EINSTEIN FIELD EQUATIONS

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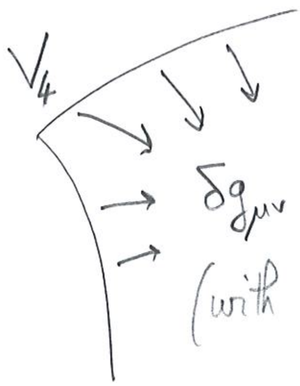
They derive from the action

$$\boxed{S = \frac{c^3}{16\pi G}\int d^4x \sqrt{-g}\, R \; + S_m[g, \Psi_m]}$$

Einstein-Hilbert action

matter action (all matter fields universally coupled to the metric $g_{\mu\nu}$)



(with $\delta g_{\mu\nu} = 0$ when $|x^\lambda| \to \infty$)

10 differential equations of second order

$$\underbrace{G^{\mu\nu}[g, \partial g, \partial^2 g]}_{\text{Einstein tensor}} = \frac{8\pi G}{c^4}\underbrace{T^{\mu\nu}[g]}_{\text{stress-energy tensor}}$$

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \qquad T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$$

4 eqs. give the evolution of matter fields

$$\boxed{\nabla_\nu G^{\mu\nu} \equiv 0 \quad \Longrightarrow \quad \nabla_\nu T^{\mu\nu} = 0}$$

contracted Bianchi identity (or Einstein identity)

Geometry is governed by 6 eqs., 4 eqs. can be imposed by a choice of coordinates

$$h^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} - \eta^{\mu\nu} \qquad \eta^{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

auxiliary Minkowski metric (signature $-+++$)

Choice of coordinates $\boxed{\partial_\nu h^{\mu\nu} = 0}$

Harmonic or de Donder

$$\boxed{\Box h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu}}$$

ordinary flat d'Alembertian $\Box = \eta^{\rho\sigma}\partial_\rho\partial_\sigma$

stress-energy pseudo tensor (actually a Lorentz tensor) of matter and gravitational fields (in harm. coordinates)

$$\tau^{\mu\nu} = |g|\, T^{\mu\nu} + \underbrace{\frac{c^4}{16\pi G} \Lambda^{\mu\nu}(h, \partial h, \partial^2 h)}_{\text{includes all non-linearities of Einstein's eqs.}}$$

$\Lambda^{\mu\nu} = O(h^2)$

Harmonic coordinate condition is equivalent to matter equation

$$\boxed{\partial_\nu h^{\mu\nu} = 0 \iff \partial_\nu \tau^{\mu\nu} = 0 \iff \nabla_\nu T^{\mu\nu} = 0}$$

NO-INCOMING RADIATION CONDITION

Boundary conditions are imposed at past null infinity (case where $T^{\mu\nu}$ has a spatially compact support)

Spatio-temporal infinities

I^+ = future temporal infinity $\begin{pmatrix} t \to +\infty \\ r = \text{const} \end{pmatrix}$

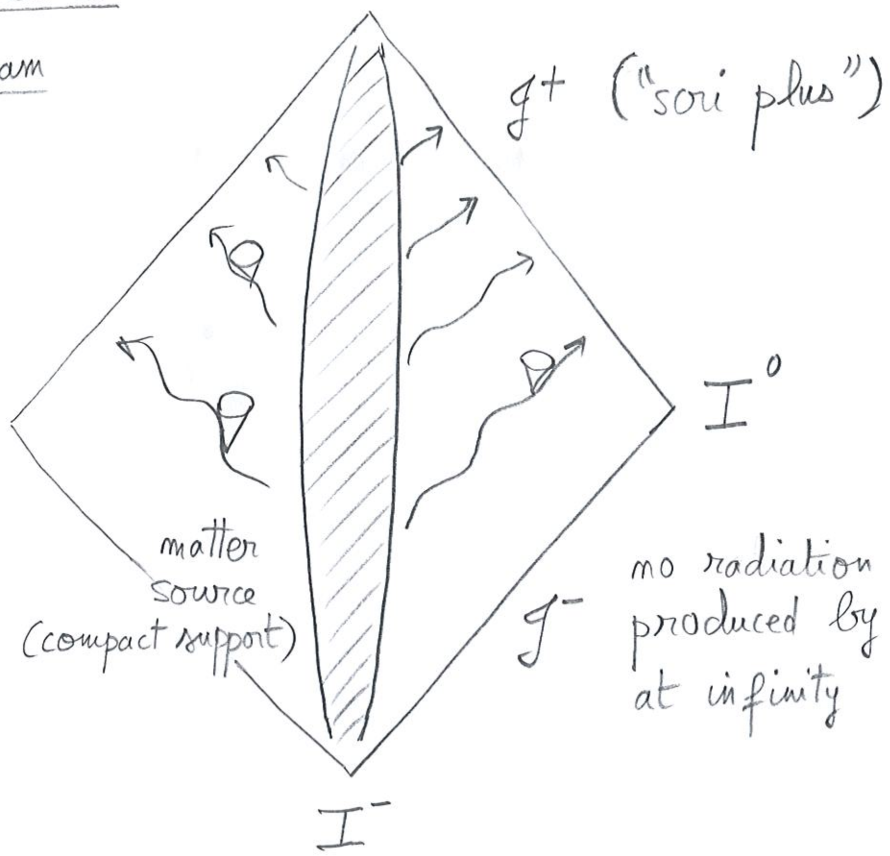
$\mathcal{J}^+$ = future null infinity $\begin{pmatrix} r \to +\infty \\ t - r/c = \text{const} \end{pmatrix}$

I^0 = spatial infinity $\begin{pmatrix} r \to +\infty \\ t = \text{const} \end{pmatrix}$

$\mathcal{J}^-$ = past null infinity $\begin{pmatrix} r \to +\infty \\ t + r/c = \text{const} \end{pmatrix}$

I^- = past temporal infinity $\begin{pmatrix} t \to -\infty \\ r = \text{const} \end{pmatrix}$

Carter-Penrose diagram

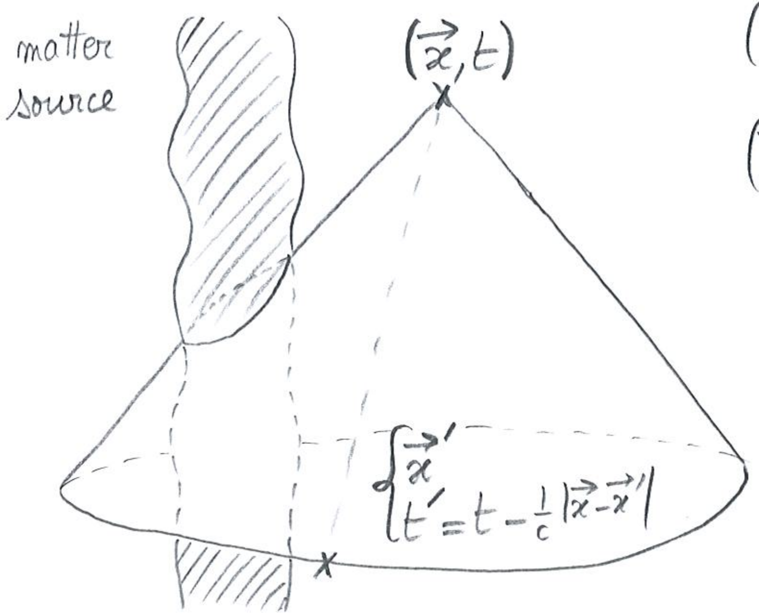


no radiation is produced by sources located at infinity

Kirchhoff's formula for the homogeneous sol. of

$$\Box\, h_{Hom} = 0$$

$$\boxed{h_{Hom}(\vec{x},t) = \lim_{|\vec{x}'|\longrightarrow\infty} \iint \frac{d\Omega'}{4\pi}\left(\frac{\partial}{\partial r} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(r\, h_{Hom}\right)\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}$$



$(\vec{x},t)$ = field point

$(\vec{x}',t')$ = source point

No-incoming rad. cond. is

$$\boxed{\lim_{\mathscr{I}^-} \left(\frac{\partial}{\partial r} + \frac{1}{c}\frac{\partial}{\partial t} \right) (r\, h^{\mu\nu}) = 0}$$

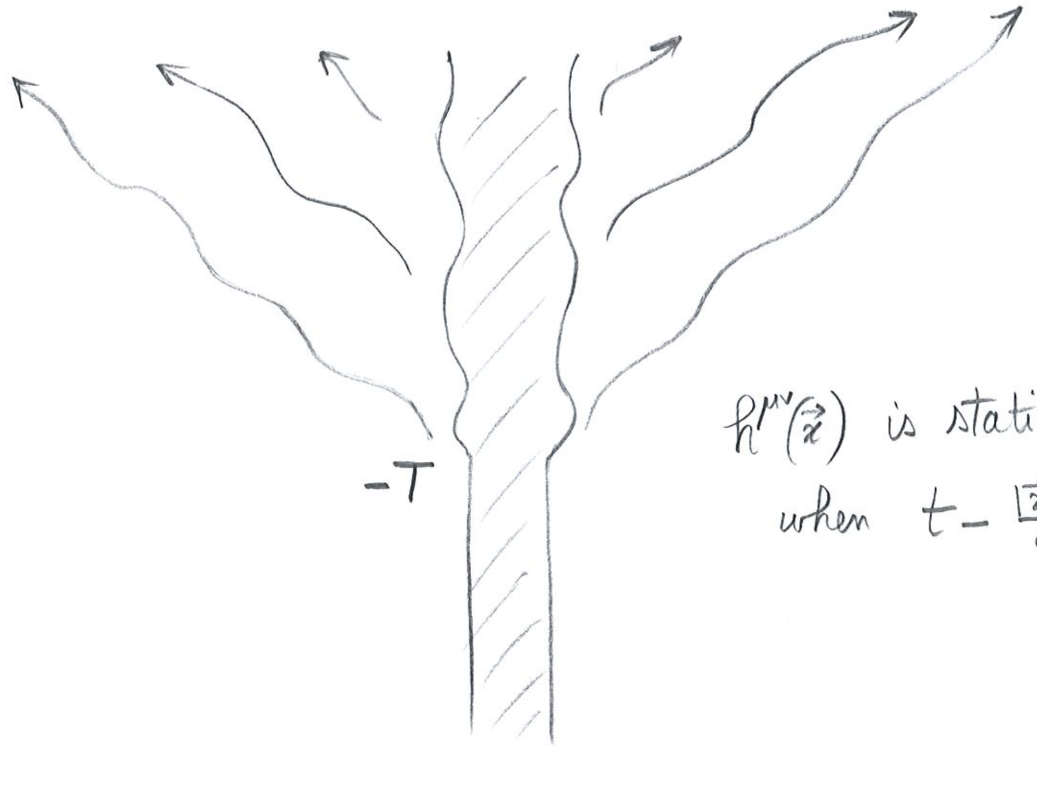
This excludes advanced waves $r\, h_{adv} \sim f(t + r/c)$ at $\mathscr{I}^-$

