BLACK HOLES
IN GENERAL RELATIVITY

Leonardo Gualtieri & Valeria Ferrari

Dipartimento di Fisica, Università degli studi di Roma "Sapienza"

(2011)

Contents

These notes have been written for a graduate course on black holes in general relativity; it is assumed that the students have already followed a basic course on general relativity. The notes are based on the extensive literature existing on the subject, in particular on R.W. Wald, *General relativity*, University of Chicago Press, Chicago, 1984; S.W. Hawking, G.F.R. Ellis, *The large scale structure of spacetime*, Cambridge University Press, Cambridge, 1973; and on the original articles cited in these books. For more basic topics on general relativity I refer the reader to C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, W.H. Freeman and co., S. Francisco, 1973; S. Weinberg, *Gravitation and cosmology*, Wiley, New York, 1972; B.F. Schutz, *A first course in general relativity*, Cambridge University Press, Cambridge, 1985; H. Stephani, *Relativity*, Cambridge University Press, Cambridge, 2004; N. Straumann, *General Relativity*, Springer Verlag, Berlin, 2004; and on the notes of the undergraduate course "General Relativity" held by Valeria Ferrari.

I use the geometric units $G = c = 1$.

Chapter 1

The Schwarzschild solution

In this chapter we discuss some features of the Schwarzschild solution; some of them are already covered by fundamental general relativity courses, and will be only mentioned; some others will be discussed in greater detail.

1.1 Line element and general properties

The Schwarzschild solution can be written as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1.1)$$

in the coordinates (t, r, θ, ϕ). This solution is asymptotically flat, since as $r \to \infty$ it tends to the Minkowski metric in polar coordinates.

The spacetime metric (1.1) depends on the positive constant M, which can be interpreted as the BH mass. A reason for this interpretation can be the following: if a body moves in the Schwarzschild spacetime, and if r if enough large to assume, with good accuracy, the weak-field approximation, then the motion of the body is described by Newton's laws, with gravitational potential

$$\Phi = -\frac{M}{r} \quad (1.2)$$

which coincides with the gravitational potential of a central body with mass M. As we will see later on, there are other reasons to give to M the interpretation of BH mass.

The Schwarzschild solution is spherically symmetric; furthermore, it is static, i.e., it admits a timelike Killing vector ($\xi^\mu = (1, 0, 0, 0)$) hypersurface orthogonal ($g_{0i} = 0$ if $i \neq 0$). As a consequence, the Schwarzschild metric is invariant for temporal translations (it is stationary) and for time reversal $t \leftrightarrow -t$. The Schwarzschild solution admits four independent Killing vectors: one timelike, and three spacelike, corresponding to the symmetries of the rotation group $SO(3)$.

It is possible to show that any asymptotically flat and spherically symmetric solution of Einstein's equations in vacuum is also static, and then it is the Schwarzschild solution. This result, named the *Birkhoff theorem*, is very important; it implies, for instance, that the exterior of spherically symmetric stars is described by the Schwarzschild solution, and that spherically symmetric objects cannot be sources of gravitational radiation.

1.2 Horizon

The metric, in these coordinates, is singular at $r = 0$ and at $r = 2M$. However, the singularity $r = 2M$ is only an artifact of the coordinate choice, and can be removed by changing coordinates (and then it is called "coordinate singularity"); the singularity $r = 0$, instead, is a true singularity of the metric (and is called "curvature singularity") [1].

Before discussing the removal of the coordinate singularity, let us consider in greater detail the surface $r = 2M$, and the other hypersurfaces $r = constant$.

The normal of a hypersurface whose equation is $\Sigma \equiv r - constant = 0$ is

$$n_\mu = \Sigma_{,\mu} = (0, 1, 0, 0) \,. \qquad (1.3)$$

Let us consider a generic hypersurface. At any point of such hypersurface we can introduce a locally inertial frame, and rotate it in such a way that the components of the normal vector are

$$n^\alpha = (n^0, n^1, 0, 0) \quad \text{and} \quad n_\alpha n^\alpha = (n^1)^2 - (n^0)^2 \,. \qquad (1.4)$$

[1] Actually, there are other coordinate singularities: $\theta = 0, \pi$ and $\phi = 0, 2\pi$; the removal of such singularities is trivial: it is sufficient to define new angular variables by performing a rotation; therefore, in the following we will not consider these singularities.

