{x} dy|u(y)| \int\limits_{-\infty}^{y} dz|u(z)||\mu_1(z,\lambda)| \\ &\leq \tfrac{1}{2} \int\limits_{-\infty}^{x} dy|u(y)| \int\limits_{-\infty}^{y} dz|u(z)|U^2(z) = \tfrac{1}{4!} U^4(z). \end{aligned} \tag{735}$$

Iterating this procedure, one can prove by induction that

$$|\mu_n(x,\lambda)| \leq \frac{(U(x))^{2n}}{(2n)!}, \quad n \geq 1, \tag{736}$$

implying that

$$|\mu_1^+(x,\lambda)| \leq \sum_{n\geq 0} \frac{(U(x))^{2n}}{(2n)!} = \cosh\left(U(x)\right) \leq \cosh\left(\|u\|_1\right). \tag{737}$$

Then the Neumann series is totally (absolutely and uniformely) convergent if $u(x) \in L^1(\mathbb{R})$, i.e., if $\int_{\mathbb{R}} |u(y)|dy < \infty$. In addition, since all terms of the series are analytic for $Im\ \lambda \geq 0$, the uniform convergence implies that the sum $\mu_1^+(x,\lambda)$ be analytic for $Im\ \lambda \geq 0$. Similar considerations imply that
i) the functions

$$\underline{\mu}^- := \underline{\phi}^{(2)} e^{-i\lambda x}, \quad \underline{\nu}^- := \underline{\psi}^{(1)} e^{i\lambda x}, \quad \underline{\nu}^+ := \underline{\psi}^{(2)} e^{-i\lambda x}, \tag{738}$$

satisfy respectively the integral equations

$$\begin{aligned} \mu_1^-(x,\lambda) &= i \int_{-\infty}^{x} u(y)\mu_2^-(y,\lambda)e^{2i\lambda(y-x)}dy, \\ \mu_2^-(x,\lambda) &= 1 + i\sigma \int_{-\infty}^{x} \bar{u}(y)\mu_1^-(y,\lambda)dy, \end{aligned} \tag{739}$$

$$\begin{aligned} \nu_1^-(x,\lambda) &= 1 - i \int_{x}^{\infty} u(y)\nu_2^-(y,\lambda)dy, \\ \nu_2^-(x,\lambda) &= -i\sigma \int_{x}^{\infty} \bar{u}(y)\nu_1^-(y,\lambda)e^{2i\lambda(x-y)}dy, \end{aligned} \tag{740}$$

$$\begin{aligned} \nu_1^+(x,\lambda) &= -i \int_{x}^{\infty} u(y)\nu_2^+(y,\lambda)e^{-2i\lambda(x-y)}dy, \\ \nu_2^+(x,\lambda) &= 1 - i\sigma \int_{x}^{\infty} \bar{u}(y)\nu_1^+(y,\lambda)dy, \end{aligned} \tag{741}$$

ii) $\underline{\mu}^-, \underline{\nu}^-$ are analytic for $Im\lambda \leq 0$, and $\underline{\nu}^+$ is analytic for $Im\lambda \geq 0$. Then the wronskian relations (722) imply that $S_{11}(\lambda)$ and $S_{22}(\lambda)$ are analytic respectively for $Im\lambda \geq 0$ and for $Im\lambda \leq 0$. The functions $S_{12}(\lambda)$ and $S_{21}(\lambda)$ do not have analyticity properties off the real axis, unless the potential $u(x)$ is exponentially localized. Verify it!

Using integration by parts one can show that the large λ limit of the Volterra integral equations (740) and (728) yields

$$\begin{aligned} \nu_1^-(x,\lambda) &= 1 + \tfrac{i\sigma}{2\lambda} \int_{x}^{\infty} |u(y)|^2 dy + O(\lambda^{-2}), \\ \nu_2^-(x,\lambda) &= -\tfrac{\sigma}{2\lambda}\overline{u(x)} + O(\lambda^{-2}), \\ \mu_1^+(x,\lambda) &= 1 - \tfrac{i\sigma}{2\lambda} \int_{-\infty}^{x} |u(y)|^2 dy + O(\lambda^{-2}), \\ \mu_2^+(x,\lambda) &= -\tfrac{\sigma}{2\lambda}\overline{u(x)} + O(\lambda^{-2}), \end{aligned} \tag{742}$$

implying that u and $|u|^2$ can be written in terms of the eigenfuntion as follows (show it):

$$\begin{array}{l}\overline{u(x,t)} = -2\sigma \lim_{\lambda\to\infty}\left(\lambda\nu_2^-(x,t,\lambda)\right) = -2\sigma \lim_{\lambda\to\infty}\left(\lambda\mu_2^+(x,t,\lambda)\right), \\ |u(x,t)|^2 = 2i\sigma \lim_{\lambda\to\infty}\left(\lambda\nu_{1\ x}^-(x,t,\lambda)\right) = 2i\sigma \lim_{\lambda\to\infty}\left(\lambda\mu_{1\ x}^+(x,t,\lambda)\right).\end{array} \tag{743}$$

In addition, since $\mu_1^+ \to S_{11}$ and $\mu_2^+ \to S_{21}e^{2i\lambda x}$ for $x \to \infty$ (see (717)), the $x \to \infty$ limit of (728) yields the following integral representation of the scattering data:

$$S_{11}(\lambda) = 1 + i\int_{\mathbb{R}} u(y)\mu_2^+(y,\lambda)dy, \quad S_{21}(\lambda) = i\sigma\int_{\mathbb{R}} \bar{u}(y)\mu_1^+(y,\lambda)e^{-2i\lambda y}dy. \tag{744}$$

Evaluating equations (744) for large λ, and using (742), we obtain

$$S_{11}(\lambda) = 1 - \frac{i\sigma}{2\lambda}\int_{\mathbb{R}} |u(y)|^2dy + O(\lambda^{-2}), \ S_{21}(\lambda) = i\sigma\int_{\mathbb{R}} \bar{u}(y)e^{-2i\lambda y}dy(1+O(\lambda^{-1}) \tag{745}$$

and, consequently

$$\|u\|_2^2 = \int_{\mathbb{R}} |u(y)|^2dy = 2i\sigma \lim_{\lambda\to\infty}\left(\lambda\left(S_{11}(\lambda) - 1\right)\right). \tag{746}$$

At last, the reality symmetry (724) for the analytic eigenfunctions reads

$$\nu_1^+(\lambda) = -\sigma\overline{\nu_2^-(\bar{\lambda})}, \qquad \nu_2^+(\lambda) = \overline{\nu_1^-(\bar{\lambda})}. \tag{747}$$

6.2.2 Bound states

Since the spectral problem $\underline{\psi}_x = X\underline{\psi}$ can be rewritten as the eigenvalue problem

$$\mathcal{L}\underline{\psi} = \lambda\underline{\psi}, \qquad \mathcal{L} := \sigma_3\left(i\partial_x + U(x)\right) = \begin{pmatrix} i\partial_x & u(x) \\ -\sigma\overline{u(x)} & -i\partial_x \end{pmatrix}, \tag{748}$$

we observe that the operator $\mathcal{L}$ is self-adjoint in the defocusing ($\sigma = -1$) case, with respect to the scalar product

$$(\underline{a},\underline{b}) := \int_{\mathbb{R}}\left(\overline{a_1(x)}b_1(x) + \overline{a_2(x)}b_2(x)\right)dx, \quad \underline{a} = \begin{pmatrix} a_1(x) \\ a_2(x)\end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1(x) \\ b_2(x)\end{pmatrix}, \tag{749}$$

Indeed, if $\sigma = -1$,

$$\begin{array}{l}(\underline{a},\mathcal{L}\underline{b}) = \int_{\mathbb{R}}\left[\overline{a_1}\left(ib_{1x} + ub_2\right) + \overline{a_2}\left(-ib_{2x} + \bar{u}b_1\right)\right]dx \\ = \int_{\mathbb{R}}\left[\left(\overline{ia_{1x} + ua_2}\right)b_1 + \left(\overline{-ia_{2x} + \bar{u}a_1}\right)b_2\right]dx \\ = \int_{\mathbb{R}}\left[\left(\overline{(\mathcal{L}\underline{a})_1}\right)b_1 + \left(\overline{(\mathcal{L}\underline{a})_2}\right)b_2\right]dx = (\mathcal{L}\underline{a},\underline{b}).\end{array} \tag{750}$$

It follows that, in the defocusing case, the spectrum is real. Since for real λ the eigenfunctions behave as monochromatic waves at $x \sim \pm\infty$, they cannot belong to $L^2(\mathbb{R})$; it follows that, in the defocusing NLS case, the discrete spectrum in absent. Discrete spectrum is generically present in the focusing case ($\sigma = 1$).

We first recall that $S_{11}(\lambda)$ is analytic for $Im\lambda \geq 0$. If $\lambda_j,\ Im\lambda_j > 0$ is a zero of $S_{11}(\lambda)$ (then it will be an isolated zero): $S_{11}(\lambda_j) = 0$, it follows from (722) that the Jost solutions $\underline{\phi}^{(1)}$ and $\underline{\psi}^{(2)}$ are proportional at λ_j:

$$\underline{\phi}^{(1)}(x,\lambda_j) = b_j \underline{\psi}^{(2)}(x,\lambda_j) \ \Rightarrow\ \underline{\mu}^{+}(x,\lambda_j) = b_j\ \underline{\nu}^{+}(x,\lambda_j)e^{2i\lambda_j x}. \tag{751}$$

Since $\underline{\phi}^{(1)}$ and $\underline{\psi}^{(2)}$ are analytic for $Im\lambda > 0$, and

$$\underline{\phi}^{(1)} \sim \begin{pmatrix}1\\0\end{pmatrix} e^{-i\lambda x}, \quad x \sim -\infty; \quad \underline{\psi}^{(2)} \sim \begin{pmatrix}0\\1\end{pmatrix} e^{i\lambda x}, \quad x \sim \infty, \quad \lambda \in \mathbb{R}, \tag{752}$$

it follows that $\underline{\phi}^{(1)}(x,\lambda_j)$ decays exponentially at $x \to \pm\infty$:

$$\underline{\phi}^{(1)}(x,\lambda_j) \sim \begin{cases} \begin{pmatrix}1\\0\end{pmatrix} e^{-i\lambda_j x}, \ x \sim -\infty, \\ b_j \begin{pmatrix}0\\1\end{pmatrix} e^{i\lambda_j x}, \ x \sim \infty, \end{cases} \tag{753}$$

and λ_j belongs to the discrete spectrum of the operator $\mathcal{L}$. One can also prove the reverse: if $\lambda_j,\ Im\lambda_j > 0$ is a discrete eigenvalue of the operator $\mathcal{L}$, then λ_j is a zero of $S_{11}(\lambda)$. The reality symmetry implies that $\bar{\lambda}_j$ is a zero of $S_{22}(\lambda)$, and belongs to the discrete spectrum as well.

Summarizing, viewing the initial condition $u_0(x) = u(x,0)$ of the NLS Cauchy problem as the potential of the spectral problem, the **direct spectral problem** consists of constructing from $u_0(x)$ the following spectral data. i) The scattering data $S_{11}(\lambda), S_{21}(\lambda)$, associated with the continuous spectrum, given by the real line $\lambda \in \mathbb{R}$, and, only in the focusing NLS case, the discrete spectrum data $\lambda_j, b_j,\ j = 1,\dots,N$, $\mathrm{Im}\lambda_j > 0$, corresponding to exponentially decaying eigenfunctions. Since $S_{11}(\lambda)$ is analytic for $\mathrm{Im}\lambda > 0$, its zeroes can cluster only for $\lambda \sim \infty$ and/or $\mathrm{Im}\lambda \to 0$. But $S_{11}(\lambda) \to 1$ for $\lambda \to \infty$ in the upper half plane, and $|S_{11}(\lambda)|^2 + \sigma|S_{21}(\lambda)|^2 = 1,\ \lambda \in \mathbb{R}$; then S_{11} cannot be zero for $\lambda \sim \infty$ and for $\lambda \in \mathbb{R}$, its zeroes cannot have cluster points and N is finite.

Can one give a condition on the initial datum $u_0(x)$ for not having discrete spectrum in the focusing case? To do it we go back to (730) with $\sigma = 1$, inferring that

$$|\mu_1(x,\lambda) - 1| \leq \int_{-\infty}^{x} dy\ |u(y)| \int_{-\infty}^{y} dz |u(z)||\mu_1(z,\lambda)|, \quad \mathrm{Im}\lambda > 0. \tag{754}$$

Using again the Neumann series representation (731) of μ_1 and the inequalities (736) we infer that

$$
\begin{aligned}
&|\mu_1(x,\lambda)-1| \leq \int_{-\infty}^{x} dy\ |u(y)| \int_{-\infty}^{y} dz|u(z)| \left(1+\sum_{j\geq 1}|\mu_j(z,\lambda)|\right) \\
&\leq \int_{-\infty}^{x} dy\ |u(y)| \int_{-\infty}^{y} dz|u(z)| \left(1+\sum_{j\geq 1}\tfrac{U^{2j}(z)}{(2j)!}\right) = \sum_{j\geq 1}\tfrac{(U(x))^{2j}}{(2j)!} \\
&= \cosh(U(x))-1 \leq \cosh\left(\|u\|_1\right)-1, \quad \mathrm{Im}\lambda > 0.
\end{aligned}
\tag{755}
$$

Recalling that $\mu_1(x,\lambda)\to S_{11}(\lambda)$ as $x\to\infty$, for $\mathrm{Im}\lambda\geq 0$, it follows that

$$|S_{11}(\lambda)-1| \leq \cosh(\|u\|_1)-1, \quad \mathrm{Im}\lambda > 0. \tag{756}$$

If the norm 1 of the initial datum is sufficiently small:

$$\cosh(\|u\|_1) < 2, \tag{757}$$

then $\cosh(\|u\|_1)-1<1$ and

$$|S_{11}(\lambda)-1| < 1, \quad \mathrm{Im}\lambda > 0, \tag{758}$$

incompatible with the existence of discrete spectrum (the existence of discrete spectrum would imply the nonsense $1<1$). The condition (757) is equivalent to

$$\|u\|_1 < \cosh^{-1}(2) = \log\left(2+\sqrt{3}\right) \sim 1.317, \tag{759}$$

where $\cosh^{-1}(\cdot)$ is here the positive branch.

6.2.3 Time evolution of the spectral data

Now we show that the time evolution of the spectral data is very simple, thus justifying the effort made to go to spectral space. We first observe that the compatibility condition (708) is not affected by the change $T\to T+c(\lambda,t)I$, where I is the 2×2 identity matrix and c is any parameter such that $c_x=0$. Therefore equation (706) at $|x|\to\infty$ reads

$$\underline{\psi}_t \sim \begin{pmatrix} -2i\lambda^2+c & 0 \\ 0 & 2i\lambda^2+c \end{pmatrix} \underline{\psi}. \tag{760}$$

For the eigenfunction $\underline{\phi}^{(1)}$, satisfying $\underline{\phi}^{(1)} \sim \begin{pmatrix} e^{-i\lambda x} \\ 0 \end{pmatrix}$, $x\sim-\infty$, and $\underline{\phi}^{(1)} \sim \begin{pmatrix} S_{11}(\lambda)e^{-i\lambda x} \\ S_{21}(\lambda)e^{i\lambda x} \end{pmatrix}$, $x\sim+\infty$, we have at $x\sim-\infty$ the condition $(-2i\lambda^2+$

$c)e^{-i\lambda x}=0$, implying $c=2i\lambda^2$. At $x\sim\infty$ we have

$$\begin{pmatrix} S_{11t}(\lambda)e^{-i\lambda x} \\ S_{21t}(\lambda)e^{i\lambda x}\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 4i\lambda^2\end{pmatrix}\begin{pmatrix} S_{11}(\lambda)e^{-i\lambda x} \\ S_{21}(\lambda)e^{i\lambda x}\end{pmatrix}, \tag{761}$$

implying the equations $S_{11t}=0$ and $S_{21t}=4i\lambda^2S_{21}$. Therefore

$$S_{11}(\lambda,t)=S_{11}(\lambda,0),\ S_{21}(\lambda,t)=S_{21}(\lambda,0)e^{4i\lambda^2t}\ \Rightarrow\ R(\lambda,t)=R(\lambda,0)e^{4i\lambda^2t}. \tag{762}$$

Since S_{11} does not evolve in time, then its zeroes are also independent of time: $\lambda_n(t)=\lambda_n(0),\ n=1,\dots,N$. In addition, we observe that, for the potentials u such that $u(x)\exp(Im(\lambda_j)|x|)\to 0,\ j=1,\dots,N$, as $|x|\to\infty$, S_{21} can be analytically extended in a strip of the upper half λ plane containing all the eigenvalues, and $S_{21}(\lambda_j)=b_j,\ j=1,\dots,N$; consequently $b_n(t)=b_n(0)e^{4i\lambda_n^2t}$. Summarizing, the t-evolution of the discrete spectrum is

$$\lambda_n(t)=\lambda_n(0),\quad b_n(t)=b_n(0)e^{4i\lambda_n^2t},\ n=1,\dots,N. \tag{763}$$

6.2.4 Inverse problem

In the inverse problem we reconstruct the potential $u(x)$ from a suitable set of spectral data, and we make essential use of the analyticity properties of the eigenfunctions. Therefore we begin with the introduction of the "analyticity projectors" $P^{\pm}$ defined by

$$P^{\pm}f(\lambda):=\pm\frac{1}{2\pi i}\lim_{\epsilon\to 0^+}\int_{\mathbb{R}}\frac{f(\lambda')}{\lambda'-(\lambda\pm i\epsilon)}d\lambda',\qquad \lambda\in\mathbb{R}. \tag{764}$$

They map a Holder function[2] $f(\lambda),\ \lambda\in\mathbb{R}$ decaying at ∞ sufficiently fast into functions analytic in the upper and lower halves of the complex λ plane respectively, and satisfy the projection properties

$$P^+P^-=P^-P^+=0,\quad {P^+}^2=P^+,\quad {P^-}^2=P^-,\quad P^++P^-=1. \tag{765}$$

$f^{\pm}(\lambda):=P^{\pm}f(\lambda)$ are indeed analytic in the upper and lower halves of the complex λ plane respectively: for $\lambda+i\epsilon$ and $\mathrm{Im}\lambda>0$, $f^+(\lambda)$ is well defined and analytic for $\mathrm{Im}\lambda>0$; analogously, for $\lambda-i\epsilon$ and $\mathrm{Im}\lambda<0$, $f^-(\lambda)$ is well defined and analytic for $\mathrm{Im}\lambda<0$. In addition:

$$P^{\pm}f(\lambda)\sim\mp\frac{1}{2i\pi\lambda}\int_{\mathbb{R}}f(\lambda')d\lambda',\quad |\lambda|\gg 1. \tag{766}$$

[2] A function $f(\lambda)$ in Holder in $[a,b]$ if there exist $c>0$ and $0<\mu<1$ such that $|f(\lambda_1)-f(\lambda_2)|<c|\lambda_1-\lambda_2|^{\mu}\ \forall\lambda_1,\lambda_2\in[a,b]$.

They also satisfy the following Plemelj-Sokhotsky formulas:

$$P^{\pm}f(\lambda) = \pm\frac{1}{2\pi i}PV\int_{\mathbb{R}}\frac{f(\lambda')}{\lambda'-\lambda}d\lambda' + \frac{1}{2}f(\lambda), \tag{767}$$

where $PV\int$ is the principal value integral. To show it quickly, we also assume that $f(\lambda)$ be analytic in a very thin horizontal strip including the real λ axis. Then

$$\begin{array}{l} P^{\pm}f(\lambda) = \pm\frac{1}{2\pi i}\lim_{\epsilon\to 0^+}\int_{\mathbb{R}}\frac{f(\lambda')}{\lambda'-(\lambda\pm i\epsilon)}d\lambda' = \pm\frac{1}{2\pi i}\lim_{\epsilon\to 0^+}\int_{\gamma^{\pm}}\frac{f(\lambda')}{\lambda'-\lambda}d\lambda' \\ = \pm\frac{1}{2\pi i}PV\int_{\mathbb{R}}\frac{f(\lambda')}{\lambda'-\lambda}d\lambda' + \frac{1}{2}f(\lambda), \end{array} \tag{768}$$

where the contours $\gamma^{\pm}$ are shown in Fig. 44.

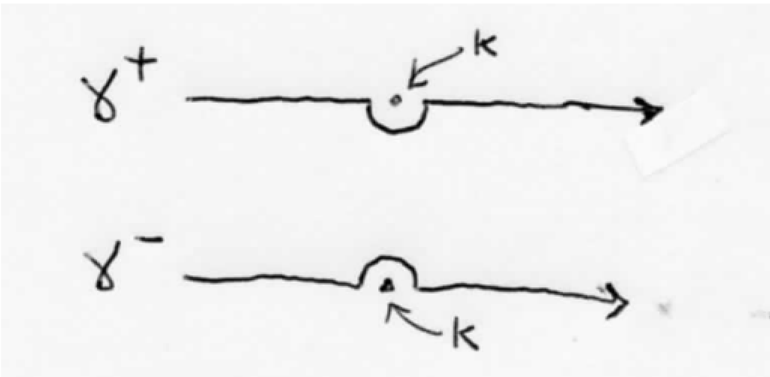


Figure 44:

Moreover the properties (765) are satisfied; f.i.:

$$P^+P^-f(\lambda) = P^+f^-(\lambda) = \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{f^-(\lambda')}{\lambda'-(\lambda+i\epsilon)}d\lambda = 0, \tag{769}$$

closing the contour downstairs, and using the analyticity properties of $f^-(k)$ and the Cauchy theorem;

$$P^{+2}f(\lambda) = P^+f^+(\lambda) = \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{f^+(\lambda')}{\lambda'-(\lambda+i\epsilon)}d\lambda = f^+(\lambda) = P^+f(\lambda), \tag{770}$$

closing the contour upstairs, and using the analyticity properties of $f^+(\lambda)$ and the residue theorem. At last the condition $P^+ + P^- = 1$ follows directly from (768).

After these preliminaries, we begin with the scattering equation $\underline{\phi}^{(1)} = S_{11}(\lambda)\underline{\psi}^{(1)} + S_{21}(\lambda)\underline{\psi}^{(2)}$ rewritten as follows

$$\frac{\underline{\mu}^+(x,\lambda)}{S_{11}(\lambda)} = \underline{\nu}^-(x,\lambda) + R(\lambda)\underline{\nu}^+(x,\lambda)e^{2i\lambda x}, \quad \lambda\in\mathbb{R}. \tag{771}$$

In (771):
i) $\frac{\underline{\mu}^+(x,\lambda)}{S_{11}(\lambda)}$ is analytic for Im$\lambda > 0$, up to the finite number N of poles $\lambda_j,\ j = 1,\dots,N$, the zeroes of $S_{11}(\lambda)$ (present only in the focusing case $\sigma = 1$), that we assume to be simple; this is generically true, and the case, for instance, of a double pole, can be obtained through the coalescence of two simple poles. Then

$$\frac{\underline{\mu}^+(x,\lambda)}{S_{11}(\lambda)} = \underline{\varphi}^+(x,\lambda) + \frac{1+\sigma}{2}\sum_{j=1}^{N}\frac{\underline{\mu}^+(x,\lambda_j)}{S'_{11}(\lambda_j)(\lambda-\lambda_j)}, \tag{772}$$

where $\underline{\varphi}^+(x,\lambda)$ is analytic for Im$\lambda > 0$ and tends to $\begin{pmatrix}1\\0\end{pmatrix}$ for $\lambda \to \infty$.

ii) $\underline{\nu}^-(x,\lambda)$ is analytic for Im$\lambda < 0$ and tends to $\begin{pmatrix}1\\0\end{pmatrix}$ for $\lambda \to \infty$.

iii) the last term is, in general, analytic nowhere.

Introducing in front of this last term the identity operator $P^+ + P^- = 1$, moving to the left all the functions analytic for Im$\lambda > 0$, moving to the right all the functions analytic for Im$\lambda < 0$, and subtracting the vector $\begin{pmatrix}1\\0\end{pmatrix}$ from both sides, we obtain

$$\begin{array}{l} \underline{\varphi}^+(x,\lambda) - \begin{pmatrix}1\\0\end{pmatrix} - P^+R\underline{\nu}^+e^{2i\lambda x} = \underline{\nu}^- - \begin{pmatrix}1\\0\end{pmatrix} + P^-R\underline{\nu}^+e^{2i\lambda x} \\ -\frac{1+\sigma}{2}\sum\limits_{j=1}^{N}\frac{\underline{\mu}^+(x,\lambda_j)}{S'_{11}(\lambda_j)(\lambda-\lambda_j)}, \quad \lambda \in \mathbb{R}. \end{array} \tag{773}$$

The LHS, analytic for Im$\lambda > 0$, is equal, for $\lambda \in \mathbb{R}$, to the RHS, analytic for Im$\lambda < 0$; therefore the LHS is the analytic continuation of the RHS to the upper half plane, and viceversa. In addition, the LHS and the RHS tend to $\underline{0}$ as $\lambda \to \infty$. From the Liouville theorem, it follows that they are $\underline{0}$:

$$\begin{array}{l} \underline{\nu}^-(x,\lambda) = \begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{R(\lambda')\underline{\nu}^+(x,\lambda')e^{2i\lambda' x}}{\lambda'-(\lambda-i\epsilon)}d\lambda' + \frac{1+\sigma}{2}\sum\limits_{j=1}^{N}\frac{\underline{\mu}^+(x,\lambda_j)}{S'_{11}(\lambda_j)(\lambda-\lambda_j)}, \\ \underline{\varphi}^+(x,\lambda) = \begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{R(\lambda')\underline{\nu}^+(x,\lambda')e^{2i\lambda' x}}{\lambda'-(\lambda+i\epsilon)}d\lambda'. \end{array} \tag{774}$$

Concentrating on the first of these two equations, we observe that it can be written as an equation for the eigenfunction $\underline{\nu}^-$ only, using the reality symmetry $\nu_1^+(\lambda) = -\sigma\overline{\nu_2^-(\bar{\lambda})},\quad \nu_2^+(\lambda) = \overline{\nu_1^-(\bar{\lambda})}$ and the bound state condition

$\underline{\mu}^+(x,\lambda_j) = b_j\ \underline{\nu}^+(x,\lambda_j)e^{2i\lambda_j x}$:

$$\begin{aligned}&\begin{pmatrix}\nu_1^-(x,t,\lambda)\\ \nu_2^-(x,t,\lambda)\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \tfrac{1}{2\pi i}\int_{\mathbb{R}} \tfrac{R(\lambda',t)e^{2i\lambda' x}}{\lambda'-(\lambda-i\epsilon)}\begin{pmatrix}-\sigma\overline{\nu_2^-(x,t,\lambda')}\\ \overline{\nu_1^-(x,t,\lambda')}\end{pmatrix}d\lambda'\\ &+\tfrac{1+\sigma}{2}\sum_{j=1}^{N}\tfrac{c_j(t)e^{2i\lambda_j x}}{\lambda-\lambda_j}\begin{pmatrix}-\overline{\nu_2^-(x,t,\bar\lambda_j)}\\ \overline{\nu_1^-(x,t,\bar\lambda_j)}\end{pmatrix},\end{aligned} \tag{775}$$

where

$$c_j(t) := \frac{b_j(t)}{S_{11}'(\lambda_j)} = \frac{b_j(0)e^{4i\lambda_j^2 t}}{S_{11}'(\lambda_j)} = c_j(0)e^{4i\lambda_j^2 t}. \tag{776}$$

To close the system (775), we evaluate these equations at $\bar\lambda_n,\ n=1,\dots,N$:

$$\begin{aligned}&\begin{pmatrix}\nu_1^-(x,t,\bar\lambda_n)\\ \nu_2^-(x,t,\bar\lambda_n)\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \tfrac{1}{2\pi i}\int_{\mathbb{R}} \tfrac{R(\lambda',t)e^{2i\lambda' x}}{\lambda'-\bar\lambda_n}\begin{pmatrix}-\sigma\overline{\nu_2^-(x,t,\lambda')}\\ \overline{\nu_1^-(x,t,\lambda')}\end{pmatrix}d\lambda'\\ &+\tfrac{1+\sigma}{2}\sum_{j=1}^{N}\tfrac{c_j(t)e^{2i\lambda_j x}}{\bar\lambda_n-\lambda_j}\begin{pmatrix}-\overline{\nu_2^-(x,t,\bar\lambda_j)}\\ \overline{\nu_1^-(x,t,\bar\lambda_j)}\end{pmatrix},\quad n=1,\dots,N.\end{aligned} \tag{777}$$

We have established the following result. Knowing the spectral data

$$\begin{aligned}&\mathcal{S}(t) = \{R(\lambda,t) = R(\lambda,0)e^{4i\lambda^2 t},\ \lambda\in\mathbb{R},\ \lambda_n(t)=\lambda_n(0),\\ &c_n(t) = c_n(0)e^{4i\lambda_n^2 t},\ n=1,\dots,N\},\end{aligned} \tag{778}$$

equations (775),(777) are a closed system of $2(N+1)$ linear equations for the vector eigenfunction $\underline{\nu}^-(x,\lambda)$ and for the vector eigenfunctions $\underline{\nu}^-(x,\bar\lambda_n),\ n = 1,\dots,N$. Then they are the main equations of the inverse problem. To complete the inverse problem, we use equations (779):

$$\begin{aligned}&\overline{u(x,t)} = -2\sigma\lim_{\lambda\to\infty}\left(\lambda\nu_2^-(x,t,\lambda)\right) = -2\sigma\lim_{\lambda\to\infty}\left(\lambda\mu_2^+(x,t,\lambda)\right),\\ &|u(x,t)|^2 = 2i\sigma\lim_{\lambda\to\infty}\left(\lambda\nu_{1\ x}^-(x,t,\lambda)\right) = 2i\sigma\lim_{\lambda\to\infty}\left(\lambda\mu_{1\ x}^+(x,t,\lambda)\right).\end{aligned} \tag{779}$$

to reconstruct the potential from the data (778) as follows

$$\begin{aligned}&\overline{u(x,t)} = \tfrac{\sigma}{i\pi}\int_{\mathbb{R}} R(\lambda,t)e^{2i\lambda x}\overline{\nu_1^-(x,t,\lambda)}d\lambda - (1+\sigma)\sum_{j=1}^{N} c_j(t)e^{2i\lambda_j x}\overline{\nu_1^-(x,t,\bar\lambda_j)},\\ &|u(x,t)|^2 = \partial_x\Big(\tfrac{1}{\pi}\int_{\mathbb{R}} R(\lambda,t)e^{2i\lambda x}\overline{\nu_2^-(x,t,\lambda)}d\lambda - i(1+\sigma)\sum_{j=1}^{N} c_j(t)e^{2i\lambda_j x}\overline{\nu_2^-(x,t,\bar\lambda_j)}\Big).\end{aligned} \tag{780}$$

We first observe that the NLS solution (780) is the sum of two terms

$$u(x,t) = u_{rad}(x,t) + u_{sol}(x,t); \tag{781}$$

$u_{rad}(x,t)$ is associated with the continuous spectrum and $u_{sol}(x,t)$ is associated with the discrete spectrum. We also observe that, if the initial datum is small: $|u(x,0)| \ll 1$, then $\nu^- \sim 1$, and u_{rad} reduces to the Fourier transform representation of the linear theory (verify it):

$$\begin{array}{l} u_{rad}(x,t) \sim \frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}(k,0) e^{i(kx-k^2t)}dk, \\ \hat{u}(k,0) = i\sigma R\left(-\frac{k}{2},0\right). \end{array} \tag{782}$$

Then one verifies, through the stationary phase method, that the solution exhibits the following longtime behavior

$$\begin{array}{l} u_{rad}(x,t) \sim \frac{1}{\sqrt{t}} A\left(\frac{x}{t}\right) e^{i\frac{x^2}{4t}}, \quad t \gg 1, \quad \xi = \frac{x}{t} = O(1), \\ A(\xi) = \frac{e^{-i\frac{\pi}{4}}}{2\sqrt{\pi}} \hat{u}(\xi/2,0). \end{array} \tag{783}$$

If the initial datum is not small, but no solitons are present, the above formula generalizes as follows

$$\begin{array}{l} u_{rad}(x,t) = \frac{1}{\sqrt{t}} R(x,t) e^{i\left(\frac{x^2}{4t} + 2\eta R_0^2\left(\frac{x}{t}\right)\log t + O(1)\right)}, \quad t \gg 1, \quad \frac{x}{t} = O(1), \\ R(x,t) = R_0\left(\frac{x}{t}\right) + O\left(\frac{\log t}{t}\right), \end{array} \tag{784}$$

where the real amplitude $R_0\left(\frac{x}{t}\right)$ can be expressed in terms of the initial data through the IST (verify (784) by direct substitution).

The amplitude $R_0\left(\frac{x}{t}\right)$ is slowly varying, since

$$\begin{array}{l} (R_0(x/t))_x = \frac{R_0'(\xi)}{t} \ll 1, \quad t \gg 1, \quad \xi = \frac{x}{t} = O(1), \\ (R_0(x/t))_t = -\frac{\xi R_0'(\xi)}{t} \ll 1, \end{array} \tag{785}$$

then the solution describes a slowly varying amplitude modulation of the carrier wave $\exp\left(i\left(\frac{x^2}{4t} + 2\eta R_0^2\left(\frac{x}{t}\right)\log t + O(1)\right)\right)$, decaying to zero as $1/\sqrt{t}$ (see Figure 45), as t grows. Then

$$u(x,t) \to u_{sol}(x,t), \quad t \to \infty. \tag{786}$$

Therefore the longtime behavior of the solution of the Cauchy problem is described by the discrete part of the spectrum, and now we concentrate on it.

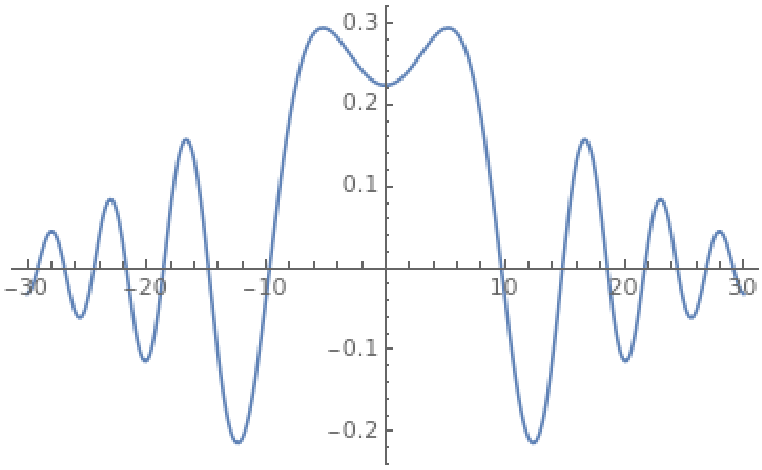


Figure 45: The graphs of the analytic formula describing the asymptotics of the real part of the solution for a gaussian initial condition at $t = 10$.

6.2.5 The N soliton solution

We concentrate on the focusing case $\sigma = 1$ and assume that the initial condition $u_0(x)$ be a reflectionless potential, such that $R(\lambda, 0) = 0$. Then the inverse equations reduce to the algebraic system

$$\begin{pmatrix}\nu_1^-(x,t,\bar\lambda_n)\\ \nu_2^-(x,t,\bar\lambda_n)\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} + \sum_{j=1}^{N} \frac{c_j(t)e^{2i\lambda_j x}}{\bar\lambda_n-\lambda_j}\begin{pmatrix}-\overline{\nu_2^-(x,t,\bar\lambda_j)}\\ \overline{\nu_1^-(x,t,\bar\lambda_j)}\end{pmatrix}. \tag{787}$$

Known its solution by Cramer's rule, then u and $|u|^2$ are reconstructed, using (743), as follows

$$\begin{aligned} u(x,t) &= -2\sum_{n=1}^{N}\overline{c_n(t)}e^{-2i\bar\lambda_n x}\nu_1^-(x,t,\bar\lambda_n),\\ |u(x,t)|^2 &= -2i\partial_x\Big(\sum_{j=1}^{N}c_j(t)e^{2i\lambda_j x}\overline{\nu_2^-(x,t,\bar\lambda_j)}\Big). \end{aligned} \tag{788}$$

To construct u, we rewrite (787) for $\nu_n(x,t) := \nu_1^-(x,t,\bar\lambda_n)$:

$$\begin{aligned} &\nu_n(x,t) + \sum_{l=1}^{N} D_{nl}(x,t)\nu_l(x,t) = 1,\\ &D_{nl}(x,t) := \sum_{k=1}^{N}\frac{c_k(0)\overline{c_l(0)}e^{2i(\lambda_k-\bar\lambda_l)x+4i\left(\lambda_k^2-\bar\lambda_l^2\right)t}}{(\bar\lambda_n-\lambda_k)(\lambda_k-\bar\lambda_l)} \end{aligned} \tag{789}$$

and

$$u(x,t) = -2\sum_{n=1}^{N}\overline{c_n(0)}e^{-2i\bar\lambda_n x-4i\bar\lambda_n^2 t}\nu_n(x,t). \tag{790}$$

To construct directly $|u|^2$, we rewrite (787) for $\chi_n(x,t) := \overline{\nu_2^-(x,t,\bar{\lambda}_n)}$:

$$\begin{array}{l}\chi_n(x,t) + \sum\limits_{l=1}^{N} C_{nl}(x,t)\chi_l(x,t) = w_n(x,t), \quad n = 1,\dots,N, \\ C_{nl}(x,t) = \overline{D_{nl}(x,t)}, \quad w_n(x,t) = \sum\limits_{k=1}^{N} \frac{\bar{c}_k(0)e^{-2i\bar{\lambda}_k x - 4i\bar{\lambda}_k^2 t}}{\lambda_n - \bar{\lambda}_k}.\end{array} \tag{791}$$

Remarkably, $|u|^2$ can be written in the compact form

$$|u(x,t)|^2 = \partial_x^2 \log\left(\det A\right), \quad A = I + C = I + \overline{D}, \tag{792}$$

entirely in terms of $\det A$.

To prove (792), we proceed as follows. By Cramer's rule, the solution of (791) reads

$$\chi_n = \frac{1}{\det A}\sum_{k=1}^{N}(-1)^{k+n} w_k M_{kn}(A), \quad n = 1,\dots,N, \quad A = I + C, \tag{793}$$

where $M_{kn}(A)$ is the determinant minor of matrix A, obtained from A eliminating the k^{th} row and the n^{th} column. On the other hand, from the Jacobi formula:

$$\left(\det A\right)_x = \mathrm{tr}\left(A_x \mathrm{Adj}(A)\right) = -2i\sum_{n,m,k=1}^{N}(-1)^{n+m}\frac{\overline{c_k(t)}c_m(t)e^{2i(\lambda_m - \bar{\lambda}_k)x}}{\lambda_n - \bar{\lambda}_k}M_{nm}(A). \tag{794}$$

Then the second of equations (788) leads to

$$\begin{array}{l}|u|^2 = -2i\partial_x\left(\frac{1}{\det A}\sum\limits_{n,m,k=1}^{N}(-1)^{n+m}\frac{\overline{c_k(t)}c_m(t)e^{2i(\lambda_m - \bar{\lambda}_k)x}}{\lambda_n - \bar{\lambda}_k}M_{nm}(A)\right) \\ = \partial_x\left(\frac{(\det A)_x}{\det A}\right) = \partial_x^2\left(\log\left(\det A\right)\right).\end{array} \tag{795}$$

We recall that the eigenvalues λ_k lie in the upper half plane:

$$\lambda_k = \xi_k + i\eta_k, \quad \xi_k, \eta_k \in \mathbb{R}, \quad \eta_k > 0, \quad k = 1,\dots,N; \tag{796}$$

it is convenient to introduce the real parameters

$$x_k = \frac{1}{2\eta_k}\log\left(\frac{|c_k(0)|}{2\eta_k}\right), \quad \varphi_k = \arg(c_k(0)), \tag{797}$$

and the quantities

$$X_k = 2\eta_k(x - x_k + 4\xi_k t), \quad Y_k = 2\xi_k x + 4(\xi_k^2 - \eta_k^2)t + \varphi_k, \quad k = 1,\dots,N, \tag{798}$$

so that D and u are rewritten as

$$D_{nm} = 4\sum_{k=1}^{N}\frac{\eta_k\eta_m e^{-(X_k + X_m) + i(Y_k - Y_m)}}{[\xi_n - \xi_k - i(\eta_n + \eta_k)][\xi_k - \xi_m + i(\eta_k + \eta_m)]}. \tag{799}$$

and

$$u(x,t) = -4\sum_{n=1}^{N} \eta_n e^{-X_n - iY_n - i\varphi_n} \nu_n. \tag{800}$$

If $N = 1$, one obtains the 1-soliton solution of focusing NLS (also called the bright soliton):

$$\begin{aligned} &D_{11} = e^{-2X_1} \quad \Rightarrow \quad \nu_1 = \left(1 + e^{-2X_1}\right)^{-1}, \\ &u_1(x,t) = -4\eta_1 e^{-X_1 - iY_1} \nu_1 = -\tfrac{(2\eta_1)e^{-2i[\xi_1 x + 2(\xi_1^2 - \eta_1^2)t] - i\varphi_1}}{\cosh[2\eta_1(x - x_1 + 4\xi_1 t)]}. \end{aligned} \tag{801}$$

It describes the amplitude modulation of a monochromatic wave (the carrier wave), whose envelope is exponentially localized, with amplitude and localization $2\eta_1$, traveling with speed $-4\xi_1$; the carrier wave travels with the independent speed $2\frac{\eta_1^2 - \xi_1^2}{\xi_1}$. If $\xi_1 = 0$, the soliton is stationary (see Figure 46).

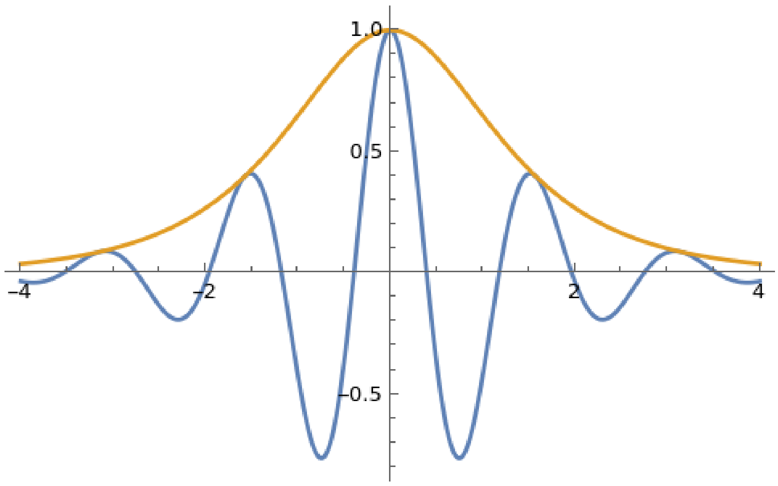


Figure 46: Plots of $|u_1(x,t)|$ in yellow, and of $\mathrm{Re}(u_1(x,t))$ in blue, for $\xi_1 = -2, \eta_1 = 0.5, x_1 = \varphi_1 = 0$.