Einstein field eqs. can be solved (in an iterative way) by means of standard retarded integral in 3+1 dimensions

$$\boxed{h^{\mu\nu}(\vec{x}, t) = -\frac{4G}{c^4} \iiint \frac{d^3\vec{x}'}{|\vec{x} - \vec{x}'|}\, \tau^{\mu\nu}\left(\vec{x}', t - \frac{1}{c}|\vec{x} - \vec{x}'|\right)}$$

note this is infact an integro-differential equation because $\tau^{\mu\nu}$ depends on $h, \partial h, \partial^2 h$

Stationarity in the past (simple way to implement the no-incoming rad. condition)



$h^{\mu\nu}(\vec{x})$ is stationary (ind. of t) when $t - \frac{|\vec{x}|}{c} \lesssim -T$

LINEARIZED GRAVITATIONAL WAVES IN VACUUM

$$\begin{cases} \Box h^{\mu\nu} = 0 \\ \partial_\nu h^{\mu\nu} = 0 \end{cases} \qquad \text{(we neglect } \mathcal{O}(h^2)\text{)}$$

Gauge transformation preserving the harmonic cond. $\partial h = 0$

$$h'^{\mu\nu} = h^{\mu\nu} + \partial^\mu \xi^\nu + \partial^\nu \xi^\mu - \eta^{\mu\nu} \partial_\rho \xi^\rho$$

where $\Box \xi^\mu = 0$

Fourier decomposition $\boxed{h^{\mu\nu}(x) = \int d^4x \, H^{\mu\nu}(k) \, e^{i k_\lambda x^\lambda}}$

$H^{\mu\nu}(k)$: Fourier amplitude of monochromatic wave $k_\lambda = \binom{\text{wave}}{\text{vector}}$

$$\boxed{\begin{aligned} k^2 &\equiv \eta_{\mu\nu} k^\mu k^\nu = 0 \\ k_\nu H^{\mu\nu} &= 0 \end{aligned}}$$

Can perform a gauge transf.

with any $\xi^\mu(x) = \int d^4x \, \mathcal{E}^\mu(k) e^{i k\cdot x}$

TT coordinates u^μ four-vector constant (independent of x) and not orthogonal to k_μ (i.e. $u_\mu k^\mu \neq 0$) for instance

u^μ = four velocity of an observer (time-like)

There exists a gauge such that (at once)

$$\boxed{\begin{aligned} u_\nu H^{\mu\nu} &= 0 \\ H \equiv \eta_{\mu\nu} H^{\mu\nu} &= 0 \end{aligned}}$$

$\longleftarrow$ transverse (T) condition

$\longleftarrow$ traceless (T) condition

Proof: perform a gauge transf. in Fourier domain

$$H^{\mu\nu} = H_0^{\mu\nu} + i k^\mu \varepsilon^\nu + i k^\nu \varepsilon^\mu - i \eta^{\mu\nu} k_\rho \varepsilon^\rho$$

Then TT conditions are satisfied with gauge vector

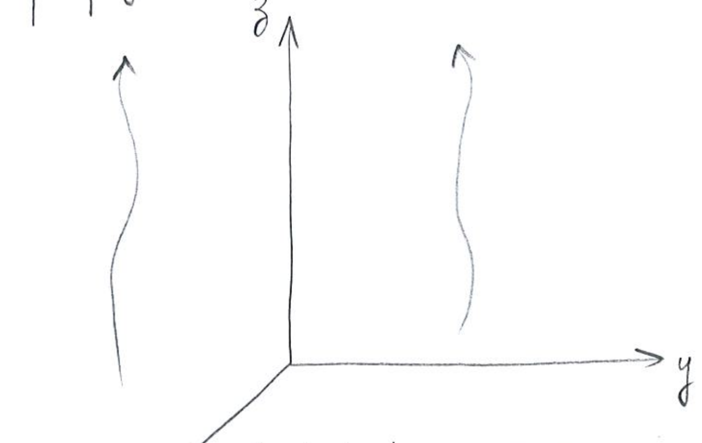
$$\varepsilon^\mu = \frac{i}{(uk)}\left[u_\nu \bar{H}_0^{\mu\nu} - \frac{k^\mu}{2(uk)} u_\rho u_\sigma \bar{H}_0^{\rho\sigma} \right]$$

where $\bar{H}_0^{\mu\nu} = H_0^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} H_0$

$$\boxed{\begin{gathered} 10 - 4 - (4-1) - 1 = 2 \text{ independent components of } H^{\mu\nu} \\ \text{2 polarization states} \end{gathered}}$$

$u^\mu = (1, \vec{0})$ in rest frame of observer

propagation in z-direction

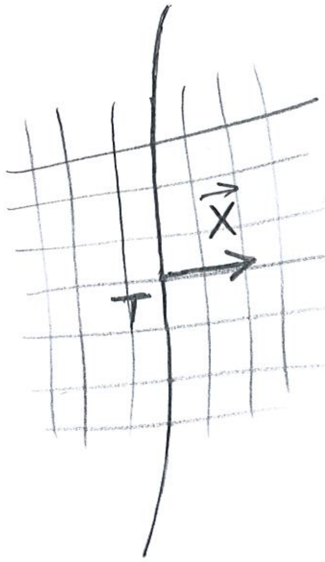


(x,y) is transverse to propagation

$$\begin{array}{cc} & \begin{array}{cccc} t & x & y & z \end{array} \\ h_{\mu\nu}^{TT} = & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(t-z/c) & h_\times(t-z/c) & 0 \\ 0 & h_\times(t-z/c) & -h_+(t-z/c) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

ACTION OF GRAVITATIONAL WAVES ON MATTER

central geodesics ($X^i=0$)



<u>Fermi coordinates</u> $(X^i;T)$ in the neighborhood of central geodesics

$T =$ proper time along central geodesics

$$\boxed{g_{\mu\nu}(\vec{X},T) = \eta_{\mu\nu} + \underbrace{F_{\mu\nu ij}(T)}_{\substack{\text{function of}\\ \text{time } T}} X^i X^j + \mathcal{O}(|\vec{X}|^3)}$$

Geodesic equ. in vicinity of central geodesic $\left(|\vec{X}| \ll \lambda^{GW}\right)$

$$\frac{d^2X^i}{dT^2} = -c^2 \frac{\partial \Gamma^i_{00}}{\partial X^j}(T,\vec{0})\,X^j = -c^2 R^i_{\cdot 0j0}(T,\vec{0})\,X^j$$

(to first order in X^i)

Riemann in Fermi coord.
($-c^2 R_{i0j0}$ is a relativistic version of the tidal tensor $\partial_i \partial_j U$)

$$R^i_{\cdot 0j0} = \frac{\partial X^i}{\partial x^\lambda}\frac{\partial x^\mu}{\partial X^0}\cdots \overset{TT}{R}{}^{\lambda}_{\cdot\mu\nu\rho} \approx \overset{TT}{R}{}^{i}_{\cdot 0j0} \approx -\frac{1}{2c^2}\frac{\partial^2 \overset{TT}{h}_{ij}}{\partial t^2}$$

Riemann in TT coordinates

$$\boxed{\frac{d^2X^i}{dT^2} = \frac{1}{2}\frac{\partial^2 \overset{TT}{h}_{ij}}{\partial t^2}(T,\vec{0})\,X^j}$$

acceleration in Fermi coord.

wave form in TT coord. evaluated on central geodesic

$$\boxed{X^i(T) = X^i(0) + \frac{1}{2}\overset{TT}{h}_{ij}(T,\vec{0})\,X^j(0)}$$

(to first order in h)

position before passage of GW

QUADRUPOLE MOMENT FORMALISM

QUADRUPOLE MOMENT FORMALISM

Matter source is

- *isolated* ($T^{\mu\nu}$ has a compact support)
- *post-Newtonian* $\boxed{\varepsilon \approx \frac{v}{c} \ll 1}$
- *self-gravitating*: internal motion is due to gravitational forces