Consider a vector t^α tangent to the surface at the same point. t^α must be orthogonal to n^β:

$$n_\alpha t^\alpha = -n^0 t^0 + n^1 t^1 = 0 \qquad \Rightarrow \qquad \frac{t^0}{t^1} = \frac{n^1}{n^0} \tag{1.5}$$

thus

$$t^\alpha = \Lambda(n^1, n^0, a, b) \qquad \text{with a, b e} \quad \Lambda \quad \text{constant and arbitrary.} \tag{1.6}$$

Consequently the norm of the tangent vector is

$$t_\alpha t^\alpha = \Lambda^2[-(n^1)^2 + (n^0)^2 + (a^2 + b^2)] = \Lambda^2[-n_\alpha n^\alpha + (a^2 + b^2)]\,. \tag{1.7}$$

We have that

- If $n_\mu n^\mu < 0$, the hypersurface is called spacelike, and t^μ is necessarily a spacelike vector.
- If $n_\mu n^\mu > 0$, the hypersurface is called timelike, and t^μ can be timelike, spacelike or null.
- If $n_\mu n^\mu = 0$, the hypersurface is called null, and t^μ can be spacelike or null.

Let us consider a point P on a surface $\Sigma = 0$. If the surface is spacelike, the tangent vectors of the surface lie all outside the light-cone in P. Therefore, a particle passing through P, whose velocity vector lies inside the cone, can cross the surface only in one direction. If the surface is null, the situation is nearly the same: the tangent vectors to the surface lie inside to the light-cone in P, or are tangent to it, thus a particle can cross the surface in one direction only. It the surface is timelike, some tangent vectors of the surface are inside the cone, some others are outside, i.e. the surface cuts the cone, and then a particle passing through P can cross the surface in both directions.

From (1.3), (1.1), in the case of an $r = constant$ surface

$$n_\mu n_\nu g^{\mu\nu} = g^{rr} = 1 - \frac{2M}{r} \tag{1.8}$$

thus the surfaces $r = constant$ are spacelike if $r < 2M$, null if $r = 2M$, timelike if $r > 2M$.

The null hypersurface $r = 2M$, then, separates regions of space-time where $r = const$ are timelike hypersurfaces from regions where

$r = const$ are spacelike hypersurfaces; therefore, an object crossing a null hypersurface $r = const$ can never come back; for this reason, the null hypersurface $r = 2M$ is called *horizon*. As we will see when studying other BH solutions, this is a general property of null hypersurfaces. The horizon $r = 2M$ separates the spacetime in:

- the region with $r > 2M$, where the $r = const.$ hypersurfaces are timelike; the $r \to \infty$ limit, where the metric becomes flat, is in this region, so we can consider this region as the exterior of the black hole;

- the region with $r < 2M$, where the $r = const.$ hypersurfaces are spacelike; an object which falls inside the horizon and enter in this region can only continue falling to decreasing values of r, until it reaches the curvature singularity $r = 0$; this region is then considered the interior of the BH; the name BH is due to the fact that nothing, neither objects nor signals of any kind, can escape from this region.

1.3 Singularities

The fact that $r = 0$ is a curvature singularity can be shown by computing the curvature scalar

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 48\frac{M^2}{r^6} \tag{1.9}$$

which diverges at $r = 0$. The fact that $r = 2M$ is a coordinate singularity is less easy to prove: the finiteness of the polynomials in the curvature tensor does not exclude, in principle, that there is a curvature singularity there. Therefore we need to study this singularity, finding the coordinate change which allows to remove it.

1.3.1 Spacetime extension

To study the structure of singularities, we must first define rigorously the concept of singularity in general relativity. This is not obvious, since a singularity does not belong to the spacetime manifold; changing coordinate frame, a singularity can be mapped to infinity, so that at a first look the metric does not appear singular (there is nothing strange in a singular behaviour at infinity).

The key notion is the length of geodesics. In Schwarzschild spacetime, for instance, an observer falling into the BH reaches the singularity $r = 0$ in a finite amount of proper time, thus its (timelike) geodesic has a finite length and cannot be extended; this means that $r = 0$ is a singularity: whatever coordinate frame we choose, that geodesic has a finite length.

