6.2 EINSTEIN'S FIELD EQUATIONS

Let us first discuss the differential order of $G_{\mu\nu}$ in terms of derivatives of $g_{\mu\nu}$. A comparison with the Poisson equation shows that $G_{\mu\nu}$ must have the dimensions of a second derivative with respect to the coordinates, i.e. the inverse of a square lenght. Furthermore, for simplicity, we seek for an operator that does not contain any fundamental constant, since G and c should only appear in our field equations 6.22 as the coupling term between the source of the gravitational field on the right-hand side ($T_{\mu\nu}$), and the tensor which describes the spacetime geometry ($G_{\mu\nu}$) on the left-hand side. The latter assumption is equivalent to require that $G_{\mu\nu}$ is scale invariant. Finally, we want the differential operator in $G_{\mu\nu}$ to be at most of second order, since higher-order field equations are generically pathological, as will be discussed in a following chapter. The above requirements strongly constrain the differential structure of $G_{\mu\nu}$. Indeed, suppose that $G_{\mu\nu}$ contains terms of the following schematic type

$$\frac{\partial^3 g_{\mu\nu}}{\partial x_\rho^3}\,, \qquad \frac{\partial^2 g_{\mu\nu}}{\partial x_\sigma^2}\,, \qquad \frac{\partial g_{\mu\nu}}{\partial x_\rho}\,, \qquad g_{\mu\nu}\,. \tag{6.24}$$

In order to be dimensionally homogeneous each term should be multiplied by a constant having the dimensions of a suitable power of a lenght, e.g.

$$\frac{\partial^3 g_{\mu\nu}}{\partial x_\rho^3} l\,, \qquad \frac{\partial^2 g_{\mu\nu}}{\partial x_\sigma^2}\,, \qquad \frac{\partial g_{\mu\nu}}{\partial x_\rho}\frac{1}{l}\,, \qquad g_{\mu\nu}\frac{1}{l^2}\,. \tag{6.25}$$

In this case, a gravitational field acting on a very small or on a very large scale would be described by equations where some of the terms would be negligible with respect to some others. That is, there would be an extra scale that is incompatible with the required scale invariance of $G_{\mu\nu}$. Consequently the only terms we can accept in $G_{\mu\nu}$ are those containing the second derivatives of $g_{\mu\nu}$ and products of first derivatives.

Let us summarize the assumptions that we need to make on $G_{\mu\nu}$ on the basis of Eqs. 6.22 and 6.23, and of the discussion above:

1. $G_{\mu\nu}$ must be a tensor and, as $T_{\mu\nu}$, must be symmetric; moreover, in the weak-field, stationary limit, it must reduce to Eq. 6.21, i.e.

$$G_{00} \sim -\nabla^2 g_{00}\,. \tag{6.26}$$

2. It must be at most linear in the second derivatives, contain at most products of first derivatives of $g_{\mu\nu}$, and no linear terms in $g_{\mu\nu}$.

3. Since $T_{\mu\nu}$ satisfies the divergenceless equation $T^{\mu\nu}{}_{;\mu} = 0$, $G_{\mu\nu}$ must satisfy the same equation

$$G^{\mu\nu}{}_{;\nu} = 0\,, \tag{6.27}$$

   for *any* $g_{\mu\nu}$.

As discussed below, there exists a theorem, due to Lovelock, which guarantees that, under the above assumptions, $G_{\mu\nu}$ is in fact unique.

In Chapter 4 we introduced the Riemann tensor and we showed that it carries the information on the curvature of the spacetime. It has the differential structure which we require for the tensor $G_{\mu\nu}$, i.e. it is linear in the second derivatives of $g_{\mu\nu}$, contains products of the first derivatives and no linear terms in $g_{\mu\nu}$.

