**PSR 1913+16**

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\[ P = 27907 \text{ s, } E_{\text{orb}} \sim -1.4 \cdot 10^{48} \text{ erg, } L_{\text{GW}} \sim 0.7 \cdot 10^{31} \text{ erg/s} \]

and

\[ \frac{dP}{dt} = \frac{3}{2} \frac{P}{E_{\text{orb}}} L_{\text{GW}} \rightarrow \frac{dP}{dt} \sim -2.2 \cdot 10^{-13}. \]

The orbit of the real system has a quite strong eccentricity \( \epsilon \simeq 0.617 \). Using the quadrupole formalism with the appropriate orbit:

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Residual differences due to Doppler corrections, due to the relative velocity between us and the pulsar induced by the differential rotation of the Galaxy.

\[
\frac{\dot{P}_{corrected}}{\dot{P}_{GR}} = 1.0013(21)
\]
For the recently discovered double pulsar PSR J0737-3039

\[ P = 8640 \text{ s}, \quad E_{\text{orb}} \sim -2.55 \cdot 10^{48} \text{ erg}, \quad L_{\text{GW}} \sim 2.24 \cdot 10^{32} \text{ erg/s} \]

and

\[ \frac{dP}{dt} \sim -1.2 \cdot 10^{-12}, \]

which also agrees with observations.
Thus, this prediction of General Relativity is confirmed by observations. This result provided the first indirect evidence of the existence of gravitational waves and for this discovery R.A. Hulse and J.H. Taylor have been awarded of the Nobel prize in 1993.
WAVEFORM: AMPLITUDE AND PHASE

Orbital separation

\[ l_0(t) = l_0^{\text{in}} \left[ 1 - \frac{t}{t_{\text{coal}}} \right]^{1/4}, \]

where \( t_{\text{coal}} = \frac{5}{256} \frac{c^5 (l_0^{\text{in}})^4}{G^3 \mu M^2} \).

Wave frequency:

\[
\nu_{GW}(t) = \frac{\nu_{GW}^{\text{in}}}{\pi} = \frac{\nu_{GW}^{\text{in}}}{\pi} \left[ 1 - \frac{t}{t_{\text{coal}}} \right]^{3/8}, \quad \nu_{GW}^{\text{in}} = \frac{1}{\pi} \sqrt{\frac{GM}{(l_0^{\text{in}})^3}},
\]

The instantaneous amplitude of the emitted signal is

\[
h_0(t) = \frac{4 \mu M G^2}{r l_0(t) c^4} = \frac{4 G^{5/3} \mu M^{2/3}}{r c^4} \cdot \omega_K^{2/3}(t), \quad \omega_K = GM/l_0^3
\]

if we define the quantity \( \mathcal{M} \), called **chirp mass**, \n
\[
\mathcal{M}^{5/3} = \mu M^{2/3}
\]

and remember that the wave frequency \( \nu_{GW} \) is twice the orbital frequency \( \nu_{K} = \omega_K / 2\pi \) we find

\[
h_0(t) = \frac{4 \pi^{2/3} G^{5/3} \mathcal{M}^{5/3}}{c^4 r} \nu_{GW}^{2/3}(t).
\]
The amplitude and the frequency of the gravitational signal emitted by a coalescing system increase with time. For this reason this peculiar waveform is called **chirp**, like the chirp of a singing bird.

\[
h_0(t) = \frac{4\pi^{2/3} G^{5/3} M^{5/3}}{c^4 r} \nu_{GW}^{2/3}(t).
\]
if we know the signal phase we can measure the chirp mass.

