Chapter 6

The Curvature Tensor

We are now in a position to introduce the curvature tensor. We will do it in two different ways.

6.1 a) A Formal Approach

Let us start writing the transformation rule for affine connections

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial x^{\tau'}} \frac{\partial x^{\rho'}}{\partial x^{\mu}} \frac{\partial x^{\sigma'}}{\partial x^{\nu}} \Gamma^{\tau'}{}_{\rho'\sigma'} + \frac{\partial x^{\lambda}}{\partial x^{\tau'}} \frac{\partial^2 x^{\tau'}}{\partial x^{\mu} \partial x^{\nu}}. \tag{6.1}$$

As we already noticed (Chapter V sec. 5) if the last term on the right-hand side would be zero $\Gamma^{\lambda}{}_{\mu\nu}$ would transform as a tensor. Let us isolate the 'bad term', by multiplying eq. (6.1) by $\frac{\partial x^{\tau'}}{\partial x^{\lambda}}$:

$$\frac{\partial^2 x^{\tau'}}{\partial x^{\mu} \partial x^{\nu}} = \frac{\partial x^{\tau'}}{\partial x^{\lambda}} \Gamma^{\lambda}{}_{\mu\nu} - \frac{\partial x^{\rho'}}{\partial x^{\mu}} \frac{\partial x^{\sigma'}}{\partial x^{\nu}} \Gamma^{\tau'}{}_{\rho'\sigma'}. \tag{6.2}$$

We now differentiate this equation with respect to x^k

$$\begin{aligned} \frac{\partial^3 x^{\tau'}}{\partial x^k \partial x^{\mu} \partial x^{\nu}} &= \frac{\partial^2 x^{\tau'}}{\partial x^k \partial x^{\lambda}} \Gamma^{\lambda}{}_{\mu\nu} + \frac{\partial x^{\tau'}}{\partial x^{\lambda}} \left(\frac{\partial}{\partial x^k} \Gamma^{\lambda}{}_{\mu\nu} \right) \\ &- \frac{\partial^2 x^{\rho'}}{\partial x^k \partial x^{\mu}} \frac{\partial x^{\sigma'}}{\partial x^{\nu}} \Gamma^{\tau'}{}_{\rho'\sigma'} - \frac{\partial x^{\rho'}}{\partial x^{\mu}} \frac{\partial^2 x^{\sigma'}}{\partial x^k \partial x^{\nu}} \Gamma^{\tau'}{}_{\rho'\sigma'} - \frac{\partial x^{\rho'}}{\partial x^{\mu}} \frac{\partial x^{\sigma'}}{\partial x^{\nu}} \left(\frac{\partial}{\partial x^k} \Gamma^{\tau'}{}_{\rho'\sigma'} \right). \end{aligned} \tag{6.3}$$

We now use eq. (6.2):

$$\begin{aligned} &\frac{\partial^3 x^{\tau'}}{\partial x^k \partial x^{\mu} \partial x^{\nu}} = \\ &+\Gamma^{\lambda}{}_{\mu\nu} \left[\frac{\partial x^{\tau'}}{\partial x^{\alpha}} \Gamma^{\alpha}{}_{k\lambda} - \frac{\partial x^{i'}}{\partial x^k} \frac{\partial x^{j'}}{\partial x^{\lambda}} \Gamma^{\tau'}{}_{i'j'} \right] + \frac{\partial x^{\tau'}}{\partial x^{\lambda}} \left[\frac{\partial}{\partial x^k} \Gamma^{\lambda}{}_{\mu\mu} \right] \\ &- \frac{\partial x^{\sigma'}}{\partial x^{\nu}} \Gamma^{\tau'}{}_{\rho'\sigma'} \left[\frac{\partial x^{\rho'}}{\partial x^{\alpha}} \Gamma^{\alpha}{}_{k\mu} - \frac{\partial x^{i'}}{\partial x^k} \frac{\partial x^{j'}}{\partial x^{\mu}} \Gamma^{\rho'}{}_{i'j'} \right] \end{aligned} \tag{6.4}$$

$$-\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\left[\frac{\partial x^{\sigma\prime}}{\partial x^{\alpha}}\Gamma^{\alpha}{}_{k\nu}-\frac{\partial x^{i\prime}}{\partial x^{k}}\frac{\partial x^{j\prime}}{\partial x^{\nu}}\Gamma^{\sigma\prime}{}_{i\prime j\prime}\right]$$
$$-\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\left(\frac{\partial}{\partial x^{k}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\right).$$

