Gravitational Waves from Neutron Stars: Rotation and Oscillations

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NSs are among the most promising sources of GWs
for ground based interferometers
(Advanced Virgo/LIGO, KAGRA, ET, etc.)

Three main emission processes:

- Compact binary coalescence
- Rotation of a non-axisymmetric NS
  ("mountains" or "wobble")
- Stellar oscillations
  (quasi-normal modes)

All these processes carry the imprint of the NS EoS!

I will discuss the last two

Rotation of a non-axisymmetric NS

NSs *do* rotate, with rotation frequencies which are of the order of hundreds of Hz: in the middle of the Virgo/LIGO bandwidth.

However, an axisymmetric source does not emit GWs. Emission require an asymmetry, like for instance:

- A tri-axial ellipsoid ("mountain")
- A symmetry axis non coincidend with a rotation axis ("wobble")

The easiest way to compute the GW signal from a rotating NS is to use the quadrupole formula

The quadrupole formula

A very simple and powerful formula, which allows to compute the GW emission from a compact source by finding an approximate solution of Einstein's equation.

This approach can be applied if the following conditions are satistied:

1) Weak gravitational field: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}|\,,\ |h_{\mu\nu,\alpha}| \ll 1$

2) Small source: lengthscale ε $\quad \varepsilon \ll \lambda_{GW} = \dfrac{2\pi c}{\omega}$

(equivalent to slow motion $\ v_{typical} \sim \varepsilon\omega \ll c$)

Rotation of a non-axisymmetric NS

The quadrupole formula

If these hypotheses are satisfied, the GW in the TT-gauge is

$$
\begin{aligned}
h^{TT}_{\mu 0} &= 0 && (\mu = 0, \ldots, 3) \\
h^{TT}_{jk}(t,r) &= \frac{2G}{c^4 r} \cdot \left[\frac{d^2}{dt^2}\, Q^{TT}_{jk}(t - \frac{r}{c}) \right] && (j,k = 1, \ldots, 3)
\end{aligned}
$$

where

$$q_{ij}(T) = \frac{1}{c^2} \int_V T^{00}(t, x^i) x^i x^j d^3x$$ quadrupole moment

$$Q_{ij} = q_{ij} - \frac{1}{3}\delta_{ij} q_{kl} \eta^{kl}$$ traceless quadrupole moment

$$Q^{TT}_{ij} = \mathcal{P}_{ijkl} Q_{kl}$$ TT quadrupole moment

Rotation of a non-axisymmetric NS

The quadrupole formula

The GW energy flux can only be defined as an average over several wavelengths:

$$\frac{dE_{GW}}{dtdS} = \frac{c^3}{32\pi G}\left\langle \sum_{jk}\left(\frac{dh^{TT}_{jk}(t,r)}{dt}\right)^2\right\rangle$$

$$= \frac{G}{8\pi c^5\ r^2}\left\langle \sum_{jk}\ \left(\dddot{Q}^{TT}_{jk}\left(t-\frac{r}{c}\right)\right)^2\right\rangle$$

$$L_{GW} = \int \frac{dE_{GW}}{dtdS}\ dS = \frac{G}{5c^5}\left\langle \sum_{k,n=1}^{3}\dddot{Q}_{kn}\left(t-\frac{r}{c}\right)\dddot{Q}_{kn}\left(t-\frac{r}{c}\right)\right\rangle$$

Rotation of a non-axisymmetric NS

GWs from a rotating star:
consider an non-rotating ellipsoid of constant density ρ

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 + \left(\frac{x_3}{c}\right)^2 = 1 \qquad V = \frac{4}{3}\pi abc$$

Quadrupole moment (traceless):

$$Q_{ij} = \int_V \rho \left(x^i x^j - \frac{1}{3} r^2 \delta_{ij}\right) d^3x = -\left(I_{ij} - \frac{1}{3}\delta_{ij} \mathrm{Tr} I\right)$$

where

$$I_{ij} = \int_V \rho (r^2 \delta_{ij} - x^i x^j) = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

inertia tensor

$$I_1 = \frac{1}{5} M(b^2 + c^2) \qquad I_2 = \frac{1}{5} M(c^2 + a^2) \qquad I_3 = \frac{1}{5} M(a^2 + b^2)$$

Rotation of a non-axisymmetric NS

If the ellipsoid rotates with Ω around one of its principal axes, e.g. I_3 define a co-rotating frame $\{x'^i\}$ and an inertial frame $\{x^i\}$

$$x^i = R_{ij} x'^{\,j} \qquad R_{ij} = \begin{pmatrix} \cos\Omega t & \sin\Omega t & 0 \\ -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I'_{ij} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \qquad I_{ij} = (R I' R^T)_{ij}$$

$$\begin{aligned} Q_{ij} &= -\left(I_{ij} - \frac{1}{3}\delta_{ij}\mathrm{Tr} I\right) \\ &= \frac{I_2 - I_1}{2} \begin{pmatrix} \cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & -\cos 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} + constant \end{aligned}$$

Rotation of a non-axisymmetric NS

$$Q_{ij} = \frac{I_2 - I_1}{2} \begin{pmatrix} \cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & -\cos 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} + constant$$

Since $\quad I_1 = \frac{1}{5} M(b^2 + c^2) \qquad I_2 = \frac{1}{5} M(c^2 + a^2)$

if $a^2=b^2$ (axisymmetry), no gravitational wave is emitted!