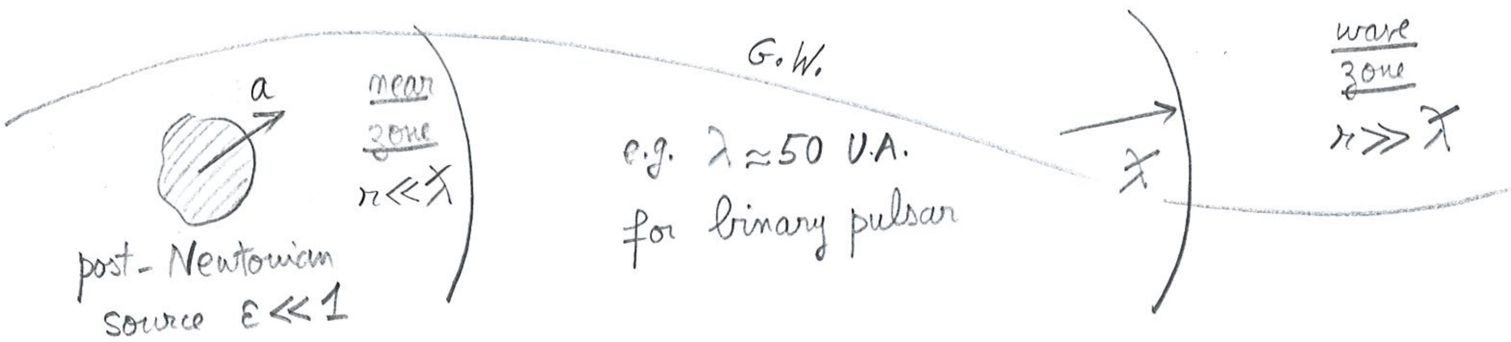
$$\gamma \sim \frac{v^2}{a} \sim \frac{GM}{a^2}$$

a = size of source
M = its mass

Period of motion $P \sim \frac{2\pi a}{v}$

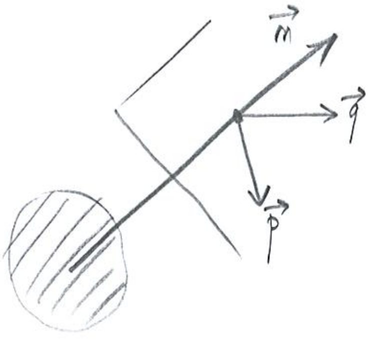
Gravitational wave length $\lambda = cP$, $\quad ƛ = \frac{\lambda}{2\pi}$

$$\boxed{\frac{a}{ƛ} \sim \frac{v}{c} \approx \varepsilon}$$



The near zone ($r \ll ƛ$) covers entirely the post-Newtonian source

$$Q_{ij}(t) = \int_{\text{source}} d^3x \, \rho(\vec{x}, t)\left(x_i x_j - \frac{1}{3}\delta_{ij}\vec{x}^{\,2}\right)$$



$$\boxed{h_{ij}^{TT} = \frac{2G}{c^4 r} P_{ijkl}(\vec{n})\left\{ \ddot{Q}_{kl}\left(t - \frac{r}{c}\right) + O(\varepsilon)\right\} + O\left(\frac{1}{r^2}\right)}$$

TT projection operator $P_{ijkl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$ where $P_{ij} = \delta_{ij} - n_i n_j$

Polarization states w.r.t. $\vec{p}, \vec{q}$

$$h_+ = \frac{p_i p_j - q_i q_j}{2} h_{ij}^{TT}$$

$\vec{p}, \vec{q}$ polarization vectors

$$h_\times = \frac{p_i q_j + p_j q_i}{2} h_{ij}^{TT}$$

$$\boxed{\mathcal{F}^{GW} \equiv \left(\frac{dE}{dt}\right)^{GW} = \underbrace{\frac{G}{5c^5}\left\{\dddot{Q}_{ij}\,\dddot{Q}_{ij} + \mathcal{O}(\varepsilon^2)\right\}}}$$

Einstein quadrupole formula

order of magnitude of radiation reaction

$\mathcal{O}(\varepsilon^5)$ called also 2.5PN

Typically $Q \sim M a^2$ $\quad \dddot{Q} \sim M a^2 \omega^3$ $\quad \omega = \frac{2\pi}{P}$

Self-gravitating source $\omega^2 \sim \frac{GM}{a^3}$

$$\boxed{\mathcal{F}^{GW} \sim \left(\frac{c^5}{G}\right)\left(\frac{GM\omega}{c^3}\right)^{10/3}}$$

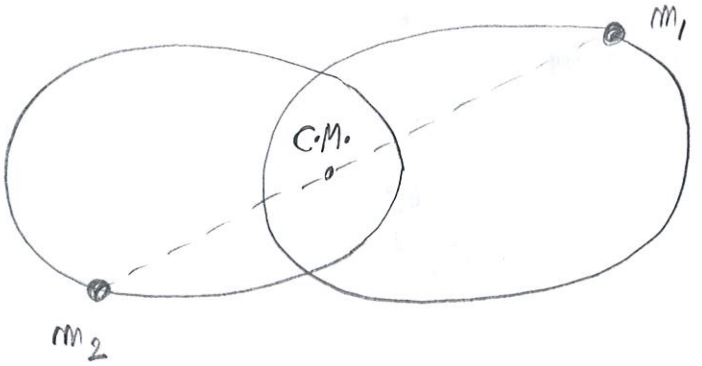
Ultra-relativistic source $v \sim c$ or $\frac{GM\omega}{c^3} \sim 1$

$$\underbrace{\mathcal{F}^{GW}\Big|_{\substack{\text{ultra}\\ \text{relativistic}}} \sim \frac{c^5}{G}}_{\text{value independent of source}} = 3.63\ 10^{52}\, W$$

GW has typically the frequency $\omega \sim \frac{c^3}{GM}$

$M \sim 1\, M_\odot$	$\omega \sim 10^3\, Hz$	bandwidth of LIGO/VIRGO
$M \sim 10^6\, M_\odot$	$\omega \sim 10^{-3}\, Hz$	bandwidth of LISA

PETERS & MATHEWS FORMULA



Two compact objects (without spin) on a Keplerian ellipse

a = semi-major axis

e = eccentricity

$$M = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{M}$$

$$\boxed{\nu = \frac{\mu}{M} \text{ such that } 0 < \nu \leq \frac{1}{4}}$$

test-mass limit (0) — equal masses ($\frac{1}{4}$)

$$\boxed{\langle \mathcal{F}^{GW} \rangle \equiv \frac{1}{P}\int_0^P dt\, \mathcal{F}^{GW}(t) = \frac{32}{5}\frac{c^5}{G}\nu^2\left(\frac{GM}{ac^2}\right)^5 \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}}}$$

eccentricity dependent "enhancement" factor $f(e)$

Energy balance argument

$$\frac{dE}{dt} = -\langle \mathcal{F}^{GW} \rangle \quad \text{with} \quad E = -\frac{GM^2\nu}{2a}$$

$$GM = \omega^2 a^3$$

$$\boxed{\dot{P} = -\frac{192\pi}{5c^5}\left(\frac{2\pi GM}{P}\right)^{5/3} \nu\, f(e) = -2.4\ 10^{-12}\ \text{s/s}}$$

Binary pulsar PSR 1913+16

in agreement with observations (Taylor et al).

INSPIRALLING COMPACT BINARIES

Evolution of eccentricity $e(t)$

Orbit's energy and angular momentum

$$\boxed{\begin{aligned} \frac{E}{\nu} &= -\frac{GM^2}{2a} \\ \frac{J}{\nu} &= \sqrt{GM^3 a(1-e^2)} \end{aligned}} \qquad \nu \equiv \frac{\mu}{M}$$

Apply quadrupole formulas for both E and J

$$\dot{E} = -\left\langle \frac{G}{5c^5} \dddot{Q}_{ij} \dddot{Q}_{ij} \right\rangle$$

$$\dot{J}^i = -\left\langle \frac{2G}{5c^5} \varepsilon_{ijk} \ddot{Q}_{jl} \dddot{Q}_{kl} \right\rangle$$

$$\boxed{\frac{e^2}{(1-e^2)^{19/6}} \left(1 + \frac{121}{304} e^2\right)^{\frac{145}{121}} = \left(\frac{\omega}{\omega_0}\right)^{-\frac{19}{9}}}$$

gives $e(t)$ as a function of $\omega(t)$ during the inspiral (ω_0 is determined from initial conditions) ($e^2 \sim P^{19/9}$ for small e)

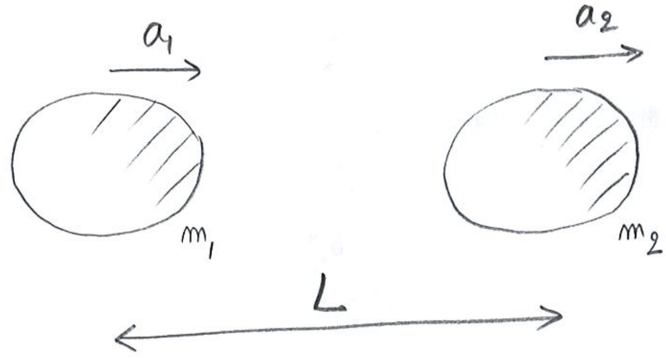
For the binary pulsar $\quad e_{now} = 0.617$
$\quad \omega_{now} = 2.24 \; 10^{-4} \, Hz$

hence GWs are visible by VIRGO/LIGO when

$$\boxed{\omega \sim 30 \, Hz \quad \Rightarrow \quad e \sim 5 \; 10^{-6}}$$

eccentricity is negligible in general.