Let us rewrite the last term as

$$\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\frac{\partial x^{\eta\prime}}{\partial x^{k}}\left(\frac{\partial}{\partial x^{\eta\prime}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\right). \tag{6.5}$$

(The reason is that the indices of Γ have a prime, thus the derivatives must be computed with respect to the $\{x^{\alpha\prime}\}$). We now rewrite eq. (6.5) in the following way

$$\frac{\partial^3 x^{\tau\prime}}{\partial x^k\partial x^\mu\partial x^\nu}= \tag{6.6}$$
$$\left[\frac{\partial x^{\tau\prime}}{\partial x^{\lambda}}\left(\frac{\partial}{\partial x^{k}}\Gamma^{\lambda}{}_{\mu\nu}\right)+\left(\frac{\partial x^{\tau\prime}}{\partial x^{\alpha}}\Gamma^{\alpha}{}_{k\lambda}\Gamma^{\lambda}{}_{\mu\nu}\right)\right]$$
$$-\left[\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\frac{\partial x^{\eta\prime}}{\partial x^{k}}\left(\frac{\partial}{\partial x^{\eta\prime}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\right)-\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\frac{\partial x^{i\prime}}{\partial x^{k}}\frac{\partial x^{j\prime}}{\partial x^{\mu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\Gamma^{\rho\prime}{}_{i\prime j\prime}\right]$$
$$-\left[\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\frac{\partial x^{i\prime}}{\partial x^{k}}\frac{\partial x^{j\prime}}{\partial x^{\nu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\Gamma^{\sigma\prime}{}_{i\prime j\prime}\right]$$
$$-\left[\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\frac{\partial x^{\rho\prime}}{\partial x^{\alpha}}\Gamma^{\alpha}{}_{k\mu}+\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\frac{\partial x^{\sigma\prime}}{\partial x^{\alpha}}\Gamma^{\alpha}{}_{k\nu}+\frac{\partial x^{i\prime}}{\partial x^{k}}\frac{\partial x^{j\prime}}{\partial x^{\lambda}}\Gamma^{\lambda}{}_{\mu\mu}\Gamma^{\tau\prime}{}_{i\prime j\prime}\right].$$

We now relabel the indices in the following way

$$\frac{\partial x^{\tau\prime}}{\partial x^{\alpha}}\Gamma^{\alpha}{}_{k\lambda}\Gamma^{\lambda}{}_{\mu\nu}\rightarrow\frac{\partial x^{\tau\prime}}{\partial x^{\lambda}}\Gamma^{\lambda}{}_{k\eta}\Gamma^{\eta}{}_{\mu\nu} \tag{6.7}$$

$$\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\frac{\partial x^{i\prime}}{\partial x^{k}}\frac{\partial x^{j\prime}}{\partial x^{\mu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\Gamma^{\rho\prime}{}_{i\prime j\prime}\rightarrow\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\frac{\partial x^{\eta\prime}}{\partial x^{k}}\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\Gamma^{\tau\prime}{}_{\lambda\prime\sigma\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\rho\prime}$$
$$\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\frac{\partial x^{i\prime}}{\partial x^{k}}\frac{\partial x^{j\prime}}{\partial x^{\nu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\Gamma^{\sigma\prime}{}_{i\prime j\prime}\rightarrow\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\frac{\partial x^{\eta\prime}}{\partial x^{k}}\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\Gamma^{\tau\prime}{}_{\rho\prime\lambda\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\sigma\prime}$$

$$\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\frac{\partial x^{\rho\prime}}{\partial x^{\alpha}}\Gamma^{\alpha}{}_{k\mu}\rightarrow\frac{\partial x^{\rho\prime}}{\partial x^{\nu}}\Gamma^{\tau\prime}{}_{\sigma\prime\rho\prime}\frac{\partial x^{\sigma\prime}}{\partial x^{\lambda}}\Gamma^{\lambda}{}_{k\mu}$$
$$\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\frac{\partial x^{\sigma\prime}}{\partial x^{\alpha}}\Gamma^{\alpha}{}_{k\nu}\rightarrow\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\frac{\partial x^{\sigma\prime}}{\partial x^{\lambda}}\Gamma^{\lambda}{}_{k\nu}$$
$$\frac{\partial x^{i\prime}}{\partial x^{k}}\frac{\partial x^{j\prime}}{\partial x^{\lambda}}\Gamma^{\lambda}{}_{\mu\mu}\Gamma^{\tau\prime}{}_{i\prime j\prime}\rightarrow\frac{\partial x^{\rho\prime}}{\partial x^{k}}\frac{\partial x^{\sigma\prime}}{\partial x^{\lambda}}\Gamma^{\lambda}{}_{\mu\mu}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}$$