We need $a \neq b$, which implies $I_1 \neq I_2$.
In practice, this breaking of axisymmetry, if present, is very small.
To parametrize the deviation from axisymmetry we define the oblateness ε

$$\varepsilon = \frac{a - b}{(a + b)/2} = \frac{I_2 - I_1}{I_3} + O(\varepsilon^3)$$

Rotation of a non-axisymmetric NS

In terms of oblateness,

$$Q_{ij} = \frac{\epsilon\ I_3}{2} \begin{pmatrix} \cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & -\cos 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} + constant$$

replacing in the quadrupole formula we find

$$h_{ij} = h_0 \left[\mathcal{P} \begin{pmatrix} -\cos 2\Omega \left(t - \frac{r}{c}\right) & -\sin 2\Omega \left(t - \frac{r}{c}\right) & 0 \\ -\sin 2\Omega \left(t - \frac{r}{c}\right) & \cos 2\Omega \left(t - \frac{r}{c}\right) & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]$$

where h_0 is the amplitude of the wave:

$$h_0 = \frac{4G\ \Omega^2}{c^4 r}\ I_3\ \epsilon = \frac{16\pi^2 G}{c^4\ r\ T^2}\ I_3$$ (T rotation period)

Note that $\nu_{GW} = 2\nu_{rot}$ $\quad (\Omega = 2\pi\nu_{rot})$

Rotation of a non-axisymmetric NS

Similar expressions for the GW wave amplitude $h_0 \simeq \frac{16\pi^2 G I \varepsilon}{c^4 r T^2}$

also arise in a "realistic" case (not ellipsoidal NS, $\rho=\rho(r)$).
The (small) departure from axisymmetry is described by the
quadrupole ellipticity $\varepsilon=Q/I$ (Q: mass quadrupole moment of the NS).
This feature is often called "mountain". But *how large* ε *can be*?

Most model suggest $\varepsilon \sim 10^{-6}$, but some of them allow larger values.

Note that $\varepsilon \sim 10^{-6}$ would mean a "mountain" as high as ~2cm!

- Detailed models of crustal strain suggest $\varepsilon \leq 2\ 10^{-6}\ (u_{break}/0.01)$ where $u_{break} \sim 0.01$ is the crustal break strain. *(Haskell et al., '06)*
  Recent studies suggest $u_{break} \sim 0.1$ *(Hotowits, Kadau, '06)*

- In case of "exotic" matter, the maximal oblateness can be larger:
  - $\varepsilon \leq 6\ 10^{-4}\ (u_{break}/0.01)$ for a strange quark star *(Owen et al., '05)*
  - $\varepsilon \leq 10^{-3}\ (u_{break}/0.01)$ for a color superconducting quark star *(Haskell et al., '07)*

Rotation of a non-axisymmetric NS

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- A different possibility: when the NS is born, a strong magnetic field reaches a stationary configuration and deforms the star *before* the crust is formed. Then, the deformation persists after crust formation, and the crust breaking bound may be overcame.

  In this case we could have $$\varepsilon \sim 10^{-4}\left(\frac{B}{10^{16}G}\right)^2$$

  (Haskell et al. '08; Colaiuda et al., '08; Ciolfi et al. '09; Lander & Jones '09)

- For a superconducting core $$\varepsilon \sim 10^{-5}\left(\frac{B}{10^{16}G}\right)\left(\frac{H_{crit}}{10^{15}G}\right)$$

Summarizing, most models predict $\varepsilon<10^{-6}$, but in some of them $\varepsilon<10^{-4}$

Rotation of a non-axisymmetric NS

The wave amplitude $h_0 = \frac{16\pi^2 G}{c^4\ r\ T^2}\ I_3\ \epsilon$ can be normalized as

$$h_0 = 4.21 \cdot 10^{-24} \left[\frac{ms}{T}\right]^2 \left[\frac{Kpc}{r}\right] \left[\frac{I_3}{10^{38} Kg\, m^2}\right] \left[\frac{\epsilon}{10^{-6}}\right]$$

Energy flux: replacing the expression for Q_{ij} into the flux formula,

$$L_{GW} = \frac{32G}{5c^5}\Omega^6 \varepsilon^2 I^2$$

Change in the rotational energy $E_{rot} = \frac{1}{2} I\Omega^2$

is due to GW emission but also to other processes (e.g. EM emission)

$$|\dot{E}_{rot}| = I\Omega|\dot{\Omega}| \geq L_{GW} \quad \Rightarrow \quad 4\pi^2 I\nu_{rot}|\dot{\nu}_{rot}| \geq \frac{32G}{5c^5}(2\pi\nu_{rot})^6\varepsilon^2 I^2$$

$$\varepsilon \leq \varepsilon_{sd} \equiv \left(\frac{5c^5|\dot{\nu}|}{512\pi^4 G\nu^5 I}\right)^{1/2} \quad \Rightarrow \quad h_0 \leq h_{0\,sd} = \left(\frac{5GI|\dot{\nu}|}{2c^3 r^2 \nu}\right)^{1/2}$$