Therefore, to characterize a singular behaviour we need the notion of *geodesic completeness*: a spacetime is geodesically complete if every timelike and null geodesic can be extended to values arbitrarily large of the affine parameter. If, instead, the spacetime admits at least one incomplete (i.e., which cannot be extended) timelike or null geodesic, we say that it is *geodesically incomplete*, and this means that there is a singularity (either a true curvature singularity or a coordinate singularity). We only consider timelike or null geodesics because they represent worldlines of massive or massless particles, and then they represent observers or signals, while spacelike geodesics do not correspond to worldlines of physical objects.

Let us consider the Schwarzschild metric in the coordinates (t, r, θ, ϕ), with line element (1.1)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \,. \quad (1.10)$$

To be rigorous, the spacetime manifold described by (1.10) is not defined at $r = 0$ and at $r = 2M$. We have then two disconnected manifolds, $\mathcal{M}_1$ with $0 < r < 2M$ (the interior of the BH) and $\mathcal{M}_2$ with $r > 2M$ (the exterior of the BH). A geodesic corresponding to an observer falling into the BH cannot be extended across the hypersurface $r = 2M$, since it does not belong to $\mathcal{M}_1 \cup \mathcal{M}_2$, and the geodesic terminates at a finite value of the affine parameter (notice that the observer arrives at $r = 2M$ with $t = +\infty$). On the other hand, since this singularity is not a true curvature singularity, it can be removed with the following two steps:

- we change the coordinate frame, for instance to the Kruskal frame (U, V, θ, ϕ), for which (as we shall show in Section 1.3.2)

  $$ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} dU dV + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1.11)$$

  (note that the r in (1.11) should be considered as a function of the coordinates, $r(U, V)$).

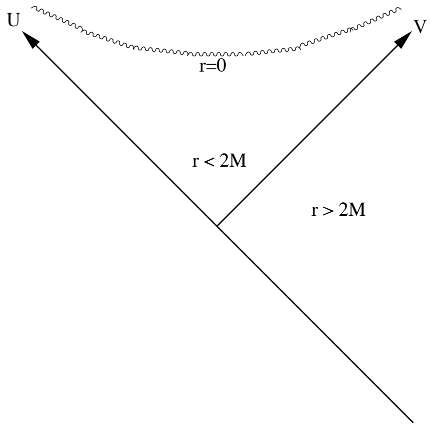

Figure 1.1: Interior and exterior of a Schwarzschild black hole in Kruskal coordinates.

In these coordinates (see Fig. 1.1), the BH exterior $r > 2M$ corresponds to ($U < 0$, $V > 0$) and the BH interior $r < 2M$ corresponds to ($U > 0$, $V > 0$ with $UV < 1$); the singularity $r = 2M$ (and $t = +\infty$) corresponds to the semiaxis ($U = 0$, $V > 0$); the curvature singularity $r = 0$ corresponds to the upper branch of the hyperbole $UV = 1$.

- In the new frame the manifold can be extended across the semiaxis ($U = 0$, $V > 0$), separating $\mathcal{M}_1$ and $\mathcal{M}_2$, since the line element (1.11) is not singular there; this means that we consider a new manifold,
$$\mathcal{M} \supset \mathcal{M}_1 \cup \mathcal{M}_2\,, \tag{1.12}$$
defined by
$$V > 0\,, \qquad UV < 1\,. \tag{1.13}$$

Therefore, in order to eliminate a coordinate singularity we have to *extend* the spacetime manifold.

Notice that when we think of a Schwarzschild BH, we are implicitly considering the extended manifold $\mathcal{M}$: indeed, we assume that an object falling inside the BH crosses the horizon $r = 2M$. Even the discussion of the previous section about the hypersurfaces $r = constant$ (and, among them, $r = 2M$), assumes that the manifold is $\mathcal{M}$; we stress that the manifold $\mathcal{M}$ is not covered by the

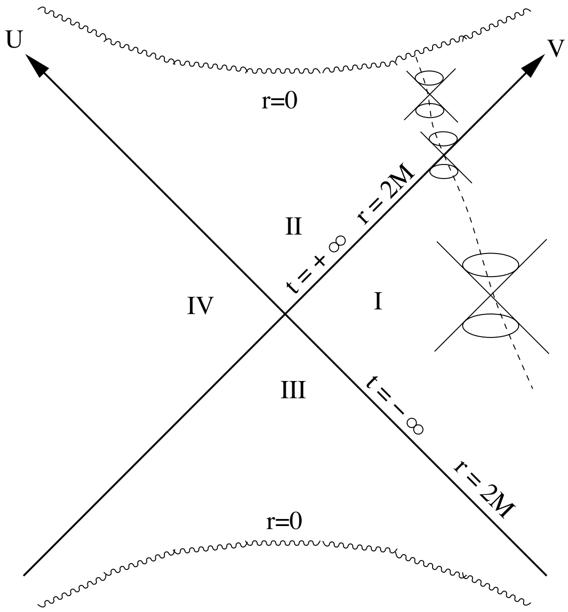

Figure 1.2: Maximal extension of Schwarzschild spacetime in Kruskal coordinates. The dashed line represents the worldline of an observer falling into the black hole.

coordinates (t, r, θ, ϕ) and the line element (1.10).