With these changes the terms can be collected in the following way

$$\frac{\partial^3 x^{\tau\prime}}{\partial x^k\partial x^\mu\partial x^\nu}=\frac{\partial x^{\tau\prime}}{\partial x^{\lambda}}\left[\left(\frac{\partial}{\partial x^{k}}\Gamma^{\lambda}{}_{\mu\nu}\right)+\Gamma^{\lambda}{}_{k\eta}\Gamma^{\eta}{}_{\mu\nu}\right] \tag{6.8}$$
$$-\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}\frac{\partial x^{\sigma\prime}}{\partial x^{\nu}}\frac{\partial x^{\eta\prime}}{\partial x^{k}}\left[\left(\frac{\partial}{\partial x^{\eta\prime}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\right)-\Gamma^{\tau\prime}{}_{\lambda\prime\sigma\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\rho\prime}-\Gamma^{\tau\prime}{}_{\rho\prime\lambda\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\sigma\prime}\right]$$
$$-\frac{\partial x^{\sigma\prime}}{\partial x^{\lambda}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\left[\Gamma^{\lambda}{}_{k\mu}\frac{\partial x^{\rho\prime}}{\partial x^{\nu}}+\Gamma^{\lambda}{}_{k\nu}\frac{\partial x^{\rho\prime}}{\partial x^{\mu}}+\Gamma^{\lambda}{}_{\mu\nu}\frac{\partial x^{\rho\prime}}{\partial x^{k}}\right].$$

We now subtract from this expression the same expression with k and ν interchanged

$$
\begin{aligned}
&\frac{\partial^3 x^{\tau\prime}}{\partial x^k \partial x^\mu \partial x^\nu} - \frac{\partial^3 x^{\tau\prime}}{\partial x^\nu \partial x^\mu \partial x^k} = 0 = \qquad (6.9)\\
&\frac{\partial x^{\tau\prime}}{\partial x^\lambda}\left[\left(\frac{\partial}{\partial x^k}\Gamma^\lambda{}_{\mu\nu}\right) + \Gamma^\lambda{}_{k\eta}\Gamma^\eta{}_{\mu\nu}\right]\\
&-\frac{\partial x^{\rho\prime}}{\partial x^\mu}\frac{\partial x^{\sigma\prime}}{\partial x^\nu}\frac{\partial x^{\eta\prime}}{\partial x^k}\left[\left(\frac{\partial}{\partial x^{\eta\prime}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\right) - \Gamma^{\tau\prime}{}_{\lambda\prime\sigma\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\rho\prime} - \Gamma^{\tau\prime}{}_{\rho\prime\lambda\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\sigma\prime}\right]\\
&-\frac{\partial x^{\sigma\prime}}{\partial x^\lambda}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\left[\Gamma^\lambda{}_{k\mu}\frac{\partial x^{\rho\prime}}{\partial x^\nu} + \Gamma^\lambda{}_{k\nu}\frac{\partial x^{\rho\prime}}{\partial x^\mu} + \Gamma^\lambda{}_{\mu\nu}\frac{\partial x^{\rho\prime}}{\partial x^k}\right] -\\
&\frac{\partial x^{\tau\prime}}{\partial x^\lambda}\left[\left(\frac{\partial}{\partial x^\nu}\Gamma^\lambda{}_{\mu k}\right) + \Gamma^\lambda{}_{\nu\eta}\Gamma^\eta{}_{\mu k}\right]\\
&+\frac{\partial x^{\rho\prime}}{\partial x^\mu}\frac{\partial x^{\sigma\prime}}{\partial x^k}\frac{\partial x^{\eta\prime}}{\partial x^\nu}\left[\left(\frac{\partial}{\partial x^{\eta\prime}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\right) - \Gamma^{\tau\prime}{}_{\lambda\prime\sigma\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\rho\prime} - \Gamma^{\tau\prime}{}_{\rho\prime\lambda\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\sigma\prime}\right]\\
&+\frac{\partial x^{\sigma\prime}}{\partial x^\lambda}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime}\left[\Gamma^\lambda{}_{\nu\mu}\frac{\partial x^{\rho\prime}}{\partial x^k} + \Gamma^\lambda{}_{\nu k}\frac{\partial x^{\rho\prime}}{\partial x^\mu} + \Gamma^\lambda{}_{\mu k}\frac{\partial x^{\rho\prime}}{\partial x^\nu}\right]
\end{aligned}
$$