Equation (792) gives directly

$$|u_1(x,t)|^2 = \partial_x^2 \log\left(1 + e^{-2X_1}\right) = \frac{4\eta_1^2}{\cosh^2[2\eta_1(x - x_1 + 4\xi_1 t)]}. \tag{802}$$

Now we investigate the elastic interaction of two solitons choosing $N = 2$, and concentrating directly on (792). Then the components of the 2×2 matrix

$A = I + \overline{D}$ are

$$\begin{aligned}
A_{11} &= 1 + e^{-2X_1} - \tfrac{4\eta_1\eta_2 e^{-(X_1+X_2)+i(Y_1-Y_2)}}{(\lambda_1-\bar\lambda_2)^2}, \\
A_{12} &= \tfrac{4\eta_1\eta_2 e^{-(X_1+X_2)-i(Y_1-Y_2)}}{(\lambda_1-\bar\lambda_1)(\bar\lambda_1-\lambda_2)} + \tfrac{4\eta_2^2 e^{-2X_2}}{(\lambda_1-\bar\lambda_2)(\bar\lambda_2-\lambda_2)}, \\
A_{21} &= \tfrac{4\eta_1^2 e^{-2X_1}}{(\lambda_2-\bar\lambda_1)(\bar\lambda_1-\lambda_1)} + \tfrac{4\eta_1\eta_2 e^{-(X_1+X_2)+i(Y_1-Y_2)}}{(\lambda_2-\bar\lambda_2)(\bar\lambda_2-\lambda_1)}, \\
A_{22} &= 1 + e^{-2X_2} - \tfrac{4\eta_1\eta_2 e^{-(X_1+X_2)-i(Y_1-Y_2)}}{(\lambda_2-\bar\lambda_1)^2}.
\end{aligned} \tag{803}$$

To specify the problem, we consider two solitons traveling with positive speed: $\xi_1, \xi_2 < 0$, and such that soliton 1 is faster than soliton 2: $|\xi_1| > |\xi_2|$, and observe the interaction in the reference frame of soliton 1:

$$x = -4\xi_1 t + O(1), \quad |t| \gg 1. \tag{804}$$

. Then

$$\begin{aligned}
&X_1 = O(1), \quad t \pm \infty, \\
&X_2 \sim 2\eta_2(x + 4\xi_2 t + const) \sim 2\eta_2(4(|\xi_1| - |\xi_2|)t + const) \sim \pm\infty, \ t \sim \pm\infty.
\end{aligned} \tag{805}$$

It follows that, when $t \gg 1$, $A_{11} \sim 1 + e^{-2X_1}$, $A_{12} \sim 0$, $A_{21} \sim O(1)$, $A_{22} \sim 1$, implying that

$$\det A = A_{11}A_{22} - A_{12}A_{21} \sim 1 + e^{-2X_1}, \tag{806}$$

the same formula as in (802). Therefore

$$|u_2(x,t)|^2 \sim \partial_x^2 \log\left(1 + e^{-2X_1}\right) = \frac{4\eta_1^2}{\cosh^2[2\eta_1(x - x_1 + 4\xi_1 t)]}, \ t \sim \infty, \tag{807}$$

and the observer sees at $t \sim \infty$ the soliton 1 in the position $x_1(\infty) = x_1$. If, instead, $t \sim -\infty$, then

$$\det A \sim e^{-2X_2}\left(1 + \left|\frac{\lambda_1 - \lambda_2}{\lambda_1 - \bar\lambda_2}\right|^4 e^{-2X_1}\right) \tag{808}$$

(verify it!), up to exponentially small corrections.

At this stage it is convenient to introduce the parameter Δx_1 as follows

$$\Delta x_1 = \frac{1}{\eta_1}\log\left|\frac{\lambda_1 - \bar\lambda_2}{\lambda_1 - \lambda_2}\right|. \tag{809}$$

It is a positive parameter, since

$$\left|\frac{\lambda_1 - \bar\lambda_2}{\lambda_1 - \lambda_2}\right|^2 = \frac{(\eta_1 + \eta_2)^2 + (\xi_1 - \xi_2)^2}{(\eta_1 - \eta_2)^2 + (\xi_1 - \xi_2)^2} > 1. \tag{810}$$

Then

$$\det A \sim e^{-2X_2}\left(1+e^{-4\eta_1(x-x_1(-\infty)+4\xi_1 t)}\right), \quad x_1(-\infty)=x_1-\Delta x_1. \tag{811}$$

Therefore (792) gives

$$|u_2(x,t)|^2=\partial_x^2\log(\det A)\sim\partial_x^2\left(-2X_2+\log\left(1+e^{-4\eta_1(x-x_1(-\infty)+4\xi_1 t)}\right)\right)$$
$$=\partial_x^2\left(\log\left(1+e^{-4\eta_1(x-x_1(-\infty)+4\xi_1 t)}\right)\right)=\frac{4\eta_1^2}{\cosh^2[2\eta_1(x-x_1(-\infty)+4\xi_1 t)]}. \tag{812}$$

Also at $t\sim-\infty$ the observer sees soliton 1, but now in the position $x_1(-\infty)$, with a global positive phase shift:

$$x_1(\infty)-x_1(-\infty)=x_1-(x_1-\Delta x_1)=\Delta x_1>0. \tag{813}$$

Therefore the fastest soliton 1 experiences a forward shift; analogously one could show that soliton 2 has a backward shift (verify it !) (see Fig. 47)

$$\Delta x_2=-\frac{1}{\eta_2}\log\left|\frac{\lambda_1-\bar{\lambda}_2}{\lambda_1-\lambda_2}\right|<0, \tag{814}$$

with the conservation of the momentum:

$$\eta_1\Delta x_1+\eta_2\Delta x_2=0. \tag{815}$$

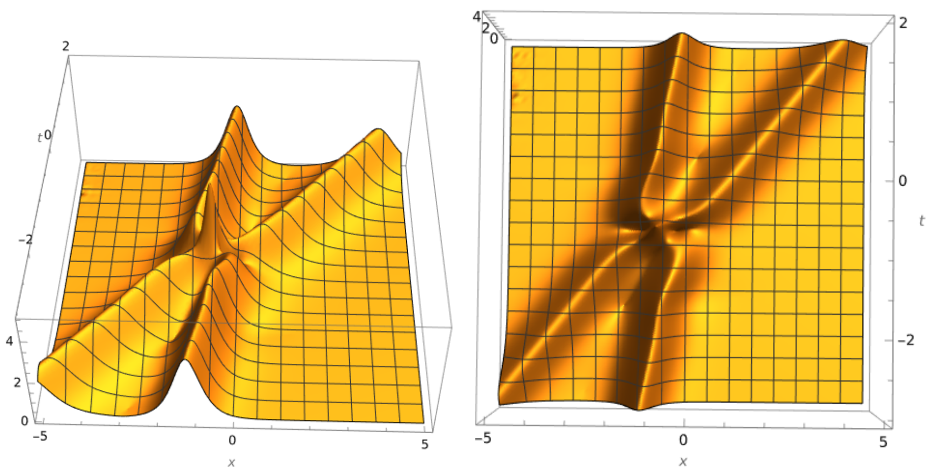


Figure 47: The evolution in space-time of the 2 soliton solution from two different perspectives (the second is the view from above) describing the interaction of 2 bright solitons with $\xi_1=-0.4,\eta_1=1,x_1=1,\varphi_1=0,\xi_2=-0.1,\eta_2=1.5,x_2=0,\varphi_2=0$. Soliton 1 has amplitude 2 and speed 1.6, while soliton 2 has amplitude 3 and speed 0.4. Therefore soliton 1 is smaller than soliton 2, but four times faster, and overcomes soliton 2 during the dynamics. The perspective from above shows clearly the elastic interaction with phase shift.

In the general case, the N-soliton solution describes the nonlinear pairwise elastic interaction of N bright solitons traveling with different speeds $-4\xi_k$, $k = 1, \ldots, N$. If $|\xi_1| > |\xi_2| > \cdots > |\xi_N|$, then soliton n is pushed forward in the interaction with the slower solitons $(n+1), \ldots, N$, and is pushed backward in the interaction with the faster solitons $1, \ldots, n-1$, experiencing the global phase shift

$$\Delta x_n = \frac{1}{\eta_n}\left(\log \prod_{k=n+1}^{N} \left|\frac{\lambda_n - \bar{\lambda}_k}{\lambda_n - \lambda_k}\right| - \log \prod_{k=1}^{n-1} \left|\frac{\lambda_n - \bar{\lambda}_k}{\lambda_n - \lambda_k}\right|\right), \tag{816}$$

with

$$\sum_{k=1}^{N} \eta_k \Delta x_k = 0. \tag{817}$$

If $\xi_k = 0$, $k = 1, \ldots, N$, the envelopes of the N solitons do not travel and the N-soliton solution describes a bound state with $N-1$ degrees of freedom, periodic in time, with period

$$\begin{array}{l} T = LCM\{T_{ij},\ i,j = 1, \ldots, N,\ i > j\}, \\ T_{ij} = \frac{2\pi}{\omega_i - \omega_j},\quad \omega_k = -4\eta_k^2,\quad \eta_1 > \eta_2 > \cdots > \eta_N. \end{array} \tag{818}$$

7 The KdV hierarchy; infinitely many symmetries and constants of motion

7.1 The KdV hierarchy

It is possible to use the Lax pair approach to construct the hierarchy of integrable PDEs associated with the Schrödinger equation $\psi_{xx} - (u+\lambda)\psi = 0$ (with $\lambda = -E$) as follows.

First we rewrite it in matrix form (with $\lambda = -E$):

$$\vec{\psi}_x = U\vec{\psi},\quad U = \left(\begin{array}{cc} 0 & 1 \\ u+\lambda & 0 \end{array}\right),\quad \vec{\psi} = \left(\begin{array}{c} \psi \\ \psi_x \end{array}\right) \tag{819}$$

and we look for a compatible time evolution

$$\vec{\psi}_t = V\vec{\psi},\quad V = \left(\begin{array}{cc} A & B \\ C & D \end{array}\right), \tag{820}$$

From the compatibility $\vec{\psi}_{xt} = \vec{\psi}_{tx}$ we obtain

$$\begin{array}{l} \vec{\psi}_{xt} = U_t\vec{\psi} + U\vec{\psi}_t = (U_t + UV)\vec{\psi},\quad \vec{\psi}_{tx} = V_x\vec{\psi} + V\vec{\psi}_x = (V_x + VU)\vec{\psi}, \\ \Rightarrow\quad (U_t - V_x + [U,V])\,\vec{\psi} = \vec{0}, \end{array} \tag{821}$$

valid for every eigenfunction, implying the following matrix equation

$$U_t - V_x + [U,V] = 0. \tag{822}$$

In components:

$$\begin{aligned}
&A_x = C - (u+\lambda)B,\\
&D_x = -C + (u+\lambda)B,\\
&B_x = D - A,\\
&u_t = C_x + (u+\lambda)(D-A).
\end{aligned} \tag{823}$$

Subtracting the first two and using the x derivative of the third:

$$(D-A)_x = -2C + 2(u+\lambda)B = B_{xx} \;\Rightarrow\; C = (u+\lambda)B - \frac{1}{2}B_{xx}. \tag{824}$$

The last equation becomes

$$\begin{aligned}
u_t &= u_x B + (u+\lambda)B_x - \tfrac{1}{2}B_{xxx} + (u+\lambda)B_x\\
&= 2\lambda B_x - \tfrac{1}{2}\left(B_{xxx} - 4uB_x - 2u_x B\right)\\
&= 2\lambda B_x - \tfrac{1}{2}\left(B_{xx} - 2uB - 2\partial_x^{-1}(uB_x)\right)_x.
\end{aligned} \tag{825}$$

It can be written in the following compact form

$$u_t = 2\lambda B_x - \tfrac{1}{2}\left(\mathcal{L}B\right)_x \tag{826}$$

introducing the integro-differential operator

$$\mathcal{L} := \partial_x^2 - 2u - 2\partial_x^{-1}u\partial_x, \qquad \partial_x^{-1}f := \int\limits_{-\infty}^{x} f(x')dx'. \tag{827}$$

The PDE (826) must be λ-independent, and we look for B as polynomial in λ

$$B = \sum_{j=0}^{n} \lambda^j B_j(x,t) \tag{828}$$

in order to achieve it. Substituting (828) into (826) we obtain

$$\begin{aligned}
u_t &= 2\sum_{j=0}^{n} \lambda^{j+1} B_{j\,x} - \tfrac{1}{2}\sum_{j=0}^{n} \lambda^j \left(\mathcal{L}B_j\right)_x\\
&= 2\sum_{j=1}^{n+1} \lambda^{j} B_{j-1\,x} - \tfrac{1}{2}\sum_{j=0}^{n} \lambda^j \left(\mathcal{L}B_j\right)_x.
\end{aligned} \tag{829}$$

Equating to zero the coefficients of all λ powers we obtain

$$\begin{aligned}
&\lambda^{n+1}: \; B_{n\,x} = 0 \;\Rightarrow\; B_n = const,\\
&\lambda^{j}: \;\; B_{j-1\,x} = \tfrac{1}{4}\left(\mathcal{L}B_j\right)_x, \;\; 1 \le j \le n\\
&\Rightarrow \;\; B_0 = \tfrac{1}{4^n}\mathcal{L}^n B_n = \tfrac{B_n}{4^n}\mathcal{L}^n \cdot 1,\\
&u_t = -\tfrac{1}{2}\left(\mathcal{L}B_0\right)_x = \alpha_n\left(\mathcal{L}^{n+1}\cdot 1\right)_x, \;\; \alpha_n := -\tfrac{B_n}{2\cdot 4^n}.
\end{aligned} \tag{830}$$

Since $\mathcal{L}\cdot 1 = -2u$ we finally construct the following hierarchy of nonlinear PDEs

$$\begin{aligned}
&u_{t_{2n+1}} = \alpha_n\left(\mathcal{L}^{n+1}\cdot 1\right)_x = c_{2n+1}\left(\mathcal{L}^n u\right)_x =: c_{2n+1}K^{(n)}, \;\;\; n \in \mathbb{N},\\
&c_{2n+1} = -2\alpha_n.
\end{aligned} \tag{831}$$

called the "KdV hierarchy", that can be integrated by the IST associated with the Schrödinger operator.

Let us introduce the operator

$$L := \partial_x^2 - 4u - 2u_x\partial_x^{-1}; \tag{832}$$

then it is possible to show that:
1) the operators $\mathcal{L}$ and L are "well coupled" by the following relation:

$$\partial_x \mathcal{L} = L\partial_x, \tag{833}$$

implying

$$\partial_x \mathcal{L}^n = L^n \partial_x, \quad n \in \mathbb{N}^+. \tag{834}$$

2) The operators $\mathcal{L}$ and L are adjoint

$$< f, Lg > = < \mathcal{L}f, g > \tag{835}$$

with respect to the bilinear form

$$< f, g > := \int_{\mathbb{R}} f(x)g(x)dx. \tag{836}$$

Both proofs are direct; for (833):

$$\begin{array}{l} \partial_x \mathcal{L} = \partial_x(\partial_x^2 - 2u - 2\partial_x^{-1} u \partial_x) = \partial_x^3 - 2\partial_x u - 2u\partial_x \\ = (\partial_x^2 - 2\partial_x u \partial_x^{-1} - 2u)\partial_x = (\partial_x^2 - 2u_x \partial_x^{-1} - 4u)\partial_x = L\partial_x \end{array} \tag{837}$$

Equation (834) follows by repeated application of (833) and by induction:

$$\partial_x \mathcal{L}^2 = \partial_x \mathcal{L}\mathcal{L} = L\partial_x \mathcal{L} = L^2 \partial_x. \tag{838}$$

For (835):

$$\begin{array}{l} < f, Lg > = \int_{\mathbb{R}} f(g_{xx} - 4ug - 2u_x \partial_x^{-1} g)dx = \int_{\mathbb{R}} [f_{xx} - 4uf + 2\partial_x^{-1}(u_x f)]gdx \\ \int_{\mathbb{R}} (f_{xx} - 2uf - \partial_x^{-1}(uf_x))g = < \mathcal{L}f, g > \end{array} \tag{839}$$

where we have used the integration by parts formula $\partial_x^{-1}(u_x f) = uf - \partial_x^{-1}(uf_x)$. Using the operators $\mathcal{L}$ and L, and (834), the KdV hierarchy of integrable equations can be written in two equivalent ways

$$u_{t_{2n+1}} = -\frac{c_{2n+1}}{2} \left(\mathcal{L}^{n+1} 1\right)_x = c_{2n+1} \left(\mathcal{L}^n u\right)_x = c_{2n+1} L^n u_x = c_{2n+1} K_n \tag{840}$$

The first three equations of the KdV hierarchy, for $n = 0, 1, 2$, are:

$$\begin{array}{l} u_{t_1} = c_1 K_0, \quad K_0 = u_x, \quad \text{the advection equation,} \\ u_{t_3} = c_3 K_1, \quad K_1 = (u_{xx} - 3u^2)_x = u_{xxx} - 6uu_x, \quad \text{the KdV equation,} \\ u_{t_5} = c_5 K_2, \quad K_2 = (u_{xxxx} - 10uu_{xx} - 5u_x^2 + 10u^3)_x = u_{xxxxx} - 10uu_{xxx} \\ -20u_x u_{xx} + 30u^2 u_x, \quad \text{the quintic KdV equation.} \end{array} \tag{841}$$

As we see, although the operators $\mathcal{L}$ is integro-differential, $\mathcal{L}^n u, \ n = 0, 1, 2$ are local functions of u and its derivatives. It is possible to show that this property is true for any $n \in \mathbb{N}^+$.

7.1.1 Symmetries and constants of motion

Now we introduce the notion of **(infinitesimal generator of a) symmetry** of a given PDE

$$u_t = K(x, u, u_x, u_{xx}, \dots). \tag{842}$$

Def. $\sigma(x, u, u_x, u_{xx}, \dots)$ is a symmetry (an infinitesimal generator of a symmetry) of equation (842) when, if $u(x,t)$ is a solution of (842), then $u + \epsilon\sigma$ is a solution of (842) up to $O(\epsilon)$. Then equation

$$(u + \epsilon\sigma)_t = K\left(x, u + \epsilon\sigma, u_x + \epsilon\sigma_x, u_{xx} + \epsilon\sigma_{xx}, \dots\right) \tag{843}$$

is satisfied up to $O(\epsilon)$:

$$\begin{array}{l} (u + \epsilon\sigma)_t = u_t + \epsilon\sigma_t + O(\epsilon^2) = u_t + \epsilon\left(\frac{\partial\sigma}{\partial u} u_t + \frac{\partial\sigma}{\partial u_x} u_{xt} + \frac{\partial\sigma}{\partial u_{xx}} u_{xxt} + \dots\right) + O(\epsilon^2) \\ = u_t + \epsilon\left(\sum\limits_{j\geq 0} \frac{\partial\sigma}{\partial(\partial_x^j u)} \partial_x^j\right) u_t + O(\epsilon^2) = u_t + \epsilon\left(\sum\limits_{j\geq 0} \frac{\partial\sigma}{\partial(\partial_x^j u)} \partial_x^j\right) K + O(\epsilon^2), \\ K\left(x, u + \epsilon\sigma, u_x + \epsilon\sigma_x, u_{xx} + \epsilon\sigma_{xx}, \dots\right) = K + \epsilon\left(\sum\limits_{j\geq 0} \frac{\partial K}{\partial(\partial_x^j u)} \partial_x^j\right)\sigma + O(\epsilon^2). \end{array} \tag{844}$$

Since $u_t = K$, to leading ($O(\epsilon)$) order we obtain the symmetry equation

$$\hat{\sigma}' \cdot K = \hat{K}' \cdot \sigma, \tag{845}$$

where $\hat{f}' \cdot g$ is the Frechét derivative of $f(x, u, u_x, u_{xx}, \dots)$ in the direction g:

$$\hat{f}' \cdot g := \left(\sum_{j \geq 0} \frac{\partial f}{\partial(\partial_x^j u)} \partial_x^j \right) g = \frac{\partial}{\partial_\epsilon} f(u + \epsilon g)\Big|_{\epsilon=0}. \tag{846}$$

Since all the equations of the KdV hierarchy are associated with the Schrödinger operator, they share the same direct and inverse problems. It is easy to show that the time evolution of the spectral data associated with the flow $M = 2n + 1$ is:

$$\mathcal{S}(k, t_M) = \{R(k, t_M), p_l(t_M), \beta_l(t_M), l = 1, \dots, N\}, \quad M = 2n + 1, \tag{847}$$

$$\begin{array}{l} R(k, t_M) = R(k, 0)e^{-i\omega_M(k)t_M}, \quad \omega_M(k) := c_M(-1)^{\frac{M+1}{2}}(2k)^M, \\ p_l(t_M) = p_l(0), \quad \beta_l(t_M) = \beta_l(0)e^{-c_M(2p_l)^M t_M}. \end{array} \tag{848}$$

We remark that two different flows of the hierarchy induce the following time evolutions of the reflection coefficients:

$$R_{t_{2n+1}} = -i\omega_{2n+1}R, \quad R_{t_{2m+1}} = -i\omega_{2m+1}R, \tag{849}$$

and these evolutions are compatible (they commute):

$$R_{t_{2n+1}t_{2m+1}} = R_{t_{2m+1}t_{2n+1}} = -\omega_{2n+1}\omega_{2m+1}R. \tag{850}$$

This commutation of the two flows in the spectral space implies the commutation of the two flows also in physical space:

$$\begin{array}{c} u_{t_{2n+1}t_{2m+1}} = u_{t_{2m+1}t_{2n+1}} \quad \Leftrightarrow \quad \partial_{t_{2m+1}} K_n = \partial_{t_{2n+1}} K_m \quad \Leftrightarrow \\ \hat{K}'_n \cdot K_m = \hat{K}'_m \cdot K_n, \end{array} \tag{851}$$

implying that the vector fields $\{K_n\}_{n \in \mathbb{N}}$ are the (infinitesimal generators of) **symmetries** of the whole KdV hierarchy. In particular, **KdV possesses the infinitely many commuting symmetries $\sigma_n = K_n,\ n \in \mathbb{N}^+$**. Equivalently, **the KdV flow commutes with the infinitely many flows of the KdV hierarchy**.

We remark that the first two symmetries $\sigma_0 = K_0 = u_x$ and $\sigma_1 = K_1 = u_t = u_{xxx} - 6uu_x$ of the hierarchy (859) are the infinitesimal generators of respectively the x- and t-translations symmetries. Indeed, if the equation is x-translation invariant and t-translation invariant (like the equations of the KdV hierarchy, that do not depend explicitly on x and t) and $u(x,t)$ is a solution, it follows that also $u(x + a, t + b)$ is solution $\forall a, b \in \mathbb{R}$. then

$$\begin{array}{l} u(x + \epsilon, t) = u(x, t) + \epsilon u_x(x, t) + O(\epsilon^2), \quad \sigma_0 = u_x(x, t), \quad a = \epsilon,\ b = 0, \\ u(x, t + \epsilon) = u(x, t) + \epsilon u_t(x, t) + O(\epsilon^2), \quad \sigma_1 = u_t(x, t) = u_{xxx} - 6uu_x, \quad a = 0,\ b = \epsilon. \end{array} \tag{852}$$

As we know from classical mechanics, the Noether theorem establishes a connection between symmetries and constants of motion. Let's see how this connection appears in this integrable field theory.

Let $\mathcal{F}$ be the functional

$$\mathcal{F}[u] := \int_{\mathbb{R}} \rho(u, u_x, u_{xx}, \dots) dx. \tag{853}$$

A variation δu of the field induces the following variation $\delta\mathcal{F}$ of the functional

$$\begin{array}{l} \delta\mathcal{F} = \int_{\mathbb{R}} \left(\frac{\partial\rho}{\partial u}\delta u + \frac{\partial\rho}{\partial u_x}\delta u_x + \frac{\partial\rho}{\partial u_{xx}}\delta u_{xx} + \dots \right) dx \\ = \int_{\mathbb{R}} \left(\frac{\partial\rho}{\partial u} - \partial_x\left(\frac{\partial\rho}{\partial u_x}\right) + \partial_x^2\left(\frac{\partial\rho}{\partial u_{xx}}\right) + \dots \right) \delta u\ dx = \int_{\mathbb{R}} \frac{\delta\mathcal{F}}{\delta u}\ \delta u\ dx, \end{array} \tag{854}$$

where

$$\frac{\delta\mathcal{F}}{\delta u} := \frac{\partial\rho}{\partial u} - \partial_x\left(\frac{\partial\rho}{\partial u_x}\right) + \partial_x^2\left(\frac{\partial\rho}{\partial u_{xx}}\right) + \dots = \sum_{k \geq 0} (-1)^k \partial_x^k \left(\frac{\partial\rho}{\partial(\partial_x^k u)}\right) \tag{855}$$

is the so-called “gradient of $\mathcal{F}$” (or variational (Euler) derivative of $\mathcal{F}$). The second step in (854) comes from integration by parts.

It is possible to show that the denumerable set of functions $\mathcal{L}^n u$ are gradients (variational derivatives) of some functionals $\mathcal{H}_n$:

$$\mathcal{L}^n u = \frac{\delta \mathcal{H}_n}{\delta u}. \tag{856}$$

For instance we have, for the first three flows:

$$\begin{array}{l} \mathcal{L}^0 u = u = \frac{\delta H_0}{\delta u}, \quad \mathcal{H}_0 := \int_{\mathbb{R}} \frac{u^2}{2} dx, \\ \mathcal{L} u = u_{xx} - 3u^2 = \frac{\delta H_1}{\delta u}, \quad \mathcal{H}_1 := -\int_{\mathbb{R}} \left(\frac{u_x^2}{2} + u^3 \right) dx, \\ \mathcal{L}^2 u = u_{xxxx} - 10uu_{xx} - 5u_x^2 + 10u^3 = \frac{\delta \mathcal{H}_2}{\delta u}, \quad \mathcal{H}_2 := \int_{\mathbb{R}} \rho_2 dx, \\ \rho_2 = \frac{u_{xx}^2}{2} + 5uu_x^2 + \frac{5}{2} u^4. \end{array} \tag{857}$$

Let’s verify, f.i., that $\frac{\delta \mathcal{H}_2}{\delta u} = \mathcal{L}^2 u$:

$$\begin{array}{l} \frac{\delta \mathcal{H}_2}{\delta u} = \frac{\partial \rho_2}{\partial u} - \partial_x \left(\frac{\partial \rho_2}{\partial u_x} \right) + \partial_x^2 \left(\frac{\partial \rho_2}{\partial u_{xx}} \right) = 10u^3 - 5u_x^2 - (10uu_x)_x \\ +u_{xxxx}. \end{array} \tag{858}$$

Using (831) and (856), the KdV hierarchy (840) can be written in the Hamiltonian form

$$u_{t_{2n+1}} = c_{2n+1} L^n u_x = c_{2n+1} \partial_x \mathcal{L}^n u = c_{2n+1} \partial_x \frac{\delta \mathcal{H}_n}{\delta u}, \quad n \in \mathbb{N} \tag{859}$$

for the Hamiltonian operator ∂_x, where $\mathcal{H}_n$ is the Hamiltonian of the n^{th} flow. For instance, the Hamiltonian of KdV is $\mathcal{H}_1$.

The denumerable set of functionals $\{\mathcal{H}_m\}_{m \in \mathbb{N}}$ are constants of motion for the whole KdV hierarchy; indeed:

$$\begin{array}{l} \frac{d\mathcal{H}_m}{dt_{2n+1}} = \int_{\mathbb{R}} \left(\frac{\partial \rho_m}{\partial u} u_{t_{2n+1}} + \frac{\partial \rho_m}{\partial u_x} u_{xt_{2n+1}} + \frac{\partial \rho_m}{\partial u_{xx}} u_{xxt_{2n+1}} + \dots \right) dx \\ = \int_{\mathbb{R}} \frac{\delta \mathcal{H}_m}{\delta u} u_{t_{2n+1}} dx = \int_{\mathbb{R}} (\mathcal{L}^{m+1} \cdot 1)(L^n u_x) dx = \int_{\mathbb{R}} L^{n+m+1} u_x dx \\ = \int_{\mathbb{R}} \left(\mathcal{L}^{n+m+1} u \right)_x dx = 0, \end{array} \tag{860}$$

using the fact that $\mathcal{L}$ is the adjoint of L, and, in the last step, integrating the exact derivative of a localized function.

Then the relations

$$\sigma_n = \partial_x \frac{\delta \mathcal{H}_n}{\delta u}, \quad n \in \mathbb{N} \tag{861}$$

connecting (infinitesimal generators of) symmetries to constants of motion are the analogues of the Noether theorem.

7.1.2 Involutivity of the constants of motion

The connections with the finite dimensional theory of Hamiltonian systems of ODEs

$$\dot{\underline{q}} = \nabla_{\underline{p}} H(\underline{q}, \underline{p}), \quad \dot{\underline{p}} = -\nabla_{\underline{q}} H(\underline{q}, \underline{p}), \tag{862}$$

where $\underline{q}, \underline{p} \in \mathbb{R}^N$, and $H(\underline{q}, \underline{p})$ is the Hamiltonian function on the $2N$-dimensional phase space, is quite clear. This system can be written in the more compact form

$$\dot{\underline{u}} = J \nabla_{\underline{u}} H(\underline{u}), \tag{863}$$

where

$$\underline{u} = \begin{pmatrix} q \\ \underline{p} \end{pmatrix} \in \mathbb{R}^{2N}, \quad J = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}, \tag{864}$$

0_N and I_N are respectively the $N \times N$ zero and identity matrices, and J is the Hamiltonian operator.

At last the Hamiltonian $H(\underline{u})$ is a constant of motion:

$$\dot{H} = \sum_{j=1}^{2N} \frac{\partial H}{\partial u_j} \dot{u_j} = \sum_{j=1}^{2N} \frac{\partial H}{\partial u_j} J_{jk} \frac{\partial H}{\partial u_k} = \nabla_{\underline{u}} H \cdot J \nabla_{\underline{u}} H = 0. \tag{865}$$

Comparing the Hamiltonian form (859) of the KdV hierarchy with (863) we see that the Hamiltonian operator J in (863) is replaced by the Hamiltonian operator ∂_x in (859), and the gradient of the Hamiltonian $H(\underline{u})$ by the Euler derivative (the gradient) of the Hamiltonian functionals $\mathcal{H}_n[u]$ of the KdV hierarchy.

Now the Liouville theorem of integrability for Hamiltonian systems (863) states that, if the system (863) possesses N independent constants of motion $H_1, H_2, \dots, H_N$, and these constants of motion are in involution, i.e, they satisfy the equations

$$\nabla_{\underline{u}} H_n \cdot J \nabla_{\underline{u}} H_m = 0, \quad \forall n, m = 1, 2, \dots, N, \tag{866}$$

then the system (863) is Liouville integrable, since there exists a change of dependent variables to action-angle variables allowing one to linearize the dynamics.

In the infinite dimensional Hamiltonian field theory described by the KdV hierarchy, the existence of N independent constants of motion for (863) is replaced by the existence of a denumerable set of constants of motion for the KdV hierarchy. As for the involutivity of these constants of motion, we observe that the scalar product in (866), defining the involutivity of the constants of motion of (863), should be replaced by the equations

$$< \frac{\delta \mathcal{H}_m}{\delta u}, \partial_x \frac{\delta \mathcal{H}_n}{\delta u} >= 0, \quad m, n \in \mathbb{N}^+ \tag{867}$$

involving the infinitely many constants of motion of the KdV hierarchy. The proof of (867), and then the proof of the Liouville integrability of the equations of the KdV hierarchy, follows from (860) and the fact that

$$0 = \frac{d\mathcal{H}_m}{dt_{2n+1}} = \int_{\mathbb{R}} \frac{\delta \mathcal{H}_m}{\delta u} u_{t_{2n+1}} dx = \int_{\mathbb{R}} \frac{\delta \mathcal{H}_m}{\delta u} \left(\mathcal{L}^n u\right)_x dx =< \frac{\delta \mathcal{H}_m}{\delta u}, \partial_x \frac{\delta \mathcal{H}_n}{\delta u} > . \tag{868}$$

7.2 Other soliton equations and their integrability scheme

Also the NLS and KP equations are distinguished examples of soliton equations, and share with KdV many features: the existence of a Lax pair and of an IST method to solve the Cauchy problem, the existence of infinitely many soliton solutions, the existence of infinitely many symmetries and constants of motion in involution. Here we just list their Lax pairs and some basic soliton solutions.

7.2.1 The KP equation

The KP equation

$$(u_t - u_{xxx} + 6uu_x)_x = 3\alpha^2 u_{yy}, \quad u = u(x, y, t) \tag{869}$$

arises from the following integrability scheme

$$\begin{aligned} &(\alpha\partial_y + \partial_x^2 - u)\psi = 0, \\ &(\partial_t + 4\partial_x^3 - 6u\partial_x - 3u_x - 3\alpha w)\psi = 0, \quad w_x = u_y, \end{aligned} \tag{870}$$

where $\alpha = i$ (the time dependent Schrödinger equation) or $\alpha = -1$ (the heat equation).

Its 1 soliton solution (for $\alpha = i$) reads:

$$u = -\frac{(p+q)^2}{2} \cosh^2 \left(\frac{(p+q)x + (q^2 - p^2)y - 4(p^3 + q^3)t - c}{2} \right) \tag{871}$$

(if $p = q$ it reduces to the 1 soliton solution of KdV).

7.3 Applicability versus integrability

We have introduced a certain number of integrable PDEs of the nonlinear mathematical physics, like the Hopf, Burgers, KdV, KP and NLS equations, that are applicable in many physical contexts and, at the same time, integrable. Is it a coincidence? It happens too many times to be a coincidence; so let's try to understand why.

Considering KdV as illustrative example, using the multiscale perturbation theory we have established that a large class of nonlinear dispersive PDEs reduces to the KdV equation under the hypothesis of weak nonlinearity and weak dispersion. Therefore the KdV equation is **the simplest nonlinear model describing weak nonlinearity and weak dispersion**, and since the class of nonlinear dispersive equations contains many physical systems exhibiting weak dispersion, the **KdV equation is largely applicable in physics**. Suppose now that, in the large class of equations reducing to KdV in the above limit, there is one integrable PDE (possessing an integrability scheme like a Lax pair, infinitely many symmetries and constants of motion). Since the multiscale procedure preserves integrability, mapping f.i. symmetries of the original integrable equation to symmetries of KdV, it follows that the KdV equation inherits the integrability properties. Therefore it is enough that one of the equations of the large class of nonlinear dispersive PDEs be integrable, to imply the integrability properties of the model equation. In the light of these considerations, the integrability of KdV is not a miracle, and it is no surprising at all. The same considerations can be made for the other nonlinear integrable PDEs studied in these notes. Therefore it is not surprising that many model equations of the nonlinear Mathematical Physics be integrable.

7.4 Exercices

1) Analyticity projectors. Show that the operators

$$P^{\pm}f(\lambda) := \pm\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{f(\lambda')}{\lambda' - (\lambda \pm i\epsilon)}d\lambda. \tag{872}$$

are analyticity projectors on the real line; i.e., they map a Holder function $f(\lambda)$, $\lambda \in \mathbb{R}$ decaying at ∞ sufficiently fast into functions analytic in the upper and lower halves of the complex λ plane respectively. ii) Show, in particular, that

$$(P^+)^2 = P^+,\ (P^-)^2 = P^-,\ P^+P^- = P^-P^+ = 0,\ P^+ + P^- = 1. \tag{873}$$

2) Given a Holder function $f(\lambda)$ for $\lambda \in \mathbb{R}$ decaying at ∞ sufficiently fast, a polynomial $P(\lambda)$, a set of complex numbers $\{k_j^+,\ R_j^+,\ j = 1,\dots,N^+,$
$k_j^-,\ R_j^-,\ j = 1,\dots,N^-\}$, where Im $k_j^+ > 0$ and Im $k_j^- < 0$, show that the unique solution of the Riemann problem

$$\psi^+(\lambda) - \psi^-(\lambda) = f(\lambda),\quad \lambda \in \mathbb{R} \tag{874}$$

where $\psi^{\pm}(\lambda)$ are analytic in the upper and lower halves of the complex λ plane respectively, except for the simple poles $k_j^{\pm}$'s with residues $R_j^{\pm}$'s, and $\psi^{\pm}(\lambda) \to P(\lambda),\ |\lambda| >> 1$, is

$$\psi^{\pm}(\lambda) = P(\lambda) + \sum_{j=1}^{N^+}\frac{R_j^+}{\lambda - k_j^+} + \sum_{j=1}^{N^-}\frac{R_j^-}{\lambda - k_j^-} \pm P^{\pm}f(\lambda). \tag{875}$$

3) Let $u(x) = -A\delta(x - x_0),\ A \in \mathbb{R}$, be the potential of the Schrödinger equation $[-\partial_x^2 + u(x)]\psi = k^2\psi$. Evaluate explicitly: i) the eigenfunctions of the continuous spectrum and the coefficients $a(k), b(k), R(k), T(k)$; ii) the discrete spectrum p_j, the corresponding eigenfunctions and the norming constants b_j. Show that the existence of discrete spectrum depends on the sign of A.

4) Assume $u(x) = O(\epsilon),\ \epsilon << 1$, and construct the first two terms of the ϵ - expansion of the eigenfunctions and of the spectral data.

5) *Scattering problem.* Study the scattering problem described by the Schrödinger equation

$$-\psi''(x,k)+u(x)\psi(x,k)=k^2\psi(x,k), \quad x\in\mathbb{R}, \quad k>0,$$

where $\psi(x,k)$, the eigenfunction of the continuous spectrum of the Schrödinger operator $-d^2/dx^2+V(x)$, represents the wave function of a particle beam scattered by the localized potential $u(x)$ e $E=k^2>0$ is the energy of the beam (the continuous spectrum $\sigma_c=\{E>0\}$), with the following boundary conditions:

$$\psi(x,k)\sim \ R(k)e^{-ikx}+e^{ikx}, \ x\sim-\infty; \qquad \psi(x,k)\sim \ T(k)e^{ikx}, \ x\sim\infty$$

describing an incoming beam of particles of wave number k and intensity 1, partially reflected and transmitted through the potential ($R(k)$ e $T(k)$ are respectively the reflection and transmission coefficients).
i) Observe that the function $\phi(x,k)=\psi(x,k)/T(k)$ satisfies a simpler scattering problem:

$$\phi''(x,k)+k^2\phi(x,k)=u(x)\phi(x,k), \quad x\in\mathbb{R},, \quad k>0$$

$$\phi(x,k)\sim\frac{R(k)}{T(k)}e^{-ikx}+\frac{e^{ikx}}{T(k)}, \ x\sim-\infty; \quad \phi(x,k)\sim \ e^{ikx}, \ x\sim\infty$$

and use the advanced Green function of the operator d^2/dx^2+k^2 to rewrite such a problem as a Volterra integral equation [9], obtaining:

$$\phi(x,k)=e^{ikx}-\int\limits_x^\infty dy\frac{\sin k(x-y)}{k}u(y)\phi(y,k)$$

and the following integral representations for the reflection and transmission coefficients:

$$\frac{1}{T(k)}=1-\int_{\mathbb{R}}dk\frac{e^{-iky}}{2ik}u(y)\phi(y,k), \quad \frac{R(k)}{T(k)}=\int_{\mathbb{R}}dk\frac{e^{iky}}{2ik}u(y)\phi(y,k).$$

Such an integral equation, equivalent to the Schrödinger differential equation + boundary conditions, is the most convenient formulation of the problem to extract informations.
ii) Use the method of successive approximations to study the properties of ϕ in the following way.
a) Rerwrite the integral equation for the unknown $f(x,k)=\phi(x,k)e^{-ikx}$, such that $f\sim 1, \ x\to\infty$:

$$f(x,k)=1+\int\limits_x^\infty\frac{e^{2ik(y-x)}-1}{2ik}u(y)f(y,k)dy$$

and look for the solution as a Neumann series:

$$f(x,k)=\sum_{i=0}^\infty h_i(x,k), \quad h_0=1, \tag{876}$$

obtaining the recursion relation:

$$h_{j+1}(x,k)=\int\limits_x^\infty\frac{e^{2ik(y-x)}-1}{2ik}u(y)h_j(y,k)dy, \quad j\ge 0. \tag{877}$$

b) From the inequality: $|e^{2ik(y-x)}-1|/|2ik|\le 1/|k|$, valid for Im $k\ge 0$, $k\neq 0$, show that

$$|h_{j+1}(x,k)|\le\tfrac{1}{|k|}\int\limits_x^\infty|u(y)||h_j(y,k)|dy, \tag{878}$$

and then that:

$$\begin{array}{l} |h_n(x,k)|\le\frac{1}{n!}\left(\frac{A(x)}{|k|}\right)^n\le\frac{1}{n!}\left(\frac{A(-\infty)}{|k|}\right)^n, \\ A(x):=\int\limits_x^\infty|V(y)|dy. \end{array} \tag{879}$$

Therefore the Neumann series representing the solution is absolutely and uniformely convergent for Im $k \geq 0,\ k \neq 0$, if $u(x) \in L_1(\mathbb{R})$. Under these conditions, the solution exists unique, and it is analytic in the upper half of the complex k plane. Analogously one can prove that $1/T(k)$ is analytic in the upper half of the complex k plane. Under more stringent conditions on u, one could show, in a similar manner, that the eigenfunction is also continuous on the real k axes, where the physics takes place.
c) Let $k_j,\ j = 1,..,N$ be the zeroes of the function $1/T(k)$ in the upper half of the complex k plane (the poles of the transmission coefficient). Then, since $\lambda_j = E_j = k_j^2 \in \mathbb{R}$, it follows that a) k_j is purely imaginary: $k_j = ip_j,\ p_j > 0,\ j = 1,..,N$, b) the functions $\phi(x, k_j),\ j = 1,..,N$ are exponentially localized:

$$\phi_j(x) := \phi(x, k_j) = O(e^{-p_j|x|}), \quad |x| \to \infty, \quad j = 1,..N$$

and then they are eigenfunctions of the Schrödinger operator in $L_2(\mathbb{R})$:

$$-\phi_j''(x) + u(x)\phi_j(x) = -p_j^2\phi_j(x), \quad x \in \mathcal{R}$$

corresponding to negative eigenvalues $\lambda_j = E_j = -p_j^2 < 0$ of the energy (the discrete spectrum: $\sigma_p = \{-p_j^2\}_1^N$). Summarizing: $\sigma = \sigma_p \cup \sigma_c = \{-p_j^2\}_1^N \cup \mathbb{R}^+$.
d) Show that the set of $\lambda_j = -p_j^2,\ j = 1,..,N$ is bounded from below.
Hint. Take the scalar product of the eigenfunction ϕ_j, normalized to 1, with the Schrödinger equation, obtaining:

$$\lambda_j - (\phi_j, u\phi_j) = (\phi_j', \phi_j') \geq 0 \quad \Rightarrow \quad |\lambda_j| \leq -(\phi_j, V\phi_j) \leq |(\phi_j, u\phi_j)| \leq ||u||_\infty.$$

e) Show that, if $u(x) = u_0\delta(x - x_0)$, the integral equation admits the solution

$$\phi(x, k) = e^{ikx} - u_0 H(x_0 - x)\frac{\sin k(x - x_0)}{k}e^{ikx_0}.$$

Then:

$$\phi(x, k) = \frac{2ik - u_0}{2ik}e^{ikx} + \frac{u_0 e^{2ikx_0}}{2ik}e^{-ikx}, \quad x < x_0$$

$$T(k) = \frac{2ik}{2ik - u_0}, \qquad R(k) = \frac{u_0 e^{2ikx_0}}{2ik - u_0}.$$

Found $\phi(x, k)$, at last reconstruct $\psi(x, k) = \frac{2ik}{2ik - u_0}\phi(x, k)$.
f) Verify that the solution we found for $k \in \mathbb{R}$, if extended outside the real k axis, diverges always at + or - infinity, unless $k = -iu_0/2 \in i\mathbb{R}^+$. Therefore, if the potential is positive ($u_0 > 0$), no eigenfunctions exist in $L_2(\mathbb{R})$; if, instead, the potential is negative, then there exists one and only one $L_2(\mathbb{R})$ eigenfunction $\psi_1(x) := \phi(x, i|u_0|/2) \in L_2(\mathbb{R})$:

$$\psi_1(x) = H(x - x_0)e^{-\frac{|u_0|}{2}x} + H(x_0 - x)e^{\frac{|u_0|}{2}x}$$

corresponding to the negative energy $E_1 = k_1^2 = -u_0^2/4$, and describing a bound state (a localized quantum particle): $\sigma_p = \{E_1\}$.
g) If $u(x) = \epsilon v(x),\ \epsilon << 1$, show that:

$$\phi(x, k) = e^{ikx} - \epsilon \int_x^\infty dy \frac{\sin k(x - y)}{k} v(y) e^{iky} + O(\epsilon^2),$$

$$T(k) = 1 + \frac{\epsilon}{2ik}\int_{\mathcal{R}} dx v(x) + O(\epsilon^2), \quad R(k) = \frac{\epsilon}{2ik}\int_{\mathcal{R}} dx v(x) e^{-2ikx} + O(\epsilon^2)$$

6) Using the above strategy, study the scattering problem

$$\phi''(x, k) + k^2\phi(x, k) = u(x)\phi(x, k), \quad x \in \mathcal{R}, \qquad \phi(x, k) \sim\ e^{-ikx}, \ x \sim -\infty$$

showing that, in this case, it is convenient to use the retarded Green function of the operator $d^2/dx^2 + k^2$.