Rotation of a non-axisymmetric NS

Spin-down limit

(assuming $I=10^{38}$kg m^2)

Limits on oblateness:

$$\varepsilon \leq \varepsilon_{sd} \equiv \left(\frac{5c^5|\dot{\nu}|}{512\pi^4 G\nu^5 I}\right)^{1/2}$$

name	ν_{GW} (Hz)	ϵ_{sd}
Vela	22	$1.8 \cdot 10^{-3}$
Crab	60	$7.5 \cdot 10^{-4}$
Geminga	8.4	$2.3 \cdot 10^{-3}$
PSR B 1509-68	13.2	$1.4 \cdot 10^{-2}$
PSR B 1706-44	20	$1.9 \cdot 10^{-3}$
PSR B 1957+20	1242	$1.6 \cdot 10^{-9}$
PSR J 0437-4715	348	$2.9 \cdot 10^{-8}$

Limits on GW amplitude:

$$h_0 \leq h_{0\,sd} = \left(\frac{5GI|\dot{\nu}|}{2c^3 r^2 \nu}\right)^{1/2}$$

Most promising sources: Crab (r=2kpc), Vela (r=300pc)

Assuming GW only: $h_{0\,sd}^{Crab}=1.4\ 10^{-24}$ $h_{0\,sd}^{Vela}=0.9\ 10^{-24}$

More refined computation: $h_{0\,sd}^{Crab}=5.5\ 10^{-25}$ $h_{0\,sd}^{Vela}=3.5\ 10^{-25}$

(taking into account EM emission)

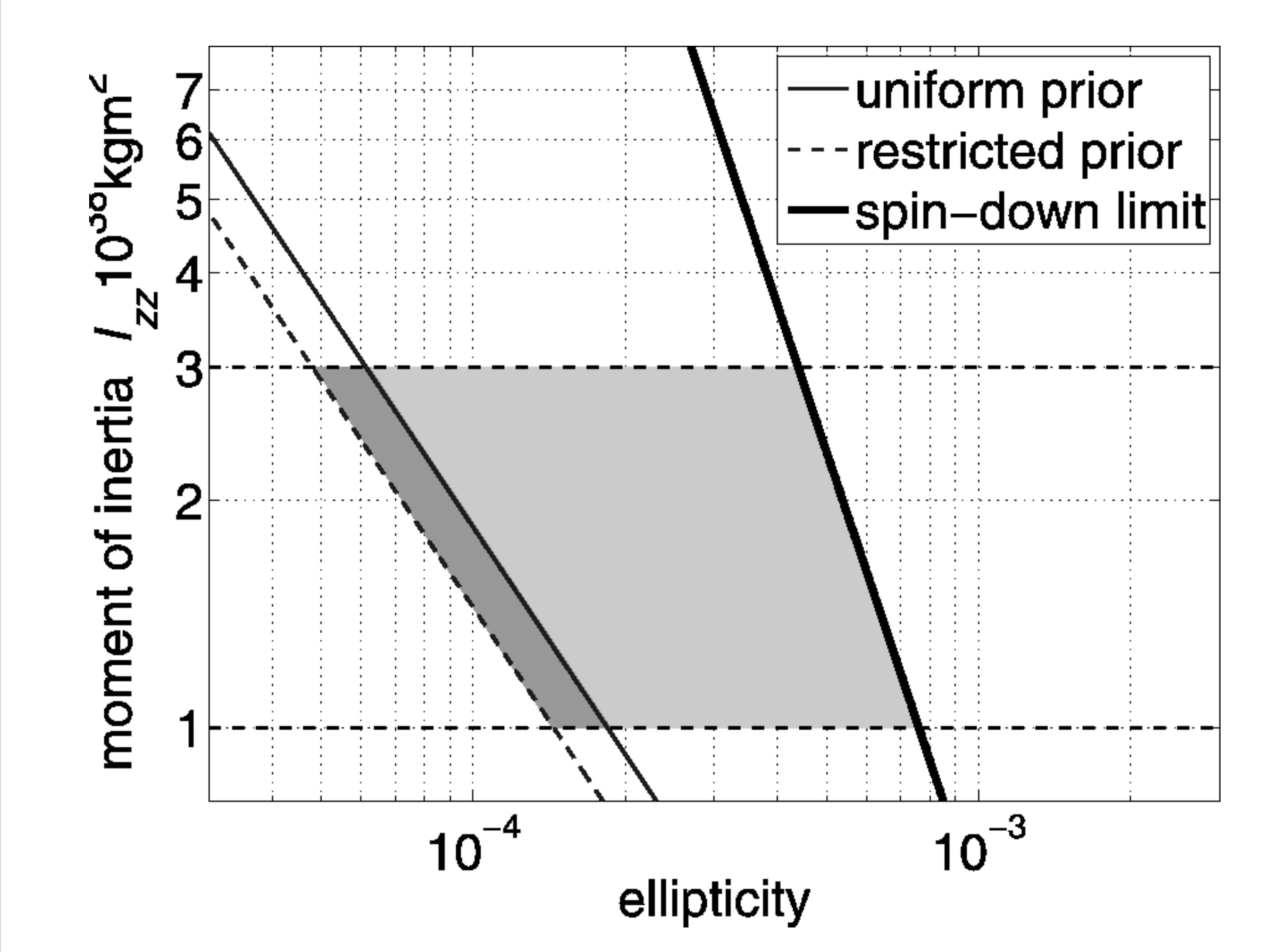
Rotation of a non-axisymmetric NS

Results from LIGO/Virgo:

"Beating the spin-down limit on GW emission from the Crab/Vela pulsar"

(Abbot et al., Ap.J. 683, L45 '08; Ap.J. 737, 93, '11)

Crab pulsar



An analogue (but different) approach fore the Vela pulsar shows that the spin-down limit can be overcome (to a smaller amount) for this pulsar as well.