collecting all terms we find

$$
\begin{aligned}
&\frac{\partial x^{\tau\prime}}{\partial x^\lambda}\left[\frac{\partial}{\partial x^k}\Gamma^\lambda{}_{\mu\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\lambda{}_{\mu k} + \Gamma^\lambda{}_{k\eta}\Gamma^\eta{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\eta}\Gamma^\eta{}_{\mu k}\right] \qquad (6.10)\\
&-\frac{\partial x^{\rho\prime}}{\partial x^\mu}\frac{\partial x^{\sigma\prime}}{\partial x^\nu}\frac{\partial x^{\eta\prime}}{\partial x^k}\left[\frac{\partial}{\partial x^{\eta\prime}}\Gamma^{\tau\prime}{}_{\rho\prime\sigma\prime} - \frac{\partial}{\partial x^{\sigma\prime}}\Gamma^{\tau\prime}{}_{\rho\prime\eta\prime} + \Gamma^{\tau\prime}{}_{\lambda\prime\eta\prime}\Gamma^{\lambda\prime}{}_{\sigma\prime\rho\prime} - \Gamma^{\tau\prime}{}_{\lambda\prime\sigma\prime}\Gamma^{\lambda\prime}{}_{\eta\prime\rho\prime}\right] = 0.
\end{aligned}
$$

If we now define the following [1]

$$
R^\lambda{}_{\mu\nu k} = -\left[\frac{\partial}{\partial x^k}\Gamma^\lambda{}_{\mu\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\lambda{}_{\mu k} + \Gamma^\lambda{}_{k\eta}\Gamma^\eta{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\eta}\Gamma^\eta{}_{\mu k}\right], \qquad (6.11)
$$

we can write eq. (6.10) as the transformation law for the tensor

$$
R^{\sigma\prime}{}_{\alpha\prime\beta\prime\gamma\prime} = \frac{\partial x^{\sigma\prime}}{\partial x^\lambda}\frac{\partial x^\mu}{\partial x^{\alpha\prime}}\frac{\partial x^\nu}{\partial x^{\beta\prime}}\frac{\partial x^k}{\partial x^{\gamma\prime}}R^\lambda{}_{\mu\nu k}. \qquad (6.12)
$$

The tensor (6.11) is **The Curvature Tensor**, also called **The Riemann Tensor**, and it can be shown that it is the only tensor that can be constructed by using the metric, its first and second derivatives, and which is linear in the second derivatives.

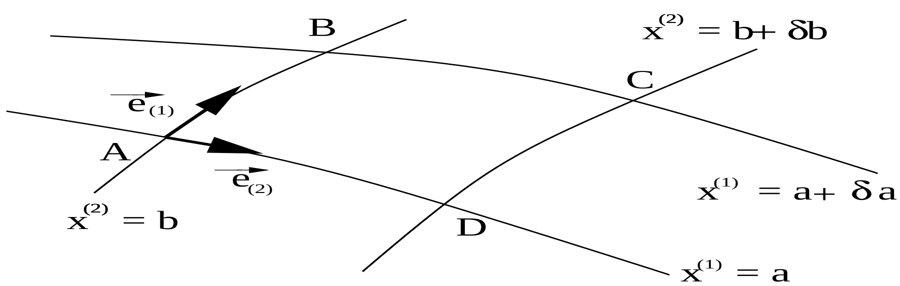
This way of defining the Riemann tensor is the "old-fashioned way": it is based on the transformation properties of the affine connections. The idea underlying this derivation is that the information about the curvature of the space must be contained in the second derivative of the metric, and therefore in the first derivative of the $\Gamma^\alpha{}_{\mu\nu}$. But since we want to find a tensor out of them, we must eliminate in eq. (6.1) the part which does not transform as a tensor, and we do this in eq. (6.9).