7) Let $\varphi(x,k)$ and $\psi(x,k)$ be the Jost eigenfunctions of the Schrödinger operator satisfying the boundary conditions:

$$\varphi(x,k) \sim e^{-ikx}, \quad x \to -\infty, \qquad \psi(x,k) \sim e^{-ikx}, \quad x \to \infty \tag{880}$$

i) Write the integral equations satisfied by them; ii) show that $\varphi(x,k)e^{ikx}$ and $\psi(x,k)e^{ikx}$ are analytic respectively in the upper and lower halves of the k plane; iii) show that

$$-2i\frac{d}{dx}[k(\psi(x,k)e^{ikx} - 1)] \to u(x), \quad |k| \gg 1. \tag{881}$$

8) Let k_0 be a zero of $a(k) = 1/T(k)$, where $T(k)$ is the transmission coefficient of the Schrödinger equation. i) Show that k_0 belongs to the discrete spectrum (therefore $k_0 = ip$, $p > 0$) and, correspondingly, that $\varphi(x,k_0) \in L^2(\mathbb{R})$, with the asymptotics

$$\varphi(x,k_0) \sim e^{px}, \quad x \sim -\infty, \quad \varphi(x,k_0) \sim be^{-px}, \quad x \sim \infty \tag{882}$$

where $b \in \mathbb{R}$.
ii) Show that the zeroes $k_0 = ip$ of $a(k)$ are simple, and that $iba'(ip) > 0$.
A. For i), use the Wronskian relation $W(\varphi, \bar{\psi}) = 2ika(k)$ to infer that $\varphi(x,k_0) = b\overline{\psi(x,k_0)} = b\psi(x,-k_0)$.

9) *Inverse Problem.* Using the analyticity properties of $\varphi(x,k), \psi(x,k), a(k)$, together with their asymptotics for large k, i) rewrite the scattering equation

$$\varphi(x,k) = a(k)\psi(x,k) + b(k)\psi(x,-k), \quad k \in \mathbb{R} \tag{883}$$

for the Schrödinger operator as a linear Riemann - Hilbert problem on the real k axis, for a given set of scattering data. ii) Express the solution of such a linear RH problem in terms of integral equations for the eigenfunctions, and iii) reconstruct the potential u(x) in terms of the scattering data.

10) *t - evolution of the scattering data.* Obtain the t evolution of the scattering data if u evolves according to KdV.

11) Consider the Cauchy problem on the line for the KdV equation $u_t + u_{xxx} - 6uu_x = 0$, with the initial condition $u(x,0) = -b\exp(-x^2)$. Show (numerically) that, i) if $b = 0.1$, the dynamics is described by a pure nonlinear dispersive waves (travelling with negative group velocity); if $b = 1$, by a nonlinear dispersive waves (traveling with negative group velocity) + one soliton, travelling with positive speed; if $b = 4$, by a nonlinear dispersive waves (traveling with negative group velocity) + two solitons, traveling with positive speeds (see the figures below). Interpret these numerical experiments in the light of the IST for KdV.

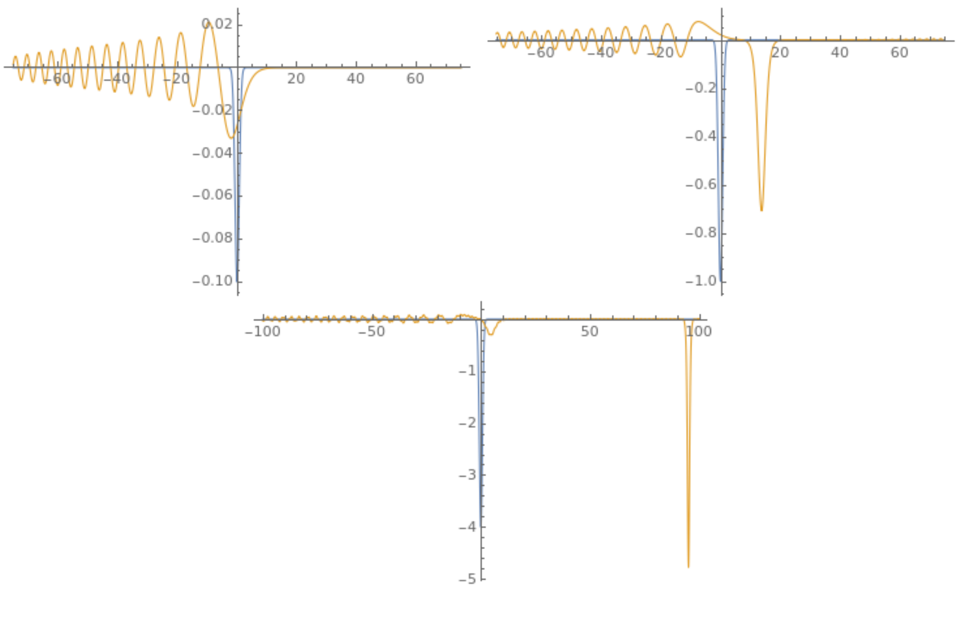


Figure 1 for $b = 0.1$: the area of the well is not large enough to support bound states $\Rightarrow$ the solution evolves into nonlinear dispersive waves; Figure 2 for $b = 1$: the area of the well is large enough to support one bound state $\Rightarrow$ the solution evolves into a one soliton + nonlinear dispersive waves; Figure 3 for $b = 4$: the area of the well is large enough to support two bound states $\Rightarrow$ the solution evolves into two solitons + nonlinear dispersive waves.

11) Construct the 2-soliton solution of KdV and study the interaction of the two solitons.

8 Darboux transformations

Darboux transformations (DTs) are algebraic techniques enabling one to construct exact solutions of nonlinear integrable PDEs from simpler exact solutions. Here we present them for the KdV and NLS equations.

8.1 DTs for KdV

We begin with the KdV equation

$$u_t + u_{xxx} - 6uu_x = 0, \quad u = u(x,t) \tag{884}$$

and its Lax pair

$$\begin{aligned} &\psi_{xx} = (u - k^2)\psi, \quad \psi = \psi(x,t;k), \\ &\psi_t = (c - u_x)\psi + (2u + 4k^2)\psi_x, \end{aligned} \tag{885}$$

and we indicate by (u,ψ) any pair of functions $u = u(x,t)$ and $\psi = \psi(x,t;k)$ satisfying equations (884) and (885). We look for a DT transformation in the matrix form

$$\vec{\psi} = \chi\vec{\psi}^{(0)}, \quad \vec{\psi} = \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}, \quad \vec{\psi}^{(0)} = \begin{pmatrix} \psi^{(0)} \\ \psi^{(0)}_x \end{pmatrix}, \quad \chi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{886}$$

implying

$$\psi(x,t;k) = A(x,t,k)\psi^{(0)}(x,t;k) + B(x,t,k)\psi^{(0)}_x(x,t;k), \tag{887}$$

mapping a given solution $(u^{(0)}, \psi^{(0)})$ of equations (884), (885) to another solution (u,ψ) of the same equations.

From (887)

$$\begin{aligned} &\psi_x = A_x\psi^{(0)} + A\psi^{(0)}_x + B_x\psi^{(0)}_x + B\psi^{(0)}_{xx} = A_x\psi^{(0)} + A\psi^{(0)}_x \\ &+B_x\psi^{(0)}_x + B(u^{(0)} - k^2)\psi^{(0)} = [A_x + B(u^{(0)} - k^2)]\psi^{(0)} + [A + B_x]\psi^{(0)}_x, \end{aligned} \tag{888}$$

$$\begin{aligned} &\psi_{xx} = [A_{xx} + B_x(u^{(0)} - k^2) + Bu^{(0)}_x]\psi^{(0)} + [A_x + B(u^{(0)} - k^2)]\psi^{(0)}_x \\ &+[A_x + B_{xx}]\psi^{(0)}_x + [A + B_x](u^{(0)} - k^2)\psi^{(0)} = (u - k^2)\psi \\ &= (u - k^2)[A\psi^{(0)} + B\psi^{(0)}_x]. \end{aligned} \tag{889}$$

Since $\psi^{(0)}$ and $\psi^{(0)}_x$ are independent, we have two equations for the unknowns A and B:

$$\begin{aligned} &A_{xx} + 2B_x(u^{(0)} - k^2) + A(u^{(0)} - u) + Bu^{(0)}_x = 0, \\ &2A_x + B_{xx} + B(u^{(0)} - u) = 0 \end{aligned} \tag{890}$$

As we did in the construction of the KdV hierarchy, the hierarchy of DTs is constructed assuming for A and B a polynomial dependence on k^2. In the simplest case, A, B do not depend on k^2; then the first of equations (890) implies that

$$B_x = 0 \quad \Rightarrow \quad B = 1 \tag{891}$$

and equations (890) simplify to

$$\begin{aligned} &A_{xx} + A(u^{(0)} - u) + u^{(0)}_x = 0, \\ &2A_x + (u^{(0)} - u) = 0. \end{aligned} \tag{892}$$

Multiplying the second equation by A, and subtracting it to the first, we obtain the equation

$$A_{xx} - 2AA_x + u^{(0)}_x = 0 \tag{893}$$

that can be integrated once to the Riccati equation

$$A_x - A^2 + u^{(0)} - k_1^2 = 0, \tag{894}$$

where k_1 is a constant. The Riccati equation can be linearized by the transformation

$$A = -\frac{\phi_x}{\phi}, \tag{895}$$

to the Schrödinger equation

$$\phi_{xx} = (u^{(0)} - k_1^2)\phi, \quad \phi = \phi(x), \tag{896}$$

for $k = k_1$.

Summarizing, given the solution $(u^{(0)}, \psi^{(0)})$ of equations (884), (885), and given a solution $\phi(x,t)$ of the Schrödinger equation for the potential $u^{(0)}(x,t)$ and $k = k_1$, then the pair (u, ψ), defined by

$$\begin{array}{l} u(x) = u^{(0)}(x) - 2\left(\frac{\phi_x}{\phi}\right)_x, \\ \psi(x,k) = \psi_x^{(0)}(x,k) - \frac{\phi_x}{\phi}\psi^{(0)}(x,k), \end{array} \tag{897}$$

is also solution of equations (884), (885). The first of the DTs (897) comes from the second of equations (892) and from (895); the second from (887), (895), and $B = 1$.

As an example, one construct the 1-soliton solution of KdV choosing $u^{(0)}(x,t) = 0$. Then $\phi(x,t) = \gamma_1 e^{p(x-4p^2t)} + \gamma_2 e^{-p(x-4p^2t)}$ solves the Schrödinger equation for $k_1 = ip$ and

$$u(x,t) = -2p\left(\frac{\gamma_1 e^{p(x-4p^2t)} - \gamma_2 e^{-p(x-4p^2t)}}{\gamma_1 e^{p(x-4p^2t)} + \gamma_2 e^{-p(x-4p^2t)}}\right)_x. \tag{898}$$

Choosing $\gamma_1 = re^{-px_0}, \ \gamma_2 = re^{px_0}$, one recovers the 1-soliton solution:

$$u(x,t) = -\frac{2p^2}{\cosh^2\left(p(x - 4p^2t - x_0)\right)}. \tag{899}$$

8.1.1 What is the spectral meaning of the DT?

For the given solution $u^{(0)}(x,t)$ of KdV, we choose $\psi^{(0)}$ to be the Jost eigenfunction $\varphi^{(0)}(x,t,k)$:

$$\varphi^{(0)} \sim e^{-ikx}, \ x \sim -\infty, \quad \varphi^{(0)} \sim a^{(0)}(k)e^{-ikx} + b^{(0)}(k)e^{ikx}, \ x \sim \infty, \tag{900}$$

and $\phi(x,t)$ to be a solution of the Lax pair for $u = u^{(0)}(x,t)$ and $k = ip, \ p > 0$ such that

$$\phi \sim \alpha_\pm e^{-px} + \beta_\pm e^{px}, \ x \sim \pm\infty, \quad \Rightarrow \quad \frac{\phi_x}{\phi} \sim \pm p, \ x \sim \pm\infty. \tag{901}$$

Then, at $x \sim -\infty$, the second of the DT equations (897)

$$\psi(x,k) = \varphi_x^{(0)}(x,k) - \frac{\phi_x(x)}{\phi(x)}\varphi^{(0)}(x,k) \sim -i(k+ip)e^{-ikx}, \tag{902}$$

Let $\varphi(x,k)$ be the Jost eigenfunction corresponding to the transformed potential u and behaving as

$$\varphi \sim e^{-ikx}, \ x \sim -\infty, \quad \varphi \sim a(k)e^{-ikx} + b(k)e^{ikx}, \ x \sim \infty; \tag{903}$$

then $\psi(x,k) = -i(k+ip)\varphi(x,k)$ and

$$\begin{array}{l} \varphi(x,k) = \frac{i}{k+ip}\left[\varphi_x^{(0)}(x,k) - \frac{\phi_x}{\phi}\varphi^{(0)}(x,k)\right] \\ \sim \frac{i}{k+ip}\left[ik\left(-a^{(0)}(k)e^{-ikx} + b^{(0)}(k)e^{ikx}\right) - p\left(a^{(0)}(k)e^{-ikx} + b^{(0)}(k)e^{ikx}\right)\right] \\ \sim \frac{k-ip}{k+ip}a^{(0)}(k)e^{-ikx} - b^{(0)}(k)e^{ikx} = a(k)e^{-ikx} + b(k)e^{ikx}, \quad x \sim \infty. \end{array} \tag{904}$$

We conclude that

$$a(k) = \frac{k-ip}{k+ip}a^{(0)}(k), \qquad b(k) = -b^{(0)}(k). \tag{905}$$

In terms of transmission and reflection coefficients:

$$T(k) = \frac{k+ip}{k-ip}T^{(0)}(k), \quad R(k) = -\frac{k+ip}{k-ip}R^{(0)}(k). \tag{906}$$

Therefore **the DT adds the zero ip to the scattering coefficient a (the pole ip to the transmission coefficient), corresponding to the bound state $k = ip, \ p > 0$.**

8.2 DTs for the NLS equation [37, 40]

Now we outline the construction of the DT for the NLS equation, showing how to add simultaneously n poles.

We recall that the focusing and defocusing NLS equations

$$iu_t + u_{xx} + 2\eta|u|^2u = 0, \quad \eta = \pm 1 \tag{907}$$

are the compatibility condition of the following Zakharov-Shabat Lax pair:

$$\vec{\Psi}_x(\lambda, x, t) = \hat{X}(\lambda, x, t)\vec{\Psi}(\lambda, x, t), \quad \vec{\Psi}_t(\lambda, x, t) = \hat{T}(\lambda, x, t)\vec{\Psi}(\lambda, x, t), \tag{908}$$

where

$$\begin{aligned} &\hat{X}(\lambda, x, t) = -i\lambda\sigma_3 + iU(x, t), \\ &\hat{T}(\lambda, x, t) = \begin{pmatrix} -2i\lambda^2 + i\eta u(x, t)\overline{u(x, t)} & 2i\lambda u(x, t) - u_x(x, t) \\ 2i\lambda\eta\overline{u(x, t)} + \eta \overline{u_x(x, t)} & 2i\lambda^2 - i\eta u(x, t)\overline{u(x, t)} \end{pmatrix}, \\ &\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & u(x, t) \\ \eta\bar{u}(x, t) & 0 \end{pmatrix}. \end{aligned} \tag{909}$$

We first observe that the vector eigenfunction $\vec{\Psi}(\lambda, x, t)$ satisfies the following symmetry.
If $\vec{\Psi}(\lambda, x, t) = (\Psi_1(\lambda, x, t), \Psi_2(\lambda, x, t))^T$ solves the Lax pair (908), then $(-\eta\overline{\Psi_2(\lambda, x, t)}, \overline{\Psi_1(\lambda, x, t)})^T$ solves the Lax pair (908) in which λ is replaced by $\bar{\lambda}$.

We also observe that, since $\mathrm{tr}X = \mathrm{tr}T = 0$, the Abel theorem implies

$$(\det\Psi)_x = \mathrm{tr}(X)\ \det(\Psi) = 0, \qquad (\det\Psi)_t = \mathrm{tr}(T)\ \det(\Psi) = 0; \tag{910}$$

then $\det(\Psi) = const$.

It is easy to verify that

$$U^\dagger = \mathcal{N}U\mathcal{N}, \quad X^\dagger(\bar{\lambda}) = -\mathcal{N}X(\lambda)\mathcal{N}, \quad T^\dagger(\bar{\lambda}) = -\mathcal{N}T(\lambda)\mathcal{N} \tag{911}$$

where $U^\dagger$ is the adjoint of U, and

$$\mathcal{N} = \begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix}. \tag{912}$$

In addition we have

$$\begin{aligned} &\left(\Psi^{-1}(\lambda)\right)_x = -\Psi^{-1}(\lambda)\Psi_x(\lambda)\Psi^{-1}(\lambda) = -\Psi^{-1}(\lambda)X(\lambda), \\ &\left(\Psi^{-1}(\lambda)\right)_t = -\Psi^{-1}(\lambda)T(\lambda) \end{aligned} \tag{913}$$

and, using (911):

$$\begin{aligned}\Psi^\dagger_x(\bar\lambda) &= \Psi^\dagger(\bar\lambda)X^\dagger(\bar\lambda) = -\Psi^\dagger(\bar\lambda)\mathcal{N}X(\lambda)\mathcal{N},\\ \Psi^\dagger_t(\bar\lambda) &= \Psi^\dagger(\bar\lambda)T^\dagger(\bar\lambda) = -\Psi^\dagger(\bar\lambda)\mathcal{N}T(\lambda)\mathcal{N}\end{aligned} \tag{914}$$

Comparing equations (913) and (914), we infer that $\Psi^{-1}(\lambda)$ and $\mathcal{N}\Psi^\dagger(\bar\lambda)\mathcal{N}$ satisfy the same matrix equations $F_x = -FX, \quad F_t = -FT$; therefore the normalization of the fundamental solution can be chosen such that

$$\Psi^{-1}(\lambda) = \mathcal{N}\Psi^\dagger(\bar\lambda)\mathcal{N} \tag{915}$$

(we often omit to indicate the dependence on x, t, if not necessary).

Let $u^{(0)}(x,t)$ be a particular solution of NLS, and let $\Psi^{(0)}(x,t,\lambda)$ be the corresponding fundamental solution of (908). Again its normalization is chosen such that

$${\Psi^{(0)}}^{-1}(\lambda) = \mathcal{N}{\Psi^{(0)}}^\dagger(\bar\lambda)\mathcal{N}. \tag{916}$$

We look for the following relation

$$\Psi(x,t,\lambda) = \chi(x,t,\lambda)\Psi^{(0)}(x,t,\lambda) \tag{917}$$

between the matrix solutions $\Psi(x,t,\lambda)$ and $\Psi^{(0)}(x,t,\lambda)$ of (908), corresponding to the particular solutions $u(x,t)$ and $u^{(0)}(x,t)$ of NLS, where $\chi(x,t,\lambda)$ is the so-called Darboux (Dressing) matrix.

We also assume that

$$\chi(x,t,\lambda) = I + \frac{\tilde\chi(x,t)}{\lambda} + O(\lambda^{-2}), \quad |\lambda| \gg 1. \tag{918}$$

If Ψ and $\Psi^{(0)}$ satisfy (915) and (916), then also the Darboux matrix satisfies the symmetry

$$\chi^{-1}(\lambda) = \mathcal{N}\chi^\dagger(\bar\lambda)\mathcal{N}. \tag{919}$$

Substituting (917) in (908) and using (918), we infer that

$$U = U^{(0)} + [\sigma_3, \tilde\chi], \tag{920}$$

implying that

$$u(x,t) = u^{(0)}(x,t) + 2\left(\tilde\chi(x,t)\right)_{12}, \tag{921}$$

where $(M)_{12}$ is the component 12 of matrix M.

From the definition (917) we have

$$\begin{aligned}&\Psi_x = \chi_x\Psi^{(0)} + \chi\Psi^{(0)}_x = (\chi_x + \chi X^{(0)})\Psi^{(0)},\\ &\Psi_x = X\Psi = X\chi\Psi^{(0)},\\ &\Rightarrow \quad \chi_x = X\chi - \chi X^{(0)}\end{aligned} \tag{922}$$

and, analogously

$$\chi_t = T\chi - \chi T^{(0)}. \tag{923}$$

In addition:

$$\begin{aligned}(\chi^{-1})_x &= -\chi^{-1}\chi_x\chi^{-1} = -\chi^{-1}(X\chi - \chi X^{(0)})\chi^{-1} = -\chi^{-1}X + X^{(0)}\chi^{-1},\\ (\chi^{-1})_t &= -\chi^{-1}\chi_t\chi^{-1} = -\chi^{-1}(T\chi - \chi T^{(0)})\chi^{-1} = -\chi^{-1}T + T^{(0)}\chi^{-1}.\end{aligned} \tag{924}$$

Extracting X or T from these equations we get

$$\begin{aligned}X(\lambda) &= -\chi(\chi^{-1})_x + \chi X^{(0)}\chi^{-1} = -\chi(\lambda)\left(\partial_x - X^{(0)}(\lambda)\right)\chi^{-1}(\lambda),\\ T(\lambda) &= -\chi(\chi^{-1})_t + \chi T^{(0)}\chi^{-1} = -\chi(\lambda)\left(\partial_t - T^{(0)}(\lambda)\right)\chi^{-1}(\lambda).\end{aligned} \tag{925}$$

We remark that the matrices $X^{(0)}, T^{(0)}, X, T$ depend on λ polinomially:

$$\begin{aligned}&X^{(0)}(\lambda; x, t) = -i\lambda\sigma_3 + iU^{(0)}(x,t), \quad X(\lambda; x, t) = -i\lambda\sigma_3 + iU(x,t),\\ &T^{(0)}(\lambda; x, t) = 2\lambda X^{(0)}(\lambda; x, t) + W^{(0)}(x,t), \quad T(\lambda; x, t) = 2\lambda X(\lambda; x, t) + W(x,t),\end{aligned} \tag{926}$$

and this will imply suitable constraints on χ.

8.2.1 Rational dependence on λ

We also assume that $\chi(\lambda)$ be a rational function of λ:

$$\chi(x,t;\lambda) = I + \sum_{m=1}^{N} \frac{A_m(x,t)}{\lambda - \lambda_m}, \quad \lambda_m \in \mathbb{C}, \tag{927}$$

implying that

$$u(x,t) = u_0(x,t) + 2\sum_{m=1}^{N} \left(A_m(x,t)\right)_{12}. \tag{928}$$

Using (919) it follows that

$$\chi^{(-1)}(\lambda) = \mathcal{N}\chi^{\dagger}(\bar{\lambda})\mathcal{N} = I + \sum_{m=1}^{N} \frac{\mathcal{N}A_m^{\dagger}(x,t)\mathcal{N}}{\lambda - \bar{\lambda}_m}. \tag{929}$$

In addition:

$$\chi(\lambda)\chi^{-1}(\lambda) = \chi(\lambda)\mathcal{N}\chi^{\dagger}(\bar{\lambda})\mathcal{N} = I, \tag{930}$$

Consequently we have

$$I = \chi(\lambda)\chi^{-1}(\lambda) \sim \chi(\bar{\lambda}_n)\frac{\mathcal{N}A_n^{\dagger}\mathcal{N}}{\lambda - \bar{\lambda}_n}, \quad \lambda \sim \bar{\lambda}_n \tag{931}$$

implying that the residue at $\bar{\lambda}_n$ must be zero:

$$\chi(\bar{\lambda}_n)\mathcal{N}A_n^\dagger = 0, \quad 1 \le n \le N. \tag{932}$$

It follows that the 2×2 matrices $A_n, A_n^\dagger$ are degenerate

$$\det(A_n) = \det(A_n^\dagger) = 0, \quad \forall n \tag{933}$$

and admit the following representation

$$A_n = \boldsymbol{p^{(n)}} \cdot \boldsymbol{r^{(n)}}^\dagger, \quad A_n^\dagger = \boldsymbol{r^{(n)}} \cdot \boldsymbol{p^{(n)}}^\dagger, \tag{934}$$

where the two component vectors $\boldsymbol{p^{(n)}}, \boldsymbol{r^{(n)}}$ will be specified later on, and

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \boldsymbol{v}^\dagger = (\overline{v}_1, \overline{v}_2). \tag{935}$$

In components:

$$(A_n)_{\alpha\beta} = p_\alpha^{(n)}\overline{r_\beta^{(n)}}, \quad \left(A_n^\dagger\right)_{\alpha\beta} = r_\alpha^{(n)}\, \overline{p_\beta^{(n)}}. \tag{936}$$

Therefore the constraint (932) is satisfied if

$$\chi(\bar{\lambda}_n)\, \mathcal{N}\boldsymbol{r^{(n)}} = \boldsymbol{0}. \tag{937}$$

In addition, equations (919) and (925) imply that, if $X^{(0)}, T^{(0)}$ had the λ-dependence indicated in (926), X, T would be singular in $\bar{\lambda}_n$:

$$X(\lambda) \sim -\chi(\bar{\lambda}_n) \left(\partial_x - X^{(0)}(\bar{\lambda}_n)\right) \mathcal{N}\boldsymbol{r^{(n)}} \cdot \overline{\boldsymbol{p^{(n)}}}^{\boldsymbol{T}} (\lambda - \bar{\lambda}_n)^{-1}, \quad \lambda \sim \bar{\lambda}_n. \tag{938}$$

But since X, T must have the λ-dependence indicated in (926) as well, it follows that the residue of the expression in (938) must be zero. Consequently

$$\begin{aligned} 0 &= \chi(\bar{\lambda}_n) \left(\partial_x - X^{(0)}(\bar{\lambda}_n)\right) \mathcal{N}\boldsymbol{r^{(n)}} \cdot \overline{\boldsymbol{p^{(n)}}}^{\boldsymbol{T}} \\ &= \chi(\bar{\lambda}_n) \left(\left(\partial_x - X^{(0)}(\bar{\lambda}_n)\right) \mathcal{N}\boldsymbol{r^{(n)}}\right) \cdot \overline{\boldsymbol{p^{(n)}}}^{\boldsymbol{T}} + \chi(\bar{\lambda}_n)\, \mathcal{N}\boldsymbol{r^{(n)}} \cdot \overline{\boldsymbol{p_x^{(n)}}}^{\boldsymbol{T}}. \end{aligned} \tag{939}$$

Using (937), we infer that

$$\chi(\bar{\lambda}_n) \left(\partial_x - X^{(0)}(\bar{\lambda}_n)\right) \mathcal{N}\boldsymbol{r^{(n)}} = \boldsymbol{0}. \tag{940}$$

Let $\boldsymbol{q^{(n)}} = (q_1^{(n)}, q_2^{(n)})^T$ be a vector solution of the Lax pair (908),(909) with $u = u^{(0)}$ and $\lambda = \lambda_n$:

$$\boldsymbol{q_x^{(n)}} = X^{(0)}(\lambda_n)\boldsymbol{q^{(n)}}; \tag{941}$$

then the complex symmetry of the ZS Lax pair implies that $\mathcal{N}\boldsymbol{r^{(n)}}$ solves the Lax pair (908),(909) with $u = u^{(0)}$ and $\lambda = \overline{\lambda_n}$:

$$\left(\mathcal{N}\boldsymbol{r^{(n)}}\right)_x = X^{(0)}(\overline{\lambda_n})\mathcal{N}\boldsymbol{r^{(n)}}, \tag{942}$$

where

$$\boldsymbol{r^{(n)}} := \begin{pmatrix} -\overline{q_2^{(n)}} \\ \\ \overline{q_1^{(n)}} \end{pmatrix}. \tag{943}$$

Once $\boldsymbol{r^{(n)}}$, $n = 1, \dots, N$ are known, equation (937):

$$\left(I + \sum_{m=1}^{N} \frac{\boldsymbol{p^{(m)}} \cdot \boldsymbol{r^{(m)}}^{\dagger}}{\lambda_n - \lambda_m}\right)\mathcal{N}\boldsymbol{r^{(n)}} = \boldsymbol{0}, \quad n = 1, \dots, N \tag{944}$$

must be viewed as a linear algebraic system for the vectors $\boldsymbol{p^{(n)}}$, $n = 1, \dots, N$. We rewrite it in the final form:

$$\begin{array}{l} \sum\limits_{m=1}^{N} B_{nm}\boldsymbol{p^{(m)}} = \mathcal{N}\boldsymbol{r^{(n)}}, \quad n = 1, \dots, N, \\ B_{nm} := \frac{\boldsymbol{r^{(m)}}^{\dagger} \cdot \mathcal{N}\boldsymbol{r^{(n)}}}{\lambda_m - \bar{\lambda}_n} = \frac{\eta \overline{q_2^{(n)}} q_2^{(m)} + \overline{q_1^{(n)}} q_1^{(m)}}{\lambda_m - \bar{\lambda}_n}. \end{array} \tag{945}$$

Known $\boldsymbol{r^{(n)}}, \boldsymbol{p^{(n)}}$, $n = 1, \dots, N$, the matrices $A_n = \boldsymbol{p^{(n)}} \cdot \boldsymbol{r^{(n)}}^{\dagger}$, $n = 1, \dots, N$ and the Darboux matrix χ are known, together with the dressed solution

$$u(x,t) = u^{(0)}(x,t) + 2\sum_{m=1}^{N} p_1^{(m)} q_1^{(m)}. \tag{946}$$

In the simplest case $N = 1$, omitting the superscript (1), we have:

$$\boldsymbol{p} = \frac{1}{B_{11}}\mathcal{N}\boldsymbol{r} = \frac{1}{B_{11}}\begin{pmatrix} -\eta\,\overline{q_2} \\ \overline{q_1} \end{pmatrix}, \qquad B_{11} = \frac{\eta\,|q_2|^2 + |q_1|^2}{2i\mathrm{Im}\lambda_1}, \tag{947}$$

$$u(x,t) = u^{(0)}(x,t) + 2p_1\overline{r_2} = u^{(0)}(x,t) - 4i\eta\mathrm{Im}\lambda_1 \frac{q_1\overline{q_2}}{\eta\,|q_2|^2 + |q_1|^2}, \tag{948}$$

and the Darboux matrix reads

$$\begin{array}{l} \chi = I + \frac{2iIm(\lambda_1)}{\lambda - \lambda_1}P_\eta, \\ P_\eta = \frac{1}{\eta|q_2|^2 + |q_1|^2}\begin{pmatrix} -\eta\bar{q}_2 \\ \bar{q}_1 \end{pmatrix}(-q_2, q_1) = \frac{1}{\eta|q_2|^2 + |q_1|^2}\begin{pmatrix} \eta|q_2|^2 & -\eta\bar{q}_2 q_1 \\ -q_2\bar{q}_1 & |q_1|^2 \end{pmatrix}. \end{array} \tag{949}$$

If $\eta = 1$, P_1 is the orthogonal projector on the subspace span$\{(-\bar{q}_2, \bar{q}_1)^T\}$.

8.2.2 The solution on the zero background

If, as for the KdV, $u^{(0)} = 0$, then the fundamental solution reads $\Psi^{(0)} = \exp[(-i\lambda x - 2i\eta\lambda^2 t)\sigma_3]$, and

$$\boldsymbol{q} := \begin{pmatrix} \xi_1 e^{-i\lambda_1 x - 2i\lambda_1^2 t} \\ \xi_2 e^{i\lambda_1 x + 2i\bar{\lambda}_1^2 t} \end{pmatrix} \tag{950}$$

solve the Lax pair for $\lambda = \lambda_1$. Choosing $\lambda_1 = -a + ib,\ a, b \in \mathbb{R}$ and $\xi_j = r_j e^{i\theta_j},\ j = 1, 2$, equation (948) gives the 1-soliton solution of NLS.

If $\eta = 1$ (the focusing NLS case), one obtains the so-called "bright soliton":

$$u(x,t) = -2ib\frac{e^{2i[ax-2(a^2-b^2)t+\theta_0]}}{\cosh[2b(x-4at-x_0)]}, \tag{951}$$

where $x_0 = \frac{1}{2b}\ln(r_2/r_1)$ and $\theta_0 = (\theta_1 - \theta_2)/2$.

If $\eta = -1$ (the defocusing NLS case), one obtains the solution

$$u(x,t) = -2ib\frac{e^{2i[ax-2(a^2-b^2)t+\theta_0]}}{\sinh[2b(x-4at-x_0)]}, \tag{952}$$

singular at $x = 4at + x_0$. The soliton solution (951) describes a smooth and exponentially localized amplitude modulation of a monochromatic carrier wave; the carrier wave travels with speed $2(a^2 - b^2)/a$, while the exponentially localized envelope travels faster with speed $4a$. We remark that amplitude and localization of the soliton envelope are linearly proportional (and proportional to b), while the envelope speed is completely independent, being proportional to a, unlike the KdV case.

8.2.3 The solution on the homogeneous background $u^{(0)} = ae^{2i|a|^2t}$

In this section we limit our considerations to the focusing NLS case $\eta = 1$, and we choose, as initial solution, the homogeneous background solution

$$u^{(0)} = ae^{2i|a|^2t}, \quad a \text{ is any constant complex parameter.} \tag{953}$$

Using the gauge symmetry (if $u(x,t)$ solves NLS, also $u(x,t)e^{i\rho}$, $\forall\rho \in \mathbb{R}$, solves NLS) and the scaling symmetry (if $u(x,t)$ solves NLS, also $bu(bx, b^2t)$, $\forall b \in \mathbb{R}$, solves NLS), it is possible to choose $a = 1$ in (953) without loss of generality.

The DTs described above allow one to construct analytic solutions over this background(with $a = 1$).

It is straightforward to verify that the corresponding fundamental solution of the Lax pair (908),(908), verifying the condition $\det \Psi^{(0)} = 1$, is

$$\begin{array}{l} \Psi^{(0)}(\lambda) = \frac{1}{\sqrt{2\mu(\mu+\lambda)}} e^{it\sigma_3} \begin{pmatrix} e^{\Theta(\lambda)} & -(\mu+\lambda)e^{-\Theta(\lambda)} \\ (\mu+\lambda)e^{\Theta(\lambda)} & e^{-\Theta(\lambda)} \end{pmatrix}, \\ \Theta(\lambda) \equiv i\mu(x + 2\lambda t), \end{array} \tag{954}$$

where $\sigma_3 = \mathrm{diag}(1, -1)$ is the Pauli matrix and μ, λ are complex parameters satisfying the constraint

$$\mu^2 = 1 + \lambda^2. \tag{955}$$

Since we want to apply the results of this construction to the study of periodic anomalous waves in nature, investigated in the next chapter, we look for solutions periodic in x and hyperbolic in t (to describe the modulation instability), we must choose $-1 < \mu < 1$ and $\lambda \in i\mathbb{R}$, with $|\lambda| < 1$. It is therefore convenient to use the following parametrization of equation (955)

$$\mu = \cos\phi, \quad \lambda = i\sin\phi, \quad \phi \in \mathbb{R}, \tag{956}$$

so that

$$\Theta(\lambda) = i(\cos\phi)x - (\sin 2\phi)t = \frac{ikx - \sigma t}{2}, \quad \mu + \lambda = e^{i\phi}, \tag{957}$$

where

$$k = 2\mu = 2\cos\phi \tag{958}$$

plays the role of wave number and

$$\sigma = 2\sin 2\phi = k\sqrt{4 - k^2} \tag{959}$$

that of growth rate. Therefore

$$\Psi^{(0)}(x, t, \lambda) = \frac{1}{\sqrt{k}} e^{it\sigma_3} \begin{pmatrix} e^{\frac{ikx - \sigma t - i\phi}{2}} & -e^{-\frac{ikx - \sigma t - i\phi}{2}} \\ e^{\frac{ikx - \sigma t + i\phi}{2}} & e^{-\frac{ikx - \sigma t + i\phi}{2}} \end{pmatrix}. \tag{960}$$

A generic vector solution of the ZS Lax pair, for $\lambda = \lambda_1 = i\sin\phi_1$ is a linear combination of the two column vectors of $\Psi^{(0)}(\lambda_1)$:

$$\boldsymbol{q} = \gamma_1 \begin{pmatrix} e^{\frac{ik_1 x - \sigma_1 t - i\phi_1}{2} + it} \\ e^{\frac{ik_1 x - \sigma_1 t + i\phi_1}{2} - it} \end{pmatrix} + \gamma_2 \begin{pmatrix} -e^{-\frac{ik_1 x - \sigma_1 t - i\phi_1}{2} + it} \\ e^{-\frac{ik_1 x - \sigma_1 t + i\phi_1}{2} - it} \end{pmatrix}, \tag{961}$$

where

$$k_1 = 2\cos\phi_1, \qquad \sigma_1 = 2\sin(2\phi_1) = k_1\sqrt{4 - k_1^2}. \tag{962}$$

Choosing

$$\gamma_1 = e^{\frac{\sigma_1 t_1 - ik_1 x_1 + i\pi/2}{2}}, \quad \gamma_2 = \gamma_1^{-1} \tag{963}$$

we get

$$\begin{aligned}\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \begin{pmatrix} e^{\frac{ik_1(x-x_1)+i\pi/2-\sigma_1(t-t_1)-i\phi_1}{2}+it} \\ e^{\frac{ik_1(x-x_1)+i\pi/2-\sigma_1(t-t_1)+i\phi_1}{2}-it} \end{pmatrix} + \begin{pmatrix} -e^{-\frac{ik_1(x-x_1)+i\pi/2-\sigma_1(t-t_1)-i\phi_1}{2}+it} \\ e^{-\frac{ik_1(x-x_1)+i\pi/2-\sigma_1(t-t_1)+i\phi_1}{2}-it} \end{pmatrix} \\ &= 2\begin{pmatrix} \sinh\left[\frac{ik_1(x-x_1)+i\pi/2-\sigma_1(t-t_1)-i\phi_1}{2}\right] e^{it} \\ \cosh\left[\frac{ik_1(x-x_1)+i\pi/2-\sigma_1(t-t_1)+i\phi_1}{2}\right] e^{-it} \end{pmatrix}\end{aligned} \tag{964}$$

At last, after some algebra (verify it!), (948) becomes

$$u(x,t) = \mathcal{A}(x,t;\phi_1,x_1,t_1), \tag{965}$$

where

$$\begin{aligned}\mathcal{A}(x,t;x_1,t_1) &\equiv e^{2it}\left(1 + 2i\sin\phi_1 \frac{\sinh[\sigma_1(t-t_1)] - i\cos[k_1(x-x_1)]}{\cosh[\sigma_1(t-t_1)] - \sin\phi_1 \cos[k_1(x-x_1)]}\right) \\ &= e^{2it}\ \frac{\cosh[\sigma_1(t-t_1)+2i\phi_1] + \sin\phi_1\cos[k_1(x-x_1)]}{\cosh[\sigma_1(t-t_1)] - \sin\phi_1\cos[k_1(x-x_1)]}\end{aligned} \tag{966}$$

is the Akhmediev breather [4, 6, 5], exact solution of NLS for all values of the real parameters ϕ_1, x_1, t_1, and k_1, σ_1 are defined in (962), see Fig. 48.

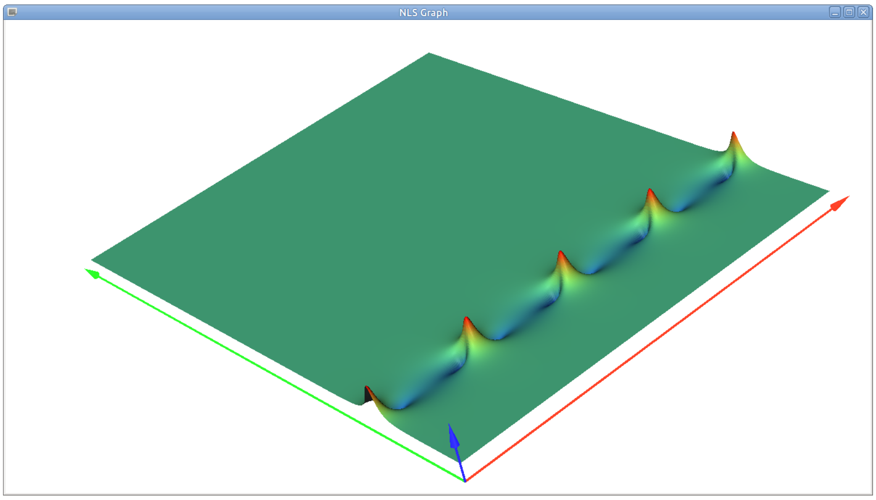


Figure 48: The Akhmediev (Akhmediev-Eleonskii-Kulagin) breather.

This solution, x-periodic with period $2\pi/k$, is exponentially localized in time over the background $u^{(0)}$, and changes it by the multiplicative phase factor $e^{4i\phi}$:

$$\mathcal{A}(x,t;\phi,x_1,t_1) \to e^{2it\pm 2i\phi}, \quad \text{as} \quad t \to \pm\infty; \tag{967}$$

in addition, its modulus takes its maximum at the point (x_1, t_1), with

$$|\mathcal{A}(x_1, t_1; \phi, x_1, t_1, \rho)| = 1 + 2\sin\phi. \tag{968}$$

As we shall see in the next chaper, it plays an important role in the theory of periodic anomalous waves in nature.

The corresponding fundamental solution of the ZS spectral problem reads

$$\Psi(x,t,\lambda) = \left[I + \frac{2iIm(\lambda_1)}{\lambda - \lambda_1} \frac{1}{|q_2|^2 + |q_1|^2} \begin{pmatrix} -\bar{q}_2 \\ \bar{q}_1 \end{pmatrix} (-q_2, q_1) \right] \Psi^{(0)}(x,t,\lambda), \tag{969}$$

where $\boldsymbol{q}$ and $\Psi^{(0)}$ are defined respectively in (964) and in (960).

8.3 Exercices

1) Use the DT of KdV to construct the exact solution of the KdV equation over the constant background solution $u^{(0)} = c$ =constant, with $c \in \mathbb{C}$.

2) Show how to derive the Akhmediev solution (965), (966) from the solution $\boldsymbol{q}$ in (964).

9 Anomalous waves in nature and the NLS model

9.1 Anomalous waves in $1+1$ dimensions

As we have seen during the course, NLS describes the amplitude modulation of quasi-monochromatic waves of small amplitude (weak nonlinearity). If X, T are the physical variables, then the physical phenomenon we want to describe in the above approximations is described by a field

$$\begin{array}{l} \eta(X,T) = \delta A(x_1,t_1,t_2)e^{i(kX-\omega(k)T)} + c.c. + O(\delta^2), \\ x_1 = \delta X, \quad t_j = \delta^j T, \ j = 1,2, \quad 0 \ll \delta \ll 1, \end{array} \tag{970}$$

$$\begin{array}{l} A(x_1,t_1,t_2) = A(\xi,t_2), \\ \xi = x_1 - \omega'(k)t_1 = \delta(X - \omega'(k)T), \end{array} \tag{971}$$

and the complex amplitude evolves according to the NLS equation

$$iA_{t_2} + \frac{\omega''(k)}{2}A_{\xi\xi} + \mu(k)|A|^2A = 0. \tag{972}$$

If $\mu(k) \in \mathbb{R}$, we distinguish two cases:
i) $\mu(k)\omega''(k) > 0$, the focusing NLS equation,
ii) $\mu(k)\omega''(k) < 0$, the defocusing NLS equation.