Finite size effects



Look for influence of quadrupole moments Q_1 and Q_2 induced by tidal interactions between non-spinning compact objects

$$\boxed{Q_1 = k_1 m_2 \frac{a_1^5}{L^3} \qquad Q_2 = k_2 m_1 \frac{a_2^5}{L^3}}$$

$k_{1,2}$ = Love numbers (depend on internal structure)

$Q_{1,2}$ scale like L^{-3} because of tidal field $\partial_{ij} U \sim \frac{1}{L^3}$

Introduce the compacity parameters

$$K_1 = \frac{2Gm_1}{a_1 c^2} \qquad K_2 = \frac{2Gm_2}{a_2 c^2}$$

The quadrupoles modify the energy and GW flux and the orbital frequency ω and phase $\phi = \int \omega dt$

$$\dot{E} = -\mathcal{F}^{GW} \Rightarrow \phi = -\int \frac{\omega dE}{\mathcal{F}^{GW}}$$

Effect of quadrupoles is

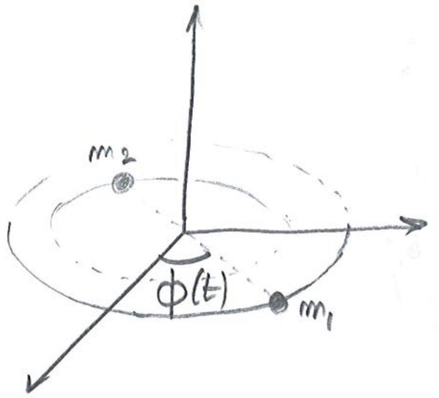
depends on internal structure

$$\boxed{\phi^{\text{finite-size}} = \underbrace{\phi_0 - \frac{1}{8x^{5/2}}\Big\{1}_{\text{point-mass result}} + (\text{const})\left(\frac{x}{K}\right)^5\Big\}}$$

$x \equiv \left(\frac{GM\omega}{c^3}\right)^{2/3}$ Since $K \sim 1$ for compact objects the formal order of magnitude of the finite-size effect is 5PN (namely $x^5 \sim \frac{1}{c^{10}}$)

Orbital phase evolution $\phi(t)$

(same as for binary pulsar, i.e. based on $\frac{dE}{dt} = -\mathcal{F}^{GW}$)



where $\frac{E}{M} = -\frac{c^2}{2}\nu x$

$$\mathcal{F}^{GW} = \frac{32}{5}\frac{c^5}{G}\nu^2 x^5$$

$$\boxed{x = \left(\frac{GM\omega}{c^3}\right)^{2/3}} = \text{PN parameter } \mathcal{O}(\varepsilon^2)$$

$$\dot{E} = -\mathcal{F}^{GW} \Rightarrow \dot{x} = \frac{64}{5}\frac{c^3}{G}\frac{\nu}{M}x^5 \Rightarrow x(t) = \left[\frac{256}{5}\frac{c^3}{G}\frac{\nu}{M}(t_c - t)\right]^{-1/4}$$

t_c = instant of coalescence

$$\phi(t) = \int \omega\, dt = \frac{5}{64\nu}\int x^{-7/2} dx \Rightarrow \boxed{\phi(t) = \phi_c - \frac{x(t)^{-5/2}}{32\nu}}$$

Number of orbital cycles left till coalescence from time t

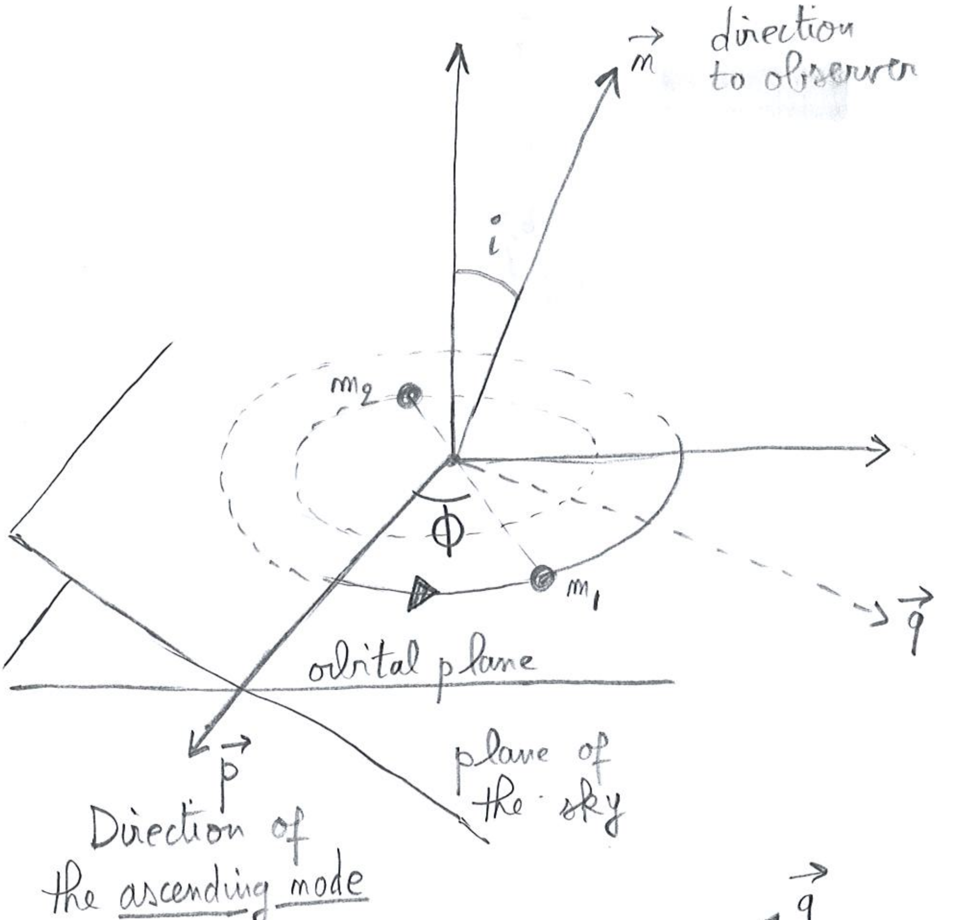
$$\mathcal{N} = \frac{\phi_0 - \phi(t)}{\pi} = \frac{1}{32\pi\nu}\left(\frac{GM\omega}{c^3}\right)^{-5/3} \underbrace{= \mathcal{O}(\varepsilon^{-5})}_{\text{inverse of order of radiation reaction } \varepsilon^{-5} \sim \left(\frac{c}{v}\right)^5}$$

But $\mathcal{N}$ should be monitored in LIGO/VIRGO with precision

$$\delta\mathcal{N} \sim 1$$

so it is evident that PN corrections in the phase will play a crucial role up to at least the 2.5PN order. Detailed analysis show that good templates for inspiralling compact binaries should have 3PN accuracy. Current theoretical prediction is 3.5PN.

Wave form of inspiralling compact binaries (ICBs)

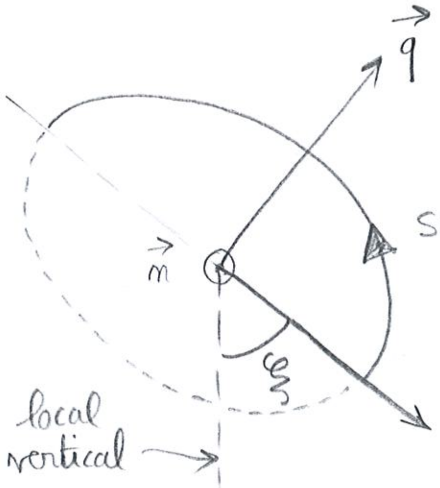


$\vec{p}, \vec{q}$ = polarization vectors (in the plane of sky)

i = inclination angle

$\phi(t)$ = orbital phase

As seen from observer:



sens of motion (bodies move toward observer when they cross $\vec{p}$)

$\vec{p}$ (ascending node)

ξ = polarization angle (between $\vec{p}$ and local vertical of observer)

Response of detector
$$\boxed{h \equiv \frac{2\delta L}{L} = F_+ h_+ + F_\times h_\times}$$

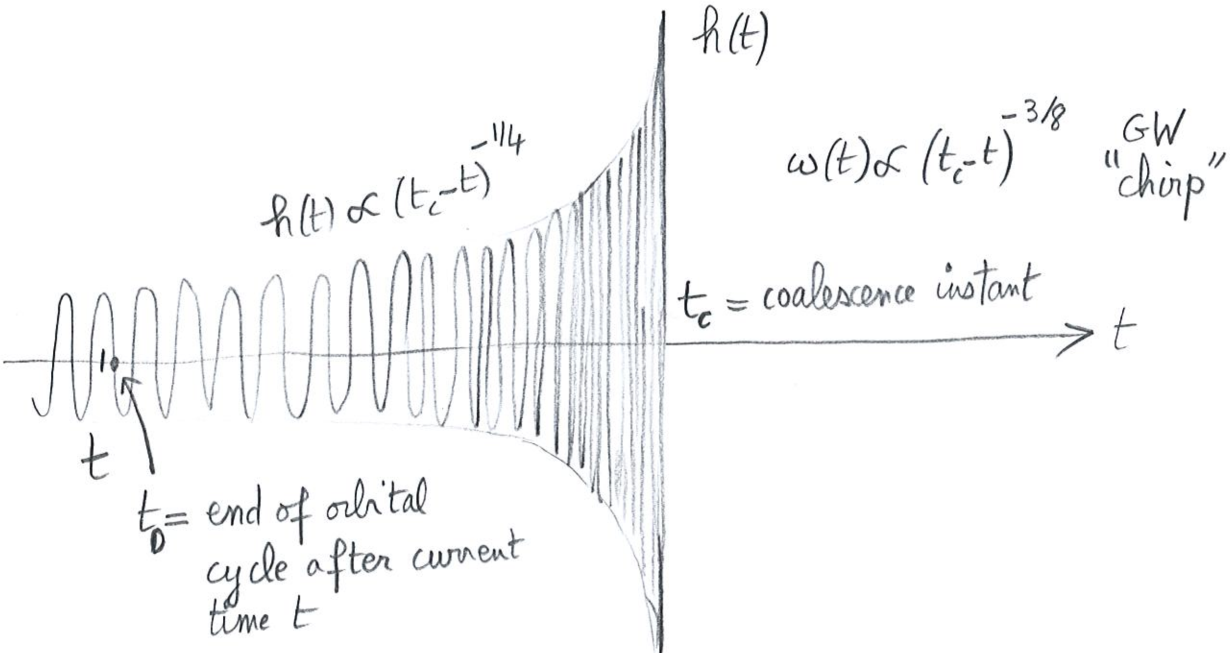
$F_{+,\times}$ = detector's pattern functions
depend on $-\vec{n}$ (direction of source) and ξ