[1]The - sign does not agree with the definition given in Weinberg, but it does agree with the definition given in many other textbooks. As we shall see in the next section it is irrelevant. What is important is to write the Einstein equations with the right signs!

6.2 b) The curvature tensor and the curvature of the spacetime

We shall now rederive the curvature tensor in a different way that explicitely shows why it espresses the curvature of a spacetime. This derivation, due to Levi Civita, will use the notion of parallel transport of a vector along a closed loop.

Consider a closed loop whose four sides are the coordinates lines $x^1 = a$, $x^1 = a + \delta a$, $x^2 = b$, $x^2 = b + \delta b$



Take a generic vector $\vec{V}$ and parallely transport $\vec{V}$ along AB, i.e. consider $\nabla_{\vec{e}_{(1)}}\vec{V} = 0$. From eq. (5.56) it follows that

$$e^{\mu}_{(1)} V^{\alpha}{}_{;\mu} = 0. \tag{6.13}$$

Since $\vec{e}_{(1)}$ has only $e^1_{(1)} \neq 0$ then

$$\frac{\partial V^\alpha}{\partial x^1} + \Gamma^\alpha{}_{\beta 1} V^\beta = 0. \tag{6.14}$$

This equation can be integrated along the line AB:

$$\delta V^\alpha_{AB} = -\int_{A(x^2=b)}^{B} \Gamma^\alpha{}_{\beta 1} V^\beta dx^1. \tag{6.15}$$

In a similar way, if we go from B to C along the line $x^1 = a + \delta a$

$$\frac{\partial V^\alpha}{\partial x^2} = -\Gamma^\alpha{}_{\beta 2} V^\beta \qquad \rightarrow \qquad \delta V^\alpha_{BC} = -\int_{B(x^1=a+\delta a)}^{C} \Gamma^\alpha{}_{\beta 2} V^\beta dx^2. \tag{6.16}$$

From C to D

$$\frac{\partial V^\alpha}{\partial x^1} = -\Gamma^\alpha{}_{\beta 1} V^\beta \qquad \rightarrow \qquad \delta V^\alpha_{CD} = -\int_{C(x^2=b+\delta b)}^{D} \Gamma^\alpha{}_{\beta 1} V^\beta dx^1, \tag{6.17}$$

and from D back to A

$$\frac{\partial V^\alpha}{\partial x^2} = -\Gamma^\alpha{}_{\beta 2} V^\beta \qquad \rightarrow \qquad \delta V^\alpha_{DA} = -\int_{D(x^1=a)}^{A} \Gamma^\alpha{}_{\beta 2} V^\beta dx^2. \tag{6.18}$$

The change in $\vec{V}$ due to this parallel transport will be a vector $\delta\vec{V}$ whose components can be found by adding eqs. (6.15)-(6.18):

$$\begin{aligned} \delta V^\alpha = & -\int_{D(x^1=a)}^{A} \Gamma^\alpha{}_{\beta 2} V^\beta dx^2 \\ & -\int_{B(x^1=a+\delta a)}^{C} \Gamma^\alpha{}_{\beta 2} V^\beta dx^2 - \int_{C(x^2=b+\delta b)}^{D} \Gamma^\alpha{}_{\beta 1} V^\beta dx^1 \\ & -\int_{A(x^2=b)}^{B} \Gamma^\alpha{}_{\beta 1} V^\beta dx^1 . \end{aligned} \tag{6.19}$$

If the spacetime is flat V^α do not change when the vector is paralleley transported, i.e. $\delta V^\alpha = 0$. **But in curved spacetime δV^α will in general be different from zero**.