For example, in the case of surface water waves in deep water, $\mu(k) = -\frac{k^2\omega(k)}{2} < 0$, and

$$\begin{array}{l} \omega^2(k) = gk\tanh(hk) \sim gk, \ hk \gg 1 \quad \Rightarrow \quad \omega(k) \sim \sqrt{gk}, \\ \omega'(k) = \frac{\omega(k)}{2k}, \quad \omega''(k) = -\frac{\omega(k)}{8k^2}, \end{array} \tag{973}$$

so that the equation becomes [39]

$$iA_{t_2} - \tfrac{\omega(k)}{8k^2}A_{\xi\xi} - \tfrac{\omega(k)k^2}{2}|A|^2A = 0. \tag{974}$$

Therefore the signs of the dispersive and nonlinear terms are both negative, and we are in the focusing NLS regime. The following change of variables

$$u(x,t) = k^2A(\xi,t_2), \quad x = \delta\frac{\sqrt{2}}{\omega'(k)}(X - \omega'(k)T), \quad t = -\delta^2\frac{T}{\omega(k)} \tag{975}$$

brings equation (972),(973) to the adimensional form used in the literature

$$iu_t + u_{xx} + 2|u|^2u = 0; \tag{976}$$

in general, taking also account of the defocusing case, we shall have

$$iu_t + u_{xx} + 2\eta|u|^2u = 0, \quad \eta = \pm 1. \tag{977}$$

If the dispersion prevails, the NLS equation (977) reduces to the linear Schrödinger equation for a free particle $iu_t + u_{xx} \sim 0$; if the nonlinearity prevails: $iu_t + 2\eta|u|^2u \sim 0$, then the general solution reads

$$u(x,t) \sim a(x)e^{2i\eta|a(x)|^2 t} \tag{978}$$

(the nonlinearity implies that the frequency is proportional to the modulus square of the amplitude).

If $a(x) = a$ constant, we obtain the exact background solution of NLS

$$u^{(0)}(x,t) = ae^{2i\eta|a|^2 t}, \tag{979}$$

used in the previous section as the starting point of the Darboux dressing construction of the Akhmediev breather.

This background solution corresponds to the first nonlinear correction, obtained by Stokes in 1847 in his effort to construct periodic solutions of the Euler equations (the Stokes waves [38]) as an asymptotic series, starting from the monochromatic wave solution of the linearized water wave theory. It describes a constant light intensity $I = |u|^2$ in nonlinear optics, and a constant boson density $\rho = |u|^2$ in a Bose condensate.

9.2 Linear stability analysis on the background solution

Since this background solution (we choose wlg $a = 1$, as discussed in the previous section) has interesting physical meanings, it is important to know if it is stable under small perturbations:

$$u(x,t) = u^{(0)}(x,t) + \epsilon w(x,t), \quad u^{(0)} = e^{2i\eta t}, \quad 0 < \epsilon \ll 1. \tag{980}$$

Substituting it into the NLS equation (977) and keeping terms up to $O(\epsilon)$, we obtain the so-called "NLS equation linearized around the exact solution $u^{(0)}$":

$$iw_t + w_{xx} + 2\eta(e^{4i\eta t}\bar{w} + 2w) = 0. \tag{981}$$

If the perturbation is a monochromatic wave of the form

$$w(x,t) = e^{2i\eta t}\left(\gamma_+(t)e^{ikx} + \gamma_-(t)e^{-ikx}\right), \quad k \in \mathbb{R}, \tag{982}$$

then equation (981) reduces to two coupled ODEs for $\gamma_{\pm}$:

$$\begin{aligned} i\dot{\gamma}_+ + (2\eta - k^2)\gamma_+ + 2\eta\overline{\gamma_-} = 0, \\ i\dot{\gamma}_- + (2\eta - k^2)\gamma_- + 2\eta\overline{\gamma_+} = 0, \end{aligned} \tag{983}$$

implying

$$-i\dot{\overline{\gamma_+}} + (2\eta - k^2)\overline{\gamma_+} + 2\eta\gamma_- = 0. \tag{984}$$

In terms of the functions

$$S(t) := \gamma_- + \overline{\gamma_+}, \quad D(t) = \gamma_- - \overline{\gamma_+}, \tag{985}$$

the above equations become simpler:

$$\begin{aligned} i\dot{S} - k^2 D = 0, \\ i\dot{D} + (4\eta - k^2)S = 0. \end{aligned} \tag{986}$$

Taking the time derivative of the second, and using the first, we obtain

$$\ddot{D} - \Omega^2(k, \eta)D = 0 \tag{987}$$

where

$$\Omega^2(k, \eta) = k^2(4\eta - k^2). \tag{988}$$

We analize the two cases.
1) If $\eta = -1$, the defocusing case, $\Omega^2 = -k^2(4 + k^2) < 0$. Then $\Omega = i\omega(k)$, $\omega(k) = k\sqrt{4 + k^2} \in \mathbb{R}$. Therefore the background is neutrally stable, since the perturbation gives rise to small oscillations around it.

2) If $\eta = 1$, the focusing case, $\Omega^2 = k^2(4 - k^2)$. Then there are two subcases:
i) if $|k| > 2$, then $\Omega^2 = -k^2(k^2 - 4) < 0$, $\Omega = i\tilde{\omega}(k)$, with $\tilde{\omega}(k) = k\sqrt{k^2 - 4} \in \mathbb{R}$, and the background is again neutrally stable, since the perturbation gives rise to small oscillations around it.
ii) If $|k| < 2$, then $\Omega^2 = k^2(4 - k^2) > 0$; then

$$\Omega(1, k) = |k|\sqrt{4 - k^2} =: \sigma(k), \tag{989}$$

and the background is unstable, since a small perturbation gives rise to exponential growth and decay with the growth rate $\sigma(k)$ (see Fig. 49).

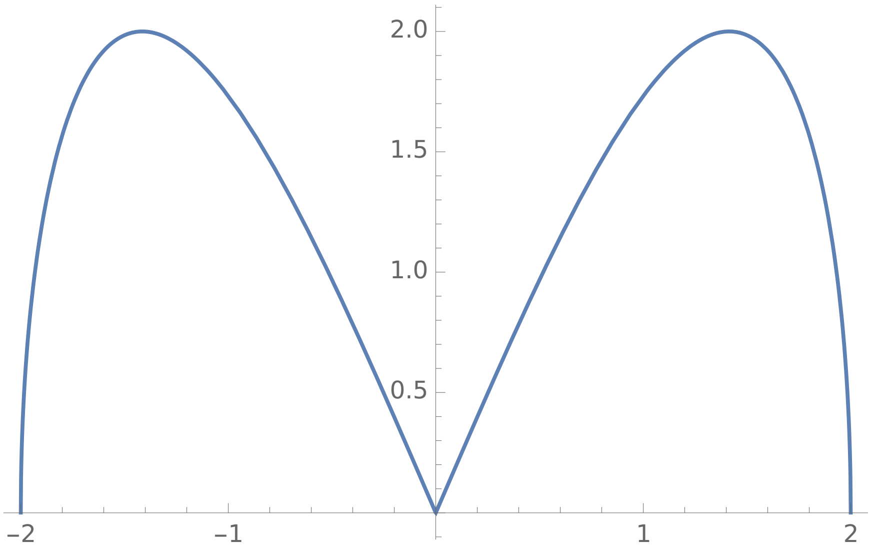


Figure 49: The graph of $\sigma(k)$ for $|k| \leq 2$ gives the instability curve of focusing NLS.

We complete the linear stability analysis in the unstable case (focusing NLS for $|k| < 2$). Then

$$\begin{aligned} &D = Ae^{\sigma t} + Be^{-\sigma t}, \\ &S = -i\tfrac{\dot{D}}{4-k^2} = -i\tfrac{k}{\sqrt{4-k^2}}\left(Ae^{\sigma t} - Be^{-\sigma t}\right), \end{aligned} \tag{990}$$

where A and B are arbitrary constants. Going back to $\gamma_\pm$, we obtain

$$\begin{aligned} \gamma_+ &= \tfrac{\bar{S}-\bar{D}}{2} = \tfrac{1}{2}\left(\tfrac{ik}{\sqrt{4-k^2}} - 1\right)\bar{A}e^{\sigma t} - \tfrac{1}{2}\left(\tfrac{ik}{\sqrt{4-k^2}} + 1\right)\bar{B}e^{-\sigma t}, \\ \gamma_- &= \tfrac{S+D}{2} = \tfrac{1}{2}\left(1 - \tfrac{ik}{\sqrt{4-k^2}}\right)Ae^{\sigma t} + \tfrac{1}{2}\left(1 + \tfrac{ik}{\sqrt{4-k^2}}\right)Be^{-\sigma t}. \end{aligned} \tag{991}$$

In the unstable region, it is convenient to introduce the angle ϕ such that

$$\begin{aligned} &\phi = \arccos\left(\tfrac{k}{2}\right), \quad \Rightarrow \\ &k = 2\cos\phi, \quad \sigma = 2\sin(2\phi). \end{aligned} \tag{992}$$

implying

$$1 \pm \frac{ik}{\sqrt{4-k^2}} = 1 \pm i\frac{\cos\phi}{\sin\phi} = \pm i\frac{e^{\mp i\phi}}{\sin\phi}. \tag{993}$$

At last, from equations (980), (982) we obtain the solution (980) of the linearized equation (981) describing the evolution of the perturbation of the unstable background:

$$
\begin{aligned}
&u(x,t) = e^{2it}\Big\{1 + \tfrac{i\epsilon}{2\sin\phi}\Big[(\bar{A}e^{\sigma t+i\phi} - \bar{B}e^{-\sigma t-i\phi})e^{ikx} + (-Ae^{\sigma t+i\phi} + Be^{-\sigma t-i\phi})e^{-ikx}\Big]\Big\} \\
&= e^{2it}\Big\{1 + \tfrac{i\epsilon}{2\sin\phi}\Big[(\bar{A}e^{ikx} - Ae^{-ikx})e^{\sigma t+i\phi} + (-\bar{B}e^{ikx} + Be^{-ikx})e^{-\sigma t-i\phi}\Big]\Big\}.
\end{aligned}
\tag{994}
$$

9.3 The Cauchy problem of the anomalous waves

In this section we want to solve the periodic Cauchy problem of the focusing NLS equation for a generic period initial perturbation of the homogeneous background solution $u^{(0)} = \exp(2it)$ of NLS (what we call the Cauchy problem of the anomalous waves AWs):

$$
\begin{aligned}
&iu_t + u_{xx} + 2|u|^2u = 0, \\
&u(x,0) = 1 + \epsilon v(x), \quad v(x+L) = v(x), \quad 0 < \epsilon \ll 1,
\end{aligned}
\tag{995}
$$

where $v(x)$ is the generic initial periodic perturbation, conveniently expanded in Fourier series:

$$
v(x) = \sum_{j\geq 1}\left(c_j e^{ik_jx} + c_{-j}e^{-ik_jx}\right), \quad k_j = \frac{2\pi}{L}j, \quad |c_j| = O(1). \tag{996}
$$

Due to the scaling properties of NLS, we have set

$$
c_0 = \int_{\mathbb{R}} v(x)dx = 0, \tag{997}
$$

without loss of generality. We also assume to be in the generic case in which π/L is not an integer.

Let

$$
N = \lfloor L/\pi \rfloor; \tag{998}
$$

then, from the instability condition $|k| < 2$ derived in the previous section, it follows that the first N Fourier modes $\pm k_j,\ j = 1, \ldots, N$ are unstable, since they give rise to exponentially growing and decaying waves of amplitudes $O(\epsilon e^{\pm\sigma_j t})$, where the growing rates σ_j are defined by

$$
\sigma_j = k_j\sqrt{4 - k_j^2} > 0, \tag{999}
$$

while the remaining infinitely many modes give rise to oscillations of amplitude $O(\epsilon e^{\pm i\omega_j t})$, where $\omega_j = k_j\sqrt{k_j^2 - 4}$, and therefore are stable.

For $t \leq O(1)$, the evolution of the unstable part of the initial condition is well described by the formulas (994) of the previous section. We just have to establish a connection between the arbitrary constants A and B in (994) and the Fourier coefficients $c_j, c_{-j},\ j \geq 1$. For each unstable mode k_j, we evaluate the perturbation in (994) at $t=0$, and we compare it the initial perturbation associated with the k_j mode, obtaining:

$$c_j = i\frac{\overline{A_j}e^{i\phi_j} - \overline{B_j}e^{-i\phi_j}}{2\sin\phi_j}, \quad c_{-j} = i\frac{-A_j e^{i\phi_j} + B_j e^{-i\phi_j}}{2\sin\phi_j} \tag{1000}$$

In terms of the convenient parameters

$$\alpha_j = -k_j A_j, \quad \beta_j = k_j \overline{B_j}, \tag{1001}$$

the relation with the Fourier coefficients is given by:

$$c_j = -\frac{i}{\sigma_j}\left(\overline{\alpha_j}e^{i\phi_j} + \beta_j e^{-i\phi_j}\right), \quad c_{-j} = \frac{i}{\sigma_j}\left(\alpha_j e^{i\phi_j} + \overline{\beta_j}e^{-i\phi_j}\right) \tag{1002}$$

and

$$\alpha_j = e^{-i\phi_j}\overline{c_j} - e^{i\phi_j}c_{-j}, \quad \beta_j = e^{i\phi_j}\overline{c_{-j}} - e^{-i\phi_j}c_j. \tag{1003}$$

At last the solution of the Cauchy problem (995),(996), for $t \leq O(1)$, is given by

$$\begin{aligned} u(x,t) = e^{2it}\Big[1 + \sum_{j=1}^{N}\Big(\tfrac{2\epsilon|\alpha_j|}{\sigma_j}e^{\sigma_j t + i\phi_j}\cos[k_j(x - X_j)] \\ + \tfrac{2\epsilon|\beta_j|}{\sigma_j}e^{-\sigma_j t - i\phi_j}\cos[k_j(x - \tilde{X}_j)] + O(\epsilon) \text{ oscillations}\Big)\Big] + O(\epsilon^2), \end{aligned} \tag{1004}$$

where the parameters α_j, β_j are the linear combination (1225) of the Fourier coefficients c_j, c_{-j} of the initial perturbation, and

$$X_j = \tfrac{\arg(\alpha_j) + \pi/2}{k_j}, \quad \tilde{X}_j = \tfrac{-\arg(\beta_j) + \pi/2}{k_j}, \quad j = 1, \dots, N. \tag{1005}$$

VERIFY IT!

Equations (1004),(1005),(1225) describe the first linear stage of MI, governed by the focusing NLS equation linearized about the background solution (953).

Therefore

the initial datum splits into exponentially growing and decaying waves, respectively the α- and β-waves, each one carrying half of the information encoded into the unstable part of the initial datum, plus small oscillations associated with the stable modes, remaining small during the evolution.

The j^{th} unstable mode becomes $O(1)$ at times of $O(\sigma_j^{-1}|\log\epsilon|)$; therefore the most unstable modes, the ones appearing first, are the modes with larger growth rate σ_j. It follows that, at logarithmically large times, one enters the (first) nonlinear stage of MI, the linearized NLS theory cannot be used anymore, and, to describe the evolution, the full integrability machinery of the finite gap method for NLS must be used.

9.4 The case of one unstable mode and the Fermi-Pasta-Ulam recurrence of anomalous waves [17, 18, 19]

From now on we concentrate on the simplest case of only one unstable mode:

$$N = 1 \quad \Leftrightarrow \quad \pi < L < 2\pi; \tag{1006}$$

then equations (1004) simplify to

$$\begin{array}{l} u(x,t) = e^{2it}\Big[1 + \Big(\frac{2\epsilon|\alpha_1|}{\sigma_1}e^{\sigma_1 t + i\phi_1}\cos[k_1(x - X_1)] \\ +\frac{2\epsilon|\beta_1|}{\sigma_1}e^{-\sigma_1 t - i\phi_1}\cos[k_1(x - \tilde{X}_1)] + O(\epsilon) \text{ oscillations}\Big)\Big] + O(\epsilon^2), \\ X_1 = \frac{\arg(\alpha_1) + \pi/2}{k_1}, \quad \tilde{X}_1 = \frac{-\arg(\beta_1) + \pi/2}{k_1}, \end{array} \tag{1007}$$

and, for

$$1 \ll t \ll \frac{1}{\sigma_1}\ln\left(\frac{1}{\epsilon}\right), \tag{1008}$$

we have the asymptotics

$$u(x,t) \sim e^{2it}\Big[1 + \frac{2\epsilon|\alpha_1|}{\sigma_1}e^{\sigma_1 t + i\phi_1}\cos(k_1(x - X_1))\Big]. \tag{1009}$$

Therefore we are looking, in the nonlinear region $t = O(\sigma_1^{-1}|\log\epsilon|)$, for an exact, x-periodic solution of NLS describing the modulation instability of a single nonlinear mode, matching with (1009) in the overlapping region $1 \ll t \ll O(|\log\epsilon|)$. The natural candidate for such a solution is, up to an arbitrary phase factor $e^{i\rho}$, $\rho \in \mathbb{R}$, the Akhmediev breather

$$\begin{array}{l} \mathcal{A}(x,t;\phi,X,T) = e^{2it}\frac{\cosh[\sigma(t-T)+2i\phi]+\sin\phi\cos[k(x-X)]}{\cosh[\sigma(t-T)]-\sin\phi\cos[k(x-X)]}, \\ \sigma = k\sqrt{4-k^2} = 2\sin(2\phi), \quad k = k_1 = \frac{2\pi}{L} = 2\cos\phi, \\ \phi = \arccos\left(\frac{k}{2}\right) = \arccos\left(\frac{\pi}{L}\right), \end{array} \tag{1010}$$

and since it should appear in the asymptotic region $t = O(\sigma_1^{-1}|\log\epsilon|)$, then t_1 must have the form

$$t_1 = \frac{1}{\sigma_1}\ln\frac{\gamma}{\epsilon}, \tag{1011}$$

where $\gamma > 0$ must be fixed by matching. Evaluating (1010) in the overlapping region $1 \ll t \ll O(\sigma_1^{-1}|\log\epsilon|)$ one obtains (now $t - t_1 \ll -1$):

$$e^{i\rho}\mathcal{A}(x,t,x_1,t_1) \sim e^{2it+i(\rho-2\phi)}\left(1 + \frac{\epsilon\sigma}{\gamma}e^{\sigma t + i\phi}\cos(k(x - x_1))\right). \tag{1012}$$

Comparing (1012) and (1009) and imposing a good matching, one fixes all the free parameters in (966) as follows

$$k = k_1,\ \sigma = \sigma_1,\ \phi = \phi_1,\ \rho = 2\phi_1,\ x_1 = X_1, \tag{1013}$$

and

$$\gamma = \frac{\sigma_1^2}{2|\alpha_1|} \ \Rightarrow\ t_1 = T_1 := \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^2}{2\epsilon|\alpha_1|}\right) = O(\sigma_1^{-1}|\log\epsilon|). \tag{1014}$$

Therefore *the first AW appears in the finite t-interval* $|t - T_1| \le O(1)$*, and is described by the Akhmediev breather solution of NLS:*

$$u(x,t) = e^{2i\phi_1}\mathcal{A}\Big(x,t;\phi_1,X_1,T_1\Big) + O(\epsilon), \tag{1015}$$

whose parameters are expressed in terms of the initial data through elementary functions. It is important to remark that the first AW contains informations only on half of the initial data (the half encoded in the parameter α_1: the α_1-wave), and that the modulus of the first AW generated by the initial condition (1221),(996) acquires its maximum at $t = T_1$ in the point $x = X_1$, mod L; and the value of this maximum is

$$|u(X_1,T_1)| = 1 + 2\sin\phi_1 < 1 + \sqrt{3} \sim 2.732. \tag{1016}$$

This upper bound, 2.732 times the background amplitude, is consequence of the formula $\sin\phi_1 = \sqrt{1-(\pi/L)^2}$, $\pi < L < 2\pi$, and is obtained when $L \to 2\pi$.

We also notice that the position $x = X_1$ of the maximum of the AW coincides with the position of the maximum of the growing sinusoidal wave of the linearized theory; this is due to the absence of nonlinear interactions with other unstable modes, in the simplest case $N = 1$.

Similar considerations can be made to study this Cauchy problem at negative times. In the time interval

$$1 \ll |t| \ll O\left(\frac{1}{\sigma_1}\ln\left(\frac{1}{\epsilon}\right)\right),\quad t < 0, \tag{1017}$$

we have the asymptotics

$$u(x,t) \sim e^{2it}\Big[1 + \frac{2\epsilon|\beta_1|}{\sigma_1}e^{-\sigma_1 t - i\phi_1}\cos(k_1(x - \tilde{X}_1))\Big] \tag{1018}$$

to be matched again with the Akhmediev solution, appearing in the asymptotic region $t = O(\sigma_1^{-1}|\log\epsilon|)$, $t < 0$. Now

$$t_1 = -\frac{1}{\sigma_1}\ln\frac{\tilde{\gamma}}{\epsilon},\quad \tilde{\gamma} > 0, \tag{1019}$$

Evaluating the Akhmediev breather (1010) in the region (1017), since now $t - t_1 \gg 1$, we obtain

$$e^{i\rho}\mathcal{A}(x,t,x_1,t_1) \sim e^{2it+i(\rho+2\phi_1)}\left(1 + \frac{\epsilon\sigma}{\tilde{\gamma}}e^{-\sigma t - i\phi}\cos(k(x-\tilde{x}_1)\right) \tag{1020}$$

Comparing now (1020) and (1018), we fix the free parameters of the Akhmediev breather as follows:

$$\begin{array}{l} k = k_1,\ \sigma = \sigma_1,\ \phi = \phi_1,\ \rho = -2\phi_1,\ x_1 = X_1, \\ t_1 = -\tilde{T}_1,\ \tilde{T}_1 := \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^2}{2\epsilon|\beta_1|}\right), \end{array} \tag{1021}$$

obtaining the following result.
The first RW appearing at negative times, in the finite t-interval $|t + \tilde{T}_1| \leq O(1)$, is described again by the Akhmediev solution of NLS, but with different parameters:

$$u(x,t) = e^{-2i\phi_1}\mathcal{A}\Big(x,t;\phi_1,\tilde{X}_1,-\tilde{T}_1\Big) + O(\epsilon), \tag{1022}$$

It is important to remark that this AW contains informations only on the second half of the initial data (the half encoded in the parameter β_1: the β_1-wave), and that the modulus of the first AW generated by the initial condition (1221),(996) acquires its maximum at $t = -\tilde{T}_1 < 0$ in the point $x = \tilde{X}_1$.

The AWs at $t = -\tilde{T}_1$ and at $t = T_1$ are two consecutive AWs, represented to leading order by the same exact solution: the Akhmediev breather, space shifted by $X_1 - \tilde{X}_1$, and shifted in time by $T_1 + \tilde{T}_1$. Since NLS is invariant under t-translations, one infers the following exact AW recurrence, in the case of a single unstable mode.

The solution of the x-periodic Cauchy problem describes, in the case of one unstable at each appearance, are expressed in terms of the initial data via elementary functions. T_1 is the first appearance time of the AW (the time at which the AW achieves the maximum of its modulus), X_1, is the position of such a maximum, $1 + 2\sin\phi_1$ is the value of the maximum,

$$\Delta T = T_1 + \tilde{T}_1 = \frac{2}{\sigma_1}\log\left(\frac{\sigma_1^2}{2\epsilon\sqrt{|\alpha_1\beta_1|}}\right), \tag{1023}$$

is the recurrence time (the time interval between two consecutive AW appearances),

$$\Delta X = X_1 - \tilde{X}_1 = \frac{\arg(\alpha_1\beta_1)}{k_1}, \quad mod\ L \tag{1024}$$

is the x-shift of the position of the maxima in the recurrence. In addition, after each appearance, the AW changes the background by the multiplicative phase factor $\exp(4i\phi_1)$ *(see Fig. 50).*

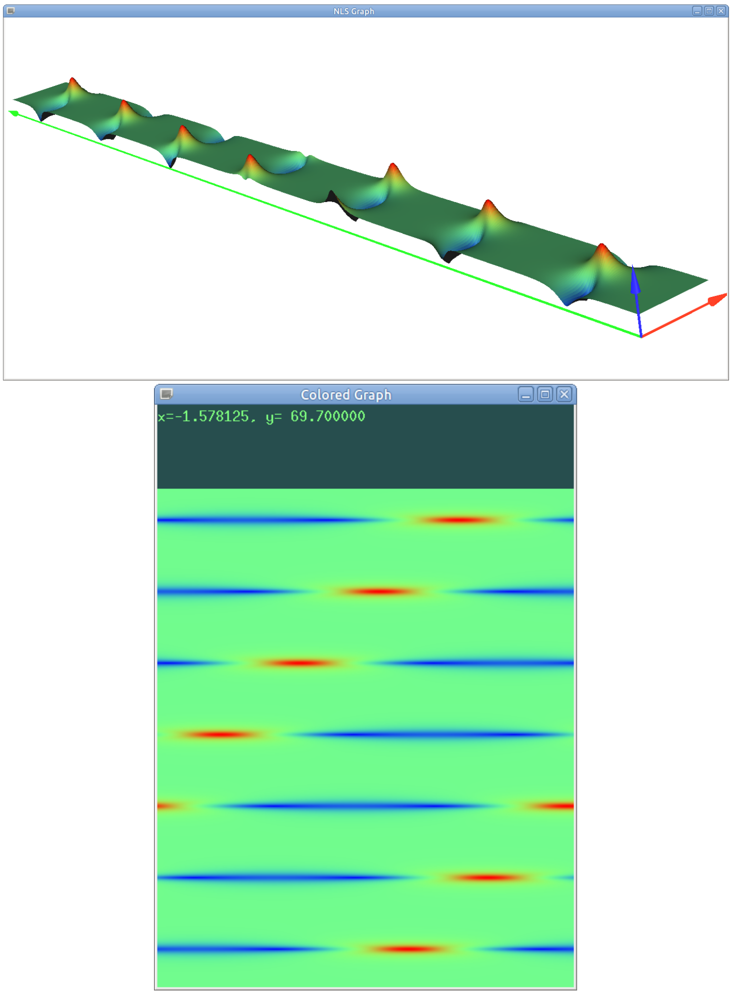


Figure 50: 3D plot and density plot of $|u(x,t)|$ describing the AW recurrence of one unstable mode.

This exact AW recurrence is an interesting example of "ideal" Fermi-Pasta-Ulam recurrence [15]. To show it, we first observe that the Akhmediev solution (1010) can be expanded in Fourier series as follows

$$\mathcal{A}(x,t;\phi,X,T) = \sum_{n\in\mathbb{Z}} C_n(t,X,T)e^{ik_n x}, \quad k_n = nk_1 = \frac{2\pi}{L}n, \tag{1025}$$

and its Fourier coefficients have a simple analytic expression, obtained via standard contour integration:

$$C_n(t,X,T)=\frac{1}{L}\int\limits_0^L e^{-ik_nx}\mathcal{A}(x,t;\phi,X,T)=$$
$$e^{2it}e^{-ik_nX}\left(-\delta_{n0}+\frac{A(t-T,\phi)+B(t-T,\phi)}{\sqrt{B^2(t-T,\phi)-1}}\left(B(t-T,\phi)-\sqrt{B^2(t-T,\phi)-1}\right)^{|n|}\right), \tag{1026}$$

where

$$A(t-T,\phi)=\frac{\cosh[\sigma(t-T)+2i\phi]}{\sin\phi},\ B(t-T,\phi)=\frac{\cosh[\sigma(t-T)]}{\sin\phi}. \tag{1027}$$

To be consistent with the condition that the background at $t=0$ is 1, while the background of (1010) in the remote past is $\exp(-2i\phi)$, then the Fourier coefficients in (1025) should be multiplied by that phase: $C_n\to\tilde{c}_n=C_ne^{-2i\phi}$.

If the initial condition excites the only unstable mode k_1: $u(x,0)=1+\epsilon(c_1\exp(ik_1x)+c_{-1}\exp(-ik_1x))$, then:
i) the energy is initially concentrated on the zero mode (the background) and on the first mode (the monochromatic perturbation):

$$|u_0(0)|^2=1,\quad |u_m(0)|^2=\delta_{m,\pm1}\epsilon^2|c_{\pm1}|^2, \tag{1028}$$

where $u_m(t),\ m\in\mathbb{Z}$ are the Fourier coefficients of the NLS solution $u(x,t)$.
ii) At the first appearance time T_1 of the AW, described by the Akhmediev solution, the energy is distributed on all the Fourier modes according to the simple law

$$\begin{aligned}&|u_0(T_1)|^2=|\tilde{c}_0(T_1)|^2=(2\cos\phi_1-1)^2+O(\epsilon),\quad 0<\phi_1<\tfrac{\pi}{2},\\&|u_m(T_1)|^2=|\tilde{c}_m(T_1)|^2=4(\cos\phi_1)^2\left(\tan\left(\tfrac{\phi_1}{2}\right)\right)^{2|m|}+O(\epsilon),\quad m\neq0.\end{aligned} \tag{1029}$$

iii) At the recurrence time ΔT, it is re-absorbed by the zero and first modes:

$$|u_0(\Delta T)|^2=1+O(\epsilon),\quad |u_m(\Delta T)|^2=\delta_{m,\pm1}\epsilon^2|c_{\pm1}|^2+O(\epsilon), \tag{1030}$$

starting an exact FPUT recurrence.

The FPUT recurrence was first observed Fermi, Pasta, Ulam, Tsingou through numerical integration, using the first computers in Los Alamos, in the study of the dynamics of N anharmonic oscillators under a nearest neighbor interaction:

$$\begin{aligned}&\ddot{q}_j(t)=-\tfrac{\partial H}{\partial q_j(t)},\ \ j=1,\dots,N,\ \ H(\underline{q})=\sum_{k=1}^{N}\left(\tfrac{1}{2}\dot{q}_j^2+V(q_{k+1}-q_k)\right),\\&V(x)=\tfrac{1}{2}x^2+\tfrac{\alpha}{3!}x^2+\tfrac{\beta}{4!}x^4,\ \ \alpha,\beta\in\mathbb{R}.\end{aligned} \tag{1031}$$

Giving energy to the first Fourier mode, part of it is distributed among the first Fourier modes, but eventually is re-absorbed by the first mode, giving rise to an approximately exact recurrence, while one expected, due to the nonlinear interaction, a redistribution of energy among all the Fourier modes (see the following Figure 51):

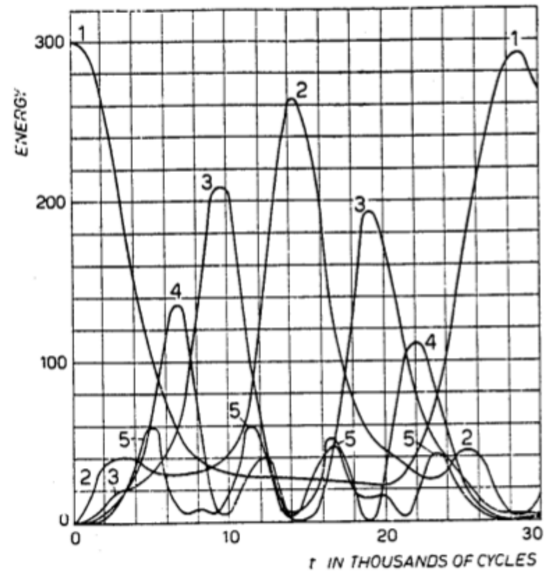


Figure 51: The figure contained in the original FPUT paper. The energy, initially given to the first Fourier mode, distributes to the first few modes, but eventually is reabsorbed by the first mode, starting a recurrence.

To explain this apparent paradox, we proceed as follows. If we are interested in long waves (the wave-length is much larger than the lattice step h or, equivalently, the lattice step is much smaller than the wave-length), the continuos limits gives a good description of the process:

$$\begin{array}{l} q_j(t) = q(jh,t) \sim u(x,t), \quad jh \sim x, \quad 0 < h \ll 1, \\ q_{j\pm 1}(t) = q(jh \pm h, t) \sim u \pm h u_x + \frac{h^2}{2} u_{xx} \pm \frac{h^3}{3!} u_{xxx} + \frac{h^4}{4!} u_{xxxx} + O(h^5), \end{array} \tag{1032}$$

implying that

$$\begin{array}{l} q_{j+1}(t) - q_j(t) = h\left(u_x + \frac{h}{2}u_{xx} + \frac{h^2}{3!}u_{xxx} + \frac{h^3}{4!}u_{xxxx} + O(h^4)\right), \\ q_j(t) - q_{j-1}(t) = h\left(u_x - \frac{h}{2}u_{xx} + \frac{h^2}{3!}u_{xxx} - \frac{h^3}{4!}u_{xxxx} + O(h^4)\right), \end{array} \tag{1033}$$

and the dynamical system becomes the following PDE (for $\alpha = h$):

$$u_{tt} - h^2 u_{xx} = h^4 u_x u_{xx} + O(h^6). \tag{1034}$$

Looking for unidirectional propagation, and using the usual multiscale expansion ansatz

$$u(x,t) = v(\xi, \tau), \quad \xi = x + ht, \quad \tau = h^3 t, \tag{1035}$$

one obtains the PDE

$$v_{\xi\tau} = \frac{1}{24} v_{\xi\xi\xi\xi} + \frac{\alpha}{2} v_\xi v_{\xi\xi}, \tag{1036}$$

reducing to the integrable KdV equation

$$w_\tau = \frac{1}{24} w_{\xi\xi\xi} + \frac{\alpha}{2} w w_\xi, \quad w = v_\xi. \tag{1037}$$

It follows that, if $t = O(1/h)$, the dynamics is described by the linear wave equation implying a rigid translation of the wave; if $t = O(1/h^3)$, the dynamics is described by the nonlinear integrable KdV equation, implying the numerically observed FPUT recurrence; at larger time scales $t = O(1/h^5)$, the dynamics is described by a nonintegrable PDE, recurrence does not apply anymore, and the energy distributes on all the Fourier modes, giving rise to an equidistribution of energy (the thermalization).

This AW recurrence is also in good agreement with nonlinear optics experiments in water waves [21] and in nonlinear optics [30]. This last experiment was performed in Roma after the above theory was developed, to test how well NLS describes the nonlinear optics of the crystal (see Fig. 52).

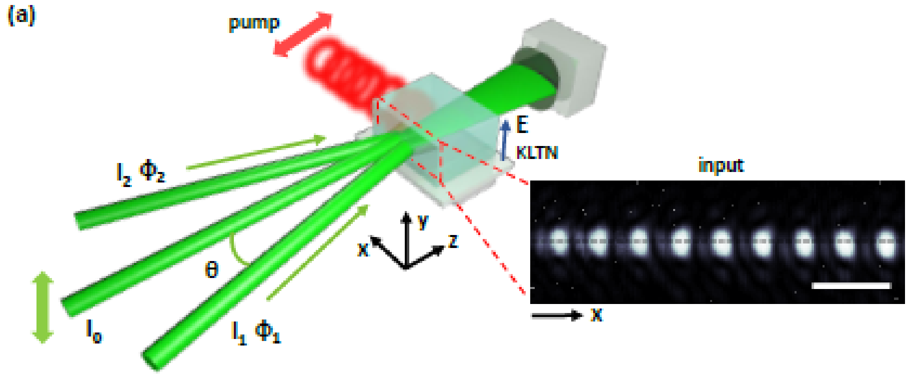


Figure 52: The symmetric 3-wave interferometric scheme used to generate the background wave with a single-mode perturbation propagating in a pumped photorefractive potassium-lithium-tantalate-niobate (KLTN) crystal.

Since NLS

$$i\psi_z + \psi_{xx} + 2|\psi|^2\psi = 0 \tag{1038}$$

is supposed to describe the above physics only to leading order, one expects that the exact NLS AW recurrence be replaced by a "FPUT" - type recurrence, before thermalization destroys the pattern.

The linearly polarized electric field at the initial face of the crystal is

$$\begin{array}{l} E = E_0 + E_1 e^{i(k_1 x + \Phi_1)} + E_2 e^{i(-k_1 x + \Phi_2)}, \quad E_0, E_1, E_2 \in \mathbb{R}^+, \\ k_1 = \frac{2\pi \tan\theta}{\lambda}. \end{array} \tag{1039}$$

The modulus square of the NLS initial condition $\psi(x,0) = 1 + \epsilon(c_1 e^{ik_1 x} + c_{-1} e^{-ik_1 x})$ is the relative light intensity

$$\frac{I}{I_0} = |\psi(x,0)|^2 = 1 + A\cos(k_1 x + B), \tag{1040}$$

where

$$
\begin{aligned}
&A = 2\epsilon|c_1 + \overline{c_{-1}}| = 2\sqrt{\tfrac{I_1}{I_0} + \tfrac{I_2}{I_0} + 2\tfrac{\sqrt{I_1 I_2}}{I_0}\cos(\Phi_1 + \Phi_2)}, \\
&B = \arg(c_1 + \overline{c_{-1}}) = \arctan\left(\tfrac{\sqrt{I_1}\sin\Phi_1 - \sqrt{I_2}\sin\Phi_2}{\sqrt{I_1}\cos\Phi_1 + \sqrt{I_2}\cos\Phi_2}\right), \\
&I = |E|^2, \quad I_j = E_j^2, \quad j = 0, 1, 2.
\end{aligned}
\tag{1041}
$$

The wave number k_1 can be varied changing the geometric angle θ; the optical power A and the phase delay B can be varied adjusting the phases Φ_1 and Φ_2 (f.i., inserting some glasses the optical path length increases). Two typical recurrence outputs, with $\Delta X = L/2$ and $\Delta X = 0$ respectively in the upper and lower pictures in Fig. 53

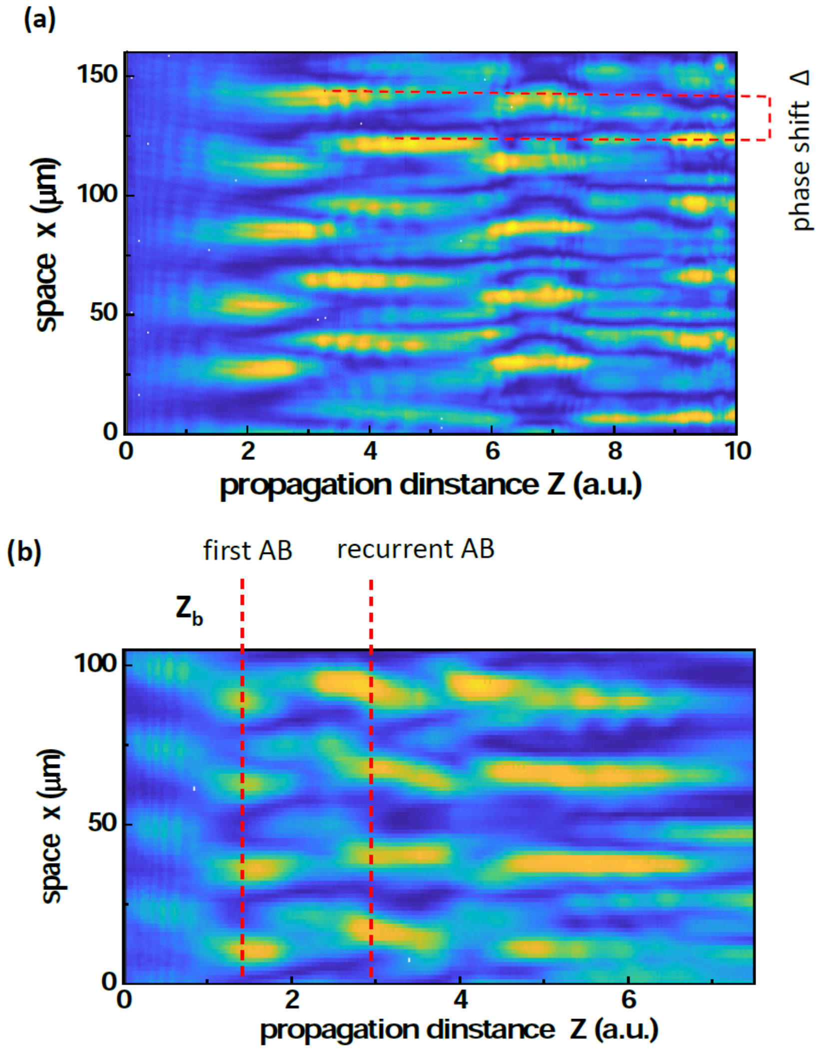


Figure 53: Two typical recurrence outputs, with $\Delta X = L/2$ (above), and $\Delta X = 0$ (below)

The intensity at the first appearance in the upper figure in Fig. 53 is in very good agreement with the Akhmediev profile, see Fig. 54:

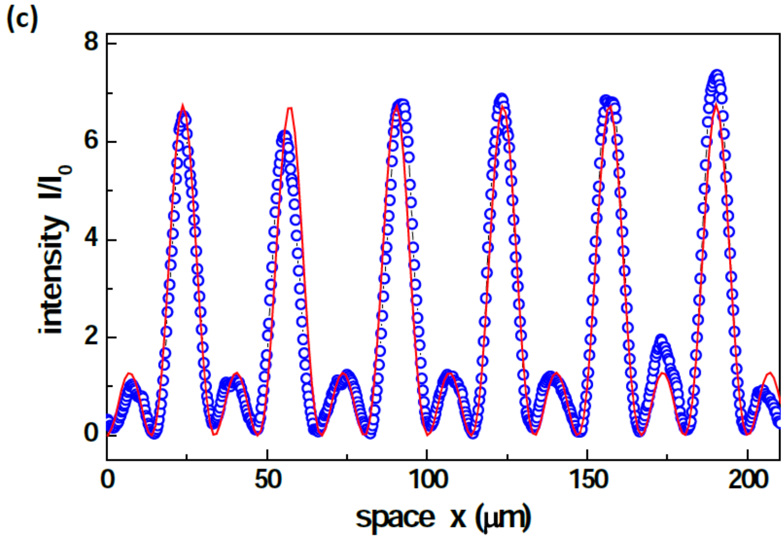


Figure 54: The light intensity at the first appearance in the upper picture of Fig. 53 is in very good agreement with the Akhmediev profile. The small circles are the light intensity measures; the continuous curve is what the theory predicts. The agreement is less and less good in the subsequent appearances, until the recurrence is completely destroyed.

AW recurrence as basic effect of nonlinear MI in the periodic setting, and the finite gap method The obtained recurrence of AWs can be predicted from simple qualitative considerations. The unstable mode grows exponentially and becomes $O(1)$ at logarithmically large times, when one enters the nonlinear stage of MI, and one expects the generation of a transient, $O(1)$, coherent structure over the unstable background (the AW). Since the Akhmediev breather describes the one-mode nonlinear instability, it is the natural candidate to describe such a stage, at the leading order. Due again to MI, this AW is expected to be destroyed in a finite time interval, and one enters the third asymptotic stage, characterized, like the first one, by the background plus an $O(\epsilon)$ perturbation. This second linear stage is expected, due again to MI, to give rise to the formation of a second AW (the second nonlinear stage of MI), described again by the Akhmediev breather, but, in general, with different parameters. And this procedure iterates forever, in the integrable NLS model, giving rise to the generation of an infinite sequence of AWs described by different Akhmediev breathers. *Then the AW recurrence is a relevant effect of nonlinear MI in the periodic setting.*

9.5 The main spectrum of the periodic problem

As we have seen, the IST is the spectral method (nonlinear analogue of the Fourier transform method for linear PDEs) used to solve the Cauchy problem on the line for soliton PDEs doing spectral theory on a suitable operator (the Schrödinger operator for KdV and the Dirac type operator for NLS). Analogously, if the Cauchy problem to solve is periodic, the method used is the so-called Finite-Gap (FG) method (nonlinear analogue of the Fourier series method). Here we just mention it, without going deeply into the theory.