As discussed in Chapter 4, by contracting the Riemann tensor with the metric we can construct the Ricci tensor (Eq. 4.21)

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} = R^{\alpha}{}_{\mu\alpha\nu}\,, \tag{6.28}$$

which is a symmetric tensor, and the scalar curvature (Eq. 4.22)

$$R = g^{\alpha\beta} R_{\alpha\beta} = R^{\alpha}{}_{\alpha}\,. \tag{6.29}$$

It can be shown, by using the symmetries of the Riemann tensor, that $R_{\mu\nu}$ and R are the *only* second rank tensor and scalar that can be constructed by contraction of $R_{\alpha\mu\beta\nu}$ with the metric. Both $R_{\mu\nu}$ and R have the same differential structure of $R_{\alpha\mu\beta\nu}$. This suggests to write the tensor $G_{\mu\nu}$ as a linear combination of $R_{\mu\nu}$ and R

$$G_{\mu\nu} = A R_{\mu\nu} + B g_{\mu\nu} R\,, \tag{6.30}$$

where A and B are constants to be determined. The tensor $G_{\mu\nu}$ is symmetric, as required by the conditions 1 in the above list, and has the differential structure imposed by condition 2. Furthermore, condition 3 requires that

$$G^{\mu\nu}{}_{;\nu} = A\left(R^{\mu\nu} + \frac{B}{A} g^{\mu\nu} R\right)_{;\nu} = 0\,. \tag{6.31}$$

In order to find whether Eq. 6.31 can be satisfied, we shall use the Bianchi identities (see Section 4.4)

$$R_{\lambda\mu\nu\beta;\eta} + R_{\lambda\mu\eta\nu;\beta} + R_{\lambda\mu\beta\eta;\nu} = 0\,. \tag{6.32}$$

By contracting these equations with $g^{\lambda\nu}$, and remembering that the covariant derivative of the metric tensor vanishes, we find

$$\begin{aligned} g^{\lambda\nu}\left(R_{\lambda\mu\nu\beta;\eta} + R_{\lambda\mu\eta\nu;\beta} + R_{\lambda\mu\beta\eta;\nu}\right) &= g^{\lambda\nu}\left(R_{\lambda\mu\nu\beta;\eta} - R_{\lambda\mu\nu\eta;\beta}\right) + g^{\lambda\nu} R_{\lambda\mu\beta\eta;\nu} \\ &= R_{\mu\beta;\eta} - R_{\mu\eta;\beta} + g^{\lambda\nu} R_{\lambda\mu\beta\eta;\nu} = 0\,. \end{aligned} \tag{6.33}$$

Contracting once more,

$$g^{\mu\beta}\left(R_{\mu\beta;\eta} - R_{\mu\eta;\beta}\right) + g^{\lambda\nu} g^{\mu\beta} R_{\lambda\mu\beta\eta;\nu} = R_{;\eta} - R^{\beta}{}_{\eta;\beta} - R^{\nu}{}_{\eta;\nu} = 0\,. \tag{6.34}$$

Contracting with $g^{\eta\alpha}$ the last expression can be rewritten in the following form

$$R^{\beta\alpha}{}_{;\beta} - \frac{1}{2} g^{\eta\alpha} R_{;\eta} = 0\,,$$

which, relabelling the indices becomes

$$\left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R\right)_{;\beta} = 0\,. \tag{6.35}$$

Therefore, the Bianchi identities imply that, if

$$\frac{B}{A} = -\frac{1}{2}\,, \tag{6.36}$$

Eq. 6.31 are satisfied *identically*. Thus

$$G_{\mu\nu} = A\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)\,. \tag{6.37}$$

We still have to find A. To this aim we will use the condition 1 of the above list, which imposes that in the limit of weak, stationary field

$$G_{00} = A\left(R_{00} - \frac{1}{2}g_{00}R\right) \sim -\nabla^2 g_{00}\,. \tag{6.38}$$