In quadrupole approximation

$$\boxed{\begin{aligned} h_+ &= \frac{2G\mu}{c^2 D}\left(\frac{GM\omega}{c^3}\right)^{2/3}(1+\cos^2 i)\cos(2\phi) \\ h_\times &= \underline{\hspace{3cm}}\;(2\cos i)\sin(2\phi) \end{aligned}}$$

D = distance of source = luminosity distance in cosmology

where
$$\boxed{\begin{aligned} \phi(t) &= \phi_c - \frac{1}{\nu}\left(\frac{\nu c^3}{5GM}(t_c-t)\right)^{5/8} \\ \omega(t) &= \frac{c^3}{8GM}\left(\frac{\nu c^3}{5GM}(t_c-t)\right)^{-3/8} \end{aligned}}$$



Suppose current time t is such that $t_c - t \gg P$ (non-relativistic limit, two bodies are well-separated)

$$t_c - t = (t_c - t_0)\left[1 + \frac{t_0-t}{t_c-t_0}\right] \quad \text{with} \quad \frac{t_0-t}{t_c-t_0} \ll 1$$

$$\phi(t) \approx \phi_c - \frac{1}{\nu}\left(\frac{\nu c^3}{5GM}(t_c-t_0)\right)^{5/8}\left[1 + \frac{5}{8}\frac{t_0-t}{t_c-t_0} + \cdots\right]$$

$$\approx \phi_0 + \frac{5}{8\nu}\left(\frac{\nu c^3}{5GM}\right)^{5/8}(t_c-t_0)^{-3/8}\,t + \cdots$$

thus $\boxed{\phi(t) \approx \phi_0 + \omega_0 t + \cdots}$ constant orbital motion in the non relativistic limit

Orders of magnitude

$$h \sim \frac{GM\nu}{c^2 D}\left(\frac{GM\omega}{c^3}\right)^{2/3}$$

Number of cycles around frequency ω

$$n = \frac{\omega^2}{\dot{\omega}} \sim \frac{1}{\nu}\left(\frac{GM\omega}{c^3}\right)^{-5/3} = \mathcal{O}(\varepsilon^{-5})$$ inverse of rad. reaction order

Effective amplitude after matched filtering

$$\boxed{h_{eff} = h\sqrt{n} \sim \frac{GM\sqrt{\nu}}{c^2 D}\left(\frac{GM\omega}{c^3}\right)^{-1/6}}$$

Example: coalescence of two supermassive BHs in LISA

Characteristic frequency $\omega_c \sim \omega_{I.C.O.}$

innermost circular orbit (defined by the minimum of the energy function)

$$\frac{GM\omega_c}{c^3} \sim 0.1 \quad \Rightarrow \quad f_c \sim 10^4 \,\mathrm{Hz}\left(\frac{M_\odot}{M}\right)$$

(from 3PN theory) For LISA $f_c \in [10^{-4}\,\mathrm{Hz}, 10^{-1}\,\mathrm{Hz}]$

Hence LISA should observe

$$\boxed{10^5 M_\odot \lesssim M \lesssim 10^8 M_\odot}$$

$$\boxed{h_{eff} \sim 10^{-14} \left(\frac{1\,\text{Gpc}}{D}\right)\left(\frac{\nu}{0.25}\right)^{1/2}\left(\frac{M}{10^7 M_\odot}\right)^{-5/6}\left(\frac{f}{10^{-4}\,\text{Hz}}\right)^{-1/6}}$$

Separation of BHs ($M \sim 10^7 M_\odot$) at entry frequency of LISA

$$r = \left(\frac{GM}{\omega^2}\right)^{1/3} \sim 1\ \text{A.U.}$$

Time left till coalescence

$$T = \frac{5GM}{\nu c^3}\left(\frac{8GM\omega}{c^3}\right)^{-8/3} \sim 10 \text{ days}$$

The signal-to-noise of the supermassive BH coalescence in LISA is enormous

$$\boxed{\frac{S}{N} = \left(\int_{-\infty}^{+\infty} d\omega\, \frac{|\tilde{h}(\omega)|^2}{S_m(\omega)}\right)^{1/2} \sim \frac{h_{eff}}{\sqrt{\omega S_m(\omega)}} \sim 10^4}$$

$S_m(\omega) \sim 10^{-34}\ \text{Hz}^{-1}$ for LISA

MODELLING SPINNING

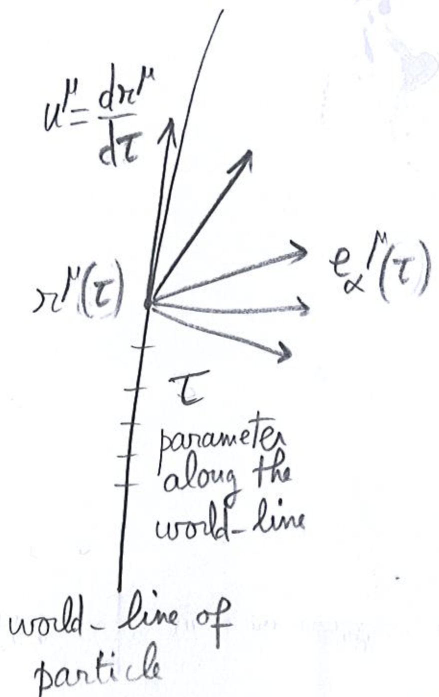
BLACK HOLES

IN POST-NEWTONIAN APPROXIMATIONS

SPINNING PARTICLES IN GR

Action of a spinning particle in a given background ($g_{\mu\nu}$).

1/4



($u^\mu = \frac{dx^\mu}{d\tau}$ is not normalized at this stage)

To describe some internal degrees of freedom associated with the spin we introduce a moving tetrad $e_a{}^\mu(s)$

$\alpha, \beta, \dots$ = Lorentz indices $= 0, 1, 2, 3$

$\mu, \nu, \dots$ = space-time indices $= 0, 1, 2, 3$

$$g_{\mu\nu}\, e_\alpha{}^\mu e_\beta{}^\nu = \eta_{\alpha\beta} \qquad \left(\text{orthonormality condition}\right)$$

Rotation tensor $\Omega^{\mu\nu}$ defined by

$$\boxed{\frac{De_\alpha{}^\mu}{ds} = -\Omega^{\mu\nu} e_{\alpha\nu}}$$

where $\frac{D}{d\tau} = u^\nu \nabla_\nu$. We have $\Omega^{\mu\nu} = -\Omega^{\nu\mu}$ because of orthonormality.

$$\boxed{S_{\text{part}}\underbrace{[x^\mu, e_\alpha{}^\mu]}_{\text{dynamical variables}} = \int_{-\infty}^{+\infty} d\tau\, L(u^\mu, \Omega^{\mu\nu}, g_{\mu\nu})} \qquad \left(\text{effective action}\right)$$

This action will be valid only at <u>first order in the spin</u>. Indeed it neglects the internal structure of the particle (e.g. a rotationally induce quadrupole moment). At linear order in the spins it is valid for BHs as well as neutron stars.

Following the spirit of <u>effective actions</u> we impose "symmetries"

1. S to be a Lorentz scalar. This is automatically verified because u^μ, $\Omega^{\mu\nu}$ and $g_{\mu\nu}$ are Lorentz scalars.

2. S to be covariant scalar. This imposes

$$\boxed{2\frac{\partial L}{\partial g_{\mu\nu}} = p^\mu u^\nu + S^\mu{}_{\cdot\rho}\,\Omega^{\nu\rho}} \qquad (*)$$

where $\quad p_\mu = \dfrac{\partial L}{\partial u^\mu}$ linear momentum (conjugate momentum of u^μ)

$S_{\mu\nu} = 2\dfrac{\partial L}{\partial \Omega^{\mu\nu}}$ spin tensor (conjugate momentum of $\Omega^{\mu\nu}$)

3. Parametrization invariance $\tau \longrightarrow \lambda\tau$

By Euler's theorem on homogeneous functions this gives

$$\boxed{L = p_\mu u^\mu + \frac{1}{2} S_{\mu\nu}\,\Omega^{\mu\nu}} \qquad (**)$$

We now vary the action w.r.t. to $e_\alpha{}^\mu$.

We must have a way to distinguish intrinsic variations of $e_\alpha{}^\mu$ from variations which are induced by a change in the background metric.

We have $\delta g^{\mu\nu} = 2\, e^{\alpha(\mu} \delta e_\alpha{}^{\nu)}$ from orthonormality condition so we have

$$\boxed{\delta e_\alpha{}^\nu = e_{\alpha\mu}\left(\delta\Theta^{\mu\nu} + \tfrac{1}{2}\delta g^{\mu\nu}\right)}$$

where $\delta\Theta^{\mu\nu} = e^{\alpha[\mu} \delta e_\alpha{}^{\nu]}$

We can consider the variations $\delta\Theta^{\mu\nu}$ and $\delta g^{\mu\nu}$ to be independant.