We recall that the ZS Lax pair associated with focusing and defocusing NLS equations $iu_t + u_{xx} + 2\eta|u|^2 u = 0,\ \eta = \pm 1$ reads

$$\vec{\psi}_x(\lambda, x, t) = \hat{X}(\lambda, x, t)\vec{\psi}(\lambda, x, t), \quad \vec{\psi}_t(\lambda, x, t) = \hat{T}(\lambda, x, t)\vec{\psi}(\lambda, x, t), \tag{1042}$$

where

$$\vec{\psi}(\lambda, x, t) = \begin{pmatrix} \psi_1(\lambda, x, t) \\ \psi_2(\lambda, x, t) \end{pmatrix}, \quad \hat{X} = -i\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i \begin{pmatrix} 0 & u(x,t) \\ \eta\overline{u(x,t)} & 0 \end{pmatrix},$$

$$\hat{T}(\lambda, x, t) = \begin{pmatrix} -2i\lambda^2 + i\eta u(x,t)\overline{u(x,t)} & 2i\lambda u(x,t) - u_x(x,t) \\ 2i\eta\lambda\overline{u(x,t)} + \eta\overline{u_x(x,t)} & 2i\lambda^2 - i\eta u(x,t)\overline{u(x,t)} \end{pmatrix}.$$

The first equation can be written as a spectral problem:

$$\mathcal{L}\vec{\psi}(\lambda, x, t) = \lambda\vec{\psi}(\lambda, x, t), \quad \mathcal{L} := \begin{pmatrix} i\partial_x & u(x,t) \\ -\eta\overline{u(x,t)} & -i\partial_x \end{pmatrix}. \tag{1043}$$

Its adjoint (hermitian conjugate) $\mathcal{L}^\dagger$ with respect to the scalar product

$$(\vec{f}, \vec{g}) = \int_0^L \sum_{j=1}^{2} \bar{f}_j(x) g_j(x) dx \tag{1044}$$

reads

$$\mathcal{L}^\dagger = \begin{pmatrix} i\partial_x & -\eta u(x,t) \\ \overline{u(x,t)} & -i\partial_x \end{pmatrix}. \tag{1045}$$

Then, in the defocusing case $\eta = -1$, the operator $\mathcal{L}$ is self-adjoint and the spectrum is real. In the focusing case $\eta = 1$, the operator $\mathcal{L}$ is not self-adjoint and the spectrum contains complex points.

From now on we concentrate on the focusing NLS equation ($\eta = 1$). We write now three basic properties of the problem.

1) If $\Psi(\lambda, x, t)$ is any fundamental matrix solution of the Lax pair, made of two independent column vector solutions of (1042), then also

$$\tilde{\Psi}(\lambda, x, t) = \Psi(\lambda, x, t)C(\lambda) \ \text{ is a fundamental solution} \tag{1046}$$

where $C(\lambda)$ is a nonsingular matrix, constant in x and t.

2) From the periodicity $u(x + L, t) = u(x, t)$:

$$\begin{array}{l} \vec{\psi}_x(\lambda, x + L, t) = \hat{X}(\lambda, x + L, t)\vec{\psi}(\lambda, x + L, t) = \hat{X}(\lambda, x, t)\vec{\psi}(\lambda, x + L, t), \\ \vec{\psi}_t(\lambda, x + L, t) = \hat{T}(\lambda, x + L, t)\vec{\psi}(\lambda, x + L, t) = \hat{T}(\lambda, x, t)\vec{\psi}(\lambda, x + L, t), \end{array} \tag{1047}$$

implying also that

$$\Psi(\lambda, x + L, t) = \Psi(\lambda, x, t)D(\lambda), \tag{1048}$$

where $D(\lambda)$ is a nonsingular matrix, constant in x and t.

3) Since tr$X$ =tr$T = 0$, the Abel theorem $(\det\Psi)_x = (\mathrm{tr}\hat{X})(\det\Psi)$ implies that $\det\Psi(\lambda, x, t)$ does not depend on x and t:

$$\left(\det\Psi(\lambda, x, t)\right)_x = \left(\det\Psi(\lambda, x, t)\right)_t = 0. \tag{1049}$$

To define the main spectrum, consider the matrix equation

$$\mathcal{L}\tilde{\Psi}(\lambda, x, x_0, t_0) = \lambda\tilde{\Psi}(\lambda, x, x_0, t_0),$$

where $\tilde{\Psi}(\lambda, x, x_0, t_0)$ is the fundamental matrix solution fixed by the initial condition $\tilde{\Psi}(\lambda, x, x_0, t_0)\Big|_{x=x_0} = I$, where I is the 2×2 identity matrix. It is obtained from any other fundamental matrix solution $\Psi(\lambda, x, t)$ as

$$\tilde{\Psi}(\lambda, x, x_0, t_0) = \Psi(\lambda, x, t_0)\Psi^{-1}(\lambda, x_0, t_0) \tag{1050}$$

using property 1). Then the monodromy matrix $T(\lambda, x_0, t_0)$ is defined by:

$$T(\lambda, x_0, t_0) := \tilde{\Psi}(\lambda, x_0 + L, x_0, t_0). \tag{1051}$$

The fundamental solution $\tilde{\Psi}(\lambda, x, x_0, t_0)$ is obtained integrating the ZS spectral problem (analytic in λ), over the finite interval $[x_0, x_0 + L]$, with the analytic initial condition I at x_0; therefore it is an entire function of λ, and so is the monodromy matrix $T(\lambda, x_0, t_0)$. In addition (1049) and the initial condition of the monodromy matrix imply that

$$\det\tilde{\Psi}(\lambda, x, x_0, t) = \det T(\lambda, x_0, t_0) = 1; \tag{1052}$$

indeed:

$$\det\tilde{\Psi}(\lambda, x, x_0, t_0) = \det\tilde{\Psi}(\lambda, x_0, x_0, t_0) = 1 = \det\tilde{\Psi}(\lambda, x_0 + L, x_0, t_0) = \det T(\lambda, x_0, t_0). \tag{1053}$$

It is easy to verify that, changing the pair of parameters x_0, t_0 to the parameters x_1, t_1, one constructs a monodromy matrix differing from the previous one by a matrix similarity transformation

$$\tilde{T}(\lambda, x_1, t_1) = M^{-1}T(\lambda, x_0, t_0)M, \quad M := \Psi(\lambda, x_0, t_0)\Psi^{-1}(\lambda, x_1, t_1) \tag{1054}$$

leaving the spectrum invariant. Therefore the spectrum of the monodromy matrix is a constant of motion. The eigenvalues and eigenvectors of $T(\lambda)$ are defined by the eigenvalue equation

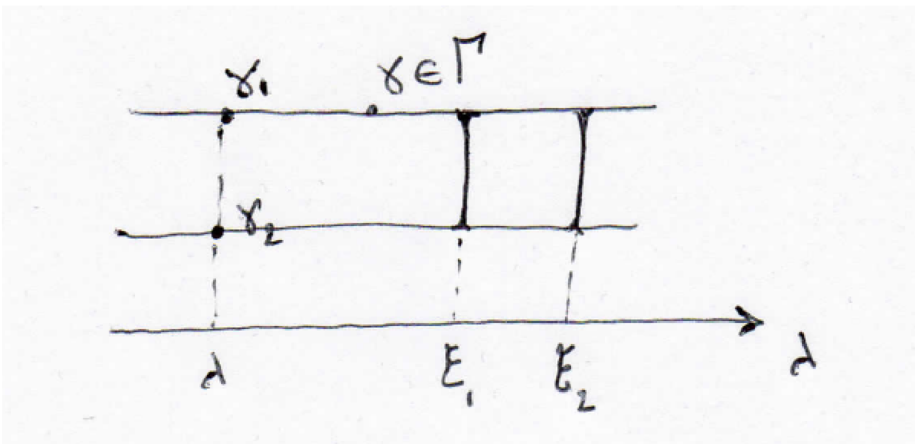
$$T(\lambda)\vec{v} = \kappa\vec{v}, \tag{1055}$$

from which

$$\kappa_{\pm} = \frac{\text{tr } T(\lambda)}{2} \pm \sqrt{\left(\frac{\text{tr } T(\lambda)}{2}\right)^2 - 1} \tag{1056}$$

since $\det T(\lambda) = 1$.

Given $\lambda \in \mathbb{C}$, the eigenvalue and eigenvector of the monodromy matrix are two-valued functions of λ, and their definition (1056) involves the square root of the entire funtion $\left(\frac{\text{tr } T(\lambda)}{2}\right)^2 - 1$; therefore eigenvalues and eigenvectors are defined on a two-sheeted covering of the λ-plane. This Riemann surface Γ is called the **spectral curve**, and is independent of time. If $\gamma \in \Gamma$, $\lambda(\gamma)$ is the projection of γ on the λ complex plane. If $\lambda(\gamma_1) = \lambda(\gamma_2)$, then $\kappa(\gamma_1)\kappa(\gamma_2) = 1$ (the condition $\kappa_+\kappa_- = 1$ coming from (1056)).



From the eigenvectors of $T(\lambda)$ one constructs the Bloch eigenfunctions of $\mathcal{L}$: $\mathcal{L}\vec{\psi}^B(x,t) = \lambda\vec{\psi}^B(x,t)$, defined by the periodicity property

$$\vec{\psi}^B(x+L,t) = \chi\vec{\psi}^B(x,t), \tag{1057}$$

where χ is a proportionality factor (the so-called Floquet multiplier) to be specified. Indeed the Bloch eigenfunction must be a suitable linear combination of the columns of the fundamental matrix $\tilde{\Psi}$:

$$\vec{\psi}^B(x,t) = \tilde{\Psi}(\lambda,x,x_0,t)\vec{A}, \quad \vec{A} \text{ constant vector.} \tag{1058}$$

Then, choosing $x_0 = 0$:

$$\begin{aligned}&\vec{\psi}^B(L,t_0) = \tilde{\Psi}(\lambda,L,0,t_0)\vec{A} = T(\lambda)\vec{A},\\ &\vec{\psi}^B(L,t_0) = \chi\vec{\psi}^B(0,t_0) = \chi\tilde{\Psi}(\lambda,0,0,t_0)\vec{A} = \chi\vec{A}\end{aligned} \tag{1059}$$

implying the eigenvalue equation $T(\lambda)\vec{A} = \chi\vec{A}$. Comparing it with (1055), we infer that the Floquet multiplier in (1057) is the eigenvalue of T and $\vec{A}$ is its eigenvector.

Summarizing, while the monodromy matrix $T(\lambda)$ is an entire function of λ, its eigenvalues and eigenfunctions, as well as the Bloch eigenfunctions, are functions of $\gamma \in \Gamma$:

$$T(\lambda)\vec{v}(\gamma) = \kappa(\gamma)\vec{v}(\gamma), \quad \gamma \in \Gamma, \tag{1060}$$

and

$$\begin{aligned}&\vec{\psi}^B(\gamma,x,t) = \tilde{\Psi}(\lambda,x,,x_0,t)\vec{v}(\gamma), \quad \gamma \in \Gamma,\\ &\vec{\psi}^B(x+L,t,\gamma) = \kappa(\gamma)\vec{\psi}^B(x,t,\gamma).\end{aligned} \tag{1061}$$

The second formula in (1061) implies that, to avoid an undesired exponential growth of the Bloch eigenfunctions at $\pm\infty$, the eigenvalue $\kappa(\gamma)$ must satisfy the condition $|\kappa(\gamma)| = 1$. Therefore it is conveniently parametrized as

$$\kappa(\gamma) = e^{ip(\gamma)L}, \tag{1062}$$

in terms of the quasi-momentum $p(\gamma)$ satisfying the condition $\mathrm{Im}p(\gamma) = 0$.

$$|\kappa(\gamma)| = 1 \quad \Leftrightarrow \quad \mathrm{Im}p(\gamma) = 0. \tag{1063}$$

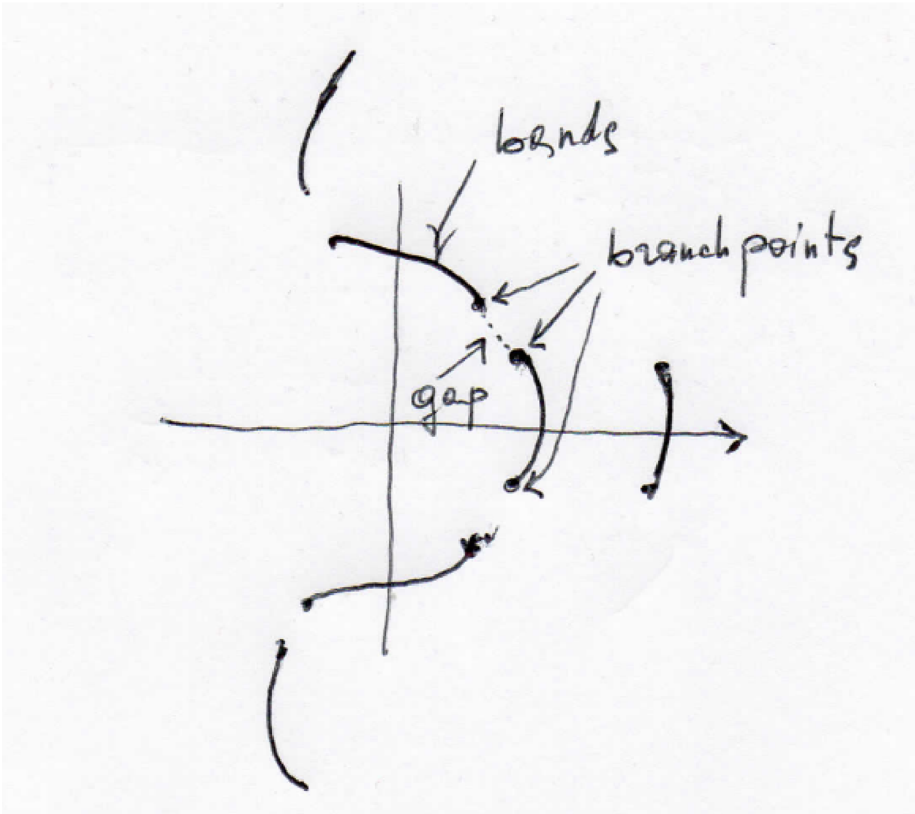
The values of $\gamma \in \Gamma$ satisfying this condition characterize the main spectrum

$$\begin{aligned}&\text{main spectrum } = \{\gamma \in \Gamma: \ \mathrm{Im}\ p(\gamma) = 0\} = \{\gamma \in \Gamma: \ |\kappa(\gamma)| = 1\}\\ &= \{\lambda \in \mathbb{C}: \ -2 \le \mathrm{tr}\ T(\lambda) \le 2\}.\end{aligned} \tag{1064}$$

In general, this spectrum consists of disconnected lines in the complex λ plane, the so-called "bands", separated by "gaps". The end points of the main spectrum (of the disconnected bands) correspond to the conditions

$$\mathrm{tr}\ T(\lambda) = \pm 2 \ \Leftrightarrow \ \kappa(\gamma) = \pm 1 \tag{1065}$$

(see (1056)); they are associated with periodic and anti-periodic Bloch eigenfunctions, and are square root branch points or multipole points (when n branch points coincide, they give rise to a multipole point of multiplicity n); see (1056).



One can verify that the eigenfunctions of the ZS spectral problem satisfy the following symmetry: if $(\psi_1(\lambda),\psi_2(\lambda))^T$ is a solution of the ZS spectral problem, then $(-\overline{\psi_2(\bar\lambda)},\overline{\psi_1(\bar\lambda)})^T$ is also a solution of the ZS spectral problem. This symmetry implies that $\mathrm{tr}T(\lambda)=T_{11}(\lambda)+\overline{T_{11}(\bar\lambda)}$, and the conditions (1064), (1065) defining the main spectrum, the branch points and multipole points imply that, if λ belongs to the main spectrum, then also $\bar\lambda$ belongs to the main spectrum: **the main spectrum is invariant under complex conjugation**.

We end this section remarking that, if $\lambda\in\mathbb{R}$, then $\hat X^\dagger(\lambda)=-\hat X(\lambda)$. Therefore $\Psi^\dagger(\lambda)$ and $\Psi^{-1}(\lambda)$ satisfy the same equation:

$$\left(\Psi^\dagger(\lambda)\right)_x=-\Psi^\dagger(\lambda)\hat X(\lambda),\quad \left(\Psi^{-1}(\lambda)\right)_x=-\Psi^{-1}(\lambda)\hat X(\lambda),\quad \lambda\in\mathbb{R}. \tag{1066}$$

It follows that, if $\lambda\in\mathbb{R}$, one can normalize $\Psi(\lambda)$ in such a way that $\Psi(\lambda)$ be a unitary matrix: $\Psi^\dagger(\lambda)=\Psi^{-1}(\lambda)$; then also the monodromy matrix is unitary and its eigenvalues satisfy the unitary condition $|\kappa_\pm|=1$. We conclude that the real axis belongs to the main spectrum.

9.5.1 The main spectrum associated with the background solution of NLS

We have already calculated in (960) a fundamental solution of the ZS Lax pair corresponding to the background $\exp(2it)$, rewritten here without normalizing its determinant:

$$\begin{array}{l}\Psi^{(0)}(x,t,\lambda)=e^{it\sigma_3}\begin{pmatrix} e^{\Theta(\lambda)} & -(\mu+\lambda)e^{-\Theta(\lambda)} \\ (\mu+\lambda)e^{\Theta(\lambda)} & e^{-\Theta(\lambda)}\end{pmatrix},\\ \Theta(\lambda)\equiv i\mu(x+2\lambda t),\\ \det\Psi^{(0)}(x,t,\lambda)=2\mu(\mu+\lambda),\end{array} \tag{1067}$$

where μ,λ satisfy the constraint

$$\mu^2=\lambda^2+1. \tag{1068}$$

The fundamental matrix solution $\tilde{\Psi}(\lambda, x, , x_0, t)$ reducing to the identity for $x = x_0$ will be

$$\begin{array}{l}\tilde{\Psi}^{(0)}(\lambda, x, x_0, t) = \Psi^{(0)}(\lambda, x, t){\Psi^{(0)}}^{-1}(\lambda, x_0, t) = \\ e^{it\sigma_3}\begin{pmatrix} \cos(\mu(x-x_0)) - i\frac{\lambda}{\mu}\sin(\mu(x-x_0)) & i\frac{\sin(\mu(x-x_0))}{\mu} \\ i\frac{\sin(\mu(x-x_0))}{\mu} & \cos(\mu(x-x_0)) + i\frac{\lambda}{\mu}\sin(\mu(x-x_0)) \end{pmatrix} e^{-it\sigma_3},\end{array} \tag{1069}$$

leading to the monodromy matrix

$$\begin{array}{l}T^{(0)}(\lambda) = \tilde{\Psi}^{(0)}(\lambda, x_0 + L, x_0, t) = \\ e^{it\sigma_3}\begin{pmatrix} \cos(\mu L) - i\frac{\lambda}{\mu}\sin(\mu L) & i\frac{\sin(\mu L)}{\mu} \\ i\frac{\sin(\mu L)}{\mu} & \cos(\mu L) + i\frac{\lambda}{\mu}\sin(\mu L) \end{pmatrix} e^{-it\sigma_3}.\end{array} \tag{1070}$$

Therefore

$$\mathrm{tr}\ T^{(0)}(\lambda) = 2\cos(\mu L) \ \Rightarrow\ \kappa_{\pm} = e^{\pm i\mu L}. \tag{1071}$$

We remark that, although $\mu = \sqrt{\lambda^2 + 1}$, the monodromy matrix (1070) is an even function of μ and depends on μ through $\mu^2 = \lambda^2 + 1$; therefore one verifies that $T(\lambda)$ in an entire function of λ. From (1071),(1064) we infer that the main spectrum associated with the background solution corresponds to $\mu \in \mathbb{R}$; consequently, from $\mu^2 = \lambda^2 + 1$, either λ is real (when $\mu^2 > 1$) or $\lambda \in i\mathbb{R}$, with $|\lambda| < 1$ (when $\mu^2 < 1$):

$$\text{main spectrum} = \{\lambda \in \mathbb{R}\} \cup \{\lambda \in i\mathbb{R},\ |\lambda| < 1\}. \tag{1072}$$

The branch points and the double points, characterized by the condition tr $T^{(0)}(\lambda) = 2\cos(\mu L) = \pm 2$, are given by

$$\mu_n = \frac{\pi}{L}n,\ \ \lambda_n^{\pm} = \pm\lambda_n,\ \lambda_n := \sqrt{\mu_n^2 - 1},\ \ n \in \mathbb{Z}. \tag{1073}$$

Since, from (1068),

$$\partial_\lambda = \frac{\lambda}{\mu}\partial_\mu,\ \ \partial_\lambda^2 = \frac{1}{\mu^3}\partial_\mu + \frac{\lambda^2}{\mu^2}\partial_\mu^2, \tag{1074}$$

it follows that

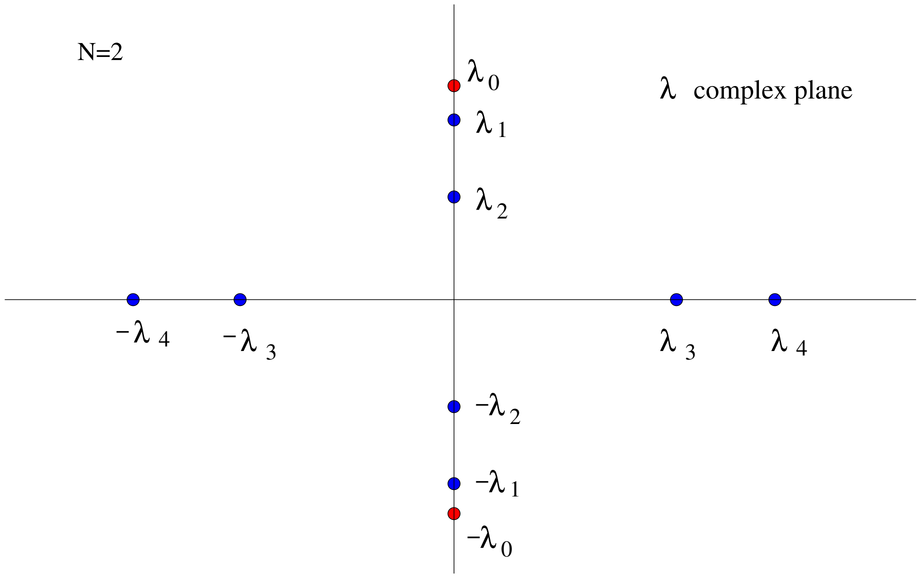
$$\begin{array}{l}\partial_\lambda \mathrm{tr}\ T^{(0)}(\lambda) = 2\frac{\lambda}{\mu}\partial_\mu\cos(\mu L) = -\frac{2L\lambda}{\mu}\sin(\mu L), \\ \partial_\lambda^2 \mathrm{tr}\ T^{(0)}(\lambda) = -2L\left(\frac{\sin(\mu L)}{\mu^3} + \frac{\lambda^2 L}{\mu^2}\cos(\mu L)\right),\end{array} \tag{1075}$$

it follows that the two points $(\mu_o, \lambda_0^{\pm}) = (0, \pm i)$ are branch points:

$$\partial_\lambda \mathrm{tr}\ T^{(0)}(\lambda)\Big|_{(0,\pm i)} = \mp 2iL^2 \neq 0, \tag{1076}$$

while the remaining end points $(\mu_n, \lambda_n^{\pm}),\ n \neq 0$ are double points:

$$\begin{array}{l}\partial_\lambda \mathrm{tr}\ T^{(0)}(\lambda)\Big|_{(\mu_n, \lambda_n^{\pm})} = 0,\ n \neq 0, \\ \partial_\lambda^2 \mathrm{tr}\ T^{(0)}(\lambda)\Big|_{(\mu_n, \lambda_n^{\pm})} = (-1)^{n+1}\frac{2L^2\lambda_n^2}{\mu_n^2} \neq 0,\ \ n \neq 0.\end{array} \tag{1077}$$



Near the double points $\pm\lambda_n,\ n>1$:

$$\mathrm{tr}\ T^{(0)}(\lambda)=(-1)^n\left[2-\frac{\lambda_n^2L^4}{\pi^2n^2}(\lambda-\lambda_n)^2+O((\lambda-\lambda_n)^4)\right]. \tag{1078}$$

9.5.2 The main spectrum associated with the perturbation of the background solution

Apart from the branch points $\pm i$, the other end points of the spectrum are doubly degenerate eigenvalues (double points). To find how a small generic perturbation resolves such a degeneration, we use a standard perturbations theory for time independent operators.

Let us compute the effect of the generic initial perturbation

$$u(x,0)=1+\epsilon v(x),\quad v(x)=\sum_{n\neq 0}c_ne^{2i\mu_nx},\quad \mu_n=\frac{\pi}{L}n, \tag{1079}$$

using the standard perturbation theory.

For the perturbed spectral operator $\mathcal{L}$:

$$\mathcal{L}=\mathcal{L}_0+\epsilon\mathcal{L}_1,\quad \mathcal{L}_0=\begin{pmatrix} i\partial_x & 1\\ -1 & -i\partial_x\end{pmatrix},\quad \mathcal{L}_1=\begin{pmatrix} 0 & v(x)\\ -\overline{v(x)} & 0\end{pmatrix} \tag{1080}$$

we consider as basis that associated with the unperturbed problem, and define

$$|n,\pm>:=\begin{pmatrix} 1\\ \mu_n\pm\lambda_n\end{pmatrix}e^{i\mu_nx},\quad n\in\mathbb{Z}, \tag{1081}$$

with the notation

$$\lambda_n=\sqrt{\mu_n^2-1},\quad \lambda_{-n}=\lambda_n,\quad \mu_{-n}=-\mu_n. \tag{1082}$$

Therefore we associate with the eigenvalue λ_n the two eigenvectors $|\pm n^{(0)},+>$, and with the eigenvalue $-\lambda_n$ the two eigenvectors $|\pm n^{(0)},->$:

$$\mathcal{L}_0|\pm n^{(0)},+>=\lambda_n|\pm n^{(0)},+>,\quad \mathcal{L}_0|\pm n^{(0)},->=-\lambda_n|\pm n^{(0)},-> \tag{1083}$$

We also introduce the dual basis $< l, \pm|$ such that

$$< l^{(0)}, \pm | m^{(0)}, \pm >= \delta_{lm}, \quad < l^{(0)}, \mp | m^{(0)}, \pm >= 0. \tag{1084}$$

We first observe that any linear combination of $|n^{(0)}, + >$ and $| - n^{(0)}, + >$ has eigenvalue λ_n:

$$\mathcal{L}_0|\psi_n^{(0)} >= \lambda_n|\psi_n^{(0)} >, \quad |\psi_n^{(0)} >= a_n|n^{(0)}, + > +a_{-n}| - n^{(0)}, + >), \tag{1085}$$

and the same for the eigenstates $| \pm n^{(0)}, - >$ of $-\lambda_n$.

We concentrate only on the perturbation of the eigenvalue problem (1083) with eigenvalue λ_n. The perturbed eigenvalue problem reads

$$(\mathcal{L}_0 + \epsilon\mathcal{L}_1)|n, + >= E_n|n, + >, \quad E_n = \lambda_n + \epsilon\Delta E_n \tag{1086}$$

and we expand the eigen-ket $|n, + >$ in the basis of the unperturbed problem:

$$|n >= a_n|n^{(0)}, + > +a_{-n}| - n^{(0)}, + > +\epsilon \sum_{k\neq 0} \left(b_k^+|k^{(0)}, + > +b_k^-|k^{(0)}, - > \right). \tag{1087}$$

Then the RHS and LHS of (1086) become

$$\begin{aligned}
&(\mathcal{L}_0 + \epsilon\mathcal{L}_1)|n, + >= \lambda_n \left(a_n|n^{(0)}, + > +a_{-n}| - n^{(0)}, + >\right) + \epsilon\Big[a_n\mathcal{L}_1|n^{(0)}, + > \\
&+a_{-n}\mathcal{L}_1| - n^{(0)}, + > + \sum_{k\neq 0} \lambda_k \left(b_k^+|k^{(0)}, + > -b_k^-|k^{(0)}, - >\right)\Big] + O(\epsilon^2), \\
&(\lambda_n + \epsilon\Delta E_n)|n, + >= \lambda_n \left(a_n|n^{(0)}, + > +a_{-n}| - n^{(0)}, + >\right) + \epsilon\Big[\Delta E_n(a_n|n^{(0)}, + > \\
&+a_{-n}| - n^{(0)}, + >) + \lambda_n \sum_{k\neq 0} \left(b_k^+|k^{(0)}, + > +b_k^-|k^{(0)}, - >\right)\Big] + O(\epsilon^2),
\end{aligned} \tag{1088}$$

implying the following equation at $O(\epsilon)$:

$$\begin{aligned}
&a_n\mathcal{L}_1|n^{(0)}, + > +a_{-n}\mathcal{L}_1| - n^{(0)}, - >= \Delta E_n \left(a_n|n^{(0)}, + > +a_{-n}| - n^{(0)}, + >\right) \\
&+ \sum_{k\neq 0} \Big[(\lambda_n - \lambda_k)b_k^+|k^{(0)}, + > +(\lambda_n + \lambda_k)b_k^-|k^{(0)}, - >\Big].
\end{aligned} \tag{1089}$$

Now we apply the bras $< \pm n^{(0)}, +|$ to this equation, using (1084), obtaining the 2×2 matrix eigenvalue equation

$$A_n\underline{a}_n = \Delta E_n\underline{a}_n, \tag{1090}$$

where

$$\begin{aligned}
&A_n := \begin{pmatrix} < -n^{(0)}|\mathcal{L}_1| - n^{(0)} > & < -n^{(0)}|\mathcal{L}_1|n^{(0)} > \\ < n^{(0)}|\mathcal{L}_1| - n^{(0)} > & < n^{(0)}|\mathcal{L}_1|n^{(0)} > \end{pmatrix}, \\
&\underline{a}_n = \begin{pmatrix} a_{-n} \\ a_n \end{pmatrix}.
\end{aligned} \tag{1091}$$

Then the two eigenvalues $\Delta E_n^\pm$ of A_n give the $O(\epsilon)$ corrections to the two branch points arising for the double point λ_n:

$$E_n^\pm = \lambda_n + \epsilon\Delta E_n^\pm, , \tag{1092}$$

and the corresponding eigenvectors $\underline{a}_n^\pm$ give the proper linear combination of the unperturbed eigenstates obtained from $|n^\pm >$ when $\epsilon \to 0$:

$$|n^\pm, + > \to \; a_n{}^\pm|n^{(0)}, + > +a_{-n}^\pm| - n^{(0)}, + >, \quad \epsilon \to 0. \tag{1093}$$

Let us construct now the branch points of the main spectrum corresponding to the perturbation given in (1079), (1080).

Recalling that

$$|m, \pm >= \begin{pmatrix} 1 \\ \mu_m \pm \lambda_m \end{pmatrix} e^{i\mu_m x}, \tag{1094}$$

we obtain

$$\mathcal{L}_1|m,\pm>=\sum_{n\neq 0}\begin{pmatrix} c_n(\mu_m\pm\lambda_m)e^{i\mu_{m+2n}x} \\ -\bar{c}_n e^{i\mu_{m-2n}x}\end{pmatrix}, \tag{1095}$$

that has to be expanded into the basis, obtaining

$$\begin{aligned}\mathcal{L}_1|m,\pm>=\sum_{n\neq 0}\Big[\tfrac{c_n(\mu_m\pm\lambda_m)}{2\lambda_{m+2n}}\Big((\lambda_{m+2n}-\mu_{m+2n})|m+2n,+>\\ +(\lambda_{m+2n}+\mu_{m+2n})|m+2n,->\Big)-\tfrac{\overline{c_n}}{2\lambda_{m-2n}}(|m-2n,+>-|m-2n,->)\Big].\end{aligned} \tag{1096}$$

applying the bras $<l,\pm|$ and recalling (1084), we obtain

$$\begin{aligned}<l,+|\mathcal{L}_1|m,\pm>=\tfrac{1}{2\lambda_l}\Big[c_{\frac{l-m}{2}}(\mu_m\pm\lambda_m)(\lambda_l-\mu_l)-\overline{c_{\frac{m-l}{2}}}\Big],\\ <l,-|\mathcal{L}_1|m,\pm>=\tfrac{1}{2\lambda_l}\Big[c_{\frac{l-m}{2}}(\mu_m\pm\lambda_m)(\mu_l+\lambda_l)+\overline{c_{\frac{m-l}{2}}}\Big].\end{aligned} \tag{1097}$$

At last

$$\begin{aligned}&<-n^{(0)},+|\mathcal{L}_1|-n^{(0)},+>=<n^{(0)},+|\mathcal{L}_1|n^{(0)},+>=0,\\ &<-n^{(0)},+|\mathcal{L}_1|n^{(0)},+>=\tfrac{1}{2\lambda_n}[c_{-n}(\mu_n+\lambda_n)^2-\bar{c}_n]=-\tfrac{e^{i\phi_n}}{2\lambda_n}\alpha_n,\\ &<n^{(0)},+|\mathcal{L}_1|n^{(0)},->=\tfrac{1}{2\lambda_n}[c_n(\mu_n-\lambda_n)^2-\bar{c}_{-n}]=-\tfrac{e^{-i\phi_n}}{2\lambda_n}\beta_n,\end{aligned} \tag{1098}$$

and

$$A_n^+=-\tfrac{1}{2\lambda_n}\begin{pmatrix}0 & \alpha_n e^{i\phi_n}\\ \beta_n e^{-i\phi_n} & 0\end{pmatrix}, \tag{1099}$$

where

$$\alpha_n=e^{-i\phi_n}\overline{c_n}-e^{i\phi_n}c_{-n},\quad \beta_n=e^{i\phi_n}\overline{c_{-n}}-e^{-i\phi_n}c_n. \tag{1100}$$

The eigenvalues and eigenvectors of A_n^+ are:

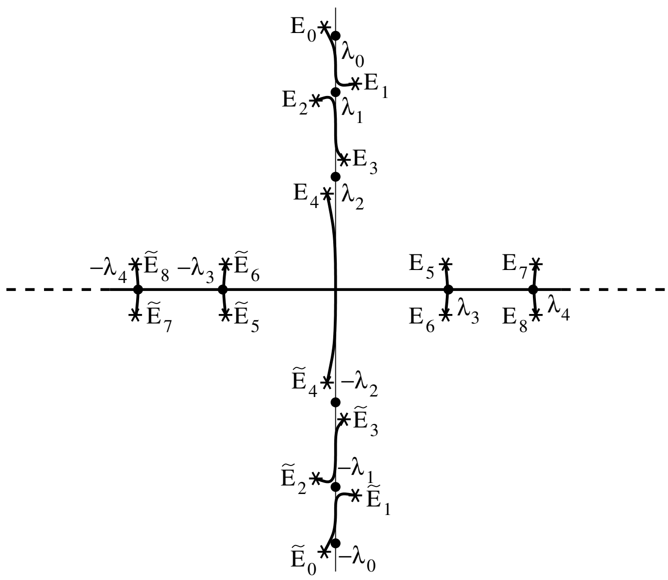
$$A_n^+:\ \ \mp\tfrac{\sqrt{\alpha_n\beta_n}}{2\lambda_n},\ \ (\pm e^{i\phi_n}\sqrt{\alpha_n/\beta_n},1)^T; \tag{1101}$$

therefore the perturbation resolves the degeneration of the double point λ_n as follows:

$$E_l=\lambda_n\mp\epsilon\tfrac{\sqrt{\alpha_n\beta_n}}{2\lambda_n}+O(\epsilon^2),\quad l=2n-1,2n. \tag{1102}$$

so that

$$(E_1-E_2)^2=-\frac{\epsilon^2\alpha\beta}{\sin^2(\phi_1)}. \tag{1103}$$



From formulas (1175), (1176) and (1103), all of them depending on the product $\alpha\beta$, one can write the following relations, to leading order, between the unstable gap $(E_1 - E_2)$ and the AW recurrence period ΔT and x-shift ΔX:

$$\begin{array}{l} \Delta T = \frac{2}{\sigma_1} \log\left(\frac{\sigma_1^2}{2\sin\phi_1 |E_1 - E_2|} \right), \\ \Delta X = \frac{\arg(-(E_1-E_2)^2)}{k_1}. \end{array} \tag{1104}$$

In the perturbed case, the analogue of equation (1078) is

$$\text{tr } T(\lambda) = \text{tr } T(\tilde{\lambda}_1) + \frac{\text{tr } T''(\tilde{\lambda}_1)}{2}(\lambda - \tilde{\lambda}_1)^2 + O((\lambda - \tilde{\lambda}_1)^3), \tag{1105}$$

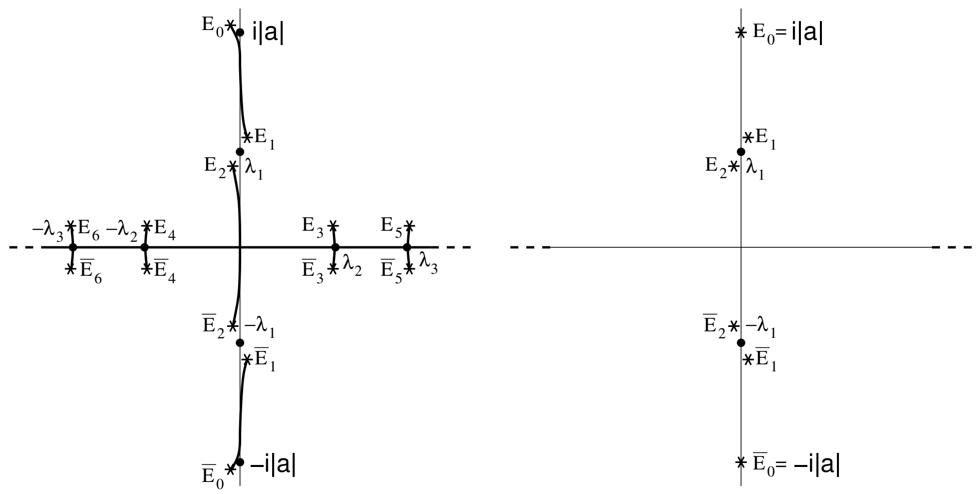
where $\tilde{\lambda}_1$ is the critical point for tr $T(\lambda)$. It is possible to show that

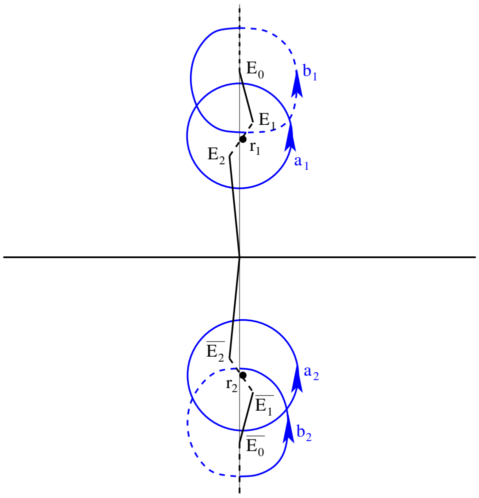
$$\tilde{\lambda}_1 - \lambda_1 = O(\epsilon^2), \ \text{tr } T(\tilde{\lambda}_1) - \text{tr } T(\lambda_1) = O(\epsilon^4), \ \text{tr } T(\lambda) - \text{tr } T^{(0)}(\lambda) = O(\epsilon^2); \tag{1106}$$

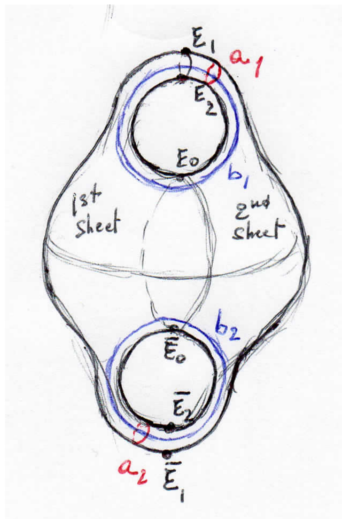
therefore, to leading order, one can replace in (1105) tr $T(\tilde{\lambda}_1)$ by tr $T(\lambda_1)$, and tr $T''(\tilde{\lambda}_1)$ by tr ${T^{(0)}}''(\lambda_1) = \frac{2\lambda_1^2 L^4}{\pi^2}$, obtaining

$$\text{tr } T(\lambda) \sim \text{tr } T(\lambda_1) + \frac{\text{tr } {T^{(0)}}''(\lambda_1)}{2}(\lambda - \lambda_1)^2 = \text{tr } T(\lambda_1) + \frac{\lambda_1^2 L^4}{\pi^2}(\lambda - \lambda_1)^2. \tag{1107}$$

As a 1st simplification, we close all gaps associated with the stable modes since the associated perturbations remain small for all times, and do not affect essentially the solution.
This finite $2N$-gap approximation is non standard: in the usual finite-gap approximation, one closes gaps smaller than a certain constant (the same criterion used when one truncates the Fourier series); in our case, all gaps are small, and the criterion for closing a gap is the stability of the corresponding mode. The spectral curve Γ of genus $2N$ is algebraic: $\nu^2 = \prod_{j=0}^{2N} (\lambda - E_j)(\lambda - \overline{E_j})$, and the FG solution $u(x,t)$ can be written in terms of the Riemann theta-functions.







9.6 The effect of dissipation on the AW dynamics

Since dissipation can hardly be avoided in all natural phenomena involving AWs, a natural question arises at this point. What is the effect of a small dissipation on the NLS periodic AW dynamics?

$$i(u_t + \nu u) + u_{xx} + 2|u|^2 u = 0, \quad 0 < \nu \ll 1 \tag{1108}$$

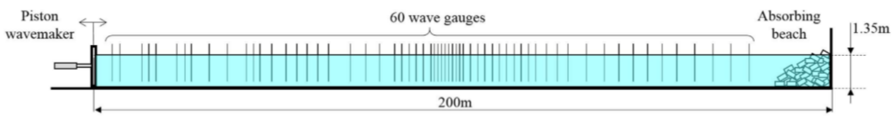
If the initial perturbation is sufficiently small, a small dissipation can quench the growth process before the nonlinear effects become relevant, stabilizing the MI, and

$$T_{diss} = \frac{1}{\nu} \log\left(\frac{2|a|}{k}\right) \tag{1109}$$

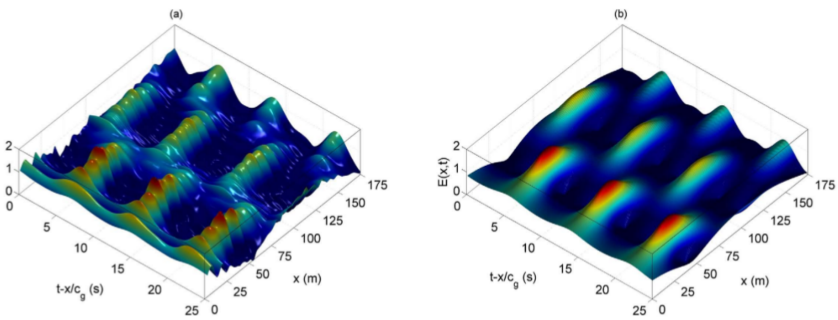
is the time at which the unstable mode k becomes stable.

But what happens in the interesting case in which dissipation is small and $T_{diss} \gg T^{(1)}$?
In [21] water tank and numerical experiments show what happens: a recurrence of ABs with $\Delta X = L/2$ shift:

The wave facility in Sydney:



The tank experiment (left) and the numerics with dissipation (right):



The effect of a small linear dissipation (loss) or of a small linear gain is described by equation

$$i(u_t + \nu u) + u_{xx} + 2|u|^2 u = 0, \quad \nu \in \mathbb{R}, \quad 0 < |\nu| \ll 1 \tag{1110}$$

($\nu > 0$ in the case of loss and $\nu < 0$ in the case of gain). In matrix form the equation becomes:

$$i(U_t + \nu U) + U_{xx} + 2U^3 = 0, \qquad U = \begin{pmatrix} 0 & u(x,t) \\ \overline{u(x,t)} & 0 \end{pmatrix}. \tag{1111}$$

If u evolves according to NLS, the trace of the monodromy matrix and the spectral curve are constants of motion. Now we calculate how the trace of the monodromy matrix, and, as a corollary, the spectral curve evolve in time in the presence of a small loss or gain. The calculation of the variation of the monodromy matrix uses a standard formula for ODEs coming from the method of variation of constants.