The manifold $\mathcal{M}$ should still be extended: timelike and null geodesics from $V = 0$ cannot be extended backwards, unless we extend the manifold to $V \leq 0$. By considering ($-\infty < U < +\infty$, $-\infty < V < +\infty$ with $UV < 1$) we have the *maximal extension* (i.e., which cannot be further extended) of the Schwarzschild spacetime. This is the usual Kruskal construction, shown in Fig. 1.2; the dashed line represents the worldline of an observer falling into the black hole, and the wave-like curves represent the curvature singularity $r = 0$.

In the Kruskal coordinates (1.11) the null worldlines with θ, ϕ constant, are straight lines at 45^o, i.e. $U = const.$ and $V = const.$; this can be seen easily from the line element (1.11): assuming θ, ϕ constant, if either $dU = 0$ or $dV = 0$, then $ds = 0$ and the infinitesimal tangent vector is null (to say in different words: any worldline with tangent vector either $(1, 0, 0, 0)$ or $(0, 1, 0, 0)$, is null). Therefore, the light-cone can be drawn as in Minkowski spacetime, and then it is easy to describe the causal connections among events (see for instance, in Fig. 1.2, the dashed line representing the possible worldline of a massive particle falling inside the black hole). In particular, we see that Sector I can send signals (and matter and energy) to sector II only, and receive signals from sector III; furthermore, there is another copy of sector I, i.e. sector IV, which is causally disconnected from I, but can receive signals from III and send signals to II.

The incomplete (timelike and null) geodesics of the maximally extended manifold correspond to a true singularity: in this case, to $r = 0$, i.e., in Kruskal coordinates, to $UV = 1$. There are geodesics reaching the singularity with finite affine parameter, and cannot be extended through it; for instance, an observer that falls inside the BH reaches the singularity in a finite amount of proper time.

It should be stressed that the maximal extension of Schwarzschild spacetime has no meaning if we consider a BH as an astrophysical object. Therefore, we do not have to worry about the meaning of the sector IV, and leave the discussion of other universes to science-fiction writers. Indeed, the above construction describes an *eternal black hole*, whereas astrophysical BHs result from stellar collapse, which happen at finite values of t. In particular, the region III cannot exist for an astrophysical BH, because the semiaxis ($U < 0$, $V = 0$) corresponds to $t = -\infty$.

A comment on the $r = 0$ singularity. The fact that we cannot say what happens to the observer as it reaches the singularity, constitutes a problem for the theory; on the other hand, such problem is not severe from an operational point of view, since no signal from the observer reaching the singularity can be sent outside the black hole: the consistency of the theory, in a certain sense, is preserved by the existence of the horizon. Roger Penrose has conjectured that there is a fundamental principle, the *cosmic censorship hypothesis*, stating that all singularities in the universe (with the exception of a possible initial singularity) are covered by horizons. In other words, there is no *naked singularity*. There is no definitive proof of this conjecture, but there are indications supporting it. Therefore, it is commonly believed that the cosmic censorship hypothesis is likely to be correct; it is worth noticing that many mathematical properties of spacetime would be challenged by the existence of a naked singularity.