Varying w.r.t. $\delta\Theta^{\mu\nu}$ and holding $\delta g^{\mu\nu} = 0$ gives

$$\frac{DS_{\mu\nu}}{d\tau} = \Omega_\mu{}^\rho S_{\nu\rho} - \Omega_\nu{}^\rho S_{\mu\rho}$$

Using (*) gives

$$\boxed{\frac{DS_{\mu\nu}}{d\tau} = p_\mu u_\nu - p_\nu u_\mu}$$

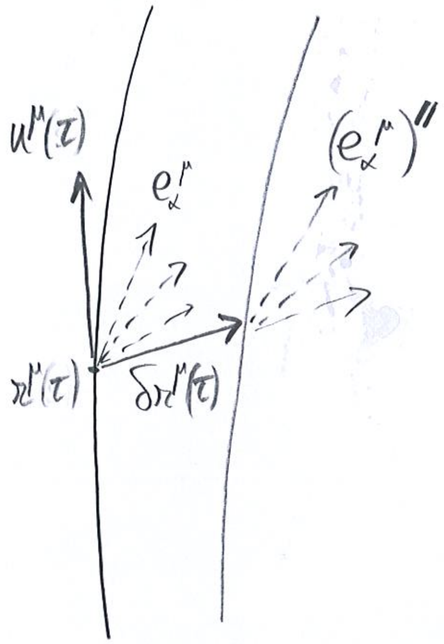
which is the equation of spin precession.

Varying w.r.t. $\delta g^{\mu\nu}$ holding $\delta\Theta^{\mu\nu} = 0$ gives the stress-energy tensor

$$\boxed{T^{\mu\nu} = \int d\tau\, p^{(\mu} u^{\nu)} \frac{\delta^{(4)}(x-r)}{\sqrt{-g}} - \nabla_\rho \int d\tau\, S^{\rho(\mu}{}_{u}{}^{\nu)} \frac{\delta^{(4)}(x-r)}{\sqrt{-g}}}$$

which is to be put on the RHS of the Einstein field equations.

Finally we vary w.r.t. the position "keeping the tetrad fixed"



This means that we have to parallel-transport the tetrad along the variation

$$\boxed{\delta x^\nu \nabla_\nu e_\alpha{}^\mu = 0}$$

To simplify the calculation we can use locally inertial coordinates ($\Gamma^\mu_{\nu\rho} = 0$ at the point x^μ)

This then means that $\delta e_\alpha{}^\mu \equiv \delta x^\nu \partial_\nu e_\alpha{}^\mu = 0$ in these coordinates

The variation yields the famous Mathisson–Papapetrou equations of motion

$$\boxed{\frac{Dp_\mu}{ds} = -\frac{1}{2} u^\nu R_{\mu\nu\rho\sigma} S^{\rho\sigma}}$$

coupling of spin to Riemann curvature

To describe correctly the 3 degrees of freedom associated with a spin vector we impose the SSC (spin supplementary conditions)

$$\boxed{p^\nu S_{\mu\nu} = 0}$$

Introducing then the mass by $g^{\mu\nu} p_\mu p_\nu = -m^2$

one can prove $m = \mathrm{const}$, $\frac{dm}{d} = 0$, as the consequence of the SSC. Introducing the (4D) magnitude $S^2 = S_{\mu\nu} S^{\mu\nu}$ of the spin one obtains also $\frac{dS}{d\tau} = 0$. Finally one obtains the relation between the momentum and the velocity as

$$\boxed{p^\mu (p u) + m^2 u^\mu = \frac{1}{2} S^{\mu\nu} S^{\lambda\rho} u^\sigma R_{\nu\sigma\lambda\rho}} \qquad (***)$$

Thus, imposing the SSC fixes from (**) the Lagrangian.

In practice one prefers to use a spin vector with constant 3D magnitude. This is done in the following way. From the SSC one introduces a spin covector S_μ by

$$S^{\mu\nu} = -\frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} \frac{p_\rho}{m} S_\sigma$$

Working at linear order in spins (***) gives

$$p_\mu = m u_\mu + \mathcal{O}(S^2)$$

and the spin precession equation becomes

$$\frac{D S_\mu}{d\tau} = \mathcal{O}(S^2)$$

We can also impose the spin vanishes in the particle's rest frame

$$u^\mu S_\mu = \mathcal{O}(S^2)$$

To define the constant-norm spin we introduce the tetrad components of S_μ, i.e.

$$S_\alpha = e_\alpha{}^\mu S_\mu$$

If we choose $e_\alpha{}^\mu = u^\mu$, then we have $S_0 = 0$ and the tetrad components S_a $(a=1,2,3)$ define a constant-norm vector. Indeed we have $S_\mu S^\mu = s^2 = \text{const}$. Since $u^\mu S_\mu = 0$ this means

$$\gamma^{\mu\nu} S_\mu S_\nu = s^2 \qquad \gamma^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$$

(projecteur $\perp$ à u^μ)

Orthonormality of the tetrad, $\gamma^{\mu\nu} = \delta^{ab} e_a^\mu e_b^\nu$, hence

$$\boxed{\delta^{ab} S_a S_b = s^2}$$

Then this vector satisfies an ordinary precession equation

$$\frac{dS_a}{dt} = \varepsilon_{abc}\, \omega_b S_c \qquad \left(\frac{d\vec{S}}{dt} = \vec{\omega} \times \vec{S}\right)$$

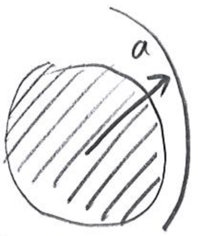
where the precession vector ω_a can be related to the rotation tensor $\Omega^{\mu\nu}$ in a precise way.

Finally the PN iteration for systems of spinning BHs proceeds with solving the Einstein field eqs. for that matter source (i.e. looking for the PN metric generated by the particles.

EXTERNAL FIELD OF AN ISOLATED SOURCE

NON-LINEARITY (POST MINKOWSKIAN) EXPANSION

In exterior region $(r > a)$



isolated source (radius a)

$$\begin{cases} \Box h^{\mu\nu}_{ext} = \Lambda^{\mu\nu}(h_{ext}) \\ \partial_\nu h^{\mu\nu}_{ext} = 0 \end{cases}$$

of order $\mathcal{O}(h_{ext}^2)$

harmonic coordinate condition

We solve these equations by means of <u>post-Minkowskian</u> (PM) or <u>non-linearity</u> expansion

$$\boxed{h^{\mu\nu}_{ext} = \sum_{n=1}^{+\infty} G^n h^{\mu\nu}_{(n)}}$$

G = Newton's constant
(viewed here as a "bookeeping" parameter to label the successive PM orders)

Insert PM expansion into <u>vacuum</u> Einstein field eqs.

$$\Box\left(G h^{\mu\nu}_{(1)} + G^2 h^{\mu\nu}_{(2)} + \dots\right) = G^2 \Lambda^{\mu\nu}_{(2)}(h_{(1)}) + G^3 \Lambda^{\mu\nu}_{(3)}(h_{(1)}, h_{(2)}) + \dots$$

$$\partial_\nu \left(\text{———————} \right) = 0$$

where $\Lambda_{(2)} \sim h_{(1)} \partial^2 h_{(1)} + \partial h_{(1)} \partial h_{(1)}$

$\Lambda_{(3)} \sim h_{(1)} \partial h_{(1)} \partial h_{(1)} + h_{(1)} \partial^2 h_{(2)} + h_{(2)} \partial^2 h_{(1)} + \partial h_{(1)} \partial h_{(2)}$

...

Hierarchy of PM equations equivalent to Einstein eqs.

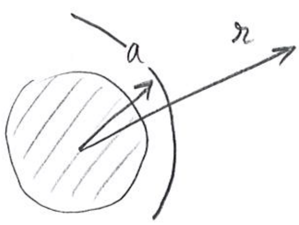
$\forall n \geq 1$

$$\boxed{\begin{aligned} \Box\, h^{\mu\nu}_{(n)} &= \Lambda^{\mu\nu}_{(n)}\left(h_{(1)}, h_{(2)} \cdots h_{(n-1)}\right) \\ \partial_\nu h^{\mu\nu}_{(n)} &= 0 \end{aligned}}$$

The source term $\Lambda_{(n)}$ is known from previous iterations

LINEARIZED SOLUTION

Solve $\Box\, h_{(1)} = 0$ by means of multipole expansion (valid in exterior $r > a$)



"Monopolar" general solution

$$h^{\text{Mono.}}_{(1)}(\vec{x}, t) = \frac{R(t - r/c) + A(t + r/c)}{r}$$

Impose no incoming rad. cond.

$$0 = \lim_{\substack{t \to -\infty \\ t + \frac{r}{c} = \text{const}}} \left[\partial_r (r\, h_{(1)}) + \partial_t (r\, h_{(1)}) \right] = 2 A'(t + \tfrac{r}{c})$$ hence $A(u)$ is constant and can be included into definition of $R(t - \frac{r}{c})$.

$$h^{\text{Mono.}}_{(1)} = \frac{R(t - r/c)}{r}$$

$(i = 1, 2, 3)$

"Dipolar" solution is obtained by applying $\partial_i \equiv \dfrac{\partial}{\partial x^i}$

hence $h^{\text{Dip.}}_{(1)} = \partial_i \left(\frac{R_i(t-r/c)}{r} \right)$. General multipolar solution is obtained by applying l spatial derivatives

$$\boxed{h^{\mu\nu}_{(1)}(\vec{x},t) = \sum_{l=0}^{+\infty} \partial_L \left(\frac{R^{\mu\nu}_L(u)}{r} \right)} \quad \left(u \equiv t - \frac{r}{c} \right)$$

$L = i_1 i_2 \cdots i_l$ a multi-index with l spatial indices

$$\partial_L \equiv \partial_{i_1 i_2 \cdots i_l} = \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_l}}$$

Without loss of generality we can assume that R_L is <u>symmetric and trace-free</u> (STF)

STF tensors

$$R_L = \hat{R}_L + \sum_{j \leq l-1} \varepsilon \underbrace{\delta \delta \ldots \delta}_{1 \text{ to } [\frac{l}{2}]} \hat{U}_J$$

0 or 1 Levi-Civita symbol

Kronecker symbol

where the $\hat{U}_J$'s are linear in the $\varepsilon \delta \ldots \delta R_K$'s.