We first observe that, given the ZS

$$\Psi_x = \hat{X}\Psi, \quad \hat{X}(x,\lambda) = -i\lambda\sigma_3 + iU, \quad U = \begin{pmatrix} 0 & u(x,t) \\ \overline{u(x,t)} & 0 \end{pmatrix} \tag{1112}$$

a small variation of $\hat{X}$:

$$\hat{X} \to \hat{X} + \delta\hat{X}, \quad \delta\hat{X} = i\delta U = i\begin{pmatrix} 0 & \delta u(x,t) \\ \overline{\delta u(x,t)} & 0 \end{pmatrix} \tag{1113}$$

induces a small variation of the solution $\Psi \to \Psi + \delta\Psi$. Solving

$$(\Psi + \delta\Psi)_x = (\hat{X} + i\delta U)(\Psi + \delta\Psi) \tag{1114}$$

up to $O(\delta)$, one obtains the following inhomogeneous version of the ZS problem for $\delta\Psi$:

$$(\delta\Psi)_x = \hat{X}\delta\Psi + i\delta U\Psi. \tag{1115}$$

Using the method of variation of the constants, we look for a solutions in the form $\delta\Psi = \Psi\delta A$, obtaining the equation

$$(\delta A)_x = i\Psi^{-1}\delta U\Psi \;\Rightarrow\; \delta A = i\int_0^x \Psi^{-1}(x')\delta U(x')\Psi(x')dx' \tag{1116}$$

(wlg we have chosen the integration constant in such a way that $\delta\Psi(0) = 0$), implying

$$\delta\Psi(x) = i\Psi(x)\int_0^x \Psi^{-1}(x')\delta U(x')\Psi(x')dx'. \tag{1117}$$

The corresponding variation of the monodromy matrix $T(\lambda) \to T(\lambda) + \delta T(\lambda)$ is defined as

$$\begin{array}{l} T(\lambda) + \delta T(\lambda) = [\Psi(L,\lambda) + \delta\Psi(L,\lambda)][\Psi(0,\lambda) + \delta\Psi(0,\lambda)]^{-1} \\ = [\Psi(L,\lambda) + \delta\Psi(L,\lambda)]\,\Psi^{-1}(0,\lambda) = T(\lambda) + \delta\Psi(L,\lambda)\Psi^{-1}(0,\lambda), \end{array} \tag{1118}$$

where we have used $\delta\Psi(0,\lambda) = 0$, from which we infer that

$$\delta T(\lambda) = i\Psi(L,\lambda)\int_0^L \Psi^{-1}(x,\lambda)\delta U(x)\Psi(x,\lambda)\Psi^{-1}(0,\lambda)dx. \tag{1119}$$

Therefore

$$\delta T(\lambda) = i\int_0^L \tilde{\Psi}(L,x,\lambda)\delta U(x)\tilde{\Psi}(x,0,\lambda)dx, \tag{1120}$$

where we have used the definition (1050) of $\tilde{\Psi}$:

$$\tilde{\Psi}(\lambda, x, x_0, t) = \Psi(\lambda, x, t)\Psi^{-1}(\lambda, x_0, t). \tag{1121}$$

Applying the trace to this matrix equation we obtain

$$\begin{aligned}
&\mathrm{tr}\delta T(\lambda, t) = i\int_0^L \mathrm{tr}\left(\tilde{\Psi}(L, x, \lambda)\delta U(x)\tilde{\Psi}(x, 0, \lambda)\right) dx \\
&= i\int_0^L \mathrm{tr}\left(\tilde{\Psi}^{-1}(\lambda, x+L, L, t)\tilde{\Psi}(\lambda, x+L, L, t)\tilde{\Psi}(L, x, \lambda)\delta U(x)\tilde{\Psi}(x, 0, \lambda)\right) dx \\
&= i\int_0^L \mathrm{tr}\left(\tilde{\Psi}(\lambda, x+L, L, t)\tilde{\Psi}(\lambda, L, x, t)\delta U(x, t)\tilde{\Psi}(\lambda, x, 0, t)\tilde{\Psi}^{-1}(\lambda, x+L, L, t)\right) dx \\
&= i\int_0^L \mathrm{tr}\left(\tilde{\Psi}(\lambda, x+L, x, t)\delta U(x, t)\right) dx.
\end{aligned} \tag{1122}$$

In the last step we used again the definition (1050) of $\tilde{\Psi}$, implying both the periodicity

$$\tilde{\Psi}(\lambda, x, 0, t)\tilde{\Psi}^{-1}(\lambda, x+L, L, t) = I \tag{1123}$$

and the property

$$\tilde{\Psi}(\lambda, x, x_1, t)\tilde{\Psi}(\lambda, x_1, x_0, t) = \tilde{\Psi}(\lambda, x, x_0, t). \tag{1124}$$

Choosing $\delta U = U_t dt = [i\sigma_3(U_{xx} + 2U^3) - \nu U]dt$, then $\delta\mathrm{tr}T = (\mathrm{tr}T)_t dt$; substituting these variations into equation (1122) and taking into account that $\int_0^L \mathrm{tr}\left[\tilde{\Psi}(\lambda, x+L, x, t)\sigma_3\left(U_{xx}(x,t) + 2U^3(x,t)\right)\right] dx = 0$, since the main spectrum is invariant with respect to the NLS dynamics, it follows that

$$(\mathrm{tr}T)_t(\lambda, t) = -i\nu\int_0^L \mathrm{tr}\left[\tilde{\Psi}(\lambda, x+L, x, t)U(x, t)\right] dx. \tag{1125}$$

At last, integrating this equation over time, from 0 to t, one obtains the variation of $\mathrm{tr}T$ in the time interval $[0, t]$:

$$\begin{aligned}
\Delta\mathrm{tr}T(\lambda, t) = -i\nu\int_0^t d\tilde{t}\left[\int_0^L \mathrm{tr}\left[\tilde{\Psi}(\lambda, L+x, x, \tilde{t})U(x, \tilde{t})\right] dx\right] = \\
= -i\nu\int_0^t d\tilde{t}\left[\int_0^L \left[\tilde{\Psi}_{21}(\lambda, x+L, x, \tilde{t})u(x, \tilde{t}) + \tilde{\Psi}_{12}(\lambda, x+L, x, \tilde{t})\overline{u(x, \tilde{t})}\right] dx\right].
\end{aligned}$$

The calculation of the above integral with high genus theta-functions is very complicated. But, to leading order, this integral can be explicitly calculated in terms elementary functions using the following properties of this solution:

1. Near each AW appearance the solution is well approximated by the Akhmediev breather.

2. Far from the AW appearance, the integral over the x-period tends to zero exponentially in t. Therefore the integral over the finite time interval of each AW appearance can be well approximated by the integral over the whole line $-\infty < t < \infty$ of the Akhmediev solution.

We conclude that, to leading order,

$$\Delta\mathrm{tr}T(\lambda, t) = n_{app}\nu J(\lambda),$$

where n_{app} is the number of AW appearances in the time interval $[0, t]$, and

$$J(\lambda) = -i\int_{-\infty}^{+\infty} dt\left[\int_0^L \left[\tilde{\Psi}_{21}(\lambda, x+L, x, t)u(x, t) + \tilde{\Psi}_{12}(\lambda, x+L, x, t)\overline{u(x, t)}\right] dx\right], \tag{1126}$$

where $u(x, t)$ is the Akhmediev breather and $\tilde{\Psi}(\lambda, x+L, x, t)$ is the corresponding fundamental matrix. Let us recall that, to calculate the variation of the curve we need both $\Delta\mathrm{tr}T(\lambda, t)$ and $J(\lambda)$ at the point $\lambda = \lambda_1$. To compute $\tilde{\Psi}(\lambda, x+L, x, t)$ it is convenient to use the classical DT for NLS, constructed in a previous chapter.

9.6.1 Darboux transformation of the constant background

We recall that, if $\Psi^{(0)}(\lambda,x,t)$ is a solution of the ZS Lax pair for $U=U^{(0)}=\begin{bmatrix} 0 & u^{(0)}(x,t) \\ \bar{u}^{(0)}(x,t) & 0 \end{bmatrix}$, then $\Psi(x,t,\lambda)$ is a solution of the ZS Lax pair for $U=\begin{bmatrix} 0 & u(x,t) \\ \bar{u}(x,t) & 0 \end{bmatrix}$, where

$$
\begin{array}{l}
\Psi(x,t,\lambda)=\left(I+\frac{2iIm(\lambda_1)}{\lambda-\lambda_1}P(x,t)\right)\Psi^{(0)}(x,t,\lambda), \\
P(x,t)=\frac{1}{|q_1|^2+|q_2|^2}\begin{pmatrix}-\bar{q}_2 \\ \bar{q}_1\end{pmatrix}(-q_2,q_1)=\frac{1}{|q_1|^2+|q_2|^2}\begin{pmatrix}|q_2|^2 & -\bar{q}_2 q_1 \\ -q_2\bar{q}_1 & |q_1|^2\end{pmatrix},
\end{array}
\tag{1127}
$$

$$
u(x,t)=u^{(0)}(x,t)-4i\mathrm{Im}\lambda_1\frac{q_1\overline{q_2}}{|q_1|^2+|q_2|^2},
\tag{1128}
$$

and $\boldsymbol{q}(x,t)=(q_1(x,t),q_2(x,t))^T$ is a vector solution of the ZS Lax pair for $\lambda=\lambda_1$. We observe that $P(x,t)$ is an orthogonal projector.

We apply this transformation to the constant background. As we have already seen, the unperturbed spectral curve Γ_0 is rational, and a point $\gamma\in\Gamma_0$ is a pair of complex numbers $\gamma=(\lambda,\mu)$ satisfying the equation $\mu^2=\lambda^2+1$, parametrized by

$$
\lambda=i\sin(\phi),\quad \mu=\cos(\phi),\quad \Rightarrow\quad \lambda+\mu=e^{i\phi}.
$$

As we have already seen, the Bloch eigenfunctions for the operator $\mathcal{L}_0$ can be easily calculated explicitly:

$$
\begin{array}{c}
\vec{\psi}^{\pm}(\gamma,x)=\begin{bmatrix} e^{it} \\ \left[\lambda(\gamma)\pm\mu(\gamma)\right]e^{-it}\end{bmatrix}e^{\pm i\mu(\gamma)x\pm 2i\lambda(\gamma)\mu(\gamma)t}, \\
\mathcal{L}_0\psi^{\pm}(\gamma,x)=\lambda(\gamma)\psi^{\pm}(\gamma,x),
\end{array}
\tag{1129}
$$

or, in a different normalization,

$$
\vec{\psi}^{\pm}(\phi,x)=\begin{bmatrix} e^{it\mp i\phi/2} \\ \pm e^{-it\pm i\phi/2}\end{bmatrix}e^{\pm i\cos(\phi)x\mp\sin(2\phi)t}.
\tag{1130}
$$

Denote by $\vec{q}$ the special solution of the ZS Lax pair, for $u=u_0$, and $\lambda=\lambda_1$, obtained adding up the two vector solutions (1130):

$$
\vec{q}(x,t)=\begin{bmatrix} q_1 \\ q_2\end{bmatrix}=\begin{bmatrix} e^{it}\left(e^{-i\phi_1/2+i\cos(\phi_1)x-\sin(2\phi_1)t}+e^{i\phi_1/2-i\cos(\phi_1)x+\sin(2\phi_1)t}\right) \\ e^{-it}\left(e^{i\phi_1/2+i\cos(\phi_1)x-\sin(2\phi_1)t}-e^{-i\phi_1/2-i\cos(\phi_1)x+\sin(2\phi_1)t}\right)\end{bmatrix}
\tag{1131}
$$

$$
=2\begin{bmatrix} e^{it}\cos(\frac{k_1}{2}x-\phi_1/2+i\frac{\sigma_1}{2}t) \\ ie^{-it}\sin(\frac{k_1}{2}x+\phi_1/2+i\frac{\sigma_1}{2}t)\end{bmatrix},
\tag{1132}
$$

where

$$
\lambda_1=i\sin(\phi_1),\quad \mu_1=\cos(\phi_1),\quad k_1=2\cos(\phi_1),\quad \sigma_1=2\sin(2\phi_1),
$$

and we assume ϕ_1 to be real. Then $\vec{q}(x+L,t)=-\vec{q}(x,t)$ is anti-periodic, $P(x+L,t)=P(x,t)$ is periodic, being quadratic in $\vec{q}(x,t)$, and the dressed potential is also periodic and reads (verify it!)

$$
u(x,t)=e^{2it}-\frac{4\lambda_1}{Den(x,t)}q_1(x,t)\overline{q_2(x,t)}=e^{2it}\frac{\cosh[\sigma_1 t+2i\phi_1]-\sin(\phi_1)\sin(k_1x)}{\cosh(\sigma_1 t)+\sin(\phi_1)\sin(k_1x)},
\tag{1133}
$$

where

$$
Den(x,t)=|q_1(x,t)|^2+|q_2(x,t)|^2=4\left[\cosh(\sigma_1 t)+\sin(\phi_1)\sin(k_1x)\right].
$$

It coincides with the Akhmediev breather solution (965),(966) introducing suitable free parameters corresponding to the x and t translation symmetries of NLS.

If $u=u^{(0)}(x,t)$, then we can choose

$$
\Psi^{(0)}(x,t,\lambda)=e^{it\sigma_3}\begin{pmatrix} e^{i\mu(x+2\lambda t)} & -(\lambda+\mu)e^{-i\mu(x+2\lambda t)} \\ (\lambda+\mu)e^{i\mu(x+2\lambda t)} & e^{-i\mu(x+2\lambda t)}\end{pmatrix}.
\tag{1134}
$$

Consequently the corresponding fundamental matrix reads:

$$
\begin{array}{l}
\tilde{\Psi}^{(0)}(x,y,t,\lambda)=\Psi^{(0)}(x,t,\lambda)\left(\Psi^{(0)}(y,t,\lambda)\right)^{-1}\\
=\begin{pmatrix}\cos(\mu(x-y))-\frac{i\lambda}{\mu}\sin(\mu(x-y)) & \frac{i}{\mu}\sin(\mu(x-y))e^{2it}\\ \frac{i}{\mu}\sin(\mu(x-y))e^{-2it} & \cos(\mu(x-y))+\frac{i\lambda}{\mu}\sin(\mu(x-y))\end{pmatrix}\\
=\frac{1}{\mu}\begin{pmatrix}\cos(\mu(y-x)-\phi) & i\sin(\mu(y-x))e^{2it}\\ i\sin(\mu(y-x))e^{-2it} & \cos(\mu(y-x)+\phi)\end{pmatrix},
\end{array}
\tag{1135}
$$

with

$$
\begin{array}{l}
\partial_\lambda\tilde{\Psi}^{(0)}(x,y,t,\lambda)=\dfrac{i\sin(\mu(y-x))}{\mu^3}\begin{pmatrix}-1 & -\lambda e^{2it}\\ -\lambda e^{-2it} & 1\end{pmatrix}+\\
+\dfrac{\lambda(x-y)}{\mu^2}\begin{pmatrix}-\sin(\mu(x-y)-\phi) & i\cos(\mu(x-y))e^{2it}\\ i\cos(\mu(x-y))e^{-2it} & -\sin(\mu(x-y)+\phi)\end{pmatrix}.
\end{array}
\tag{1136}
$$

Consequently:

$$
\begin{array}{l}
\tilde{\Psi}^{(0)}(\lambda_n,x+L,x,t)=(-1)^nI,\\
\partial_\lambda\tilde{\Psi}^{(0)}(\lambda,x+L,x,t)\Big|_{\lambda=\lambda_n}=(-1)^n n\frac{i\lambda_n\pi}{\mu_n^3}\begin{pmatrix}-\lambda_n & e^{2it}\\ e^{-2it} & \lambda_n\end{pmatrix}.
\end{array}
\tag{1137}
$$

If u is the potential (1133), dressed from the background $u^{(0)}$, then the corresponding transition matrix reads (from (1127))

$$
\begin{array}{l}
\tilde{\Psi}(\lambda,x,y,t)=\Psi(\lambda,x,t)\Psi^{(0)}(\lambda,y,t)\\
=\left(I+\frac{2i\sin\phi_1}{\lambda-\lambda_1}P(x,t)\right)\Psi^{(0)}(x,t,\lambda)(\Psi^{(0)}(\lambda,y,t))^{-1}\left(I+\frac{2i\sin\phi_1}{\lambda-\lambda_1}P(y,t)\right)^{-1}\\
=\left(I+\frac{2i\sin\phi_1}{\lambda-\lambda_1}P(x,t)\right)\tilde{\Psi}^{(0)}(\lambda,x,y,t)\left(I-\frac{2i\sin\phi_1}{\lambda-\lambda_1+2i\sin\phi_1}P(y,t)\right).
\end{array}
\tag{1138}
$$

Choosing $x\to x+L,\ y\to x$, and $\lambda\sim\lambda_1$, ... Evaluating (1138) at $y=x+L$, we observe that $q_j(x+L)=-q_j(x),\ j=1,2$, implying $\hat{\Phi}^{(0)}(\lambda_1,x+L,t)=-\hat{\Phi}^{(0)}(\lambda_1,x,t)$ and $\lambda-\tau(x+L,t)=\lambda-\tau(x,t)$. In addition, if $\lambda\sim\lambda_1$:

$$
\begin{array}{l}
\tilde{\Psi}^{(0)}(\lambda,x+L,x,t)=\tilde{\Psi}^{(0)}(\lambda_1,x+L,x,t)+\frac{\partial\tilde{\Psi}^{(0)}(\lambda,x+L,x,t)}{\partial\lambda}|_{\lambda=\lambda_1}(\lambda-\lambda_1)+O(\lambda-\lambda_1)^2\\
=-I+\frac{i\lambda_1\pi}{\mu_1^3}\begin{pmatrix}\lambda_1 & -e^{2it}\\ -e^{-2it} & -\lambda_1\end{pmatrix}(\lambda-\lambda_1)+O(\lambda-\lambda_1)^2,\\
\lambda-\tau(y,t)=\frac{\lambda_1-\overline{\lambda_1}}{Den(y,t)}\begin{bmatrix}-\overline{q_2(y,t)}\\ \overline{q_1(y,t)}\end{bmatrix}[-q_2(y,t),q_1(y,t)]+O(\lambda-\lambda_1),\\
(\lambda-\tau(x,t))^{-1}=\frac{1}{(\lambda-\lambda_1)Den(x,t)}\begin{bmatrix}q_1(x,t)\\ q_2(x,t)\end{bmatrix}[\overline{q_1(x,t)},\overline{q_2(x,t)}]+O(1).
\end{array}
\tag{1139}
$$

Therefore

$$
\begin{array}{l}
\tilde{\Psi}(\lambda,x+L,x,t)=(\lambda-\tau(x,t))\left(-I-\frac{i\lambda_1\pi}{\mu_1^3}\begin{pmatrix}-\lambda_1 & e^{2it}\\ e^{-2it} & \lambda_1\end{pmatrix}(\lambda-\lambda_1)\right)(\lambda-\tau(x,t)))^{-1}\\
=-I+\frac{\lambda_1-\overline{\lambda_1}}{Den(x,t)}\begin{bmatrix}-\overline{q_2(x,t)}\\ \overline{q_1(x,t)}\end{bmatrix}[-q_2(x,t),q_1(x,t)]\frac{i\pi\lambda_1}{\mu_1^3}\begin{pmatrix}\lambda_1 & -e^{2it}\\ -e^{-2it} & -\lambda_1\end{pmatrix}\times\\
\begin{bmatrix}-\overline{q_1(x,t)}\\ \overline{q_2(x,t)}\end{bmatrix}[\overline{q_1(x,t)},\overline{q_2(x,t)}]\\
=-I+\frac{2\pi i\lambda_1^2}{\mu_1^3}\frac{f(x,t)}{Den^2(x,t)}\begin{bmatrix}-\overline{q_2(x,t)q_1(x,t)} & -\overline{q_2(x,t)q_2(x,t)}\\ \overline{q_1(x,t)q_1(x,t)} & \overline{q_1(x,t)q_2(x,t)}\end{bmatrix},
\end{array}
\tag{1140}
$$

where

$$
f(x,t):=e^{2it}q_2^2(x,t)-e^{-2it}q_1^2(x,t)-2i\sin\phi_1\ q_1(x,t)q_2(x,t).
\tag{1141}
$$

9.7 The effect of dissipation on the main spectrum and on the dynamics [11]

Using (1126) we finally write explicitely the integral $J(\lambda_1)$, defined in (1126), describing the variation of tr$T(\lambda_1)$ at each AW appearance:

$$\begin{aligned} & J(\lambda_1) = -\tfrac{2\pi \sin^2\phi_1}{|a|\cos^3\phi_1} \int\limits_{-\infty}^{+\infty} dt \int\limits_{0}^{L} dx \tfrac{f(x,t)g(x,t)}{Den^2(x,t)}, \\ & g(x,t) = u(x,t)\bar{q}_1(x,t)^2 - \bar{u}(x,t)\bar{q}_2(x,t)^2. \end{aligned} \tag{1142}$$

The following two important simplifications

$$\begin{aligned} & f(x,t) = e^{2it}q_2^2(x,t) - e^{-2it}q_1^2(x,t) - 2i\sin\phi_1\ q_1(x,t)q_2(x,t) = -4\cos^2(\phi_1), \\ & g(x,t) = u(x,t)\bar{q}_1(x,t)^2 - \bar{u}(x,t)\bar{q}_2(x,t)^2 = 4\left(\cos(2\phi_1) - \sin(\phi_1)\sin(k_1 x - i\sigma_1 t)\right), \end{aligned} \tag{1143}$$

lead to the double integral

$$\begin{aligned} J(\lambda_1) &= 2\pi \frac{\sin^2\phi_1}{\cos\phi_1} \int\limits_{-\infty}^{+\infty} dt \int\limits_{0}^{L} dx \frac{\cos(2\phi_1) - \sin(\phi_1)\sin(k_1 x - i\sigma_1 t)}{(\cosh(\sigma_1 t) + \sin(\phi_1)\sin(k_1 x))^2} = \\ &= 2\pi^2 \sin^2(\phi_1) \int\limits_{-\infty}^{+\infty} \frac{\cosh(\sigma_1 t)}{(\cosh^2(\sigma_1 t) - \sin^2(\phi_1))^{3/2}} dt \\ &= \frac{2\pi^2\sin^2(\phi_1)}{\sigma_1} \int\limits_{-\infty}^{\infty} \frac{d(\sinh(\sigma_1 t))}{(\sinh^2(\sigma_1 t) + \cos^2(\phi_1))^{3/2}} = \frac{\pi^2 \sin\phi}{\cos^3\phi}. \end{aligned} \tag{1144}$$

The integration with respect to x has been done using contour integration. The integration wrt t is even more elementary. Therefore the variation $\Delta_1(\text{tr}T(\lambda_1))$ of tr$T(\lambda_1)$, due to a single appearance of the AW, is given by

$$\Delta_1(\text{tr}T(\lambda_1)) = \nu J(\lambda_1) = \nu \tfrac{\pi^2 \sin\phi_1}{\cos^3\phi_1}. \tag{1145}$$

Evaluating (1107), rewritten here for completeness:

$$\text{tr } T(\lambda) \sim \text{tr } T(\lambda_1) + \frac{\lambda_1^2 L^4}{\pi^2}(\lambda - \lambda_1)^2 \tag{1146}$$

at $\lambda = E_1$ and recalling that tr $T(E_1) = -2$ we obtain

$$-2 \sim \text{tr } T(\lambda_1) + \frac{\lambda_1^2 L^4}{\pi^2}(E_1 - \lambda_1)^2. \tag{1147}$$

Since $E_1 - E_2 = 2(E_1 - \lambda_1)$, we have

$$\text{tr}T(\lambda_1) \sim -2 + \frac{\sin^2(\phi_1)L^4}{4\pi^2}(E_1 - E_2)^2 \left(= -2 - \epsilon^2 \frac{L^4}{4\pi^2}\alpha\beta,\ \text{at } t = 0\right). \tag{1148}$$

Then, due to (1145) and (1148), after each appearance:

$$\nu \frac{\pi^2 \sin\phi_1}{\cos^3\phi_1} = \Delta_1(\text{tr}T(\lambda_1)) = \frac{L^2 \sin^2\phi_1}{4\cos^2\phi_1} \Delta_1\left((E_1 - E_2)^2\right). \tag{1149}$$

Therefore we have established that, after each appearance of the AW, the square of the gap varies of the same $O(\nu)$ quantity:

$$\Delta_1\left((E_1 - E_2)^2\right) = 4\nu\cot(\phi_1) \tag{1150}$$

with (see (1103))

$$(E_1 - E_2)^2\Big|_{t=0} = -\frac{\epsilon^2 \alpha\beta}{\sin^2\phi_1}. \tag{1151}$$

We conclude that

$$(E_1^{(m)} - E_2^{(m)})^2 = -\frac{\epsilon^2 \alpha\beta}{\sin^2\phi_1} + 4m\nu\cot\phi_1, \quad m \geq 0, \tag{1152}$$

where $E_1^{(m)}, E_2^{(m)}$ are the positions of the branch points of the gap after the m^{th} AW appearance. This formula implies that, as m increases, the gap tends to become horizontal if $\nu > 0$ (loss), or vertical if $\nu < 0$ (gain) (see Figure 55). The spectral description of the two asymptotics states is therefore given by the elementary formula

$$(E_1^{(m)} - E_2^{(m)})^2 = 4m\nu\cot\phi_1 \tag{1153}$$

showing that the length of the gap grows through the law

$$|E_1^{(m)} - E_2^{(m)}| = 2\sqrt{m|\nu|\cot\phi_1}. \tag{1154}$$

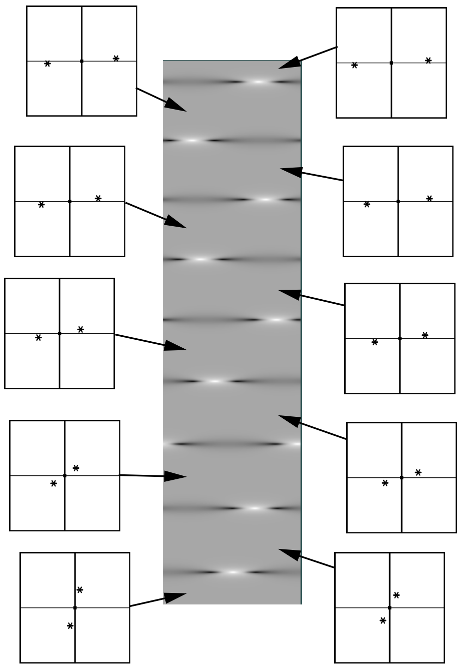


Figure 55: The figure contains the numerical experiment illustrated in the central picture of Figure 2, together with the corresponding time evolution of the gap $E_1 - E_2$, due to each AW appearance. It shows how the gap $E_1 - E_2$ tends to become horizontal as the number of AW appearances increases, in the case of loss (in the case of gain, the gap would tend to become vertical). The quantitative agreement among the numerical output, the analytic formulas (1160)-(1157) describing the AW dynamics, and the analytic formula (1152) describing the position of the gap after each AW appearance is extremely good.

It is also convenient to introduce the quantity Q such that

$$\epsilon^2 \Delta_1 Q = -\sin^2(\phi_1)\ \Delta_1 (E_1 - E_2)^2, \quad Q\Big|_{t=0} = \alpha\beta. \tag{1155}$$

Then also the variation of Q after each appearance of the AW is constant:

$$\Delta_1 Q = Q_{m+1} - Q_m = -\frac{\nu}{\epsilon^2} 2\sin(2\phi_1) = -\frac{\nu\sigma_1}{\epsilon^2}, \qquad Q_0 = \alpha\beta \tag{1156}$$

implying

$$Q_m = \alpha\beta - \frac{\nu}{\epsilon^2}\sigma_1 m, \quad m \geq 1. \tag{1157}$$

Therefore **the AW recurrence without loss or gain:**

$$u(x,t) = \sum_{m=0}^{n} \mathcal{A}\Big(x,t;\phi_1, x^{(m)}, t^{(m)}\Big) e^{i\rho^{(m)}} - \tfrac{1-e^{4in\phi_1}}{1-e^{4i\phi_1}} e^{2it}, \quad x \in [0,L], \tag{1158}$$

where the parameters $x^{(m)},\ t^{(m)},\ \rho^{(m)},\ m \geq 0$, are defined in terms of the initial data by the following elementary functions

$$\begin{array}{l} x^{(m)} = X^{(1)} + (m-1)\Delta X, \quad t^{(m)} = T^{(1)} + (m-1)\Delta T, \\ X^{(1)} = \frac{\arg\alpha}{k_1} + \frac{L}{4}, \quad \Delta X = \frac{\arg(\alpha\beta)}{k_1}, \quad (\bmod L), \\ T^{(1)} \equiv \frac{1}{\sigma_1}\log\left(\frac{2\sin^2(2\phi_1)}{\epsilon|\alpha|}\right) = \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^2}{2\epsilon|\alpha|}\right), \\ \Delta T = \frac{1}{\sigma_1}\log\left(\frac{4\sin^4(2\phi_1)}{\epsilon^2|\alpha\beta|}\right) = \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^4}{4\epsilon^2|\alpha\beta|}\right), \\ \rho^{(m)} = 2\phi_1 + (m-1)4\phi_1, \end{array} \tag{1159}$$

is significantly modified by the small loss/gain in the following way. The solution is still described by a recurrence of Akhmediev breathers

$$u(x,t) = \sum_{m=0}^{\tilde{n}} \mathcal{A}\Big(x,t;\phi_1, \tilde{x}^{(m)}, \tilde{t}^{(m)}\Big) e^{i\rho^{(m)}} - \tfrac{1-e^{4in\phi_1}}{1-e^{4i\phi_1}} e^{2i|a|^2 t}, \quad x \in [0,L], \tag{1160}$$

where $\tilde{x}^{(1)}$, $\tilde{t}^{(1)}$ are essentially the same as in (1159):

$$\tilde{x}^{(1)} = x^{(1)}, \quad \tilde{t}^{(1)} = t^{(1)}; \tag{1161}$$

but now

$$\begin{array}{l} \Delta X_m := \tilde{x}^{(m+1)} - \tilde{x}^{(m)} = \frac{\arg(Q_m)}{k_1} \quad (\bmod L), \\ \Delta T_m := \tilde{t}^{(m+1)} - \tilde{t}^{(m)} = \frac{1}{\sigma_1}\log\left(\frac{4\sin^4(2\phi_1)}{\epsilon^2|Q_m|}\right) = \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^4}{4\epsilon^2|Q_m|}\right), \end{array} \tag{1162}$$

where Q_m is defined in (1157).

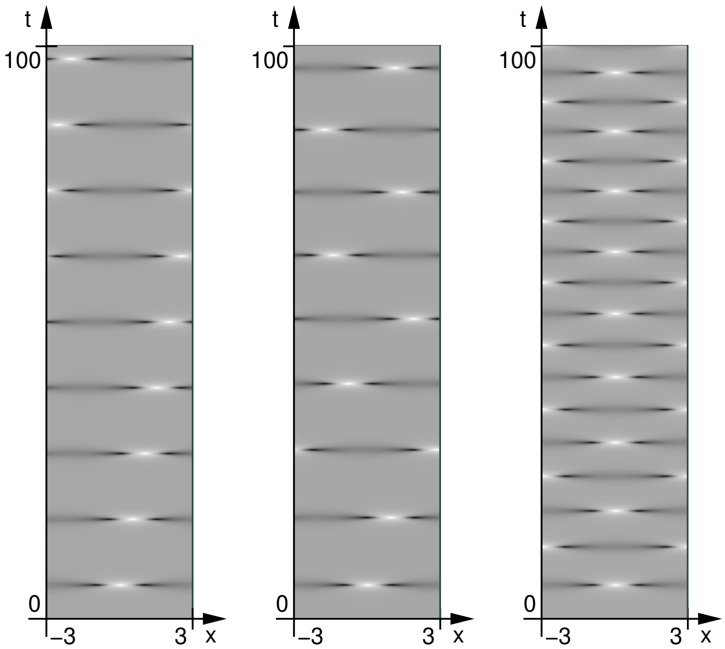


Figure 56: The density plot of $|u(x,t)|$, with $-L/2 \leq x \leq L/2$, $0 \leq t \leq 100$, $L = 6$, $\epsilon = 10^{-4}$, $a = 1$, for generic initial data: $c_1 = 0.5$ and $c_{-1} = 0.15-0.2i$, obtained using the refined split-step method [?]. From left to right: $\nu = 0$, $\nu = 10^{-9} < \epsilon^2 = 10^{-8}$, and $\nu = 10^{-5} \gg \epsilon^2$. In the left figure we have the usual AW recurrence described by formulas (1158)-(1159). In the central figure, the solution tends to the asymptotic state with $\Delta X_m \to L/2$, after a relatively long transient. In the right figure, after the first appearance, the solution enters, without any transient, the asymptotic state with $\Delta X_m = L/2, \ m \geq 1$. The first appearance is essentially the same in all the three cases.

From (1157) we infer that, if $\nu = O(\epsilon^2)$, the change in the AW dynamics is $O(1)$, no matter how small is the dissipation. A qualitative explanation of this phenomenon is the following: when the AW appears, it determines a small, $O(\nu)$ change in the dymanics; but, due to modulation instability, this small change amplifies and becomes $O(1)$ at the next AW appearance.

9.8 Anomalous waves in multidimensions

9.8.1 Physically relevant $2+1$ dimensional generalizations of NLS and their modulation instability

In most of the physical applications, we have the following two non integrable generalizations of the focusing NLS equation $iu_t + u_{xx} + 2|u|^2u = 0,\ u = u(x,t) \in \mathbb{C}$ in multidimensions.
The "elliptic saturated NLS equation" (567) in $n+1$ dimensions:

$$\begin{array}{l} iA_t + \Delta A + \varphi(|A|^2)A = 0, \\ A = A(\vec{x},t) \in \mathbb{C}, \quad \vec{x} = (x_1, x_2, \dots, x_n), \quad \Delta = \sum\limits_{j=1}^{n} \partial^2_{x_j}, \end{array} \tag{1163}$$

relevant, for instance, in a time independent nonlinear optics, where $n = 1,2$, $t \to z$ is the paraxial direction of propagation, $(x_1 = x, x_2 = y)$ is the transversal plane, and $\varphi(\zeta) : \mathbb{R}^+ \to \mathbb{R}^+$ is a saturation potential behaving as

$$\begin{array}{l} \varphi(\zeta) = O(\zeta),\ 0 < \zeta \ll 1, \quad \varphi(\zeta) \to const,\ \zeta \to \infty, \\ \varphi(\zeta) \geq 0, \quad \varphi'(\zeta) > 0, \quad \varphi''(\zeta) < 0. \end{array} \tag{1164}$$

The cubic "hyperbolic NLS equation" (549) in $n+1$ dimensions:

$$\begin{array}{l} iA_t + \sum\limits_{j=1}^{n_1} A_{x_j x_j} - \sum\limits_{j=1}^{n_2} A_{y_j y_j} + 2|A|^2 A = 0, \quad n_1 + n_2 = n, \\ A = A(\vec{x},\vec{y},t) \in \mathbb{C}, \quad \vec{x} = (x_1, \dots, x_{n_1}), \quad \vec{y} = (y_1, \dots, y_{n_2}), \end{array} \tag{1165}$$

relevant for instance in the following contexts. i) Surface water waves in deep water; in this case $n_1 = n_2 = 1$, $x_1 = x$ is the direction of propagation of the wave and $y_1 = y$ is the transversal direction. ii) Nonlinear optics in the paraxial approximation. In this case $t \to z$ is the paraxial direction of propagation of the wave; $n_2 = 1$ and $y_1 \to t$ is the time variable; $n_1 = 1,2$ are the transversal space variables (if $n_1 = 1$, $x_1 \to x$ is the transversal direction; if $n_1 = 2$, $(x_1, x_2) \to (x,y)$ is the transversal plane).

In the rest of this chapter we limit our considerations to the simplest case of 3 dimensions: $d = 2,\ x_1 = x,\ x_2 = y$ in the elliptic case (1163), and $n_1 = n_2 = 1,\ x_1 = x,\ y_1 = y$ in the hyperbolic case (1165)

$$\begin{array}{l} iA_t + A_{xx} + A_{yy} + \varphi(|A|^2)A = 0, \\ iA_t + A_{xx} - A_{yy} + 2|A|^2 A = 0. \end{array} \tag{1166}$$

Their homogeneous background solutions are

$$\begin{array}{l} A_0(x,y,t) = a e^{i\varphi(|a|^2)t}, \quad \text{elliptic case} \\ A_0(x,y,t) = a e^{2i|a|^2 t}, \quad \text{hyperbolic case,} \end{array} \tag{1167}$$

where a is an arbitrary complex parameter, and it is natural to investigate their linear stability properties under perturbations.

We observe that in the elliptic cubic NLS case in which $\varphi(|A|^2) = 2|A|^2$, we have focusing effects in all the space directions, and one can show that a generic smooth perturbation of the background containing unstable modes blows up at finite time. The saturation potential (1164) prevents such a blow up, since in the large field limit it tends to a constant, and the equation becomes linear. In the hyperbolic cubic NLS case the situation is very different, since there is focusing in the x direction and defocusing in the y direction, and one could show that the solution does not blow up.

The only examples of integrable and physically relevant generalizations of the focusing NLS equation in $2+1$ dimensions are the following Davey - Stewartson (DS) equations

$$\begin{array}{l} iA_t + A_{xx} + \nu A_{yy} + 2qA = 0, \quad q_{xx} - \nu q_{yy} = (|A|^2)_{xx} + \nu(|A|^2)_{yy}, \\ A = A(x,y,t) \in \mathbb{C}, \quad q = q(x,y,t) \in \mathbb{R}, \end{array} \tag{1168}$$

where $\nu = 1$ in the DS1 case, and $\nu = -1$ in the DS2 case, reducing both, in the $1+1$ dimensional limits ($\partial_y A = \partial_y q = 0$), to the focusing NLS equation. We remark that the second equation in (1168) implies the existence of the potential W such that

$$\begin{array}{l} W_x = (|A|^2 + q)_y \,, \\ W_y = -\nu\left(|A|^2 - q\right)_x \end{array} \tag{1169}$$

The integrability scheme of the DS equations (1168) is given by the following Lax pair:

$$\begin{array}{l} \vec{\psi}_y = i\sigma_3 \vec{\psi}_x + U\vec{\psi}, \quad \vec{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathbb{C}^2 \\ \vec{\psi}_t = 2i\sigma_3 \vec{\psi}_{xx} + 2U\vec{\psi}_x + V\vec{\psi}, \\ U = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} W + iq & u_x - iu_y \\ -(\bar{u}_x + i\bar{u}_y) & W - iq \end{pmatrix}. \end{array} \tag{1170}$$

As in the $1+1$ dimensional case, we have the following symmetry: if $(\psi_1, \psi_2)^T$ is solution of (1170), also $(-\bar{\psi}_2, \bar{\psi}_1)^T$ is solution. Therefore $\Psi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}$ is a fundamental matrix solution of (1170).

9.8.2 Quasi one dimensional AWs and the transversal fission as a critical phenomenon

The simplest way to get interesting informations concerning $(n+1)$ dimensional AWs is to consider the case of quasi one dimensional (Q1D) AWs, i.e., AWs in which the dependence on the extra space variables $\underline{y} = (y, z, \dots) \in \mathbb{R}^{n-1}$ is slow. If an $O(\epsilon)$ perturbation of the constant background is slowly varying in this sense, then the terms of the NLS type equation in multidimension depending on the derivatives wrt y, z will be relevant at large time scales (the multidimensionality scale), and if the logarithmically large time scales $\ln(1/\epsilon)$ describing the first appearance and the recurrence of the 1D AW is smaller than the multidimensionality time scale δ^{-2}, it is clear that the AW appearance and first few recurrences are ruled by the $1+1$ dimensional NLS equation, for which the theory has been constructed in the previous chapter.

Consider the simplest possible situation of Q1D AWs in $n+1$ dimensions, and the case of a single unstable mode of the corresponding 1D NLS theory. For the following initial datum

$$u(x, \underline{y}, 0) = 1 + \epsilon \left[c_+(\delta \underline{y}) e^{ikx} + c_-(\delta \underline{y}) e^{-ikx} \right], \ \epsilon, \delta \ll 1, \ \underline{y} \in \mathbb{R}^{n-1} \tag{1171}$$

the above condition for a 1D NLS AW dynamics is equivalent to

$$t \sim \log \left(\frac{1}{\epsilon} \right) \ll \frac{1}{\delta^2}. \tag{1172}$$

Then the 1D theory of AWs illustrated in the previous section implies that the first appearance and the recurrence of AWs for times smaller than $\frac{1}{\delta^2}$ are described by the Akhmediev solution slowly varying in the extra space variables $\underline{y}$:

$$\begin{array}{l} \mathcal{A}(x, t; x_1, t_1) \equiv e^{2it} \left(1 + 2i \sin \phi_1 \frac{\sinh[\sigma(t - T(\delta \underline{y}))] - i \cos[k(x - X(\delta \underline{y})]}{\cosh[\sigma(t - T(\delta \underline{y}))] - \sin \phi \cos[k(x - X(\delta \underline{y})]} \right) \\ = e^{2it} \ \frac{\cosh[\sigma(t - T(\delta \underline{y})) + 2i\phi] + \sin \phi \cos[k(x - X(\delta \underline{y}))]}{\cosh[\sigma(t - T(\delta \underline{y}))] - \sin \phi \cos[k(x - X(\delta \underline{y}))]}, \\ k = 2 \cos \phi, \quad \sigma = k\sqrt{4 - k^2} = 2 \sin(2\phi), \quad 1 < k < 2, \ k = 2\pi / L_x. \end{array} \tag{1173}$$

More precisely, let

$$\begin{array}{l} \alpha(\delta \underline{y}) = e^{-i\phi_j} \overline{c_+(\delta \underline{y})} - e^{i\phi_j} c_-(\delta \underline{y}), \\ \beta(\delta \underline{y}) = e^{i\phi_j} \overline{c_-(\delta \underline{y})} - e^{-i\phi_j} c_+(\delta \underline{y}); \end{array} \tag{1174}$$

then the solution of the x-periodic Cauchy problem describes, in the case of one unstable mode $k = 2\pi/L_x$, an exact recurrence of Akhmediev breathers, whose parameters, changing at each appearance, are expressed in terms of the initial data via elementary functions. $T_1 = \frac{1}{\sigma} \log \left(\frac{\sigma^2}{2\epsilon |\alpha(\delta \underline{y})|} \right)$ is the first appearance time of the AW (the time at which the AW achieves the maximum of its modulus), $X_1 = \frac{\arg(\alpha(\delta \underline{y})) + \pi/2}{k}$, is the position of such a maximum,

$1 + 2\sin\phi$ is the value of the maximum,

$$\Delta T = \frac{2}{\sigma_1} \log \left(\frac{\sigma_1^2}{2\epsilon\sqrt{|\alpha(\delta \underline{y})\beta(\delta \underline{y})|}} \right), \tag{1175}$$

is the recurrence time (the time interval between two consecutive AW appearances),

$$\Delta X = \frac{\arg(\alpha(\delta \underline{y})\beta(\delta \underline{y}))}{k_1}, \quad \bmod L_x \tag{1176}$$

is the x-shift of the position of the maxima in the recurrence. In addition, after each appearance, the AW changes the background by the multiplicative phase factor $\exp(4i\phi)$ (see Fig. 50).