Rotation of a non-axisymmetric NS

Other possibility: a=b ($I_1=I_2$) but wobble angle $\theta \ll 1$ between symmetry axis (I_3) and rotation axis.

This is the relevant process for magnetic field-induced deformation!

In this case $$R = \begin{pmatrix} \cos\Omega t & -\sin\Omega t & -\theta\sin\Omega t \\ \sin\Omega t & \cos\Omega t & \theta\cos\Omega t \\ 0 & -\theta & 1 \end{pmatrix} + O(\theta^2)$$

and quadrupole formula yields

$$h_{ij}^{TT} = h_0 \left[\mathcal{P} \begin{pmatrix} 0 & 0 & \sin\Omega\left(t-\frac{r}{c}\right) \\ 0 & 0 & -\cos\Omega\left(t-\frac{r}{c}\right) \\ \sin\Omega\left(t-\frac{r}{c}\right) & -\cos\Omega\left(t-\frac{r}{c}\right) & 0 \end{pmatrix} \right]$$

with $\nu_{GW} = \nu_{rot} \quad (\Omega = 2\pi\nu_{rot})$ and $h_0 = \dfrac{8\pi^2 G}{c^4\ r\ T^2}\,(I_1 - I_3)\theta$

GWs are emitted for this process at small frequencies (11 Hz for Vela, 30 Hz for Crab), at which detectors are less sensitive.

However, it is not clear whether θ is rapidly damped.

Non-radial oscillations of NSs

When a neutron star (or a black hole) is perturbed
by an internal or an external event,
it can be set into *non-radial oscillations*,
emitting GWs at the characteristic frequencies of its
quasi-normal modes (QNMs) : $\omega=2\pi\upsilon+i/\tau$
They are *damped oscillations* (=> complex frequency) due to GWs

Several kinds of processes can excite NS oscillations:

- glitches (see Ian's lecture)
- gravitational collapse giving birth to the NS
- compact binary inspiral and coalescence (ringing phase)
- phase transition of the matter composing the star
- accretion from a companion star
- EM activity, as in magnetar giant flares (see Ian's lecture)

Non-radial oscillations of NSs

Many sets of general relativistic equations have been derived in the years

(Thorne & Campolattaro '67; Lindblom & Detweiler '85; Chandrasekhar & Ferrari '90).

We follow the notation of Lindblom & Detweiler (LD).

The perturbed spacetime metric is expanded

in tensor spherical harmonics, in the frequency domain:

$$
\begin{aligned}
ds^2 \quad = \quad & -e^{\nu}\left(1 + r^{\ell} H_0^{\ell m} Y_{\ell m} e^{\mathrm{i}\omega t}\right) dt^2 + e^{\lambda}\left(1 + r^{\ell} H_2^{\ell m} Y_{\ell m} e^{\mathrm{i}\omega t}\right) dr^2 - 2\mathrm{i}\omega r^{\ell+1} H_1^{\ell m} Y_{\ell m} e^{\mathrm{i}\omega t} dt dr \\
& + r^2\left(1 - r^{\ell} K^{\ell m} Y_{\ell m} e^{\mathrm{i}\omega t}\right)\left(d\vartheta^2 + \sin^2\theta d\varphi^2\right)
\end{aligned}
$$

$$
u^{\mu} = u_0^{\mu} + \delta u^{\mu} = (e^{-\nu/2}, 0, 0, 0) + \mathrm{i}\omega e^{-\nu/2}(0, \xi_r, \xi_\theta, \xi_\phi),
$$

$$
\begin{aligned}
\xi_r(t,r,\vartheta,\varphi) &= e^{\lambda/2} r^{\ell-1} W^{\ell m}(r) Y_{\ell m}(\vartheta,\varphi) e^{\mathrm{i}\omega t} \\
\xi_\vartheta(t,r,\vartheta,\varphi) &= -r^{\ell} V^{\ell m}(r)\partial_\vartheta Y_{\ell m}(\vartheta,\varphi) e^{\mathrm{i}\omega t}, \\
\xi_\varphi(t,r,\vartheta,\varphi) &= -r^{\ell} V^{\ell m}(r)\partial_\varphi Y_{\ell m}(\vartheta,\varphi) e^{\mathrm{i}\omega t}.
\end{aligned}
$$

Einstein's equations, linearized in the metric perturbations,

yield a 4$^{\text{th}}$-order system of ODEs inside the star,

a single 2$^{\text{nd}}$-order equation (the Zerilli equation) in vacuum.

They are solved assuming regularity at the center, continuity at the surface

(together with $\Delta p=0$ as $r=R$), outgoing wave boundary conditions at infinity.