1.3.2 Kruskal and Eddington-Finkelstein coordinates

The Kruskal coordinates are defined as follows. Let us consider the $r > 2M$ manifold, and define

$$u \equiv t - r_* \,, \qquad v \equiv t + r_* \tag{1.14}$$

with

$$r_* \equiv r + 2M \ln\left(\frac{r}{2M} - 1\right) \tag{1.15}$$

tortoise coordinate, which tends to $-\infty$ as $r \to 2M$. Notice that $r > 2M$ corresponds to

$$-\infty < u < +\infty \qquad -\infty < v < +\infty \,. \tag{1.16}$$

and the limit $r \to 2M$, $t \to +\infty$ corresponds to $u \to +\infty$, with finite v. Since

$$\frac{dr_*}{dr} = \frac{1}{1 - \frac{2M}{r}} \,, \tag{1.17}$$

the metric is

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right)(dt^2 - dr_*^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= -\left(1 - \frac{2M}{r}\right) du dv + r^2(d\theta^2 + \sin^2\theta d\phi^2) \,. \end{aligned} \tag{1.18}$$

We have

$$\frac{r_* - r}{2M} = \ln\left(\frac{r}{2M} - 1\right) \quad \Rightarrow \quad 1 - \frac{2M}{r} = \frac{2M}{r} e^{\frac{r_* - r}{2M}} = \frac{2M}{r} e^{\frac{-r}{2M}} e^{\frac{v-u}{4M}} \,. \tag{1.19}$$

Then, defining

$$U \equiv -e^{-\frac{u}{4M}} \,, \qquad V \equiv e^{\frac{v}{4M}} \tag{1.20}$$

we have the Kruskal metric:

$$ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} dU dV + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.21}$$

with $U < 0$, $V > 0$.

Let us now consider the manifold $0 < r < 2M$, and define u, v as in (1.14),

$$u \equiv t - r_* \,, \qquad v \equiv t + r_* \tag{1.22}$$

but with a new tortoise coordinate

$$r_* \equiv r + 2M \ln\left(1 - \frac{r}{2M}\right) \tag{1.23}$$

which is always negative, tends to zero as $r \to 0$, and to $-\infty$ as $r \to 2M$. Differentiating this new tortoise coordinate we get the same expression as (1.17),

$$\frac{dr_*}{dr} = \frac{1}{1 - \frac{2M}{r}}\,, \tag{1.24}$$

then the metric in u, v is still given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dudv + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.25}$$

but

$$1 - \frac{2M}{r} = -\frac{2M}{r} e^{\frac{-r}{2M}} e^{\frac{v-u}{4M}} \tag{1.26}$$

thus, defining

$$U \equiv +e^{-\frac{u}{4M}}\,, \qquad V \equiv e^{\frac{v}{4M}}\,, \tag{1.27}$$

we have the same metric of eq. (1.21),

$$ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} dUdV + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{1.28}$$

but with $U > 0$, $V > 0$.

This metric, with $V > 0$ and extended to $-\infty < U < +\infty$, describes then the exterior and the interior of the BH, as anticipated.

Actually, there is a simpler coordinate system that covers the region I and II: the Eddington-Finkelstein (EF) coordinates

$$(v, r, \theta, \phi) \qquad -\infty < v < +\infty \qquad 0 < r < +\infty\,. \tag{1.29}$$

The two definitions (1.15), (1.23) can be put together as

$$r_* \equiv r + 2M \ln\left|\frac{r}{2M} - 1\right| \tag{1.30}$$

and $v = t + r_*$. Then, since

$$dt^2 - dr_*^2 = dv^2 - 2dvdr_* = dv^2 - 2\frac{dr_*}{dr} dvdr = dv^2 - 2\frac{dvdr}{1 - \frac{2M}{r}}\,, \tag{1.31}$$

the metric in the EF coordinates is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \,. \qquad (1.32)$$

This metric covers both the interior and the exterior of the BH, i.e. the sectors I and II of the Kruskal construction, and is not singular at the horizon, which is simply $r = 2M$. Notice that on the horizon v is finite because $t \to +\infty$ and $r_* \to -\infty$, while u is instead divergent. All the computations and derivations involving the interior and the exterior of the BH, like for instance the study of the $r = constant$ surfaces of Section 1.2, and the study of what happens to a spaceship falling inside the BH, can be rigorously performed in the EF coordinates.

In principle, one could also build the maximal extension of the Schwarzschild metric by patching together the EF chart, which covers sectors I,II, with other three (similar) charts, one covering sectors II,III, one covering sectors III,IV, and one covering sectors IV,II; fortunately, we do not need to discuss this complicate construction, since there exists a single coordinate frame, the Kruskal frame, which covers all the four sectors of the maximal extension. But for other BH solutions there is no equivalent to the Kruskal coordinates, and the only way to build the maximal extension is to patch together different coordinate frames (analogue to the EF frame), which eliminate the coordinate singularities but do not provide, alone, the maximal extension.