To illustrate the result, consider the simplest case of 2+1 dimensions ($n = 2$), even dependence on y, and $c_+ = c_{+0} f(\delta y),\ c_- = c_{-0} f(\delta y^2)$, with $0 < f(y) \le 1$. Then

$$\begin{array}{l} \alpha(y) = \alpha_0 f(y), \quad \alpha_0 = e^{-i\phi}\overline{c}_{+0} - e^{i\phi} c_{-0}, \\ X = \frac{\arg(\alpha_0) + \pi/2}{k}, \\ T_1(y) = T_0 + \frac{1}{\sigma} \log\left(\frac{1}{f(y)}\right), \quad T_0 = \frac{1}{\sigma}\log\left(\frac{\sigma^2}{2\epsilon|\alpha_0|}\right). \end{array} \tag{1177}$$

If, in particular, $f(y) = \exp(-b^2 y^2),\ b > 0$, then

$$T(y) = T_0 + \frac{b^2}{\sigma} y^2, \tag{1178}$$

and the first max is located at $x = X$ at $t = T(\delta y) = T_0 + b^2\delta^2 y^2/\sigma$. It describes a 2+1 dimensional AW, periodic in x and localized on the background like a gaussian in y. Its first max is located at $(x, y) = (X, 0)$ at $t = T_0$. At this time the AW undergoes a fission into two AWs whose maxima are located at $(X, \pm\frac{\sqrt{\sigma(t-T_0)}}{\delta b}),\ \ t \ge T_0$. The two products of the fission travel along the transversal y axis with speeds $\pm\frac{\sigma}{2\delta b\sqrt{\sigma(t-T_0)}}$; then their speed is infinite at the fission time T_0 and decreases to zero when $t \to \infty$. When $t - t_0 \gg 1$ the two fission products $u^\pm$ are well separated and described by

$$\begin{array}{l} u^\pm \sim \frac{\cosh\left(2\sqrt{\sigma(t-t_0)}\xi^\pm \mp 2i\phi\right) + \sin\phi\cos(k(x-x_0))}{\cosh\left(2\sqrt{\sigma(t-t_0)}\xi^\pm\right) - \sin\phi\cos(k(x-x_0))}, \\ \xi^\pm = y \mp \sqrt{\sigma(t-t_0)}, \quad t - t_0 \gg 1. \end{array} \tag{1179}$$

It follows that their width in the y direction decreases like $(t - t_0)^{-1/2}$ as t increases.

To show this shrinking we follow, say, u^+ introducing the variable $\xi^+ = y - \sqrt{\sigma(t-t_0)}$ for $t - t_0 \gg 1$. Then $\sigma(t-t_0) - y^2 \sim -2\sqrt{\sigma(t-t_0)}\xi^+$, and (1179) follows. To get u_-, we introduce the variable ξ^-, and proceed accordingly.

While the behavior of the fission products when $t - t_0 \gg 1$ depends on the slowly varying functions chosen in the initial condition, the proprties of the fission process are generic. Indeed consider any even function f of y with negative concavity at $y = 0$, and Taylor expansion $f(y) = 1 - b^2y^2 + O(y^4)$. Then $T(y) = T_0 + b^2y^2/\sigma + O(y^4)$, and again the speed of the two fission products is infinity at $t = T_0$, with the law $\pm\frac{1}{2b\sqrt{t-T_0}} + O(1)$. Therefore the fission process appears to be a universal critical phenomenon in nature, and resembles to a phase transition of type 2, with critical exponent $1/2$.

Three generalizations and their combinations are straightforward.

1) In the long wave (Peregrine) limit $\phi = \pi/2 + \delta$, $\delta \ll 1$ ($k \sim -2\delta$), and gaussian f, we get the Peregrine type solution

$$\mathcal{P}(x,y,t) = e^{2it}\left(1 - \frac{4 + 16i(t - T_0 - a^2y^2)}{1 + 4x^2 + 16(t - T_0 - a^2y^2)^2}\right), \tag{1180}$$

exhibiting the same fission properties as the Akhmediev one.

2) In the previous example we consider initial data for which $\arg(\alpha), \arg(\beta)$ do not depend on y. Then fission takes place on the trasversal y axis. If instead $\arg(\alpha), \arg(\beta)$ are slowly varying even functions of y, then the two fission products start moving on a small curvature parabola of the (x,y) plane. We study here this process on the very distinguished example in which $|c_+(y)| = |c_-(y)|$, then

$$\begin{array}{l} \alpha(y) = -2i|c_+(y)|\sin\left(\phi + \frac{\arg(c_+)+\arg(c_-)}{2}\right)e^{i\frac{\arg(c_-)-\arg(c_+)}{2}}, \\ \beta(y) = 2i|c_+(y)|\sin\left(\phi - \frac{\arg(c_+)+\arg(c_-)}{2}\right)e^{i\frac{\arg(c_+)-\arg(c_-)}{2}}, \\ \Rightarrow\ \alpha(y)\beta(y) = 4|c_+(y)|^2\sin\left(\phi - \frac{\arg(c_+)+\arg(c_-)}{2}\right)\sin\left(\phi + \frac{\arg(c_+)+\arg(c_-)}{2}\right) \\ = 2|c_+(y)|^2\left[\cos\left(\arg(c_+) + \arg(c_-)\right) - \cos(2\phi)\right]. \end{array} \tag{1181}$$

Consequently $\alpha\beta \in \mathbb{R}$ and

$$\begin{array}{l} \alpha\beta > 0 \ \Leftrightarrow\ arg(\alpha\beta) = 0 \ \Leftrightarrow\ \Delta X = 0, \\ \alpha\beta < 0 \ \Leftrightarrow\ arg(\alpha\beta) = \pi \ \Leftrightarrow\ \Delta X = \frac{L_x}{2}. \end{array} \tag{1182}$$

Both $|\alpha|$ and $\arg \alpha$ depend on y:

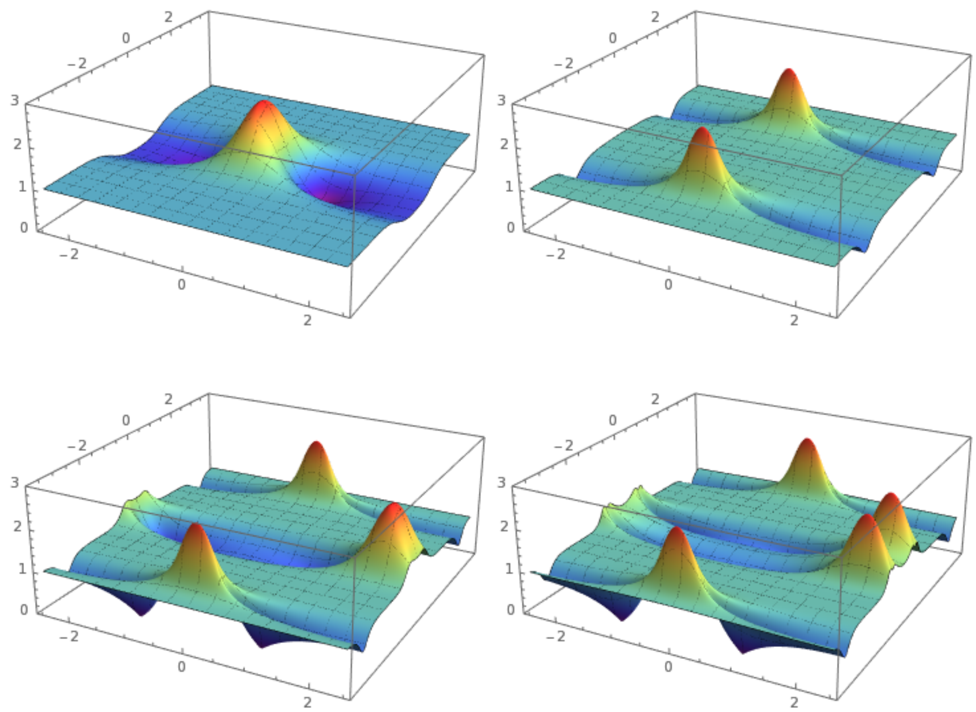
$$\begin{array}{l} |\alpha(y)| = 2|c_+(y)|| \sin\left(\phi + \frac{arg(c_+(y)c_-(y))}{2}\right)|, \\ arg(\alpha(y)) = arg\left(\frac{c_-(y)}{c_+(y)}\right) - \frac{\pi}{2} sign\left(\sin\left(\phi + \frac{\arg(c_+(y)c_-(y))}{2}\right)\right), \end{array} \tag{1183}$$

where $H(\cdot)$ is the step function. Consequently, the first appearance time and the position of the AW are generically slowly varying functions of y. In particular, if $arg(c_+(y)c_-(y)) = \theta$ independent of y, then

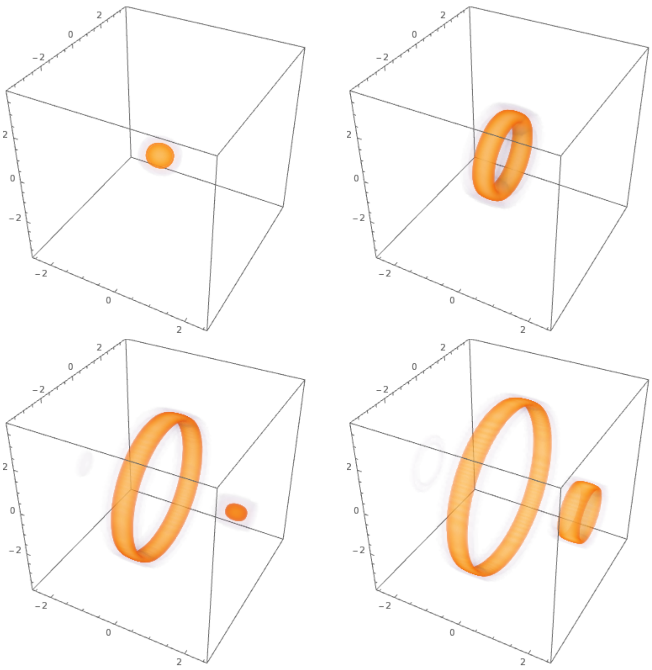
$$\begin{array}{l} |\alpha(y)| = 2|c_+(y)|| \sin\left(\phi + \theta/2\right)|, \\ arg\left(\alpha(y)\right) = -2arg(c_+(y)) + \pi H\left(-\sin(\phi + \theta/2)\right). \end{array} \tag{1184}$$

If again $arg(c_\pm(y))$ are even functions of y, then the two fission products will separate in the (x, y) plane approximately on a small curvature parabola ...

3) The case of $n+1$ dimensional Q1D AWs with the choice (1177) $f(y) \to f(\vec{y}) = \exp(-\sum\limits_{j=1}^{n-1} b_j^2 y_j^2)$. In this case fission takes place on the ellipsoid $\sigma(t - T_0) = \sum\limits_{j=1}^{n-1} b_j^2 y_j^2$, $t \geq T_0$ of the $(n-1)$-dimensional transversal hyperplane. CONSTRUCT THE SLOWLY VARYING AKHMEDIEV AND PEREGRINE SOLUTIONS IN THIS CASE (see figures B).



Figures A. Four snapshots at consecutive times illustrating the AW recurrence and fission in 2+1 dimensions. Top left: the first appearance of the Q1D AW; in the x main direction it exhibits the usual one dimensional RW features: a steep elevation (for with $|\varphi| > 1$) preceeded and followed by two holes (for with $|\varphi| >< 1$). The localization in the y direction is a simple consequence of the y localization of the initial data. Due to the existence of the additional y transversal direction, instead of desappearing, the Q1D AW experiences a fission into two Q1D AWs separating in the y direction (a transversal particle fission). Bottom left: while the two products of the first fission keep separating, the second appearance of the recurrence occurs. Bottom right: also the second AW undergoes a fission, with the same features.



Figures B. Four snapshots at consecutive times illustrating the AW recurrence and fission of Q1D AWs in 3+1 dimensions, when $b_j = 1$. The x axis is that orthogonal to the plane of the circular smoke rings (the (y, z) plane). Top left: the first appearance of the Q1D AW in 3D space; in the x main direction it exhibits the usual RW picture: an intense color preceeded and followed by two holes (here we chose to draw only the parts of the perturbation such that $|\varphi| > 1$ (the holes do not appear), to make the picture more clear. Top right: fission of the Q1D AW in the transversal (y, z) plane, generating an opening smoke ring shaped Q1D AW. Bottom left: while the smoke ring of the first fission evolves increasing its radius, the second appearance of the recurrence occurs. Bottom right: also the second Q1D AW undergoes a fission, generating another smoke ring. If $arg\,(\alpha(\vec{y})) = b_0 + b_1^2 y_1^2 + b_2^2 y_2^2 + \ldots$, then the smoke rings open up initially on a small curvature paraboloid whose axis is the x axis.

9.8.3 Modulation instability in $2+1$ dimensions

The elliptic NLS equation with saturation potential. For the elliptic NLS equation (1163) we look for a perturbation of the background solution:

$$A = ae^{i\varphi(|a|^2)t}[1 + \epsilon(u + iv)], \quad u, v \in \mathbb{R}, \quad 0 < \epsilon \ll 1. \tag{1185}$$

Then the linearized equations about the background solution read

$$u_t + v_{xx} + v_{yy} = 0, \quad v_t - u_{xx} - u_{yy} - 2|a|^2\varphi'(|a|^2)u = 0. \tag{1186}$$

If the perturbation is monochromatic:

$$u = Ue^{i(kx+ly)+\sigma t} + cc, \quad v = Ve^{i(kx+ly)+\sigma t} + cc, \quad k, l \in \mathbb{R}, \tag{1187}$$

then

$$\begin{pmatrix} \sigma & -(k^2 + l^2) \\ k^2 + l^2 - 2|a|^2\varphi'(|a|^2) & \sigma \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \underline{0}, \tag{1188}$$

The existence of nontrivial solutions ($U, V \neq 0$) implies that

$$\sigma^2 = (k^2 + l^2)[2|a|^2\varphi'(|a|^2) - (k^2 + l^2)], \quad \sigma^2 = \bar{\sigma}^2, \tag{1189}$$

implying that σ is either real or purely imaginary. In the first case

$$k^2 + l^2 > 2|a|^2\varphi'(|a|^2) \;\Leftrightarrow\; \sigma \in i\mathbb{R} \;\Leftrightarrow\; \text{neutral stability}, \tag{1190}$$

and in the second case

$$k^2 + l^2 < 2|a|^2\varphi'(|a|^2) \;\Leftrightarrow\; \sigma \in \mathbb{R} \;\Leftrightarrow\; \text{instability}. \tag{1191}$$

Therefore the background is linearly unstable when the length of the wave vector of the monochromatic perturbation is smaller than $|a|\sqrt{2\varphi'(|a|^2)}$, and the growth rate reads

$$\sigma(k, l) = \sqrt{(k^2 + l^2)[2|a|^2\varphi'(|a|^2) - (k^2 + l^2)]}, \tag{1192}$$

see Figures 57 and 58.

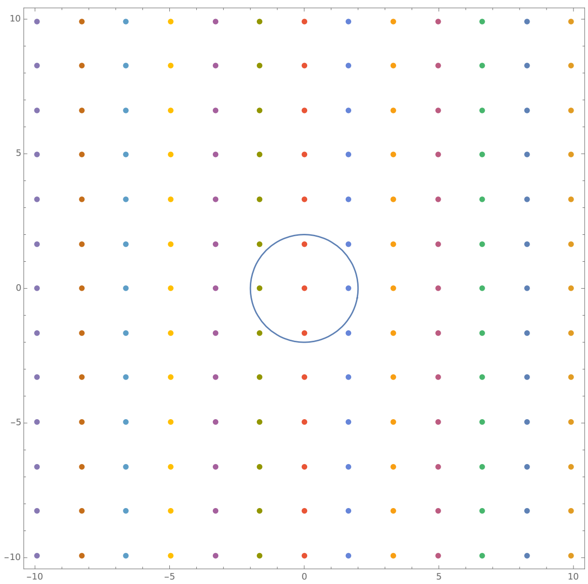


Figure 57: For the elliptic 2D NLS equation, the instability region in the $\vec{k} = (k, l)$ plane is bounded by the circle $k^2 + l^2 = 2|a|^2\varphi'(|a|^2)$. The Fourier modes of the linearized theory are $\vec{k}_{m,n} = 2\pi(\frac{m}{L_x}, \frac{n}{L_y})$, where $m, n \in \mathbb{Z}$, and L_x and L_y are respectively the periods in the x and y directions. In this picture there are only the unstable modes $\pm\vec{k}_{1,0}, \pm\vec{k}_{0,1}$, with $\vec{k}_{-m,-n} = -\vec{k}_{m,n}$.

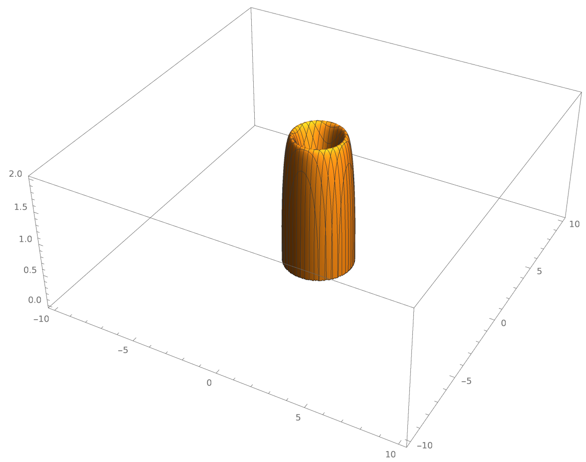


Figure 58: For the cubic elliptic 2D NLS equation ($\varphi(\zeta) = 2\zeta$), the instability circle $k^2 + l^2 = 4|a|^2$ and the corresponding growth rate $\sigma(k, l) = \sqrt{(k^2 + l^2)[4|a|^2 - (k^2 + l^2)]}$. $\sigma(k, l)$ is zero at the origin $\vec{k} = \vec{0}$ and at the boundary $|\vec{k}| = 2|a|$.

The hyperbolic cubic 2D NLS equation. For the hyperbolic NLS equation (1165), repeating the above considerations, one obtains the following scenario.

$$\begin{array}{ccc} \text{neutral stability} & \Leftrightarrow & \left\{ \begin{array}{ll} i) & k^2 > l^2 + 4|a|^2, \\ ii) & k^2 < l^2, \end{array} \right. , \\ \text{instability} & \Leftrightarrow & 0 < k^2 - l^2 < 4|a|^2. \end{array} \tag{1193}$$

The instability region in the $\vec{k} = (k, l)$ plane is bounded by the hyperbola $k^2 - l^2 = 4$ and by its asymptotes $k \pm l = 0$, and the growth rate reads

$$\sigma(k, l) = \sqrt{(k^2 - l^2)[4|a|^2 - (k^2 - l^2)]}, \tag{1194}$$

see Figures 59 and 60:

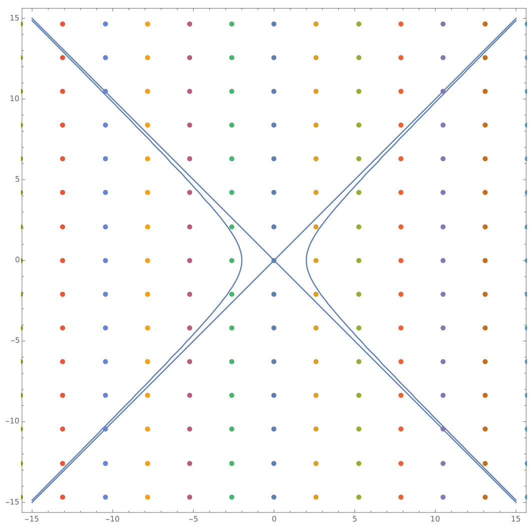


Figure 59: For the hyperbolic 2D NLS equation, the instability region in the $\vec{k} = (k, l)$ plane is bounded by the hyperbola $k^2 - l^2 - 4|a|^2 = 0$ and by its asymptotes $k \pm l = 0$. The Fourier modes of the linearized theory are $\vec{k}_{m,n} = 2\pi(\frac{m}{L_x}, \frac{n}{L_y})$, where $m, n \in \mathbb{Z}$, and L_x and L_y are respectively the periods in the x and y directions. In this graph $L_x = 2.4$, $L_y = 3.0$, and there are only the unstable modes $\pm\vec{k}_{1,1}$, $\pm\vec{k}_{1,-1}$, with $\vec{k}_{-m,-n} = -\vec{k}_{m,n}$.

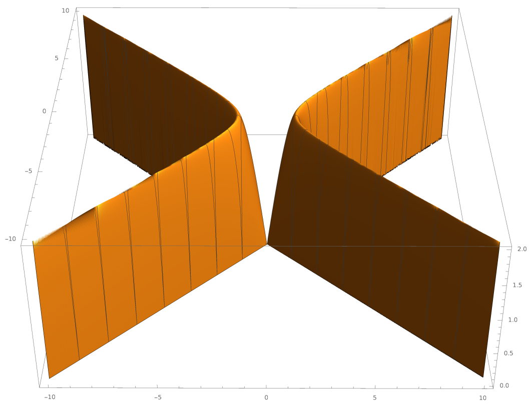


Figure 60: For the hyperbolic 2D NLS equation, the instability region in the (k, l) plane, and the growth rate $\sigma(k, l) = \sqrt{(k^2 - l^2)[4|a|^2 - (k^2 - l^2)]}$. $\sigma(k, l)$ is zero on the hyperbola and on it asymptotes.

The DS equations. Analogous considerations can be made for the focusing DS equations

$$\begin{array}{l} iA_t + A_{xx} \pm A_{yy} + 2qA = 0, \quad q_{xx} \mp \mu^2 q_{yy} = (|A|^2)_{xx} \pm \nu^2 (|A|^2)_{yy}, \\ A = A(x, y, t) \in \mathbb{C}, \quad q = q(x, y, t) \in \mathbb{R}, \quad \mu, \nu \in \mathbb{R}, \end{array} \tag{1195}$$

where DS1 and DS2 correspond respectively to the upper and lower signs, integrable if and only if $\mu = \nu = 1$.

Their homogeneous background solution

$$A_0 = a e^{2i|a|^2 t}, \quad q_0 = |a|^2 \tag{1196}$$

can be simplified to $A_0 = e^{2it}$, $q_0 = 1$ using the elementary gauge and scaling symmetries. Since equations (1195) are also invariant under the Coulomb gauge:

$$A(x, y, t) \to A(x, y, t) e^{2i \int_0^t f(t') dt'}, \quad q(x, y, t) \to q(x, y, t) + f(t), \tag{1197}$$

the background solution can be simplified further to

$$A_0 = 1, \quad q_0 = 0. \tag{1198}$$

To study the modulation instability properties of the background solution (1198), we slightly perturb it as follows

$$A = 1 + \epsilon(u + iv), \quad q = \epsilon w, \quad \epsilon \ll 1, \; u, v, w \in \mathbb{R}. \tag{1199}$$

Then u, v, w satisfy the linear PDEs

$$\begin{array}{l} u_t + v_{xx} \pm v_{yy} = 0, \quad v_t - (u_{xx} \pm u_{yy}) - 2w = 0, \\ w_{xx} \mp \mu^2 w_{yy} = 2(u_{xx} \pm \nu^2 v_{yy}). \end{array} \tag{1200}$$

Looking for a monochromatic perturbation

$$\begin{array}{l} u = U e^{i(kx+ly)+\sigma t} + cc, \; v = V e^{i(kx+ly)+\sigma t} + cc, \\ w = W e^{i(kx+ly)+\sigma t} + cc, \quad k, l \in \mathbb{R}, \end{array} \tag{1201}$$

one obtains the following system of homogeneous equations

$$\begin{pmatrix} \sigma & -(k^2 \pm l^2) & 0 \\ k^2 \pm l^2 & \sigma & -2 \\ 2(k^2 \pm \nu^2 l^2) & 0 & -(k^2 \mp \mu^2 l^2) \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix} = 0, \tag{1202}$$

and the condition for the existence of nontrivial solutions gives

$$\sigma_{\pm}^2(k, l) = \frac{(k^2 \pm l^2)[4(k^2 \pm \nu^2 l^2) - (k^2 \pm l^2)(k^2 \mp \mu^2 l^2)]}{k^2 \mp \mu^2 l^2}, \tag{1203}$$

reducing to

$$\sigma_{\pm}^2(k, l) = \frac{(k^2 \pm l^2)^2 [4 - (k^2 \mp l^2)]}{k^2 \mp l^2} \tag{1204}$$

in the integrable cases $\mu = \nu = 1$. Since we have MI when $\sigma^2 > 0$, we distinguish the following cases.

1) For DS1 (the upper sign case), we have (SHOW IT) the MI domain $Inst_{DS1}$ is the domain in between the straight lines $k^2 = \mu^2 l^2$ and the curves $4(k^2 + \nu^2 l^2) = (k^2 + l^2)(k^2 - \mu^2 l^2)$, having the straight lines $k^2 = \mu^2 l^2$ as asymptotes (see figures ...)

$$Inst_{DS1} = \{k^2 > \mu^2 l^2 \text{ and } 4(k^2 + \nu^2 l^2) > (k^2 + l^2)(k^2 - \mu^2 l^2)\}. \tag{1205}$$

2) For DS2 (the lower sign case), we have the union of two disjoint domains $Inst_{DS2} = Inst^+_{DS2} \cup \; Inst^-_{DS2}$ (see figure ...):

$$\begin{array}{l} Inst^+_{DS2} = \{k^2 > l^2 \text{ and } 4(k^2 - \nu^2 l^2) > (k^2 - l^2)(k^2 + \mu^2 l^2)\}, \\ Inst^-_{DS2} = \{k^2 < l^2 \text{ and } 4(k^2 - \nu^2 l^2) < (k^2 - l^2)(k^2 + \mu^2 l^2)\}. \end{array} \tag{1206}$$

In the integrable DS2 case it becomes the compact domain

$$k^2 + l^2 < 4, \quad \text{and} \quad k^2 \neq l^2, \tag{1207}$$

with growth rate

$$\sigma(k, l) = |\Omega(k, l)|, \quad \Omega(k, l) = \frac{(k^2 - l^2)\sqrt{4 - (k^2 + l^2)}}{\sqrt{k^2 + l^2}}. \tag{1208}$$

Therefore AWs are present in the focusing DS2 equation for sufficiently small wave vectors $\vec{k}$, in perfect analogy with the NLS case (see Figure 61).

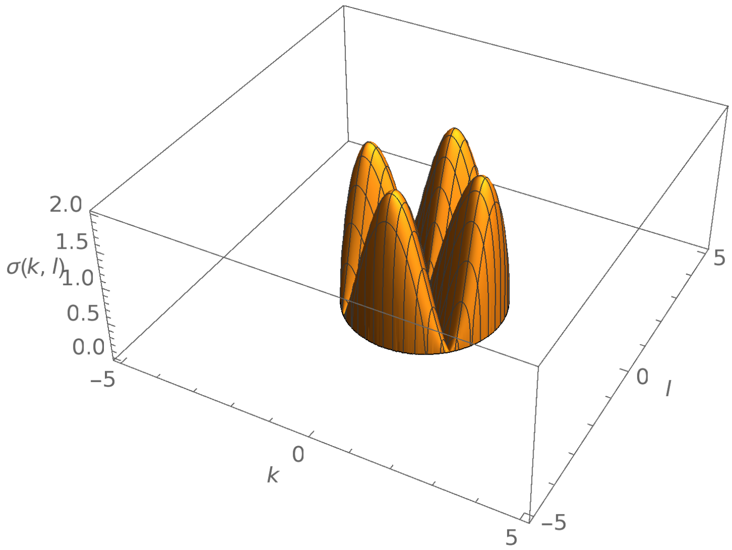


Figure 61: For the DS2 equation, the growth rate $\sigma(k,l)$ in the $\vec{k}=(k,l)$ plane; the instability region is inside the circle $k^2+l^2=4|a|^2$; it is zero outside and on the lines $k^2-l^2=0$. Here $a=1$.

9.8.4 The integrable DS2 case [20]

In the well-posed doubly-periodic DS2 Cauchy problem of AWs with periods L_x and L_y, the wave vectors of the above N-breather solution are quantized as follows

$$\vec{k}_{m,n}=(k_m,l_n),\quad k_m=\tfrac{2\pi}{L_x}m,\ l_n=\tfrac{2\pi}{L_y}n,\quad m,n\in\mathbb{Z},\quad L_x\neq L_y, \tag{1209}$$

and lie on the rectangular lattice of Figure 62, constrained by the instability condition

$$k_m^2+l_n^2<4\quad\Leftrightarrow\quad\left(\frac{m}{L_x}\right)^2+\left(\frac{n}{L_y}\right)^2<\frac{1}{\pi^2},\quad L_x\neq L_y. \tag{1210}$$

The simplest possible instability configurations are, in order of complication, the following.
1) The case in which there is only one unstable mode, the mode $\pm\vec{k}_{1,0}=\pm(k_1,0)$ on the k axis, with:

$$1<k_1<2,\quad l_1>2\quad\Leftrightarrow\quad\pi<L_x<2\pi,\quad L_y<\pi, \tag{1211}$$

or the mode $\pm\vec{k}_{0,1} = \pm(0, l_1)$ on the l axis, with:

$$1 < l_1 < 2, \quad k_1 > 2 \quad \Leftrightarrow \quad \pi < L_y < 2\pi, \quad L_x < \pi; \tag{1212}$$

see respectively the top left and top right pictures of Figure 62.
2) The case in which there are only the two unstable modes $\pm\vec{k}_{1,0}, \pm\vec{k}_{0,1}$, with

$$1 < k_1, l_1 < 2, \;\; k_1^2 + l_1^2 > 4 \quad \Leftrightarrow \quad \pi < L_x, L_y < 2\pi, \;\; \frac{1}{L_x^2} + \frac{1}{L_y^2} > \frac{1}{\pi^2}; \tag{1213}$$

see the bottom left picture of Figure 62.
3) The case in which there are only the four unstable modes $\pm\vec{k}_{1,0}$,$\pm\vec{k}_{0,1}$,$\pm\vec{k}_{1,1}$, $\pm\vec{k}_{1,-1}$, with

$$1 < k_1, l_1 < 2, \;\; k_1^2 + l_1^2 < 4 \quad \Leftrightarrow \quad \pi < L_x, L_y < 2\pi, \;\; \frac{1}{L_x^2} + \frac{1}{L_y^2} < \frac{1}{\pi^2}; \tag{1214}$$

see the bottom right picture of Figure 62. Increasing the periods L_x and L_y, higher order modes enter the instability region and the picture becomes more and more complicated. In this paper we limit our considerations to the first two cases 1) and 2), postponing to a subsequent paper the study of a higher number of unstable modes.

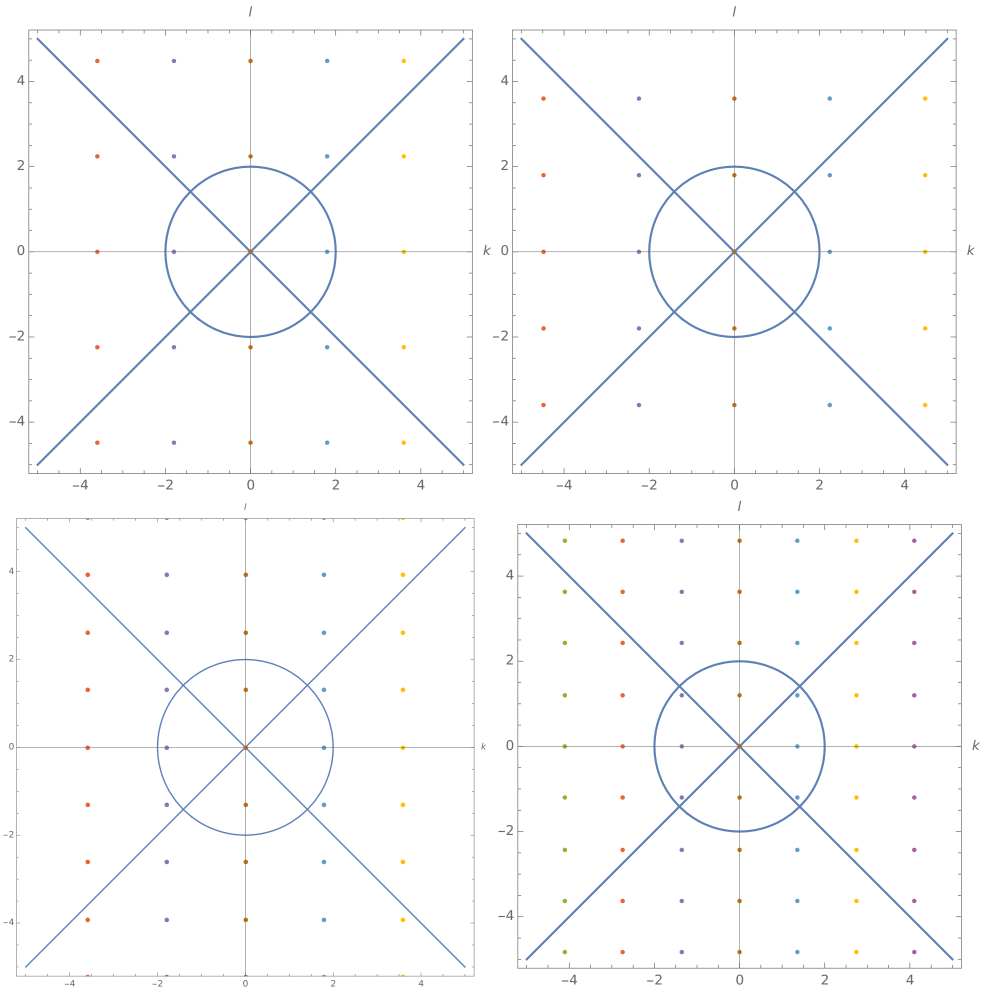


Figure 62: For the focusing DS2 equation, the instability region in the $\vec{k} = (k, l)$ plane consists of the 4 sectors inside the circle $k^2 + l^2 = 4|a|^2$ and delimited by the lines $k^2 = l^2$. In the doubly periodic case, the Fourier modes of the linearized theory are $\pm\vec{k}_{m,n}$ in (1209); hereafter $a = 1$. In the top left picture $L_x = 3.5,\ L_y = 2.8$, and there is only the unstable mode $\pm\vec{k}_{1,0}$. In the top right picture $L_x = 2.8,\ L_y = 3.5$, and there is only the unstable mode $\pm\vec{k}_{0,1}$. In the bottom left picture $L_x = 3.5,\ L_y = 4.8$, and there are only the 2 unstable modes $\pm\vec{k}_{1,0}, \pm\vec{k}_{0,1}$. Increasing more the periods one jumps from the two unstable modes $\pm\vec{k}_{1,0}, \pm\vec{k}_{0,1}$ directly to the four unstable modes $\pm\vec{k}_{1,0}, \pm\vec{k}_{0,1}, \pm\vec{k}_{1,1}, \pm\vec{k}_{1,-1}$, like in the bottom right picture, where $L_x = 4.6,\ L_y = 5.2$.

The simplest truly two dimensional AW describes the interaction of the horizontal $\pm\vec{k}_{1,0} = (k_1, 0)$ and vertical $\pm\vec{k}_{0,1} = (0, l_1)$ unstable modes, where

$$k_1 = \frac{2\pi}{L_x} = 2\cos\phi_{1,0}, \quad l_1 = \frac{2\pi}{L_y} = 2\cos\phi_{0,1}, \quad \theta_{1,0} = 0, \ \theta_{0,1} = \pi/2, \tag{1215}$$

corresponding to the conditions

$$\pi < L_x, L_y < 2\pi \ \Leftrightarrow \ 1 < k_1, l_1 < 2 \ \Leftrightarrow \ 0 < \phi_{1,0}, \phi_{0,1} < \pi/3. \tag{1216}$$

Then the solution reads:

$$u_2(x, y, t; \phi_{1,0}, \phi_{0,1}, x_0, y_0, t_{1,0}, t_{0,1}, \rho) = \frac{N(x, y, t)}{D(x, y, t)} e^{i\rho}, \tag{1217}$$

$$\begin{array}{l} N(x,y,t) = \cosh\left[\sigma_{1,0}(t - t_{1,0}) + \sigma_{0,1}(t - t_{0,1}) + 2i(\phi_{1,0} - \phi_{0,1}))\right] \\ +b_{12}^2 \cosh\left[\sigma_{1,0}(t - t_{1,0}) - \sigma_{0,1}(t - t_{0,1}) + 2i(\phi_{1,0} + \phi_{0,1}))\right] \\ -2b_{12}\Big(\sin\phi_{1,0}\cos(X_{1,0})\cosh\left[\sigma_{0,1}(t - t_{0,1}) - 2i\phi_{0,1}\right] \\ + \sin\phi_{0,1}\cos(Y_{0,1})\cosh\left[\sigma_{1,0}(t - t_{1,0}) + 2i\phi_{1,0}\right] + \sin\phi_{1,0}\sin\phi_{0,1}\cos(X_{1,0})\cos(Y_{0,1})\Big), \end{array} \tag{1218}$$

$$\begin{array}{l} D(x,y,t) = \cosh\left[\sigma_{1,0}(t - t_{1,0}) + \sigma_{0,1}(t - t_{0,1})\right] \\ +b_{12}^2 \cosh\left[\sigma_{1,0}(t - t_{1,0}) - \sigma_{0,1}(t - t_{0,1}))\right] \\ +2b_{12}\Big(\sin\phi_{1,0}\cos(X_{1,0})\cosh\left[\sigma_{0,1}(t - t_{0,1})\right] \\ + \sin\phi_{0,1}\cos(Y_{0,1})\cosh\left[\sigma_{1,0}(t - t_{1,0})\right] - \sin\phi_{1,0}\sin\phi_{0,1}\cos(X_{1,0})\cos(Y_{0,1})\Big), \end{array} \tag{1219}$$

where

$$\begin{array}{l} X_{1,0} = k_1(x - x_0) = 2\cos(\phi_{1,0})(x - x_0), \quad Y_{0,1} = l_1(y - y_0) = 2\cos(\phi_{0,1})(y - y_0), \\ \sigma_{1,0} = k_1\sqrt{4 - k_1^2} = 2\sin(2\phi_{1,0}), \quad \sigma_{0,1} = l_1\sqrt{4 - l_1^2} = 2\sin(2\phi_{0,1}) = -\Omega_{0,1}, \\ b_{12} = \frac{\cos(\phi_{1,0} - \phi_{0,1})}{\cos(\phi_{1,0} + \phi_{0,1})}, \end{array} \tag{1220}$$

and $\rho, x_0, y_0, t_{1,0}, t_{0,1}$ are arbitrary real parameters.

For generic parameters, the solution (1217)-(1220) decays to the background (953) as $t \to \pm\infty$, and describes two consecutive appearances in time of $2+1$ dimensional doubly-periodic smooth bumps, both located at (x_0, y_0) (see Figure 63).

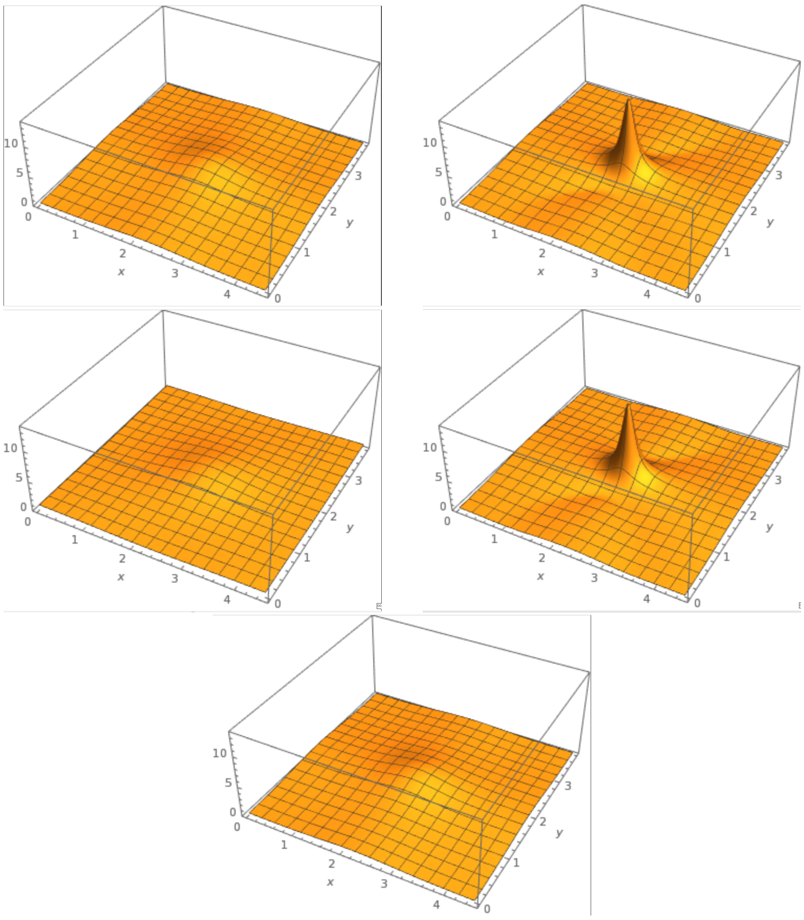


Figure 63: Five snapshots of the evolution of the 2-breather AW solution (1217) in a basic period, describing the nonlinear interaction of two unstable modes, one parallel to the x axis and the other parallel to the y axis. Top left: the growth of the AW from the background; top right: the first appearance; medium left: between two appearances; medium right: second appearance; bottom: the disappearance of the AW. For generic parameters, the solution is smooth and the two appearances are different.

To show the relevance of this solution in a Cauchy problem of anomalous waves, we study now the doubly periodic Cauchy problem of AWs for the focusing DS2 equation (1168):

$$\begin{array}{l} u(x+L_x,y,t)=u(x,y+L_y,t)=u(x,y,t), \\ q(x+L_x,y,t)=q(x,y+L_y,t)=q(x,y,t), \\ u(x,y,0)=1+\epsilon\ v(x,y), \quad q(x,y,0)=\epsilon\ w(x,y), \quad 0\le\epsilon\ll 1, \end{array} \tag{1221}$$

where the initial perturbations can be expanded in Fourier modes as follows:

$$v(x,y) = \sum_{\mu,\nu\in\mathbb{Z}} c_{\mu,\nu} e^{i(k_\mu x + l_\nu y)}, \tag{1222}$$

and $c_{0,0}$ is set to be zero without loss of generality using the scaling symmetry.