Non-radial oscillations of NSs

Inside the star: Lindblom-Detweiler equations

$$
\begin{aligned}
H_1^{lm\,\prime} &= -\frac{1}{r}\left[\ell+1+\frac{2Me^{\lambda}}{r}+4\pi r^2 e^{\lambda}(p-\epsilon)\right]+\frac{e^{\lambda}}{r}\left[H_0^{lm}+K^{lm}-16\pi(\epsilon+p)V^{lm}\right] \\
K^{lm\,\prime} &= \frac{1}{r}H_0^{lm}+\frac{\ell(\ell+1)}{2r}H_1^{lm}-\left[\frac{\ell+1}{r}+\frac{\psi'}{2}\right]K^{lm}-8\pi(\epsilon+p)\frac{e^{\lambda/2}}{r}W^{lm} \\
W^{lm\,\prime} &= -\frac{\ell+1}{r}W^{lm}+re^{\lambda/2}\left[\frac{e^{-\psi/2}}{(\epsilon+p)c_s^2}X^{lm}-\frac{\ell(\ell+1)}{r^2}V^{lm}+\frac{1}{2}H_0^{lm}+K^{lm}\right] \\
X^{lm\,\prime} &= -\frac{\ell}{r}X^{lm}+\frac{(\epsilon+p)e^{\psi/2}}{2}\left[\left(\frac{1}{r}+\frac{\psi'}{2}\right)+\left(r\omega^2e^{-\psi}+\frac{\ell(\ell+1)}{2r}\right)H_1^{lm}+\left(\frac{3}{2}\psi'-\frac{1}{r}\right)K^{lm}\right. \\
&\quad \left.-\frac{\ell(\ell+1)}{r^2}\psi' V^{lm}-\frac{2}{r}\left(4\pi(\epsilon+p)e^{\lambda/2}+\omega^2e^{\lambda/2-\psi}-\frac{r^2}{2}\left(\frac{e^{-\lambda/2}}{r^2}\psi'\right)'\right)W^{lm}\right]
\end{aligned}
$$

Outside the star: Zerilli equation

$$
\frac{d^2Z^{lm}}{dr_*^2}+\left[\omega^2-V_Z(r)\right]Z^{lm}=0 \qquad \left(V_Z\equiv e^{-\lambda}\frac{2n^2(n+1)r^3+6n^2Mr^2+18nM^2r+18M^3}{r^3(nr+3M)^2}\right)
$$

Solutions only exist for a discrete set of complex frequencies

ω = 2πν + i/τ :

the QNMs of the star.

Non-radial oscillations of NSs

Detection of the GW emission from a NS in radial oscillations will allow us to measure the frequencies and damping times of its QNMs which would give us unvaluable information on the matter composing the star.

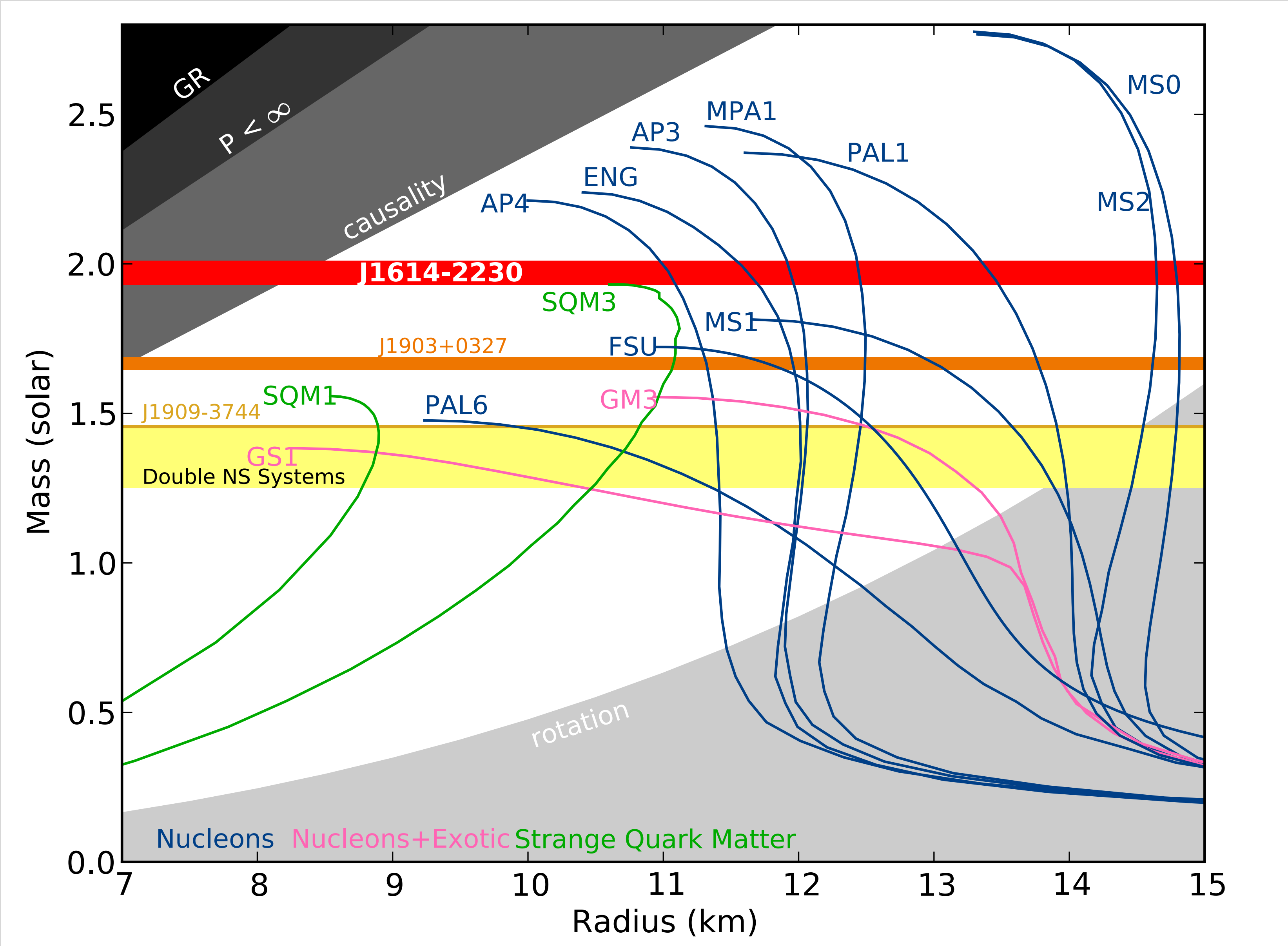
We probably know how it is organized matter in the crust of a NS, maybe also in the outer core, but we do not know the behaviour of matter in the inner core of a NS where it reaches supranuclear densities $\rho \sim 10^{15}$ g/cm^3 which cannot be reproduced in the laboratory.