\[
\begin{align*}
\mathcal{h}_{ij}^{TT} &= -\frac{4\pi^{2/3}}{c^{4\tau}} \frac{G^{5/3}}{M^{5/3}} \nu_{GW}^{2/3}(t) \left[ \mathcal{P}_{ijkl} A_{kl}(t - \frac{r}{c}) \right] \\
\end{align*}
\]

the signal phase must be integrated

\[
\Phi(t) = \int_{t}^{t_2} 2\omega_K(t) dt = \int_{t}^{t_2} 2\pi \nu_{GW}(t) dt + \Phi_{in},
\]

where \( \Phi_{in} = \Phi(t = 0) \)

\[
A_{ij}(t - \frac{r}{c}) = \begin{pmatrix}
\cos \Phi(t - \frac{r}{c}) & \sin \Phi(t - \frac{r}{c}) & 0 \\
\sin \Phi(t - \frac{r}{c}) & -\cos \Phi(t - \frac{r}{c}) & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
LIGO $[40 \text{ Hz} - 1 - 2 \text{ kHz}]$  
LISA $[10^{-4} - 10^{-1}] \text{ Hz}$  
VIRGO $[10 \text{ Hz} - 1 - 2 \text{ kHz}]$

Let us consider 3 binary system

a) $m_1 = m_2 = 1.4 \, M_\odot$

b) $m_1 = m_2 = 10 \, M_\odot$

c) $m_1 = m_2 = 10^6 \, M_\odot$
Let us consider 3 binary systems:

a) $m_1 = m_2 = 1.4 \, M_\odot$

b) $m_1 = m_2 = 10 \, M_\odot$

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Let us first calculate what is the orbital distance between the two bodies on the Innermost Stable Circular Orbit (ISCO) and the corresponding emission frequency.

$$l_{ISCO} \sim \frac{6GM}{c^2}, \quad \omega_K = \sqrt{\frac{GM}{l_{ISCO}^3}} = \pi \, \nu_{GW}$$

$$\rightarrow \nu_{GW}^{ISCO} = \frac{1}{\pi} \sqrt{\frac{GM}{(l_{ISCO})^3}}$$
LIGO [40 Hz – 1 – 2 kHz]  LISA [10^{-4} – 10^{-1}] Hz
VIRGO [10 Hz – 1 – 2 kHz]

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\[
\ell_0^{ISCO} \sim \frac{6GM}{c^2}, \quad \omega_K = \sqrt{\frac{GM}{\ell_0^3}} = \pi \nu_{GW}
\]

\[ \rightarrow \nu_{GW} = \frac{1}{\pi} \sqrt{\frac{GM}{(\ell_0^{ISCO})^3}} \]

a) \( \ell_0^{ISCO} = 24,8 \ km \quad \nu_{GW} = 1570.4 \ Hz \)
b) \( \ell_0^{ISCO} = 177,2 \ km \quad \nu_{GW} = 219.8 \ Hz \)
c) \( \ell_0^{ISCO} = 17.720.415,3 \ km \quad \nu_{GW} = 2.2 \cdot 10^{-3} \ Hz \)
LIGO $[40 \text{ Hz} - 1 - 2 \text{ kHz}]$ \quad \text{LISA} $[10^{-4} - 10^{-1}] \text{ Hz}$ \quad \text{VIRGO} $[10 \text{ Hz} - 1 - 2 \text{ kHz}]$

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$$\Rightarrow \nu_{GW} = \frac{1}{\pi} \sqrt{\frac{GM}{(l_0^{ISCO})^3}}$$

a) $l_0^{ISCO} = 24.8 \text{ km}$ \quad \nu_{GW} = 1570.4 \text{ Hz}$

b) $l_0^{ISCO} = 177.2 \text{ km}$ \quad \nu_{GW} = 219.8 \text{ Hz}$

c) $l_0^{ISCO} = 17.720.415,3 \text{ km}$ \quad \nu_{GW} = 2.2 \cdot 10^{-3} \text{ Hz}$

a) and b) are interesting for LIGO and VIRGO, c) will be detected by LISA
Let us consider LIGO and VIRGO; we want to compute the time a given signal spends in the detector bandwidth before coalescence.