As in the $1+1$ dimensional case, for $|t| \le O(1)$, the solution is described, through Fourier analysis and up to $O(\epsilon^2)$ terms, by the following formulas

$$u(x,y,t) = 1 + \epsilon \sum_{m,n\in\mathcal{D}} \Big(\tfrac{|\alpha_{m,n}|}{\sin(2\phi_{m,n})} \cos\left(k_m x + l_n y - \arg(\alpha_{m,n}) - \pi/2\right) e^{\Omega_{m,n}t + i\phi_{m,n}}$$
$$+ \tfrac{|\beta_{m,n}|}{\sin(2\phi_{m,n})} \cos\left(k_m x + l_n y + \arg(\beta_{m,n}) - \pi/2\right) e^{-\Omega_{m,n}t - i\phi_{m,n}} \Big) + O(\epsilon)\text{-oscillations}, \tag{1223}$$

$$q(x,y,t) = \epsilon \sum_{m,n\in\mathcal{D}} \tfrac{\cos(2\theta_{m,n})}{\sin(\phi_{m,n})} \Big[|\alpha_{m,n}| \cos\left(k_m x + l_n y - \arg(\alpha_{m,n}) - \pi/2\right) e^{\Omega_{m,n}t}$$
$$+ |\beta_{m,n}| \cos\left(k_m x + l_n y + \arg(\beta_{m,n}) - \pi/2\right) e^{-\Omega_{m,n}t} \Big] + O(\epsilon)\text{-oscillations}, \tag{1224}$$

where

$$\begin{array}{l} k_m = 2\cos\phi_{m,n}\cos\theta_{m,n}, \quad l_n = 2\cos\phi_{m,n}\sin\theta_{m,n}, \\ \Rightarrow \quad \phi_{m,n} = \arccos\left(\frac{\sqrt{k_m^2 + l_n^2}}{2}\right), \quad \theta_{m,n} = \arctan\left(\frac{l_n}{k_m}\right), \\ \alpha_{m,n} = e^{-i\phi_{m,n}} \bar{c}_{m,n} - e^{i\phi_{m,n}} c_{-m,-n}, \\ \beta_{m,n} = e^{i\phi_{m,n}} \bar{c}_{-m,-n} - e^{-i\phi_{m,n}} c_{m,n}, \end{array} \tag{1225}$$

and

$$\mathcal{D} = \left\{ m \ge 1,\ n \in \mathbb{Z},\ \left(\tfrac{m}{L_x}\right)^2 + \left(\tfrac{n}{L_y}\right)^2 < \tfrac{1}{\pi^2} \right\} \cup \left\{ m = 0,\ n \ge 1,\ \left(\tfrac{n}{L_y}\right)^2 < \tfrac{1}{\pi^2} \right\}. \tag{1226}$$

As time increases, the perturbation in (1223) grows exponentially and, at $t = O(\log(1/\epsilon))$, it becomes order one and the dynamics is described by the fully nonlinear theory. It is when the exact solutions we constructed play a relevant role.

From now on we consider the case in which only the modes $\pm\vec{k}_{1,0}$ and $\pm\vec{k}_{0,1}$ are unstable, where $\vec{k}_{1,0} = (k_1, 0)$ and $\vec{k}_{0,1} = (0, l_1)$, $k_1 = \frac{2\pi}{L_x}$, $l_1 = \frac{2\pi}{L_y}$ are unstable, see the bottom left picture in Figure 62. This case is characterized by the constraints

$$\begin{array}{l} \pi < L_x, L_y < 2\pi, \quad \frac{1}{L_x^2} + \frac{1}{L_y^2} > \frac{1}{\pi^2}, \ \Leftrightarrow \\ 1 < k_1, l_1 < 2, \qquad k_1^2 + l_1^2 > 4, \ \Leftrightarrow \\ 0 < \phi_{1,0}, \phi_{0,1} < \pi/3, \quad \cos^2\phi_{1,0} + \cos^2\phi_{0,1} > 1. \end{array} \tag{1227}$$

Then the linear stage of MI (1223), for $|t| \leq O(1)$, reduces to

$$
\begin{aligned}
& u(x,y,t) = 1 + \epsilon \Big[\tfrac{1}{\sin(2\phi_{1,0})} \Big(|\alpha_{1,0}| \cos\left(2\cos\phi_{1,0} x - \arg(\alpha_{1,0}) - \pi/2\right) e^{i\phi_{1,0} + \sigma_{1,0} t} \\
& + |\beta_{1,0}| \cos\left(2\cos\phi_{1,0} x + \arg(\beta_{1,0}) - \pi/2\right) e^{-i\phi_{1,0} - \sigma_{1,0} t} \Big) \\
& + \tfrac{1}{\sin(2\phi_{0,1})} \Big[|\alpha_{0,1}| \cos\left(l_1 y - \arg(\alpha_{0,1}) - \pi/2\right) e^{-\sigma_{0,1} t + i\phi_{0,1}} \\
& + |\beta_{0,1}| \cos\left(l_1 y + \arg(\beta_{1,0}) - \pi/2\right) e^{\sigma_{0,1} t - i\phi_{0,1}} \Big] + O(\epsilon)\text{-oscillations}
\end{aligned}
\tag{1228}
$$

(notice that $\Omega_{1,0} = \sigma_{1,0}$ and $\Omega_{0,1} = -\sigma_{0,1}$).

Reasoning as for NLS, since the exact solution $u_2(x,y,t)$ of DS2 in (1217)-(1220) describes the nonlinear interaction of the unstable modes $\pm\vec{k}_{1,0}$, $\pm\vec{k}_{0,1}$, it is the natural candidate to characterize this nonlinear stage, and following exactly the same strategy as before, we find that the first AW appearance is described to leading order by the solution (1217)

$$
u(x,y,t) = u_2(x,y,t;\phi_{1,0},\phi_{0,1},x^{(1)},y^{(1)},t^{(1)}_{1,0},t^{(1)}_{0,1},\rho^{(1)}) + O(\epsilon), \tag{1229}
$$

where the solution parameters are expressed in terms of the initial data as follows

$$
\begin{aligned}
& \rho^{(1)} = 2\left(\phi_{1,0} - \phi_{0,1}\right), \; x^{(1)} = \tfrac{\arg(\alpha_{1,0}) + \pi/2}{k_1}, \; y^{(1)} = \tfrac{-\arg(\beta_{0,1}) + \pi/2}{l_1}, \\
& t^{(1)}_{1,0} = \tfrac{1}{\sigma_{1,0}} \log\left(\tfrac{2b_{12}\sin^2(2\phi_{1,0})}{\epsilon|\alpha_{1,0}|}\right), \; t^{(1)}_{0,1} = \tfrac{1}{\sigma_{0,1}} \log\left(\tfrac{2b_{12}\sin^2(2\phi_{0,1})}{\epsilon|\beta_{0,1}|}\right).
\end{aligned}
\tag{1230}
$$

Therefore the first appearance of the AW in the Cauchy problem consists of the two emergences described by the exact solution (1217)-(1220) (see Figure 63), whose parameters are expressed in terms of the initial data through elementary functions.

As for the case of one unstable mode, we remark that, since we have only two growing modes in the overlapping time region, and since the exact solution (1217)-(1220), describing the growth and the nonlinear interaction of these unstable modes, contains enough free parameters for a successful matching, the remaining mismatch cannot affect the leading order behavior. Therefore this stability argument plus uniqueness of the DS2 evolution imply that the first appearance of the AW is described by the solution (1229),(1230), an elementary function of the initial data.

To have an idea of how well the analytic solution u_2 in (1229),(1230) describe the first appearance of the AW in the AW Cauchy problem, we evaluate the uniform distance between u_2 and the numerical solution u_{num}:

$$
\|u_{num} - u_2\|_\infty(t) := \sup_{x\in[0,L_x],y\in[0,L_y]} |u_{num}(x,y,t) - u_2(x,y,t)|, \tag{1231}
$$

in the time interval in which the AW first appears, see Figure 64. The agreement is excellent, since the error is much smaller than expected from theoretical considerations.

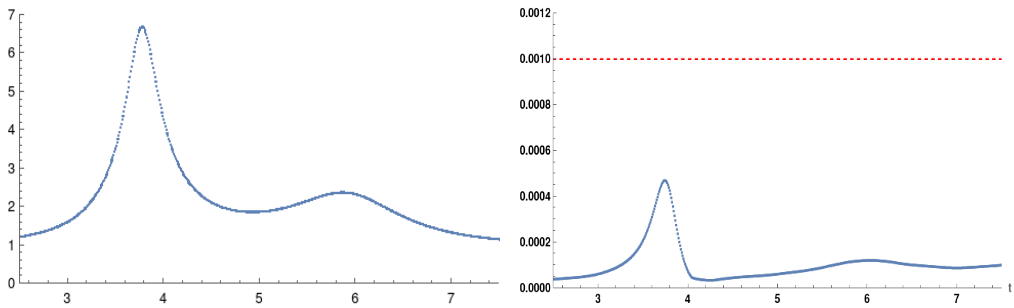


Figure 64: Here we study the two emergences of AWs in the time interval of the first appearance, for the initial data $\epsilon = 10^{-3}$, $c_{1,0} = 0.8 + i0.4$, $c_{-1,0} = 1.2 - i0.1$, $c_{0,1} = -0.64 - i0.3$, $c_{0,-1} = 0.5 + i0.2$. Left picture: the max of the amplitude of the AW $\|u_{num}\|_\infty(t)$ as function of time, where u_{num} is the numerical solution; the first emergence at $(x, y, t) = (3.24019, 1.227442, 3.780)$ with a peak of height 6.6786; the second emergence at $(x, y, t) = (3.24019, 1.227442, 5.868)$ with a peak of smaller height 2.3631. Right picture: the uniform distance $\|u_{num} - u_2\|_\infty(t)$ between the analytic solution u_2 (1229),(1230) and the numerical solution u_{num}; the two peaks of the distance correspond exactly to the two AW emergences of the left picture, and the distance remains always $\leq 5 \cdot 10^{-4}$, smaller than the estimated error from theoretical considerations $O(10^{-3})$, indicated by the horizontal dotted line.

9.9 Exercices

1) i) Verify that the formula (1004) describes to leading order the solution of the Cauchy problem of the anomalous waves (995),(996) for $0 \leq t = O(1)$, and ii) use the Akhmediev solution and matched asymptotic expansions to derive to leading order the NLS recurrence of anomalous waves in the case of one unstable mode only.

2) Derive formulas (1193) and (1194).

3) Using matched asymptotic expansions show that equations (1228) and (1217)-(1220) imply the first appearance of AWs described by (1229),(1230).

10 Appendix

10.1 A1. Similarity Solutions

Similarity solutions of a PDE are often relevant in the description of asymptotic regions of the solution space of particular importance, like the longtime behavior, the blow up, and the gradient catastrophe. They are special solutions of the form

$$u_{sim}(x,t) = \frac{1}{t^p}\, g\left(z\right), \quad z = \frac{x}{t^q}, \tag{1232}$$

where the parameters p, q and function g have to be fixed.

10.1.1 Similarity solutions of the Schrödinger equation

In the case of $iu_t + u_{xx} = 0$, VERIFY that $q = 1/2$, $\forall p$, and $g(z)$ satisfies the ordinary differential equation (ODE)

$$g'' - i(pg + \frac{z}{2}g') = 0. \tag{1233}$$

Choosing $p = 1/2$, then (1233) simplifies to

$$g'' - \frac{i}{2}(zg)' = 0, \tag{1234}$$

whose general solution is

$$g(z) = c_1 \int^z e^{\frac{i}{4}(z^2 - z'^2)} dz' + c_2 e^{iz^2/4}. \tag{1235}$$

At last, if $c_1 = 0$, one obtains the simple similarity solution

$$u_{sim}(x,t) = \frac{1}{\sqrt{t}} e^{i\frac{x^2}{4t}} \tag{1236}$$

10.1.2 Similarity solutions of the linearized KdV equation

In the case of of the linear KdV equation $u_t + u_{xxx} = 0$, VERIFY that (1232) is solution if $q = 1/3$, $\forall p$, and $g(z)$ satisfies the ODE

$$g'''(z) - \frac{z}{3}g'(z) - pg(z) = 0. \tag{1237}$$

If $p = 1/3$, then the equation simplifies to

$$(g''(z) - \frac{1}{3}(zg(z)))' = 0 \;\Rightarrow\; g''(z) - \frac{1}{3}(zg(z)) = c. \tag{1238}$$

Choosing the integration constant $c = 0$, we obtain $g''(z) - \frac{z}{3}g(z) = 0$, and changing variable: $z = \sqrt[3]{3}\xi$, the ODE becomes the ODE (39) defining the Airy functions Ai and Bi. Therefore the bounded and decaying similarity solution is

$$u_{sim}(x,t) = \frac{1}{\sqrt[3]{t}} Ai\left(\frac{x}{\sqrt[3]{3t}}\right). \tag{1239}$$

10.2 A2. Integrals depending on a large parameter: integration by parts, Laplace and the Steepest Descent Methods [36, 9, 29]

We want to estimate the leading order contribution of the Laplace integral

$$L(p) = \int_a^b e^{pf(t)} g(t) dt \tag{1240}$$

where p is a positive large parameter $p \gg 1$. We distinguish three different cases.

10.2.1 Integration by parts

If $a < b$, $f(t), g(t) : \mathbb{R} \to \mathbb{R}$ are smooth functions and $f'(t) \neq 0$ in $[a, b]$, the main contribution comes from the integration by parts:

$$\begin{aligned} L(p) &= \int_a^b \left(pf'(t)e^{pf(t)}\right)\left(\tfrac{g(t)}{pf'(t)}\right)dt = \int_a^b \left(e^{pf(t)}\right)'\left(\tfrac{g(t)}{pf'(t)}\right)dt \\ &= \tfrac{1}{p}\left(\tfrac{g(b)}{f'(b)}e^{pf(b)} - \tfrac{g(a)}{f'(a)}e^{pf(a)}\right) - \tfrac{1}{p}\int_a^b e^{pf(t)}\left(\tfrac{g(t)}{f'(t)}\right)' dt \\ &= \tfrac{1}{p}\left(\tfrac{g(b)}{f'(b)}e^{pf(b)} - \tfrac{g(a)}{f'(a)}e^{pf(a)}\right)\left(1 + O(\tfrac{1}{p})\right), \quad p \gg 1. \end{aligned}$$

In the first step we construct $(pf'(t)e^{pf(t)}) = d(e^{pf(t)})/dt$; in the second step we integrate by parts. Assuming that $f'(t), g(t)$ be smooth functions, we repeat the same integration by parts, arriving to the above estimate.

The same result holds if one integrates over a contour $\mathcal{C}$ of the complex z plane with end points $a, b \in \mathbb{C}$, where $f(z), g(z)$ are analytic in a domain $\mathcal{D}$ containing $\mathcal{C}$ and $f'(z) \neq 0$ in $\mathcal{D}$:

$$\begin{aligned} I(p) &= \int_a^b e^{pf(z)} g(z) dz \\ &= \tfrac{1}{p}\left(\tfrac{g(b)}{f'(b)}e^{pf(b)} - \tfrac{g(a)}{f'(a)}e^{pf(a)}\right)\left(1 + O(\tfrac{1}{p})\right), \quad p \gg 1. \end{aligned} \tag{1241}$$

10.2.2 The Laplace method

We consider the Laplace integral (1240) with $a < b$, and $f(t) \in \mathbb{R}$ has now a single max in $t_0 \in (a, b)$.

If $p \gg 1$, the leading order contribution to the integral comes from a neighborhood of t_0, where:

$$\begin{array}{l} f(t) = f(t_0) + \frac{f''(t_0)}{2}(t-t_0)^2 + O(t-t_0)^3, \quad f''(t_0) < 0, \\ g(t) = g(t_0) + O(t-t_0). \end{array} \tag{1242}$$

Therefore, following the same qualitative considerations made in the case of the stationary phase method, we construct the leading order contribution

$$\begin{aligned} L(p) &\sim g(t_0)e^{pf(t_0)} \int\limits_{t_0-\epsilon}^{t_0+\epsilon} e^{-p|f''(t_0)|\frac{(t-t_0)^2}{2}} dt \\ &= \frac{\sqrt{2}g(t_0)e^{pf(t_0)}}{\sqrt{p|f''(t_0)|}} \int\limits_{-\sqrt{\frac{p|f''(t_0)|}{2}}\epsilon}^{\sqrt{\frac{p|f''(t_0)|}{2}}\epsilon} e^{-s^2} ds \\ &\sim \frac{\sqrt{2}g(t_0)e^{pf(t_0)}}{\sqrt{p|f''(t_0)|}} \int_{\mathbb{R}} e^{-s^2} ds = \frac{\sqrt{2\pi}g(t_0)}{\sqrt{p|f''(t_0)|}} e^{pf(t_0)}, \end{aligned} \tag{1243}$$

where in the first step we use the fact that the integral takes its main contribution about t_0, in the second step we make the change of variables $t \to s$ given by $t - t_0 = \frac{\sqrt{2}}{\sqrt{p|f''(t_0)|}} s$, in the third step we approximate the integral of the gaussian around its $O(1)$ region by the integral over $\mathbb{R}$.

If $t_0 = a$ or $t_0 = b$, the procedure is the same and the leading order contribution is half of the above contribution. For instance, if $t_0 = a$:

$$\begin{aligned} L(p) &\sim g(t_0)e^{pf(t_0)} \int\limits_{t_0}^{t_0+\epsilon} e^{-p|f''(t_0)|\frac{(t-t_0)^2}{2}} dt \\ &= \frac{\sqrt{2}g(t_0)e^{pf(t_0)}}{\sqrt{p|f''(t_0)|}} \int\limits_{0}^{\sqrt{\frac{p|f''(t_0)|}{2}}\epsilon} e^{-s^2} ds \\ &\sim \frac{\sqrt{2}g(t_0)e^{pf(t_0)}}{\sqrt{p|f''(t_0)|}} \int\limits_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}g(t_0)}{\sqrt{2p|f''(t_0)|}} e^{pf(t_0)}, \end{aligned} \tag{1244}$$

As we shall see, this is indeed the leading order contribution of an asymptotic expansion of the following type.

If $p \gg 1$ and $t_0 \in (a, b)$:

$$L(p) = \frac{\sqrt{2\pi}g(t_0)}{\sqrt{p|f''(t_0)|}} e^{pf(t_0)} \left(1 + O\left(\frac{1}{p}\right)\right); \tag{1245}$$

if, instead, $p \gg 1$ and $t_0 = a$ or $t_0 = b$, then

$$L(p) = \frac{\sqrt{\pi} g(t_0)}{\sqrt{2p|f''(t_0)|}} e^{pf(t_0)} \left(1 + O\left(\frac{1}{\sqrt{p}}\right)\right). \tag{1246}$$

To prove these formulas, we evaluate the integral first in a right neighborhood $[t_0, t_0+\delta]$ of t_0:

$$\begin{aligned} I_+ &:= \int\limits_{t_0}^{t_0+\delta} e^{pf(t)} g(t) dt = e^{pf(t_0)} \int\limits_{t_0}^{t_0+\delta} e^{-p[f(t_0)-f(t)]} g(t) dt \\ &= e^{pf(t_0)} \int\limits_{0}^{\delta_+} e^{-py} g(\varphi(y)) \tfrac{dt}{dy} dy = e^{pf(t_0)} \int\limits_{0}^{\delta_+} e^{-py} \frac{g(\varphi(y))}{\frac{dy}{dt}\Big|_{t=\varphi(y)}} dy, \end{aligned} \tag{1247}$$

where in the first step we add and subtract $f(t_0)$ in the argument of the exponential; in the second step we change variables $t \to y(t) := f(t_0) - f(t) \geq 0$, since $y(t)$ is a monotonically decreasing function and its inverse $t = \varphi(y)$ exists, where $\delta_+ = f(t_0) - f(t_0 + \delta)$; in the third step we use the inverse function theorem. Analogously, for the left neighborhood $[t_0 - \delta, t_0]$:

$$\begin{aligned} I_- &:= \int\limits_{t_0-\delta}^{t_0} e^{pf(t)} g(t) dt = -e^{pf(t_0)} \int\limits_{0}^{\delta_-} e^{-py} \frac{g(\varphi(y))}{\frac{dy}{dt}\Big|_{t=\varphi(y)}} dy, \\ \delta_- &:= f(t_0) - f(t_0 - \delta). \end{aligned} \tag{1248}$$

Our goal now is to expand the integrand with respect to the new variable y. Since

$$y = f(t_0) - f(t) = \frac{|f''(t_0)|}{2}(t-t_0)^2 - \frac{f'''(t_0)}{6}(t-t_0)^3 + O\left((t-t_0)^4\right), \tag{1249}$$

to leading order we have:

$$y \sim \frac{|f''(t_0)|}{2}(t-t_0)^2 \quad \Rightarrow \quad t - t_0 \sim \pm\sqrt{\frac{2}{|f''(t_0)|}}\sqrt{y}, \tag{1250}$$

suggesting the following expansion

$$t - t_0 = \operatorname{sign}(t-t_0)\sqrt{\frac{2}{|f''(t_0)|}}\sqrt{y} + \beta y + O(y^{3/2}). \tag{1251}$$

Substituting this ansatz into (1249), one verifies that, to $O(y^{3/2})$, $\beta = \frac{f'''(t_0)}{3(f''(t_0))^2}$; therefore

$$t - t_0 = \operatorname{sign}(t-t_0)\sqrt{\frac{2}{|f''(t_0)|}}\sqrt{y} + \frac{f'''(t_0)}{3\left(f''(t_0)\right)^2} y + O\left(y^{3/2}\right). \tag{1252}$$

It is also possible to show that $\operatorname{sign}(t-t_0)$ is present only in terms corresponding to fractional powers $y^{n+1/2},\ n\in\mathbb{N}^+$ of y in the expansion of $t-t_o$ and of the integrand:

$$\begin{aligned}
&\tfrac{dy}{dt} = \operatorname{sign}(t-t_0)|f''(t_0)|(t-t_0) - \tfrac{f'''(t_0)}{2}(t-t_0)^2 + \dots\\
&= \operatorname{sign}(t-t_0)\sqrt{2|f''(t_0)|}\sqrt{y} + \beta_1 y + \operatorname{sign}(t-t_0)\beta_2 y^{3/2} + \dots\\
&\tfrac{g(t)}{dy/dt} = \operatorname{sign}(t-t_0)\tfrac{g(t_0)}{\sqrt{2|f''(t_0)|}}\tfrac{1}{\sqrt{y}} + c_0 + \operatorname{sign}(t-t_0)c_1\sqrt{y} + \dots
\end{aligned} \tag{1253}$$

It follows that

$$\begin{aligned}
I_+ &= e^{pf(t_0)}\int_0^{\delta_+} e^{-py}\left(\tfrac{g(t_0)}{\sqrt{2|f''(t_0)|}}\tfrac{1}{\sqrt{y}} + c_0 + c_1\sqrt{y} + O(y)\right)dy\\
&= e^{pf(t_0)}\int_0^{\infty} e^{-py}\left(\tfrac{g(t_0)}{\sqrt{2|f''(t_0)|}}\tfrac{1}{\sqrt{y}} + c_0 + c_1\sqrt{y} + O(y)\right)dy\\
&= e^{pf(t_0)}\left(\sqrt{\tfrac{\pi}{2|f''(t_0)|p}}g(t_0) + \tfrac{c_0}{p} + c_1\tfrac{\Gamma(3/2)}{p^{3/2}} + O\left(p^{-2}\right)\right),
\end{aligned} \tag{1254}$$

$$\begin{aligned}
I_- &= -e^{pf(t_0)}\int_0^{\delta_-} e^{-py}\left(-\tfrac{g(t_0)}{\sqrt{2|f''(t_0)|}}\tfrac{1}{\sqrt{y}} + c_0 - c_1\sqrt{y} + O(y)\right)dy\\
&= e^{pf(t_0)}\int_0^{\infty} e^{-py}\left(\tfrac{g(t_0)}{\sqrt{2|f''(t_0)|}}\tfrac{1}{\sqrt{y}} - c_0 + c_1\sqrt{y} + O(y)\right)dy\\
&= e^{pf(t_0)}\left(\sqrt{\tfrac{\pi}{2|f''(t_0)|p}}g(t_0) - \tfrac{c_0}{p} + c_1\tfrac{\Gamma(3/2)}{p^{3/2}} + O\left(p^{-2}\right)\right).
\end{aligned} \tag{1255}$$

In the first step we used the second of equations (1253); in the second step we replaced the finite integral by the integral over $\mathbb{R}^+$ with an exponentially small error, as we shall see below; in the third step we used the following formulas

$$\int_0^\infty e^{-py}y^{-1/2}dy = \sqrt{\frac{\pi}{p}},\quad \int_0^\infty e^{-py}dy = \frac{1}{p},\quad \int_0^\infty e^{-py}y^{1/2}dy = \sqrt{\frac{\Gamma(3/2)}{p^{3/2}}}, \tag{1256}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

If $t\in[a,t_0-\delta]$ or $t\in[t_0+\delta,b]$, there exists $\eta>0$ such that $f(t_0)-f(t)\geq\eta>0$; consequently

$$pf(t) = (p-\sigma)f(t) + \sigma f(t) \leq (p-\sigma)(f(t_0)-\eta) + \sigma f(t), \tag{1257}$$

where $0<\sigma\ll p$. It follows that the contribution of the integral outside the neighborhood of t_0 is exponentially small:

$$\begin{aligned}
&\left|\int_a^{t_0-\delta} e^{pf(t)}g(t)dt\right| \leq e^{(p-\sigma)(f(t_0)-\eta)}\int_a^{t_0-\delta} e^{\sigma f(t)}|g(t)|dt\\
&= O\left(e^{(p-\sigma)(f(t_0)-\eta)}\right) = e^{pf(t_0)}O(e^{-\eta p}).
\end{aligned} \tag{1258}$$

At last, equations (1254) and (1255) imply formulas (1245) and (1246).

10.2.3 The steepest descent method

Consider now the case of the Laplace integral over a complex contour

$$I(p) = \int_{\mathcal{C}} e^{pf(z)} g(z) dz, \quad p \gg 1, \tag{1259}$$

where $f(z)$ and $g(z)$ are analytic in a domain $\mathcal{D}$ of the complex z plane containing the curve $\mathcal{C}$, and suppose that $\exists!\ z_0 \in \mathcal{D}$ such that $f'(z_0) = 0$ and $f''(z_0) \neq 0$. We use the standard notation: $z = x + iy$, $u(x,y) =$Re$f(z) = f_R(z)$ and $v(x,y) =$Im$f(z) = f_I(z)$.

We look for a contour $\gamma \subset \mathcal{D}$ such that:
1) $u(x,y)$ has its max on γ in $z_0 = x_0 + iy_0 \in \gamma$.
2) $v(x,y)$ is constant on γ: $v(x,y) = v(x_0,y_0)$, to avoid rapid oscillations of the integrand.
Does this contour exist? Since $f(z)$ is analytic, then $u_{xx} + u_{yy} = 0$, implying that $u_{xx}u_{yy} < 0$ and that $u_{xx}u_{yy} < u_{xy}^2$. Therefore z_0 is a saddle point (see figure 3). Along γ, v is constant, implying that $\bigtriangledown v \perp \gamma$. From the Cauchy-Riemann conditions it follows that $\bigtriangledown v \cdot \bigtriangledown u = 0$; i.e., γ is tangent to $\bigtriangledown u$. Therefore **the curve on which $v(x,y) = v(x_0,y_0)$ is the curve of steepest variation of u** (see Fig. 65).

How many curves of this type pass through z_0? Can one select, among them, the steepest descent curves? Around z_0 we have:

$$f(z) = f(z_0) + \frac{f''(z_0)}{2}(z - z_0)^2 + O(z - z_0)^3. \tag{1260}$$

Therefore

$$\begin{array}{l} f(z) = f(z_0) + (z - z_0)^2 h(z), \\ h(z) \text{ analytic, with } h(z_0) \neq 0, \\ h(z_0) = \frac{f''(z_0)}{2} = \frac{\rho_0}{2} e^{i\varphi_0}, \\ \rho_0 = |f''(z_0)|, \quad \varphi_0 = \arg(f''(z_0)) \end{array} \tag{1261}$$

Let

$$z - z_0 = re^{i\theta}, \quad h(z) = \frac{\rho}{2} e^{i\varphi}, \tag{1262}$$

then

$$\begin{array}{l} f(z) - f(z_0) = \frac{r^2\rho}{2} e^{i(2\theta+\varphi)}, \\ u(x,y) - u(x_0,y_0) = \frac{r^2\rho}{2}\cos(2\theta + \varphi), \\ v(x,y) - v(x_0,y_0) = \frac{r^2\rho}{2}\sin(2\theta + \varphi). \end{array} \tag{1263}$$

Since $2\theta + \varphi$ has a variation equal to 4π when z rotates once around z_0 and θ varies from 0 to 2π, four curvilinear sectors meet in z_0, and $u(x,y) - u(x_0,y_0)$ changes sign four times. Therefore we define 4 sectors in which

$u(x, y) - u(x_0, y_0)$ has constant sign inside each sector, and opposite sign in the neighboring sector (see Fig.s 65 and 66):

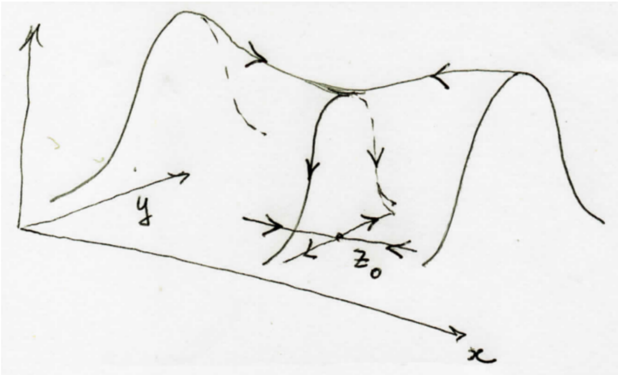


Figure 65: The plot of $u(x, y)$ and the saddle point at $z_0 = x_0 + iy_0$, in the case $f''(z_0) \neq 0$

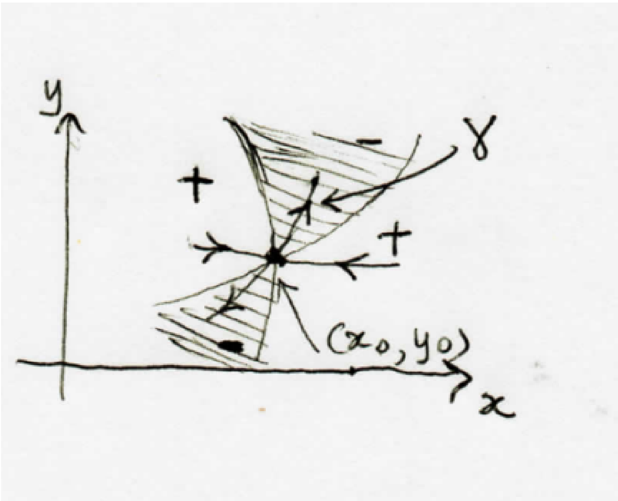


Figure 66: The four curvilinear sectors emanating from the saddle point z_0 and the two curves of steepest variation, in the case $f''(z_0) \neq 0$. In the two dark sectors $u(x, y) < u(x_0, y_0)$ and the curve γ contained there is the steepest descent curve. In the two white sectors $u(x, y) > u(x_0, y_0)$ and the curve contained there is the steepest ascent curve.

The sectors with negative sign contain the curve γ of steepest descent we are looking for, defined by the equations:

$$\begin{array}{l} u(x, y) - u(x_0, y_0) = -\frac{r^2 \rho}{2} < 0, \\ v(x, y) - v(x_0, y_0) = 0 \end{array} \tag{1264}$$

coming from $\cos(2\theta+\varphi) = -1$ and $\sin(2\theta+\varphi) = 0$. Therefore we obtain the two angles

$$\theta_{0k} = -\frac{\varphi_0}{2} + \left(k - \frac{1}{2}\right)\pi, \quad k = 1, 2, \tag{1265}$$

defining the two directions of steepest descent at z_0.

Then the integral over γ can be evaluated using the Laplace method:

$$\begin{array}{l} I_\gamma(p) = \int_\gamma e^{pf(z)} g(z) dz = e^{ipf_I(z_0)} \int_\gamma e^{pf_R(z)} g(z) dz \\ = e^{ipf_I(z_0)} \int\limits_a^b e^{pf_R(z(t))} g(z(t)) \frac{dz}{dt} dt = \\ \begin{cases} g(z_0) \frac{dz(t_0)}{dt} \sqrt{\frac{2\pi}{p \left| \frac{d^2 f_R(z(t))}{dt^2} \right|_{t_0} \right|}} e^{pf(z_0)}(1 + O(p^{-1})), & z_0 \text{ inside } \gamma, \\ g(z_0) \frac{dz(t_0)}{dt} \sqrt{\frac{\pi}{2p \left| \frac{d^2 f_R(z(t))}{dt^2} \right|_{t_0} \right|}} e^{pf(z_0)}(1 + O(p^{-1/2})), & z_0 \text{ end point of } \gamma, \end{cases} \end{array} \tag{1266}$$

where $t \in [a, b] \to z(t) \in \gamma$ is a parametrization of the steepest descent curve, with $z(t_0) = z_0$ and $t_0 \in [a, b]$. In the neighborhood of z_0 on γ:

$$\begin{array}{l} z(t) - z_0 = r(t) e^{i\theta(t)} \sim c(t - t_0) e^{i\theta(t)}, \ c > 0, \ t \sim t_0 \\ \Rightarrow \quad \frac{dz}{dt} \to c e^{i\theta_0}, \quad \left(\frac{dz}{dt}\right)^2 \to c^2 e^{i\theta_0}, \quad t \to t_0, \end{array} \tag{1267}$$

where θ_0 is one of the two steepest descent directions (1265); the choice between these two values depends on how one chooses the travel direction along the curve, and will be decided later on. In addition, since $df = df_R$ on γ:

$$\begin{array}{l} \frac{d^2 f_R(z(t))}{dt^2} = \frac{d^2 f(z(t))}{dt^2} = f''(z(t)) \left(\frac{dz(t)}{dt}\right)^2 + f'(z(t)) \frac{d^2 z(t)}{dt^2} \\ \to f''(z_0) \left(\frac{dz(t_0)}{dt}\right)^2 = \rho_0 c^2 e^{i(2\theta_0 + \phi_0)} = -\rho_0 c^2 < 0, \quad t \to t_0. \end{array} \tag{1268}$$

Replacing these results into (1266), we conclude that, for $p \gg 1$, (VERIFY IT)

$$I_\gamma(p) = \begin{cases} \sqrt{\frac{2\pi}{p|f''(z_0)|}} g(z_0) e^{pf(z_0) + i\theta_0} \left(1 + O(p^{-1})\right), & z_0 \text{ inside } \gamma, \\ \sqrt{\frac{\pi}{2p|f''(z_0)|}} g(z_0) e^{pf(z_0) + i\theta_0} \left(1 + O(p^{-1/2})\right), & z_0 \text{ end point of } \gamma. \end{cases} \tag{1269}$$

The asymptotics (1269) are useful if one can connect the original integral (1259) to the steepest descent integral $I_\gamma(p)$, like in the KdV example of §1.4, without involving regions where the integral gives a contribution bigger than that in (1269).

10.2.4 Examples

To be more explicit, let us consider the simplest possible case of the linear Schrödinger equation $iu_t + u_{xx} = 0$ and its Fourier transform solution

$$u(x,t) = \int_{\mathbb{R}} \frac{dk}{2\pi} \hat{u}_0(k) e^{i(kx-k^2t)} = \int_{\mathbb{R}} \frac{dk}{2\pi} \hat{u}_0(k) e^{f(k,x/t)t}, \tag{1270}$$

with

$$\begin{array}{l} f\left(k, \frac{x}{t}\right) := i\left(k\frac{x}{t} - k^2\right) = u + iv, \\ u := k_I(2k_R - \frac{x}{t}), \quad v := k_I^2 - k_R^2 + k_R\frac{x}{t}, \\ f'\left(k, \frac{x}{t}\right) = i\left(\frac{x}{t} - 2k\right) \ \Rightarrow \ k_0 = \frac{x}{2t} \in \mathbb{R}. \\ f''\left(k, \frac{x}{t}\right) = -2i, \\ u(k_0) = 0, \quad v(k_0) = \left(\frac{x}{2t}\right)^2 > 0. \end{array} \tag{1271}$$

The curves of steepest variation $v(k) - v(k_0) = 0$ are the main and secondary diagonals passing through the saddle point k_0:

$$v(k) - v(k_0) = -(k_R - k_I - \frac{x}{2t})(k_R + k_I - \frac{x}{2t}) = 0. \tag{1272}$$

and the steepest descent curve is the secondary diagonal $k_R + k_I - \frac{x}{2t} = 0$ (see Fig. 67)(VERIFY IT).

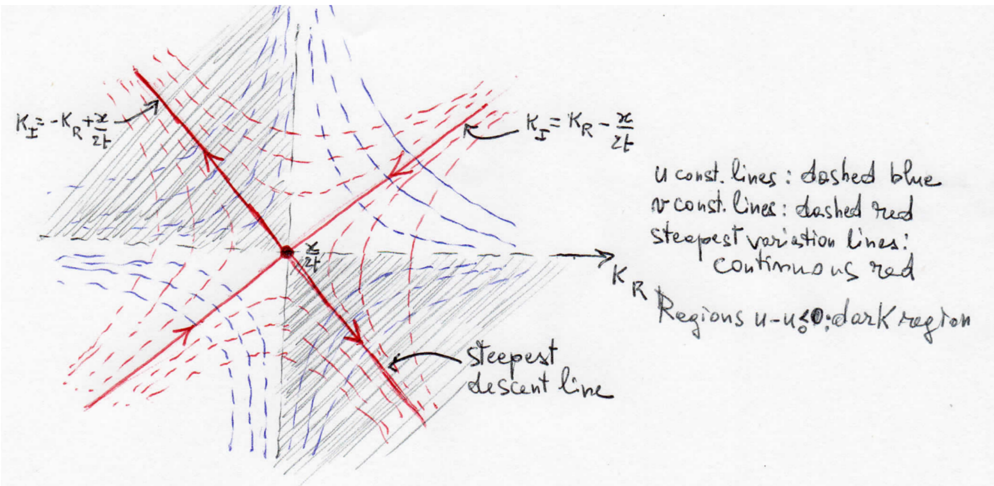


Figure 67: In the k complex plane centered at $k_0 = x/(2t)$, the lines of steepest variation are the main and secondary diagonals (in continuous red); the steepest descent line is the secondary diagonal; the lines v =const are in dashed red; the lines u =const are in dashed blue; the gray regions are the regions in which $u(k_R, k_I) - u(x/(2t), 0) < 0$.

For the convergence at $k \sim \infty$: $f(k) \sim -ik^2 = |k|^2(\sin(2\varphi) - i\cos(2\varphi))$, $k = |k|e^{i\varphi}$; then $\sin(2\varphi) < 0$, implying $\pi/2 < \varphi < \pi$ and $3\pi/2 < \varphi < 2\pi$. It follows that the integral converges to zero on infinite arcs of the second and fourth quadrants.

Therefore the integral (1270) over the real line is equal to the integral over the contour $\tilde{\gamma}$ consisting of the union of the infinite arc $(-\infty, \infty\exp(3i\pi/4))$, of the steepest descent straight line γ (from $\infty\exp(3i\pi/4)$ to $\infty\exp(-i\pi/4)$), and of the infinite arc $(\infty\exp(-i\pi/4), \infty)$, by the Cauchy theorem, assuming that $\hat{u}_0(k)$ be analytic inside the closed contour $\mathbb{R} \cup (-\tilde{\gamma})$. Since the contribution of the two infinite arcs is zero, we have

$$\begin{aligned} &\int_{\mathbb{R}} \tfrac{dk}{2\pi} \hat{u}_0(k) e^{i(kx-k^2t)} = \int_{\gamma} \tfrac{dk}{2\pi} \hat{u}_0(k) e^{i(kx-k^2t)} \\ &= \tfrac{\hat{u}_0\left(\frac{x}{2t}\right)}{\sqrt{4\pi t}} e^{i\left(\frac{x^2}{4t}-\frac{\pi}{4}\right)} \left(1 + O\left(\tfrac{1}{t}\right)\right), \quad t \gg 1, \ \tfrac{x}{t} = O(1), \end{aligned} \tag{1273}$$

coinciding with the result (32) of the qualitative analysis made using the stationary phase method. Therefore in this case the leading order term comes indeed from the steepest descent integral.

As a second example, we consider the integral (1270), but on a different contour $\mathcal{C}$ with end points a, b:

$$I(x,t) = \int_a^b \frac{dk}{2\pi} \hat{u}_0(k) e^{f(k,x/t)t}, \tag{1274}$$

where $a < x/(2t)$ and $b \in \mathbb{C}$ with $0 < \arg(b - x/(2t)) < \pi/4$. Using the Cauchy theorem, we can replace the original integral with the contour $\tilde{\gamma}$ in figure, consisting of the union of the $v =$const line $\tilde{\gamma}_1$ connecting a to $\infty\exp(3i\pi/4)$, of the steepest descent line γ, and of the $v =$const line $\tilde{\gamma}_2$ connecting $\infty\exp(-i\pi/4)$ to b, having assumed that $\hat{u}_0(k)$ is analytic inside the closed contour $\mathcal{C} \cup \tilde{\gamma}$ (see Fig. 68).

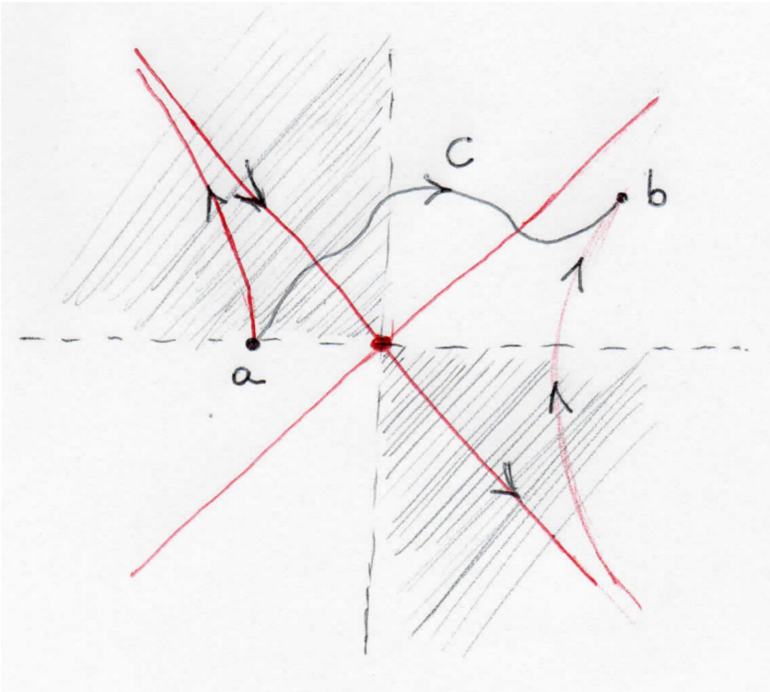


Figure 68: Due to the Cauchy theorem, the integration curve $\mathcal{C}$ is replaced by the red curve from a to b; in this case the main contribution does not come from the steepest descent curve

By the Cauchy theorem the contour $\mathcal{C}$ is replaced by the contour $\tilde{\gamma}$. The main contribution comes from the last part of the red contour and is obtained using integration by parts.

We see that the integrals over $\tilde{\gamma}_1$ and over $\tilde{\gamma}_2$ can be evaluated using the integration by parts (since $f'(k) \neq 0$ on these contours), obtaining respectively:

$$\int_a^{\infty e^{3i\pi/4}} \frac{dk}{2\pi} \hat{u}_0(k) e^{f(k,x/t)t} = -\frac{\hat{u}_0(a)}{2\pi i(x-2at)} e^{i(ax-a^2t)} \left(1 + O\left(\frac{1}{t}\right)\right), \ t \gg 1, \frac{x}{t} = O(1), \tag{1275}$$

and

$$\int_{\infty e^{-i\pi/4}}^{b} \frac{dk}{2\pi} \hat{u}_0(k) e^{f(k,x/t)t} = \frac{\hat{u}_0(b)}{2\pi i(x-2bt)} e^{i(bx-b^2t)} \left(1 + O\left(\frac{1}{t}\right)\right), \ t \gg 1, \frac{x}{t} = O(1), \tag{1276}$$

while the integral over the steepest descent contour has the leading contribution (1273). While the contribution of $\tilde{\gamma}_1$ is much smaller than the contribution of (1273), the contribution of $\tilde{\gamma}_2$ is much larger than the contribution (1273), given by the steepest descent path γ, since $\mathrm{Re}(i(bx - b^2t)) =$

$2b_I(b_R - x/(2t))t \gg 1$. Therefore in this example **the leading order term does not come from the saddle point contribution**, and

$$I(x,t) = \frac{\hat{u}_0(b)}{2\pi i(x-2bt)} e^{i(bx-b^2t)} \left(1 + O\left(\frac{1}{t}\right)\right), \quad t \gg 1, \quad \frac{x}{t} = O(1). \quad (1277)$$

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