Hadron interactions play a crucial role [a simplified NS model based on based on the Fermi pressure of neutrons alone, predicts M_{max}=0.7M_o, while we observe M=1.4M_o]

Our lack of knowledge on the NS Equation of State (EoS) (we do not even know the particle content in the core: Hadrons? Hyperons? Meson condensates? Deconfined quark matter [i.e. Strange Stars as in Witten '84]?) reflects our ignorance on the non-perturbative regime of QCD.

Even a simple information such as the value of the NS radius R (a "clean" observation of R in the EM spectrum is very difficult) would be important.

Non-radial oscillations of NSs



(Demorest et al., Nature '10)

Nuclear physicists have proposed several possible EoS describing the matter in the stellar core, which differ in the assumptions (different particle content, nuclear many body vs. mean field) and in the computational techniques. Astrophysical observations are useful to constrain the EoS, but only GW detection could give us a definite answer to these questions!

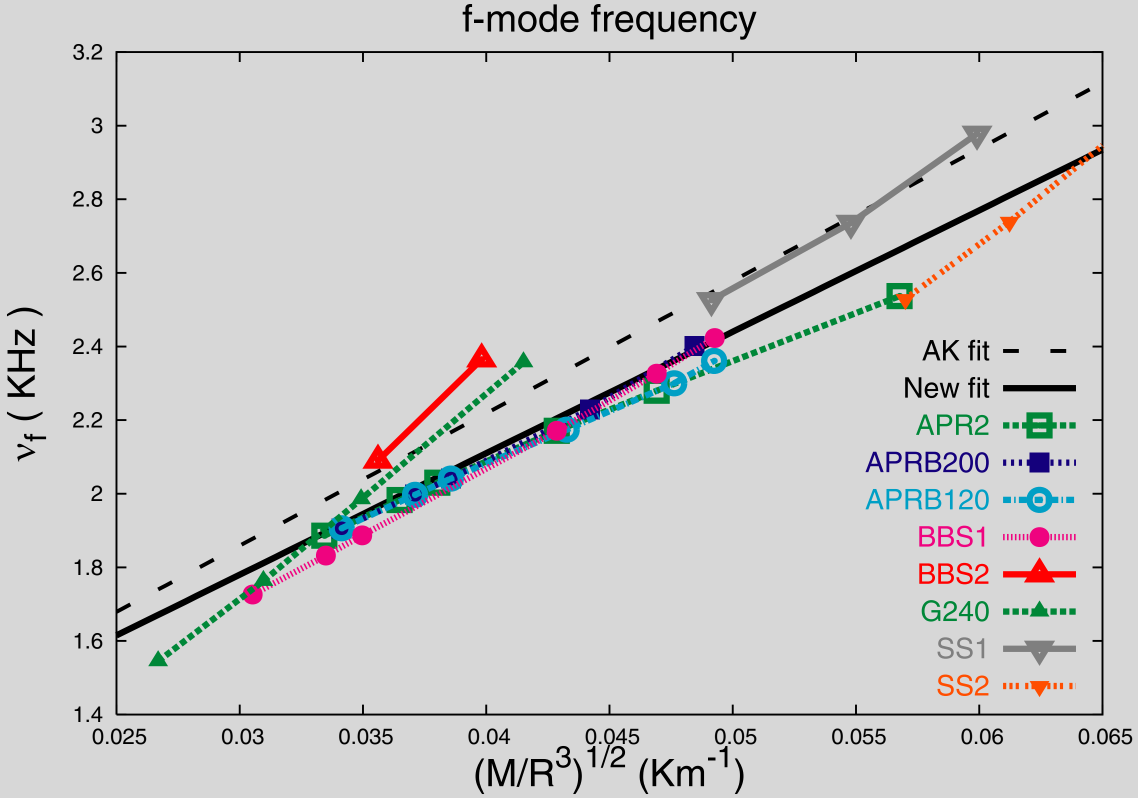
Non-radial oscillations of NSs

A GW-detection from a NS pulsating in its QNMs could allow us:

- to infer the value of the NS radius R, strongly constraining the EoS

  (N. *Andersson & K. Kokkotas, '98, O. Benhar et al. '04, '07*)
- to discriminate between different possible EoS
- to establish whether the emitting source is a NS or a quark star,
- if it is a quark star, to costrain the quark star EoS.