1.3.3 The Finkelstein diagram

A useful way to visualize the Schwarzschild spacetime is the Finkelstein diagram, in which the axes are $(\tilde{t}, r)$ where

$$\tilde{t} \equiv v - r = t + 2M \ln\left|\frac{r}{2M} - 1\right| \,. \qquad (1.33)$$

In the Finkelstein diagram the null lines $v = const.$, corresponding to ingoing massless particles, are straight lines at 45^o; the null lines $u = const.$, corresponding to outgoing massless particles, are hyperbolic curves. These two sets of curves define the light cones centered in any point of the spacetime, and then allow to establish the causal relationship among different events. Differently from the (t, r) diagram, the light-cones behave regularly at the horizon.

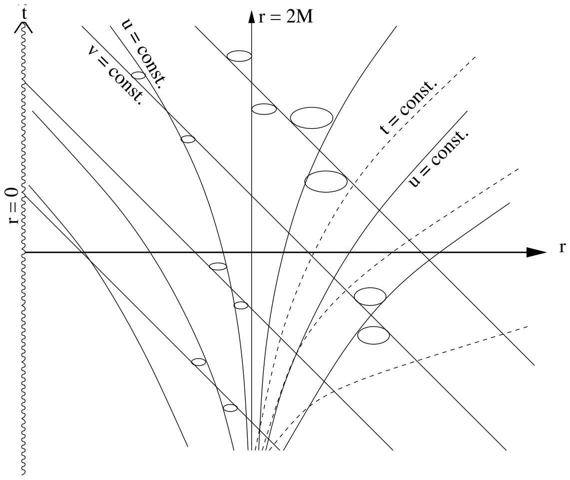

Figure 1.3: Eddington-Finkelstein diagram of a Schwarzschild black hole.

Since $\tilde{t} \simeq t$ for $r \ll 2M$ and $r \gg 2M$, the coordinate $\tilde{t}$ coincides with t far away from the horizon, but they are very different close to the horizon (where, anyway, the coordinate t has not a clear physical meaning). Still, inside the horizon $\tilde{t}$ cannot be considered a "time", because the vector $\partial/\partial\tilde{t}$ is spacelike; this is clear from Fig. 1.3, where $\partial/\partial\tilde{t}$ (directed vertically) falls outside the null cones in the interior of the BH.

Notice that the $r = \textit{constant}$ lines are vertical lines, while the $t = \textit{constant}$ lines are hyperbolic curves (dashed lines in Fig. 1.3); the $\tilde{t}$-axis represents the singularity, and for this reason it is drawn wave-like.

As we mentioned above, astrophysical BHs are the product of gravitational collapse, and then are not eternal black holes. A qualitative view of the spacetime of a realistic black hole is shown in the Finkelstein diagram in Fig. 1.4, where the shadowed area represents the interior of the star. The $r = 0$ axis is a curvature singularity for $\tilde{t} \geq \tilde{t}_0$, i.e. after the singularity formation; $\tilde{t} < \tilde{t}_0$, the $r = 0$ axis is simply the (trivial) coordinate singularity at the origin of polar coordinates. Also the horizon, represented by the dashed line, is formed during the collapse.

It is important to stress that, although we have discussed the entire Schwarzschild solution, only the $r > 2M$ region is directly

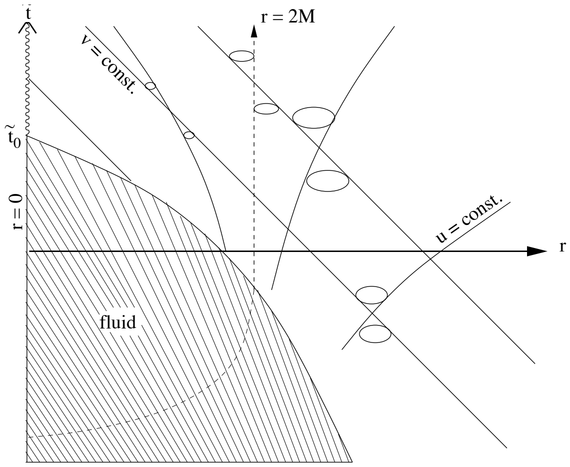

Figure 1.4: Eddington-Finkelstein diagram of a stellar collapse originating a Schwarzschild black hole. The shadowed area represents the fluid interior of the star. The curvature singularity (wave-like line) is formed at $\tilde{t} = \tilde{t}_0$. The horizon is represented by the dashed line.

relevant for astrophysical observations: no signal can come from the interior of the BH. Therefore, the most physically relevant properties of black holes are the properties of the exterior region. On the other hand, it is impossible to have a general understanding of the physics of black holes (and then of the behaviour of astrophysical black holes) without having at least a general idea of what's going on inside; for this reason we have briefly discussed the features of the entire solution.