If we consider an infinitesimal loop, i.e. δa and δb tend to zero, we can take an expansion of eq. (6.19) to first order in δa and δb:

$$\begin{aligned} \delta V^\alpha \simeq & -\int_{D(x^1=a)}^{A} \Gamma^\alpha{}_{\beta 2} V^\beta dx^2 - \\ & \left[\int_{B(x^1=a)}^{C} \Gamma^\alpha{}_{\beta 2} V^\beta dx^2 + \frac{\partial}{\partial x^1}\left(\int_B^C \Gamma^\alpha{}_{\beta 2} V^\beta dx^2\right)\delta a\right] \\ & -\left[\int_{C(x^2=b)}^{D} \Gamma^\alpha{}_{\beta 1} V^\beta dx^1 + \frac{\partial}{\partial x^2}\left(\int_C^D \Gamma^\alpha{}_{\beta 1} V^\beta dx^1\right)\delta b\right] \\ & -\int_{A(x^2=b)}^{B} \Gamma^\alpha{}_{\beta 1} V^\beta dx^1 , \end{aligned} \tag{6.20}$$

Since

$$A = (a,b), \quad C = (a+\delta a, b+\delta b), \quad B = (a+\delta a, b), \quad \text{and} \quad D = (a, b+\delta b), \tag{6.21}$$

the previous equation becomes

$$\begin{aligned} \delta V^\alpha \simeq & +\int_b^{b+\delta b} \Gamma^\alpha{}_{\beta 2} V^\beta dx^2 \\ & -\int_b^{b+\delta b} \Gamma^\alpha{}_{\beta 2} V^\beta dx^2 - \left[\int_b^{b+\delta b} \frac{\partial}{\partial x^1}\left(\Gamma^\alpha{}_{\beta 2} V^\beta\right) dx^2\right]\delta a \\ & +\int_a^{a+\delta a} \Gamma^\alpha{}_{\beta 1} V^\beta dx^1 + \left[\int_a^{a+\delta a} \frac{\partial}{\partial x^2}\left(\Gamma^\alpha{}_{\beta 1} V^\beta\right) dx^1\right]\delta b \\ & -\int_a^{a+\delta a} \Gamma^\alpha{}_{\beta 1} V^\beta dx^1 , \end{aligned} \tag{6.22}$$

i.e.

$$\begin{aligned} \delta V^\alpha \simeq & -\delta a \int_b^{b+\delta b} \frac{\partial}{\partial x^1}\left(\Gamma^\alpha{}_{\beta 2} V^\beta\right) dx^2 \\ & +\delta b \int_a^{a+\delta a} \frac{\partial}{\partial x^2}\left(\Gamma^\alpha{}_{\beta 1} V^\beta\right) dx^1 \simeq \delta a \delta b \left[-\frac{\partial}{\partial x^1}\left(\Gamma^\alpha{}_{\beta 2} V^\beta\right) + \frac{\partial}{\partial x^2}\left(\Gamma^\alpha{}_{\beta 1} V^\beta\right)\right]. \end{aligned} \tag{6.23}$$

Eq. (6.23) can be further developed by using eq. (6.14)

$$\frac{\partial V^k}{\partial x^1} = -\Gamma^k{}_{\beta 1} V^\beta, \qquad \frac{\partial V^k}{\partial x^2} = -\Gamma^k{}_{\beta 2} V^\beta \, ; \tag{6.24}$$

it becomes

$$\begin{aligned} \delta V^\alpha &= \delta a \delta b \left[\frac{\partial \Gamma^\alpha{}_{\beta 1}}{\partial x^2} V^\beta + \Gamma^\alpha{}_{k1} \frac{\partial V^k}{\partial x^2} - \frac{\partial \Gamma^\alpha{}_{\beta 2}}{\partial x^1} V^\beta - \Gamma^\alpha{}_{k2} \frac{\partial V^k}{\partial x^1} \right] \\ &= \delta a \delta b \left[\frac{\partial \Gamma^\alpha{}_{\beta 1}}{\partial x^2} - \frac{\partial \Gamma^\alpha{}_{\beta 2}}{\partial x^1} - \Gamma^\alpha{}_{k1} \Gamma^k{}_{\beta 2} + \Gamma^\alpha{}_{k2} \Gamma^k{}_{\beta 1} \right] V^\beta . \end{aligned} \tag{6.25}$$

Note that:

- δa and δb are the non vanishing components of the displacement vectors $\vec{\delta x}_{(1)}$ and $\vec{\delta x}_{(2)}$ along the direction of the basis vectors $\vec{e}_{(1)}$ and $\vec{e}_{(2)}$, i.e.