QNMs of NSs are functions of their mass and radius, irrespective of the EoS. If we measure (through GW detection) the frequencies and damping time of the f- and p_1- modes, we know M and R, useful to understand the NS EoS!



$$\nu_f = a + b\sqrt{\frac{M}{R^3}}, \quad a = 0.79 \pm 0.09, \quad b = 33 \pm 2,$$

$$\tau_f = \frac{R^4}{cM^3}\left[a + b\frac{M}{R}\right]^{-1},$$

$$a = [8.7 \pm 0.2] \times 10^{-2}, \qquad b = -0.271 \pm 0.009.$$

Non-radial oscillations of NSs

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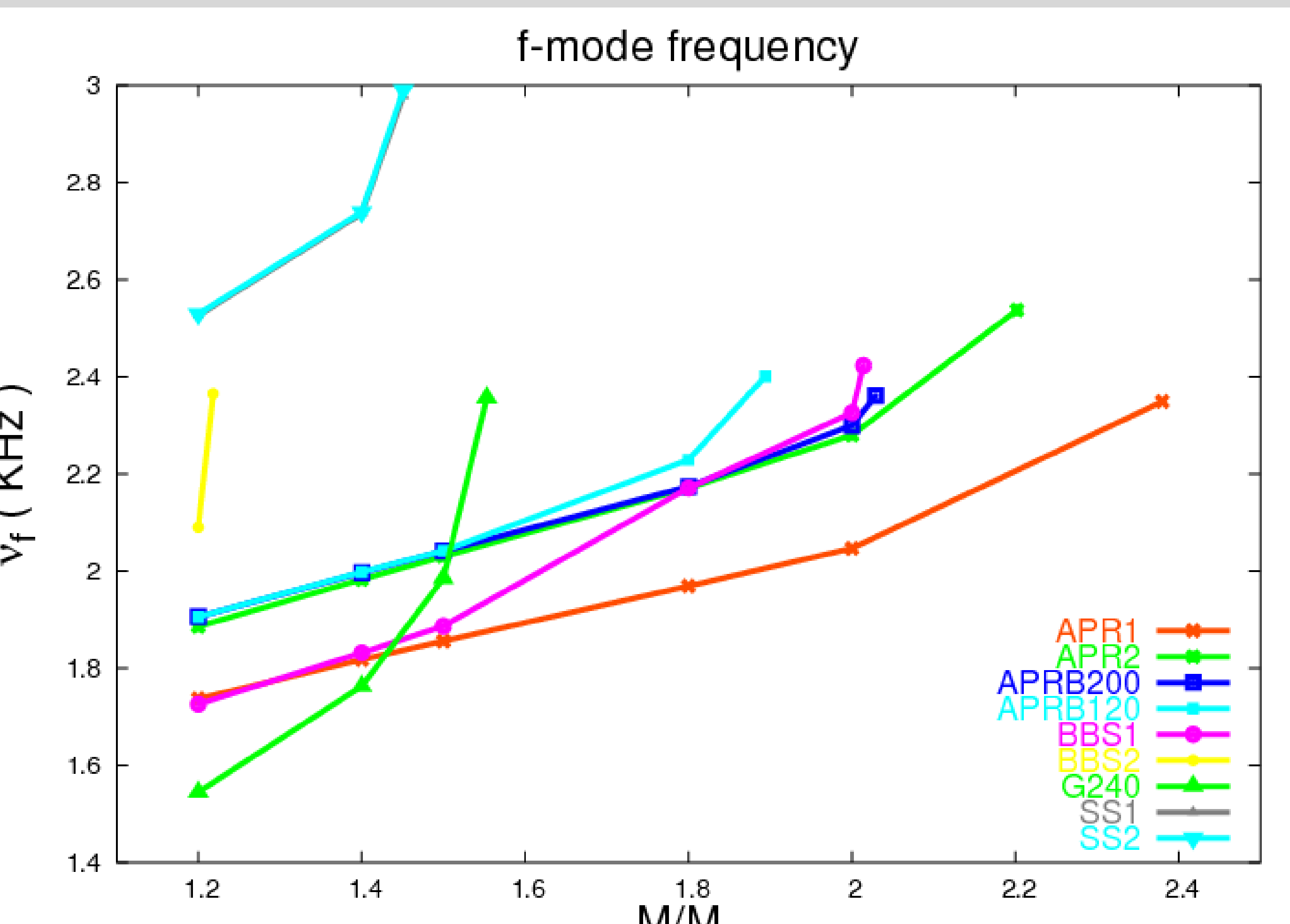