Since the field is weak, as in Section 6.1 we shall assume that $g_{\mu\nu} = \eta_{\mu\nu}+h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$ and that $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2)$. Under these conditions, the Christoffel symbols become

$$\Gamma^{\alpha}_{\mu\nu} = \eta^{\alpha\sigma}\left(h_{\sigma\mu,\nu} + h_{\sigma\nu,\mu} - h_{\mu\nu,\sigma}\right) + O(h^2)\,. \tag{6.39}$$

The expression of the Ricci tensor is (Eq. 4.56)

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\mu\alpha} + \Gamma^{\alpha}_{\sigma\alpha}\Gamma^{\sigma}{}_{\mu\nu}\,, \tag{6.40}$$

and from Eq. 6.39 it follows that the terms which contain products of Γ's in Eq. 6.40 are of order $O(h^2)$, and can be neglected. Thus, in the weak field limit only the terms linear in the second derivatives of the metric tensor survive, and the Ricci tensor can be written as

$$R_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left[g_{\mu\sigma,\nu\rho} + g_{\nu\sigma,\mu\rho} - g_{\mu\nu,\rho\sigma} - g_{\rho\sigma,\mu\nu}\right] + O(h^2)\,. \tag{6.41}$$

The 00 component of $R_{\mu\nu}$ therefore is

$$R_{00} = \frac{1}{2}g^{\sigma\rho}\left[2g_{0\sigma,0\rho} - g_{00,\rho\sigma} - g_{\rho\sigma,00}\right] + O(h^2)\,. \tag{6.42}$$

To hereafter we shall omit the term $O(h^2)$ for simplicity. If the field is stationary, the time derivatives of the metric tensor vanish, and Eq. 6.42 becomes

$$R_{00} = -\frac{1}{2}\eta^{ij}g_{00,ij} = -\frac{1}{2}\nabla^2 g_{00} \qquad i,j = 1,2,3\,. \tag{6.43}$$

In order to compute G_{00} we still need to compute R. In the weak-field limit[1]

$$|T_{ij}| \ll |T_{00}|\,. \tag{6.46}$$

We shall now compute the *trace* of $T_{\mu\nu}$, which is found by contracting the tensor with the metric tensor, i.e.

$$T = g^{\mu\nu}T_{\mu\nu} \simeq \eta^{\mu\nu}T_{\mu\nu} = -T_{00} + \delta^{ij}T_{ij} \simeq -T_{00}\,. \tag{6.47}$$

In this equation we have assumed that the stress-energy tensor, which is the source of the gravitational field, is of order h. If we now take the trace of the equation $G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$ we get

$$g^{\mu\nu}A\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = \frac{8\pi G}{c^4}T \quad \Rightarrow \quad -AR = \frac{8\pi G}{c^4}T\,. \tag{6.48}$$

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[1]Consider for example the system on non-interacting particles discussed in Chapter 5. Be ρ the mass density,

$$\rho = \sum_n m_n \delta^3(\mathbf{r} - \mathbf{r}_n)\,, \tag{6.44}$$

where $\mathbf{r}_n$ indicates the position of the n-th particle; in the weak-field limit the stress-energy tensor given in Eq. 5.15 can be written as

$$T^{\mu\nu} = \rho c^2 \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,. \tag{6.45}$$

Since $\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau}$ with $i = 1,2,3$ (see Eq. 6.4), the dominant term in $T^{\mu\nu}$ is T^{00}.

In Eq. 6.48 we have used the property $g^{\mu\nu}g_{\mu\nu} = 4$, which can easily be proved in a LIF, where $g_{\mu\nu} = \eta_{\mu\nu}$. Since $g^{\mu\nu}g_{\mu\nu}$ is a scalar quantity its value is the same in any frame. Using Eq. 6.47, Eq. 6.48 gives

$$-AR = -\frac{8\pi G}{c^4} T_{00}\,, \tag{6.49}$$

and since the right-hand side of this equation is $-G_{00}$ we find

$$-AR = -A\left(R_{00} - \frac{1}{2}\eta_{00}R\right), \qquad \Rightarrow \qquad R = 2R_{00}\,; \tag{6.50}$$

