I ESONERO 25-11-2008 - Compito A

Sia dato uno spazio-tempo descritto, nel riferimento O coordinate $x^\mu = (t, x, y, r)$, dalla metrica

$$ds^2 = \frac{1}{4}\frac{r^2}{R^2}[-dt^2 + dx^2 + dy^2] + \frac{R^2 dr^2}{4r^2}$$

con $R > 0$ costante reale. I simboli di Christoffel non nulli sono:

$$\Gamma^r_{xx} = \Gamma^r_{yy} = -\frac{r^3}{R^4}$$
$$\Gamma^x_{rx} = \Gamma^x_{xr} = \frac{1}{r}$$
$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{1}{r}$$
$$\Gamma^y_{ry} = \Gamma^y_{yr} = \frac{1}{r}$$

oltre a

$$\Gamma^r_{tt},\ \Gamma^r_{rr}.$$

1. Calcolare $\Gamma_{tt}^{\ r}$.

2. Dato il tensore $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ T, di componenti nel riferimento O

$$T_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -r & 0 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & x & 0 & 0 \end{pmatrix},$$

   calcolare le componenti del tensore $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, ad esso associato, $T^\mu_{\ \nu}$.

3. Calcolare

$$T_{rx;x}\ , \qquad T_{ry;y}\ .$$

4. Sia dato il riferimento O', di coordinate $\{x^{\alpha'}\} = (t', \rho, \phi, r')$, definito dalla trasformazione di coordinate $x^\mu = x^\mu(x^{\alpha'})$

$$\begin{aligned} t &= t' \\ x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ r &= r'\,. \end{aligned}$$

   Determinare la metrica nel riferimento O'.

5. Calcolare le componenti del tensore $T_{\mu\nu}$ nel riferimento O'.

I ESONERO 25-11-2008 - Compito B

Sia dato uno spazio-tempo descritto, nel riferimento O coordinate $x^\mu = (t, x, y, r)$, dalla metrica

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1. Calcolare $\Gamma^{\ r}_{rr}$.
2. Dato il tensore $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ T, di componenti nel riferimento O

$$T_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ r & 0 & 0 & x \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

   calcolare le componenti del tensore $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, ad esso associato, $T_\mu^{\ \nu}$.
3. Calcolare

$$T_{xr;x}\,, \qquad T_{yr;y}\,.$$

4. Sia dato il riferimento O', di coordinate $\{x^{\alpha'}\} = (t', \rho, \phi, r')$, definito dalla trasformazione di coordinate $x^\mu = x^\mu(x^{\alpha'})$

$$\begin{aligned} t &= t' \\ x &= \rho \sin\phi \\ y &= \rho\cos\phi \\ r &= r'\,. \end{aligned}$$

   Determinare la metrica nel riferimento O'.
5. Calcolare le componenti del tensore $T_{\mu\nu}$ nel riferimento O'.

Soluzioni compito A

La metrica e la metrica inversa sono

$$g_{\mu\nu} = \text{diag}\left(-\frac{r^2}{4R^2}, \frac{r^2}{4R^2}, \frac{r^2}{4R^2}, \frac{R^2}{4r^2}\right)$$

$$g^{\mu\nu} = \text{diag}\left(-\frac{4R^2}{r^2}, \frac{4R^2}{r^2}, \frac{4R^2}{r^2}, \frac{4r^2}{R^2}\right).$$

1.

$$\Gamma^r_{tt} = \frac{1}{2}g^{r\alpha}(g_{t\alpha,t} + g_{t\alpha,t} - g_{tt,\alpha}) = -\frac{1}{2}g^{rr}g_{tt,r} = \frac{r^3}{R^4}$$

2.

$$\begin{aligned} T^\mu{}_\nu &= g^{\mu\alpha}T_{\alpha\nu} = \begin{pmatrix} -\frac{4R^2}{r^2} & 0 & 0 & 0 \\ 0 & \frac{4R^2}{r^2} & 0 & 0 \\ 0 & 0 & \frac{4R^2}{r^2} & 0 \\ 0 & 0 & 0 & \frac{4r^2}{R^2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -r & 0 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & x & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{4R^2}{r} & 0 & 0 & 0 \\ 0 & 0 & \frac{4R^2}{r^2}y & 0 \\ 0 & \frac{4r^2}{R^2}x & 0 & 0 \end{pmatrix}. \end{aligned}$$

3.

$$\begin{aligned} T_{rx;x} &= T_{rx,x} - \Gamma^\alpha_{rx}T_{\alpha x} - \Gamma^\alpha_{xx}T_{r\alpha} \\ &= T_{rx,x} - \Gamma^x_{rx}T_{xx} - \Gamma^r_{xx}T_{rr} = 1 \\ T_{ry;y} &= T_{ry,y} - \Gamma^\alpha_{ry}T_{\alpha y} - \Gamma^\alpha_{yy}T_{r\alpha} \\ &= T_{ry,y} - \Gamma^y_{ry}T_{yy} - \Gamma^r_{yy}T_{rr} = -\frac{y}{r}. \end{aligned}$$