$$\begin{aligned} \delta x^\mu_{(1)} &= (0, \delta a, 0, 0) = \delta a \; \delta^\mu_1 \, , \\ \delta x^\mu_{(2)} &= (0, 0, \delta b, 0) = \delta b \; \delta^\mu_2 \, . \end{aligned} \tag{6.26}$$

  Thus, we can write eq. (6.25) as follows

$$\delta V^\alpha = \delta x^\nu_{(1)} \; \delta x^\mu_{(2)} \; \left[\frac{\partial \Gamma^\alpha{}_{\beta\nu}}{\partial x^\mu} - \frac{\partial \Gamma^\alpha{}_{\beta\mu}}{\partial x^\nu} - \Gamma^\alpha{}_{k\nu} \Gamma^k{}_{\beta\mu} + \Gamma^\alpha{}_{k\mu} \Gamma^k{}_{\beta\nu} \right] V^\beta \, . \tag{6.27}$$

- The term in square brackets is the curvature tensor which we have already defined in eq. (6.11):

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} - \Gamma^\alpha{}_{k\nu} \Gamma^k{}_{\beta\mu} + \Gamma^\alpha{}_{k\mu} \Gamma^k{}_{\beta\nu} \, . \tag{6.28}$$

  Note that it is antisymmetric in ν and μ; indeed, it must be because, if we interchange $\vec{\delta x}_{(1)}$ and $\vec{\delta x}_{(2)}$, δV^α changes sign, because we would go around the loop in the opposite direction. This shows that the sign of (6.28) can be chosen arbitrarily, and this is the reason why the definitions of the Riemann tensor given in textbooks may differ for a sign.

We have already shown that the object given in eq. (6.28) is a tensor, by looking at the way it transforms under a coordinate transformation (eq. 6.12). However, we want to see if it also agrees with the definition of tensors given in chapter 4. Let us contract eq. (6.27) with V_α.

$$\delta V^\alpha V_\alpha = \delta x^\nu_{(1)} \; \delta x^\mu_{(2)} \; \left[\frac{\partial \Gamma^\alpha{}_{\beta\nu}}{\partial x^\mu} - \frac{\partial \Gamma^\alpha{}_{\beta\mu}}{\partial x^\nu} - \Gamma^\alpha{}_{k\nu} \Gamma^k{}_{\beta\mu} + \Gamma^\alpha{}_{k\mu} \Gamma^k{}_{\beta\nu} \right] V^\beta V_\alpha . \tag{6.29}$$

The result of this contraction is, of course, a number. On the right-hand side there are the components of 3 vectors i.e.: $\delta x^\nu_{(1)}$, $\delta x^\mu_{(2)}$ and V^β; moreover there are the components of the one-form V_α. The four geometrical objects (three vectors and one one-form) are contracted with the quantity within brackets, and the result is a number. In addition, we note that (6.29) is linear in $V^\beta, V_\alpha, \delta x^\nu_{(1)} \; \delta x^\mu_{(2)}$. For instance, if we consider a displacement $\delta x^\nu_{(1a)} + \delta x^\nu_{(1b)}$ along $\vec{e}_{(1)}$ it is immediate to see that

$$\delta V^\alpha V_\alpha = \delta x^\nu_{(1a)} \delta x^\mu_{(2)} \, [...] \, V^\beta V_\alpha + \delta x^\nu_{(1b)} \delta x^\mu_{(2)} \, [...] \, V^\beta V_\alpha \, , \tag{6.30}$$

and similarly for the other quantities. If we consider a generic $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ tensor, $T^{\alpha}{}_{\beta\gamma\delta}$, since by definition it is a linear function of one one-form and three vectors, when supplied with these arguments (for example the one-form $\tilde{V}$, and the three vectors $\vec{V}$, $\vec{\delta x}_{(1)}$ and $\vec{\delta x}_{(2)}$ it will produce the following number

$$T(\tilde{V}, \vec{V}, \vec{\delta x}_{(1)}, \vec{\delta x}_{(2)}) = T^{\alpha}{}_{\beta\rho\delta} V_{\alpha} V^{\beta} \delta x^{\rho}_{(1)}\ \delta x^{\delta}_{(2)}\ . \tag{6.31}$$

Eq. (6.31) has the same structure of eq. (6.29). Therefore we are entitled to define the components of the Riemann tensor as in eq. (6.28).