Using this relation and Eq. 6.43 we can finally compute G_{00}

$$G_{00} = A\left(R_{00} - \frac{1}{2}\eta_{00}R\right) = 2AR_{00} \qquad \Rightarrow \qquad G_{00} = -A\nabla^2 g_{00}\,. \tag{6.51}$$

Comparing this equation with the Eq. 6.26, we find that the relativistic field equations reduce, in the weak-field, stationary limit, to the Newtonian equations if

$$A = 1\,. \tag{6.52}$$

Thus, in conclusion, **Einstein's equations** are[2]

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}\,, \tag{6.53}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\,, \tag{6.54}$$

is called the **Einstein tensor**.

Einstein's equations can also be written in an alternative form. If we take the trace of Eq. 6.53

$$R = -\frac{8\pi G}{c^4} T\,, \tag{6.55}$$

and replace this expression in the Einstein tensor, Eq. 6.53 becomes

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)\,. \tag{6.56}$$

In vacuum $T_{\mu\nu} = 0$, and Eq. (6.53) and (6.56) provide two equivalent forms of the Einstein equations

$$G_{\mu\nu} = 0\,, \qquad R_{\mu\nu} = 0\,. \tag{6.57}$$

The two forms are equivalent because in vacuum $R = 0$ (see Eq. 6.55), and the Einstein and the Ricci tensors coincide. Therefore, in vacuum the Ricci scalar, the Ricci tensor and the Einstein tensor all vanish, but the Riemann tensor does not, unless the gravitational field vanishes or is constant and uniform.

The above heuristic derivation of Einstein's equations might seem "ad hoc" and one might wonder whether there are other geometrical quantities that can be used in place of the Einstein tensor on the left-hand side of Eq. 6.53. A remarkable theorem, due to Lovelock (1972), proves that the above expression of the Einstein tensor is unique. Lovelock's theorem can be stated as follows

[2] Although we call these equations the Einstein equations, they were derived independently, using the variational approach, by D. Hilbert in the same year (see Chapter 7). However, Einstein showed the implications of these equations for the theory of the solar system, and in particular that the precession of the perihelion of Mercury has a relativistic origin (see Chapter ??). This led to the theory's acceptance, and since then the equations have been called the Einstein equations.

In four spacetime dimensions the only divergence-free symmetric rank-2 tensor constructed solely from the metric $g_{\mu\nu}$ and its derivatives up to second differential order, and preserving coordinate invariance, is the Einstein tensor plus a cosmological term.

In other words, Lovelock's theorem shows that general relativity emerges as the *unique* theory of gravity under the above assumptions.

Box 6-A

The cosmological constant

As proved by Lovelock's theorem, one may add to the Einstein tensor given in Eq. 6.54 a term proportional to $g_{\mu\nu}$, such that Einstein's equations would become

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,, \tag{6.58}$$

where λ is a constant. With this term $G_{\mu\nu}$ violates the condition 2 in the list of constraints that the Einstein tensor must satisfy; in addition Eq. 6.58 does not reduce to Newton's equations in the weak-field, stationary limit, as required by Eq. 6.26, unless λ is extremely small. Such term is related to the *cosmological constant* and plays a crucial role in cosmology. After being initially introduced by Einstein himself, this term was disregarded for many decades. It was only in 1998 that two independent experiments, using the observations of distant supernovae, discovered that the Universe is expanding at an increasing rate. This result (that was awarded the Nobel Prize in Physics in 2011) can be explained by a positive cosmological constant. The current measured value is $\lambda \approx 1.11 \times 10^{-52}\,\mathrm{m}^{-2}$. The value of the cosmological constant is so small that it plays a role only on cosmological scales, whereas it can be safely neglected at astrophysical scales and in the study of compact objects. We shall therefore neglect such term in the rest of the book. For an introduction of the cosmological implications of Einstein's equations, we refer to other monographs, e.g. Carrol (2004).