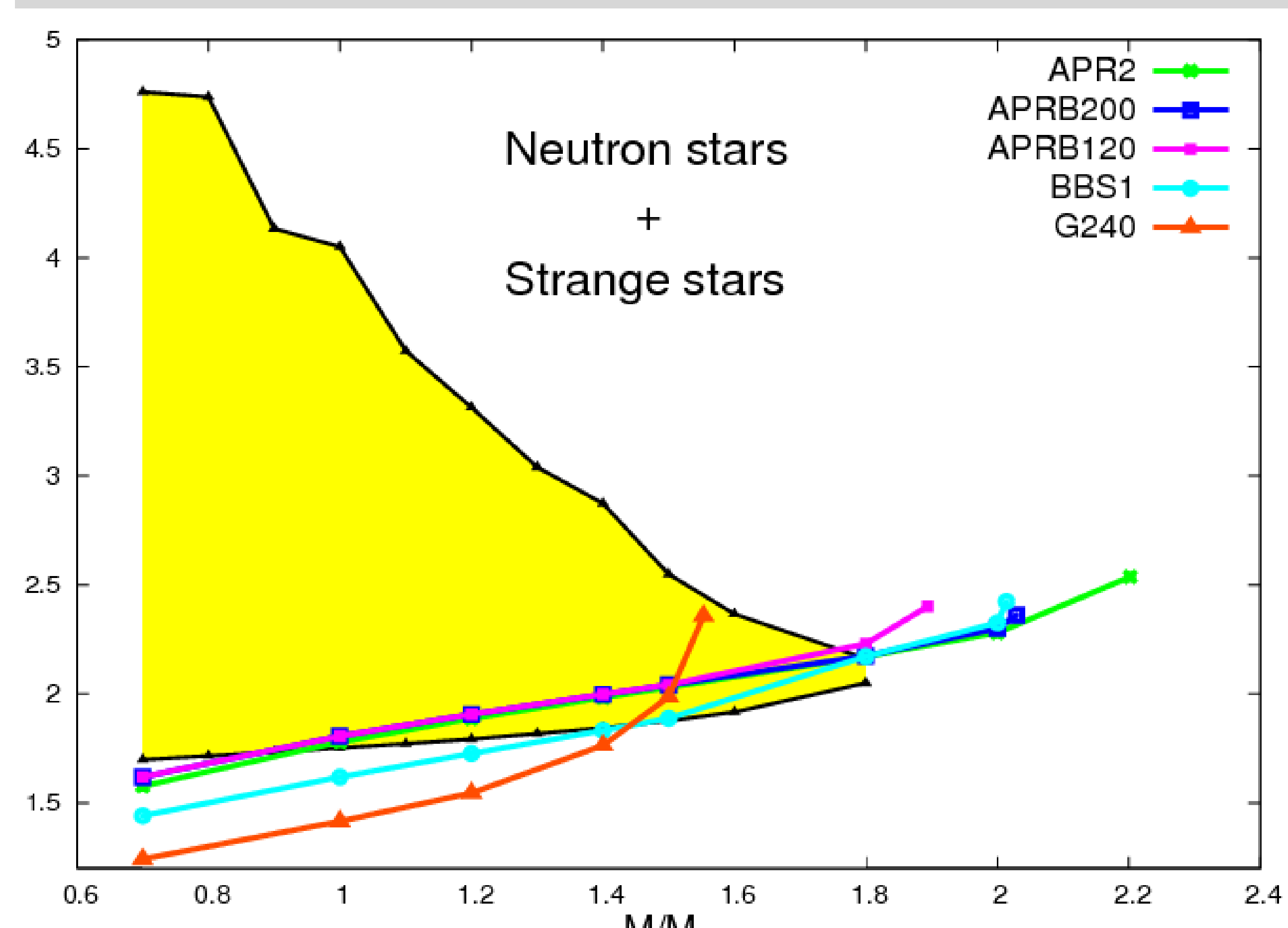

Non-radial oscillations of NSs

However, there is much more than the f-mode of non-rotating cold stars. In recent years, a lot of effort has been devoted in many groups to model different kinds of NS oscillations including more and more physics in the game:

- QNMs of rotating NSs
  They are extremely impotant, becauase these modes can become unstable, with a large gravitational emission
- QNMs of magnetized NSs
  Expecially oscillations of the crust, eventually coupled with the core. Interplay of magnetic field and crustal strain. Results can be compares with observational data (giant flares in magnetars)
- QNMs of hot, newly born NSs
  In the first tens of seconds after the bounce, thermodynamics and neutrinos strongly affect the stellar structure and its QNM spectrum
- QNMs of superfluid NSs
  NSs (if not too young) are superfluid (see Ian's lecture). This feature (in particular, having two fluids) significantly affect the QNM spectrum.

Conclusions

A neutron star can radiate gravitational waves
in various astrophysical processes.

In particular, deformed rotating NSs and oscillating NSs
are promising sources for ground based interferometers
such as Advanced LIGO/Virgo, KAGRA, ET.

Gravitational waves will hopefully be detected soon.

Such a detection would provide unvaluble information:

- on the astrophysical processes involving NSs
- on the behaviour of matter in their cores
(and then, on the nature of hadronic interaction)
- finally (even though it was not discussed in this lecture)
on the nature of the gravitational interaction