For example:

$$\begin{cases} R_{ij} = \hat{R}_{ij} + \varepsilon_{ijk} \hat{U}_k + \delta_{ij} \hat{U} \\ \hat{U}_k = \frac{1}{2} \varepsilon_{kab} R_{ab} \\ \hat{U} = \frac{1}{3} R_{kk} \end{cases}$$

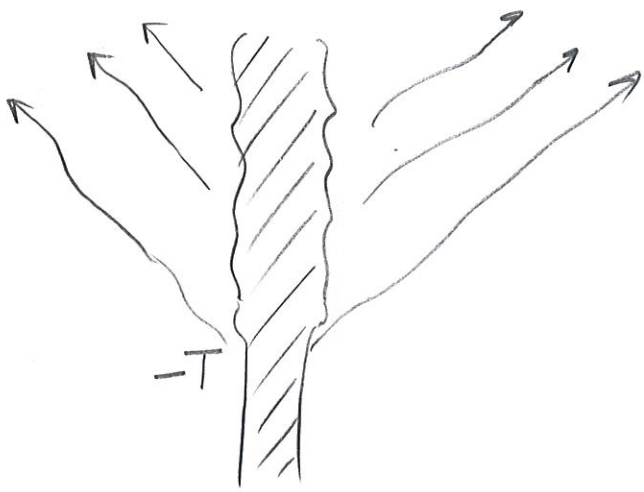
$\hat{R}_{ij} = \frac{R_{ij} + R_{ji}}{2} - \frac{1}{3} \delta_{ij} R_{kk}$ is the STF part of R_{ij}.

$$\partial_L\left(\frac{1}{r}R_L\right) = \partial_L\left(\frac{1}{r}\hat{R}_L\right) + \sum_{k\geq 1} \underset{\uparrow}{\Delta^k}\, \partial_{L-2K}\left(\frac{1}{r}\hat{U}_{L-2K}\right)$$

because of k Kronecker δs
(terms with one ε cancelled by symmetry of ∂_L)

$$\Delta^k \partial\left(\frac{1}{r}\hat{U}(u)\right) = \partial\left(\frac{1}{r}\frac{d^{2k}\hat{U}}{c^{2k}du^{2k}}(u)\right)$$ takes same structure

For simplicity assume that source emits GWs only from some finite instant $-T$ in the past (stationarity in the past)



$h^{\mu\nu}_{ext}(\vec{x})$ is independent of time when $t \leq -T$

(and even when
$\underbrace{t - \frac{r}{c} - \frac{2GM}{c^3}\ln\left(\frac{r}{r_0}\right) + \dots}_{\text{"light cone" in coordinates } (t,r)} \leq -T$)

There are 10 independent functions $R_L^{\mu\nu}(u)$ (for each multi-index L) at this stage.

We impose now the harmonicity condition $\partial_\nu h^{\mu\nu}_{(1)} = 0$ which gives 4 differential relations between the R_L's. Hence we end up with 6 independent functions (6 types of "source" multipole moments).

Most general solution of $\Box h_{(1)} = 0 = \partial h_{(1)}$ is (Thorne 1980)

$$\boxed{h^{\mu\nu}_{(1)} = k^{\mu\nu}_{(1)} + \underbrace{\partial^\mu \varphi^\nu_{(1)} + \partial^\nu \varphi^\mu_{(1)} - \eta^{\mu\nu}\partial_\rho \varphi^\rho_{(1)}}_{\text{linearized gauge transformation}}}$$

where $k^{\mu\nu}_{(1)}$ depends on two sets of <u>STF multipole moments</u>

$$\boxed{I_L(u) \quad \text{and} \quad J_L(u)}$$

<u>mass-moment of order l</u> <u>current-moment of order l</u>

and $\varphi^\mu_{(1)}$ depends on four sets of moments (for its four components $\mu = 0,1,2,3$)

$$W_L(u) \quad X_L(u) \quad Y_L(u) \quad \text{and} \quad Z_L(u)$$

$$\boxed{\begin{aligned}
k^{00}_{(1)} &= -\frac{4}{c^2}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\partial_L\left(\frac{1}{r} I_L(u)\right) \\
k^{0i}_{(1)} &= \frac{4}{c^3}\sum_{l=1}^{+\infty}\frac{(-)^l}{l!}\left\{\partial_{L-1}\left(\frac{1}{r}\dot{I}_{iL-1}(u)\right) + \frac{l}{l+1}\varepsilon_{iab}\,\partial_{aL-1}\left(\frac{1}{r}J_{bL-1}(u)\right)\right\} \\
k^{ij}_{(1)} &= -\frac{4}{c^4}\sum_{l=2}^{+\infty}\frac{(-)^l}{l!}\left\{\partial_{L-2}\left(\frac{1}{r}\ddot{I}_{ijL-2}\right) + \frac{2l}{l+1}\partial_{aL-2}\left(\frac{1}{r}\varepsilon_{ab(i}\dot{J}_{j)L-2}\right)\right\}
\end{aligned}}$$

Dots mean derivative w.r.t. time $u = t - r/c$

MATCHING TO A POST-NEWTONIAN SOURCE

THE MATCHING EQUATION

We have contructed the exterior field (physically valid when $r > a$) of <u>any isolated source</u>

$$h_{ext} = \sum_{n=1}^{+\infty} G^n \, h_{(n)} \Big[\underbrace{I_L \; J_L \; W_L \cdots Z_L}_{\text{source moments (for the moment arbitrary)}} \Big]$$

We suppose that h_{ext} comes from the <u>multipole expansion</u> of h defined everywhere inside and outside the source (for any r)

$$\boxed{h_{ext} = \mathcal{M}(h)}$$

operation of taking the multipole expansion

Note that $\mathcal{M}(h)$ is defined of any $r > 0$ but agrees with the "true" field h only when $r > a$

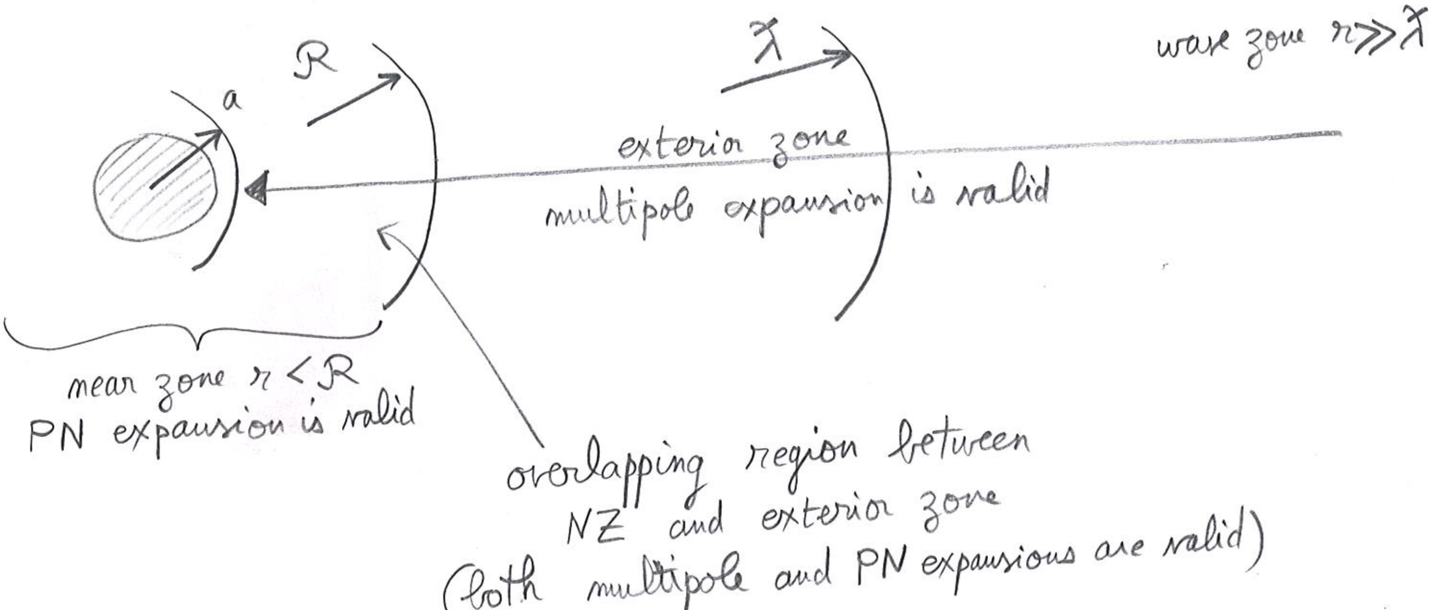
$$\boxed{r > a \quad \Rightarrow \quad \mathcal{M}(h) = h \qquad \text{(numerically)}}$$

But when $r \to 0$ $\mathcal{M}(h)$ diverges while h is a perfectly smooth solution of Einstein field eqs. inside the matter (of the extended source).

Suppose the source is <u>post-Newtonian</u> (existence of the PN parameter $\varepsilon = \frac{v}{c} \ll 1$). We know that the near zone $r < \mathcal{R}$ where $\mathcal{R} \ll \lambda\!\!\!^{-}$ encloses totally the PN source ($\mathcal{R} > a$).