From

$$\nu_{GW}(t) = \frac{\nu_{in}^{GW} t_{coal}^{3/8}}{[t_{coal} - t]^{3/8}}$$

we get

$$t = t_{coal} \left[ 1 - \left( \frac{\nu_{in}^{GW}}{\nu_{GW}(t)} \right)^{8/3} \right].$$
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Putting:

\( \nu_{GW}^{in} = \) lowest frequency detectable by the antenna, and \( \nu_{GW}^{max} = \nu_{ISCO}^{GW} \)

we find

**LIGO**

\( a \) \( (m_1 = m_2 = 1.4 \, M_\odot) \) \[ [40 - 1570.4 \, Hz] \] \[ [10 - 1570.4 \, kHz] \]

\[ \Delta t = 24.86 \, s \quad \Delta t = 16.7 \, m \]

**VIRGO**

\( a \) \( (m_1 = m_2 = 10 \, M_\odot) \) \[ [40 - 219.8 \, Hz] \] \[ [10 - 219.8 \, kHz] \]

\[ \Delta t = 0.93 \, s \quad \Delta t = 37.82 \, s \]
Let us consider **LIGO** and **VIRGO**; we want to compute the time a given signal spends in the detector bandwidth before coalescence.

From

\[ \nu_{GW}(t) = \frac{\nu_{GW}^{in} \, t_{coal}^{3/8}}{[t_{coal} - t]^{3/8}} \]

we get

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Putting:

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<table>
<thead>
<tr>
<th></th>
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<th><strong>VIRGO</strong></th>
</tr>
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**VIRGO** catches the signal for a longer time.
Planned sensitivity curve for VIRGO+ and ADVANCED VIRGO

The plotted signal is the

\[ \text{strain amplitude} = \nu^{1/2} h(\nu) , \]

evaluated for the chirp.

Source located at a distance of 100 Mpc.
Planned sensitivity curve for VIRGO+ and ADVANCED VIRGO

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Source located at a distance of 100 Mpc.

The chirp is only a part of the signal emitted during the binary coalescence.
This hybrid waveform is obtained by matching three signals:

1) chirp waveform for $\nu < \nu_{ISCO}$

2) waveform emitted during the merger phase of two black holes, obtained by numerical integration of Einstein’s equations.

3) waveform emitted after the final black hole is formed, due to ringdown oscillations.

*P. Ajith et al., Phys. Rev. D 77, 104017 2008*
WHAT ABOUT LISA?

LISA $[10^{-4} - 10^{-1}]$ Hz

Let us consider 2 BH-BH binary systems

a) $m_1 = m_2 = 10^2 \ M_\odot$

b) $m_1 = m_2 = 10^6 \ M_\odot$

Orbital distance between the two bodies on the innermost stable circular orbit (ISCO) and the corresponding emission frequency

$$l_{ISCO}^0 \sim \frac{6GM}{c^2}, \quad \omega_K = \sqrt{\frac{GM}{l_{ISCO}^3}} = \pi \nu_{GW} \rightarrow \nu_{GW}^{ISCO} = \frac{1}{\pi} \sqrt{\frac{GM}{l_{ISCO}^3}}$$
a) $l_{ISCO}^0 = 1772 \text{ km}$

b) $l_{ISCO}^0 = 17.720.415,3 \text{ km}$

\[ \nu_{GW} = 21.98 \text{ Hz} \]

\[ \nu_{GW} = 2.2 \times 10^{-3} \text{ Hz} \]

Time a given signal spends in the detector bandwidth before coalescence.

a) $m_1 = m_2 = 10^2 \, M_\odot$

b) $m_1 = m_2 = 10^6 \, M_\odot$

\[ t = t_{coal} \left[ 1 - \left( \frac{\nu_{in}}{\nu_{GW}(t)} \right)^{8/3} \right] \]

LISA

a) $[10^{-4} \ - 10^{-1} \ \text{Hz}]$

\[ \Delta t = 556.885 \text{ years} \]

b) $[10^{-4} \ - 2.2 \times 10^{-3} \ \text{Hz}]$

\[ \Delta t = 0.12 \text{ years} = 43 \text{ d } 18 \text{ h } 43 \text{ m } 24 \text{ s} \]
$S_n^{1/2}, \nu^{1/2} h(\nu)$ [Hz$^{-1/2}$]

- LISA
- $10 M_{\odot}$ BH-BH
- $10^6 M_{\odot}$ BH-BH