1.4 Geodesics

We remind that the geodesic equations are equivalent to the Euler-Lagrange equations with Lagrangian $\mathcal{L} = 1/2 g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ where a dot indicates differentiation with respect to the affine parameter λ (we call the four-velocity of the particle $\dot{x}^\mu$ or, equivalently, u^μ). For the Schwarzschild metric, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\left[-\left(1-\frac{2M}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{\left(1-\frac{2M}{r}\right)} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right] . \quad (1.34)$$

The equations of motion for $\dot{t}$, $\dot{\phi}$ and $\dot{\theta}$ are:

- Equation for $\dot{t}$:

$$\frac{\partial\mathcal{L}}{\partial t} - \frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial(\dot{t})} = 0 \quad \rightarrow \quad \frac{d}{d\lambda}\left[\left(1 - \frac{2M}{r}\right)2\dot{t}\right] = 0 \qquad (1.35)$$

i.e.

$$\dot{t} = \frac{const}{\left(1 - \frac{2M}{r}\right)} = \frac{E}{\left(1 - \frac{2M}{r}\right)}\,. \qquad (1.36)$$

In the case of a massive particle, the constant E can be interpreted as the energy of the particle per unit mass at infinity, since at $r \rightarrow \infty$, where the spacetime becomes flat and special relativity applies, the energy of the particle is $P^0 = mu^0 = mE$. It should be reminded that, since the Schwarzschild metric admits a timelike Killing vector $\frac{\partial}{\partial t} \rightarrow \xi^\alpha_{(t)} = (1, 0, 0, 0)$, then

$$g_{\alpha\beta}\xi^\alpha_{(t)}u^\beta = const \quad \rightarrow \quad -\left(1 - \frac{2M}{r}\right)\dot{t} = const = -E \quad (1.37)$$

thus E is the constant of motion associated with invariance for time translations.

- Equation for $\dot{\phi}$: in a similar way it is easy to show that

$$\dot{\phi} = \frac{const}{r^2\sin^2\theta} = \frac{L}{r^2\sin^2\theta}\,. \qquad (1.38)$$

Since there is a spacelike Killing vector $\frac{\partial}{\partial\phi} \rightarrow \xi^\alpha_{(\phi)} = (0, 0, 0, 1)$, such that

$$g_{\alpha\beta}\xi^\alpha_{(\phi)}u^\beta = const \quad \rightarrow \quad r^2\sin^2\theta\dot{\phi} = const \quad \rightarrow \quad L = const, \qquad (1.39)$$

thus L is the constant of motion associated with invariance for ϕ-rotations. It can be interpreted as the angular momentum of the particle per unit mass.

- Equation for $\dot{\theta}$:

$$\frac{\partial\mathcal{L}}{\partial\theta} - \frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial(\dot{\theta})} = 0 \quad \rightarrow \quad \frac{d}{d\lambda}(r^2\dot{\theta}) = r^2\sin\theta\cos\theta\dot{\phi}^2\,. \quad (1.40)$$

Therefore the equation for θ is

$$\ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta} + \sin\theta\cos\theta\dot{\phi}^2\,. \qquad (1.41)$$

Given the spherical symmetry, polar axes can be chosen in arbitrary way. We choose them such that, for a given value of the affine parameter, say $\lambda = 0$, the particle is on the equatorial plane $\theta = \frac{\pi}{2}$ and its three-velocity $(\dot{r}, \dot{\theta}, \dot{\phi})$ lays on the same plane, i.e. $\dot{\theta}(\lambda = 0) = 0$. Thus, we have to solve the following Cauchy problem

$$
\begin{aligned}
&\ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta} + \sin\theta\cos\theta\dot{\phi}^2 \\
&\dot{\theta}(\lambda = 0) = 0 \\
&\theta(\lambda = 0) = \frac{\pi}{2}
\end{aligned}
\tag{1.42}
$$

which admits an unique solution. Since

$$\theta(\lambda) \equiv \frac{\pi}{2} \tag{1.43}$$

satisfies the differential equation and the initial conditions, it must be *the* solution. Thus the orbit is plane and to hereafter we shall assume $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$.