4. Differenziando la legge di trasformazione delle coordinate,

$$\begin{aligned} dt &= dt' \\ dx &= \cos\phi d\rho - \rho\sin\phi d\phi \\ dy &= \sin\phi d\rho + \rho\cos\phi d\phi \\ dr &= dr' \end{aligned}$$

quindi

$$dx^2 + dy^2 = d\rho^2 + \rho^2 d\phi^2$$

e sostituendo nell'espressione della metrica si ha

$$ds^2 = \frac{1}{4}\frac{r'^2}{R^2}[-dt'^2 + d\rho^2 + \rho^2 d\phi^2] + \frac{R^2 dr'^2}{4r'^2}.$$

5.

$$T_{\alpha'\beta'} = \Lambda^{\mu}{}_{\alpha'}\Lambda^{\nu}{}_{\beta'}T_{\mu\nu} = \Lambda^{\mu}{}_{\alpha'}T_{\mu\nu}\Lambda^{\nu}{}_{\beta'}$$

e

$$\Lambda = (\Lambda^{\mu}{}_{\alpha'}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\rho\sin\phi & 0 \\ 0 & \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In forma matriciale, definendo le matrici $T = (T_{\mu\nu})$, $T' = (T_{\alpha'\beta'})$,

$$\begin{aligned}
T' &= \Lambda^T T \Lambda = \Lambda^T \begin{pmatrix} 0 & 0 & 0 & 0 \\ -r & 0 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & x & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\rho\sin\phi & 0 \\ 0 & \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\rho\sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -r & 0 & 0 & 0 \\ 0 & y\sin\phi & y\rho\cos\phi & 0 \\ 0 & x\cos\phi & -\rho x\sin\phi & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -r\cos\phi & y\sin^2\phi & y\rho\sin\phi\cos\phi & 0 \\ \rho r\sin\phi & y\rho\sin\phi\cos\phi & y\rho^2\cos^2\phi & 0 \\ 0 & x\cos\phi & -\rho x\sin\phi & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -r'\cos\phi & \rho\sin^3\phi & \rho^2\sin^2\phi\cos\phi & 0 \\ \rho r'\sin\phi & \rho^2\sin^2\phi\cos\phi & \rho^3\sin\phi\cos^2\phi & 0 \\ 0 & \rho\cos^2\phi & -\rho^2\cos\phi\sin\phi & 0 \end{pmatrix}.
\end{aligned}$$

Soluzioni compito B

La metrica e la metrica inversa sono

$$g_{\mu\nu} = \text{diag}\left(-\frac{r^2}{4R^2}, \frac{r^2}{4R^2}, \frac{r^2}{4R^2}, \frac{R^2}{4r^2}\right)$$

$$g^{\mu\nu} = \text{diag}\left(-\frac{4R^2}{r^2}, \frac{4R^2}{r^2}, \frac{4R^2}{r^2}, \frac{4r^2}{R^2}\right).$$

1.

$$\Gamma^r_{rr} = \frac{1}{2}g^{r\alpha}(g_{r\alpha,r} + g_{r\alpha,r} - g_{rr,\alpha}) = \frac{1}{2}g^{rr}g_{rr,r} = -\frac{1}{r}$$

2.

$$\begin{aligned} T_\mu{}^\nu &= T_{\mu\alpha}g^{\alpha\mu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ r & 0 & 0 & x \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{4R^2}{r^2} & 0 & 0 & 0 \\ 0 & \frac{4R^2}{r^2} & 0 & 0 \\ 0 & 0 & \frac{4R^2}{r^2} & 0 \\ 0 & 0 & 0 & \frac{4r^2}{R^2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{4R^2}{r} & 0 & 0 & \frac{4r^2}{R^2}x \\ 0 & 0 & \frac{4R^2}{r^2}y & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

3.

$$\begin{aligned} T_{xr;x} &= T_{xr,x} - \Gamma^\alpha_{rx}T_{x\alpha} - \Gamma^\alpha_{xx}T_{\alpha r} \\ &= T_{xr,x} - \Gamma^x_{rx}T_{xx} - \Gamma^r_{xx}T_{rr} = 1 \\ T_{yr;y} &= T_{yr,y} - \Gamma^\alpha_{ry}T_{y\alpha} - \Gamma^\alpha_{yy}T_{\alpha r} \\ &= T_{yr,y} - \Gamma^y_{ry}T_{yy} - \Gamma^r_{yy}T_{rr} = -\frac{y}{r}. \end{aligned}$$

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In forma matriciale, definendo le matrici $T = (T_{\mu\nu})$, $T' = (T_{\alpha'\beta'})$,

$$\begin{aligned} T' &= \Lambda^T T \Lambda = \Lambda^T \begin{pmatrix} 0 & 0 & 0 & 0 \\ r & 0 & 0 & x \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\phi & \rho\cos\phi & 0 \\ 0 & \cos\phi & -\rho\sin\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & \rho\cos\phi & -\rho\sin\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ r & 0 & 0 & x \\ 0 & y\cos\phi & -y\rho\sin\phi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ r\sin\phi & y\cos^2\phi & -y\rho\sin\phi\cos\phi & x\sin\phi \\ \rho r\cos\phi & -y\rho\sin\phi\cos\phi & y\rho^2\sin^2\phi & \rho x\cos\phi \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ r'\sin\phi & \rho\cos^3\phi & -\rho^2\sin\phi\cos^2\phi & \rho\sin^2\phi \\ \rho r'\cos\phi & -\rho^2\sin\phi\cos^2\phi & \rho^3\sin^2\phi\cos\phi & \rho^2\sin\phi\cos\phi \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$