In the NZ the field h can be expanded as a PN expansion ($\bar{h} = \sum c^{-p} (\ln c)^q$)

$$\boxed{r < \mathcal{R} \quad \Rightarrow \quad h = \bar{h} \quad \text{(numerically)}}$$



$$\boxed{a < r < \mathcal{R} \quad \Rightarrow \quad \mathcal{M}(h) = \bar{h} \quad \text{(numerically)}}$$

The <u>matching equation</u> follows from transforming the latter numerical equality in a <u>functional identity</u> (valid $\forall (\vec{x}, t)$ in $\mathbb{R}^3_* \times \mathbb{R}$) between two <u>formal asymptotic series</u>

Matching equation: $\boxed{\overline{\mathcal{M}(h)} \equiv \mathcal{M}(\overline{h})}$

NZ expansion $\left(\frac{r}{c} \to 0\right)$ of each multipolar coeff. of $\mathcal{M}(h)$

multipole expansion of each PN coefficient of $\overline{h}$

We assume (as part of our fundamental assumptions) that the matching eq. is correct (in the sense of formal series)

$$\boxed{\text{NZ expansion } \frac{r}{c} \to 0 \begin{pmatrix} \text{multipolar} \\ \text{expansion} \\ \frac{a}{r} \to 0 \end{pmatrix} \equiv \text{FZ expansion } r \to \infty \begin{pmatrix} \text{PN series} \\ c \to \infty \end{pmatrix}}$$

The NZ expansion $\frac{r}{c} \to 0$ is "equivalent" to the PN expansion $c \to +\infty$ for fixed r

The multipole expansion $\frac{a}{r} \to 0$ is "equivalent" to the FZ expansion $r \to +\infty$ for a given source (fixed a)

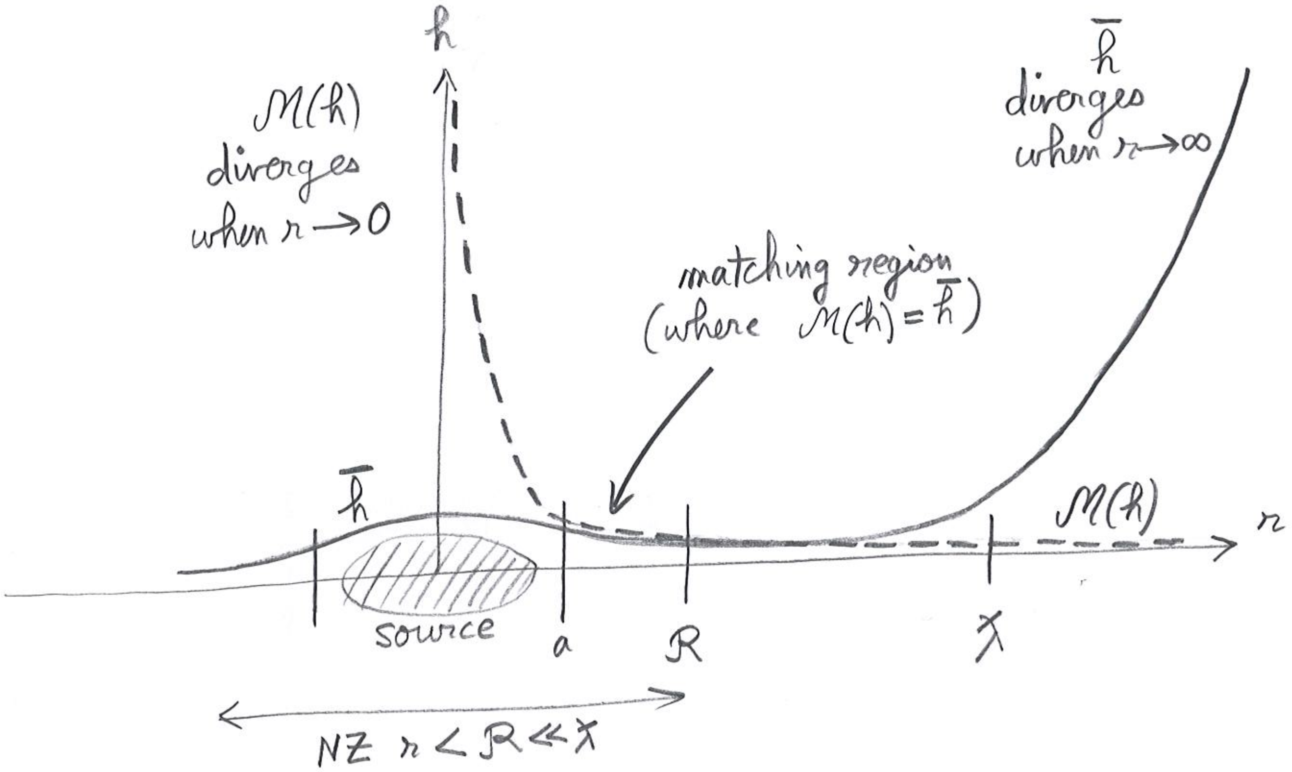
The matching equation says basically the NZ and multipole expansions can be commuted.

Thus there is a common structure for the formal NZ and FZ expansions

$$\overline{\mathcal{M}(h)} \equiv \underbrace{\sum \hat{n}_L\, r^p (\ln r)^q F(t)} \equiv \mathcal{M}(\bar{h})$$

can be interpreted either as

- NZ singular expansion when $r \to 0$
- FZ ——————— $r \to \infty$



GENERAL EXPRESSION OF THE MULTIPOLE MOMENTS

h is the sol. of Einstein eqs (in harmonic coord. $\partial h = 0$) valid everywhere inside and outside the source

$$h = \frac{16\pi G}{c^4} \Box^{-1}_{Ret} \tau \qquad \begin{pmatrix}\text{suppress} \\ \text{indices } \mu\nu\end{pmatrix}$$

where $\quad \tau = |g|\, T + \underbrace{\frac{c^4}{16\pi G} \Lambda}_{\text{gravitational source-term}}$

(non-linearities in h)

Define $$\boxed{\Delta \equiv h - \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \mathcal{M}(\Lambda)}$$

where $\mathcal{M}(\Lambda) = \Lambda[\mathcal{M}(\Lambda)] = \Lambda_{ext}$ and FP is the finite part when $B \to 0$ (plays a crucial role because Λ_{ext} diverges when $r \to 0$)

$$\Delta = \underbrace{\frac{16\pi G}{c^4} \Box^{-1}_{\mathrm{Ret}} \tau}_{\substack{\text{no FP here} \\ \text{since } \tau \text{ is regular } (C^\infty)}} - \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \mathcal{M}(\Lambda)$$

However we can add FP on the first term (do not change the value because it converges). Using also $\mathcal{M}(\tau) = 0$ since τ has a compact support

$$\Delta = \frac{16\pi G}{c^4} \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \left[\tau - \mathcal{M}(\tau)\right]$$

Hence Δ appears as the retarded integral of a source with compact support. Indeed

$$\tau = \mathcal{M}(\tau) \quad \text{when } r > a$$

$$\boxed{\mathcal{M}(\Delta) = -\frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left(\frac{1}{r} \mathcal{H}_L(u)\right)}$$

This is standard expression of multipolar expansion outside a compact-support source. Here the moments are

$$\mathcal{H}_L = \mathrm{FP}\int d^3x\, x_L \underbrace{\left[\tau - \mathcal{M}(\tau)\right]}$$

since this has compact support ($r < a$, inside the NZ) we can replace by the NZ or PN expansion

$$\mathcal{H}_L = \mathrm{FP}\int d^3x\, x_L \left[\bar{\tau} - \overline{\mathcal{M}(\tau)}\right]$$

But we know the structure $\overline{\mathcal{M}(\tau)} = \sum \hat{n}_L r^p (\ln r)^q F(t)$ which is sufficient to prove that the second term is <u>zero</u> by analytic continuation

$$\mathrm{FP}\int d^3x\, x_L \overline{\mathcal{M}(\tau)} = \sum \mathrm{FP}\int d^3x\, x_L \hat{n}_Q r^p (\ln r)^q$$

$$= \sum \underset{B\to 0}{\text{Finite Part}} \int dr\, r^{B+S} (\ln r)^p$$

integrate over angles

$$= \sum \underset{B\to 0}{\mathrm{FP}} \left(\frac{d}{dB}\right)^p \int_0^{+\infty} dr\, r^{B+S}$$

$$\int_0^{+\infty} dr\, r^{B+S} = \underbrace{\int_0^{R} dr\, r^{B+S}}_{\substack{\text{computed}\\ \text{when } \mathrm{Re}\, B > -S-1}} + \underbrace{\int_R^{+\infty} dr\, r^{B+S}}_{\substack{\text{computed}\\ \text{when } \mathrm{Re}\, B < -S-1}}$$

$$= \underbrace{\frac{R^{B+S+1}}{B+S+1}}_{\text{by analytic continuation}} \qquad = \underbrace{-\frac{R^{B+S+1}}{B+S+1}}_{\text{by analytic continuation}}$$

$$\boxed{\text{Analytic Continuation} \quad \int_0^{+\infty} dr\, r^{B+s} (\ln r)^p = 0 \qquad \forall B \in \mathbb{C}}$$

The <u>general multipole expansion</u> outside the domain of a <u>PN isolated source</u> reads (Blanchet 1995, 1998)

$$\boxed{\mathcal{M}(h) = \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}}\, \mathcal{M}(\Lambda) - \frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_L \left(\frac{1}{r} \mathcal{H}_L(u) \right)}$$

where

$$\boxed{\mathcal{H}_L(u) = \mathrm{FP} \int d^3x\, x_L\, \overline{\tau}(\vec{x}, u)}$$

PN expansion crucial here
(this is where the formalism applies only to PN sources)

Same result but in STF guise

$$\mathcal{M}(h) = \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}}\, \mathcal{M}(\Lambda) - \frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_L \left(\frac{1}{r} \mathcal{F}_L(u) \right)$$

where

$$\mathcal{F}_L(u) = \mathrm{FP} \int d^3x\, \hat{x}_L \int_{-1}^{1} dz\, \delta_\ell(z)\, \overline{\tau}(\vec{x}, u + z|\vec{x}|/c)$$

$$\delta_\ell(z) = \frac{(2\ell+1)!!}{2^{\ell+1}\, \ell!} (1-z^2)^\ell \quad \text{such that} \quad \int_{-1}^{1} dz\, \delta_\ell(z) = 1, \qquad \lim_{\ell \to +\infty} \delta_\ell(z) = \delta(z)$$