It should now be clear why the Riemann tensor deserves its name of **Curvature Tensor**: it tells us how does a vector change when it is parallely transported along a loop, due to the curvature of the spacetime. If the spacetime is flat

$$\delta V^{\alpha} = 0 \qquad \text{along any closed loop} \qquad \rightarrow \qquad R^{\alpha}{}_{\beta\gamma\delta} = 0, \tag{6.32}$$

in any reference frame. Indeed, if a tensor vanishes in a given frame, then it vanishes in any other frame.

The components of the Riemann tensor assume a very nice form when computed in a locally inertial frame:

$$R^{\alpha}{}_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left[g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu} \right], \tag{6.33}$$

or lowering the index α

$$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda} R^{\lambda}{}_{\beta\mu\nu} = \frac{1}{2} \left[g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu} \right]. \tag{6.34}$$

It should be stressed that

1) The Riemann tensor is linear in the second derivatives of $g_{\mu\nu}$, and non linear in the first derivatives.

2) In a locally inertial frame the $\Gamma^{\alpha}{}_{\nu\sigma}$ vanish and therefore the non-linear part of the Riemann tensor vanishes as well.

6.3 Symmetries

From eq. (6.34) it is easy to verify that

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}, \tag{6.35}$$

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0. \tag{6.36}$$

Since $R_{\alpha\beta\mu\nu}$ is a tensor, these symmetry properties are valid in any reference frame. The symmetries of the Riemann tensor reduce the number of independent components to 20.

6.4 The Riemann tensor gives the commutator of covariant derivatives

Let us consider the second covariant derivatives of a vector field $\vec{V}$

$$\nabla_\alpha \nabla_\beta V^\mu = \nabla_\alpha (V^\mu{}_{;\beta}) = (V^\mu{}_{;\beta})_{,\alpha} + \Gamma^\mu{}_{\sigma\alpha} V^\sigma{}_{;\beta} - \Gamma^\sigma{}_{\beta\alpha} V^\mu{}_{;\sigma}. \tag{6.37}$$

In a locally inertial frame $\Gamma^\mu{}_{\sigma\alpha} = 0$, and eq. (6.37) becomes

$$\nabla_\alpha \nabla_\beta V^\mu = (V^\mu{}_{;\beta})_{,\alpha} = V^\mu{}_{,\beta,\alpha} + \Gamma^\mu{}_{\nu\beta,\alpha} V^\nu. \tag{6.38}$$

By interchanging α and β

$$\nabla_\beta \nabla_\alpha V^\mu = (V^\mu{}_{;\alpha})_{,\beta} = V^\mu{}_{,\alpha,\beta} + \Gamma^\mu{}_{\nu\alpha,\beta} V^\nu. \tag{6.39}$$

The commutator of the covariant derivatives then is

$$[\nabla_\alpha, \nabla_\beta]\, V^\mu = \nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu = \left(\Gamma^\mu{}_{\nu\beta,\alpha} - \Gamma^\mu{}_{\nu\alpha,\beta}\right) V^\nu. \tag{6.40}$$

Since in a locally inertial frame

$$R^\lambda{}_{\mu\nu k} = \Gamma^\lambda{}_{\mu k,\nu} - \Gamma^\lambda{}_{\mu\nu,k} \tag{6.41}$$

(equivalent to eq. 6.33), eq. (6.40) becomes

$$[\nabla_\alpha, \nabla_\beta]\, V^\mu = R^\mu{}_{\nu\alpha\beta} V^\nu. \tag{6.42}$$

This is a tensor equation and since it is valid in a given reference frame, it will be valid in **any** frame. Eq. (6.42) implies that in curved spacetime covariant derivatives **do not commute** and therefore the order in which they appear is important.

6.5 The Bianchi identities

Let us differentiate eq. (6.34) with respect to x^λ (and rememeber that it is valid in a locally inertial frame)

$$R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2}\left[g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda}\right]. \tag{6.43}$$

By using the fact that $g_{\alpha\beta}$ is symmetric and eq. (6.43) one can show that

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0. \tag{6.44}$$

Since it is valid in a locally inertial frame and it is a tensor equation, it will be valid in any frame:

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0, \tag{6.45}$$

where we have replaced the ordinary derivative with the covariant derivative. **These are the Bianchi identities that, as we shall see, play an important role in the development of the theory.**