- Equation for $\dot{r}$: it is convenient to derive this equation from the condition $u_\alpha u^\alpha = -1$, or $u_\alpha u^\alpha = 0$, respectively valid for massive and massless particles.

  A) massive particles:

$$g_{\alpha\beta}u^\alpha u^\beta = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2M}{r}\right)} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 = -1 \tag{1.44}$$

which becomes, by substituting the equations for $\dot{t}$ and $\dot{\phi}$

$$\dot{r}^2 + \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right) = E^2 \tag{1.45}$$

  B) massless particles:

$$g_{\alpha\beta}u^\alpha u^\beta = -\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2M}{r}\right)} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 = 0 \tag{1.46}$$

which becomes

$$\dot{r}^2 + \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right) = E^2\,. \tag{1.47}$$

Summarizing, the geodesic equations are:

$$\begin{aligned}\theta &= \frac{\pi}{2}\\ \dot{t} &= \frac{E}{\left(1-\frac{2M}{r}\right)}\\ \dot{\phi} &= \frac{L}{r^2}\\ \dot{r}^2 &= E^2 - V(r)\end{aligned} \tag{1.48}$$

with

$$V(r) = \left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{r^2}\right) \tag{1.49}$$

for massive particles and

$$V(r) = \frac{L^2}{r^2}\left(1-\frac{2M}{r}\right) \tag{1.50}$$

for massless particles; it is worth noticing that the potential of massless particles depends on L as a multiplication constant, and then the location of maxima and minima of V (and, more generally, the shape of V) does not depend on L, while for massive particles the dependence on L is non trivial.

We also point out that for massive particles (for which $u^\mu u_\mu = -1$ and the affine parameter is the proper time) E, L are, as we said above, the energy (at infinity) and the angular momentum per unit mass, but this is not the case for massless particles. If the particle is massless, there is an arbitrariness in the definition of the affine parameter ($\lambda \to a\lambda$ preserve the normalization $u^\mu u_\mu = 0$), and E, L scale with λ; then, we can normalize λ so that E, L are the energy (at infinity) and the angular momentum of the massless particle.

The radial equations

$$\dot{r}^2 = E^2 - V(r) \tag{1.51}$$

allow a qualitative study of the motion analogue to the study performed in Newtonian mechanics. Some results of this qualitative study are the following:

- geodesics of massless particles describe either open orbits, corresponding to deflection of the ingoing particle, or unstable circular orbits with $r = 3M$;

- geodesic of massive particles describe either open orbits, corresponding to deflection of the ingoing particle, or closed orbits, with r ranging between a minimum and a maximum value (these are not really closed curves due to periastron advance);
- circular orbits of massive particles are allowed only if $r \geq 3M$; if $3M \leq r < 6M$ they are unstable orbits; if $r \geq 6M$ they are stable orbits; therefore, the innermost stable circular orbit (ISCO) has $r = 6M$.

It is important to stress that the classical tests of general relativity are based on the properties of Schwarzschild's solution: the gravitational redshift, the perihelion advance, the deflection of light signals. But, as we will discuss later on, astrophysical observations are starting to test more complex spacetimes, like the Kerr metric which describes a rotating BH.

Chapter 2

The far field limit of an isolated, stationary object

In this chapter we study the gravitational field generated by an isolated, stationary object.

We require that the source is an isolated object, thus we are assuming that the spacetime outside the source is vacuum ($T_{\mu\nu} = 0$). Therefore, it is reasonable to assume that the spacetime, far away from the source, tends to the Minkowski spacetime; this condition is called *asymptotic flatness*.

If the spacetime is asymptotically flat, we can define, in an appropriate coordinate frame, a space coordinate r such that[1]

$$\lim_{r\to\infty} g_{\mu\nu} = \eta_{\mu\nu} \,. \tag{2.1}$$

We call *far field limit* the region of spacetime in which $r \gg R$, where R is a lengthscale of the source. In the far field limit,

$$g_{\mu\nu} = \eta_{\mu\nu} + O\left(\frac{1}{r}\right) \,. \tag{2.2}$$

We also assume that the source is stationary, i.e. it does not depend on time. In this case, it can be shown that Einstein's equations imply that the spacetime is also stationary.

We want to determine the metric in the far field limit of an isolated, stationary object. In other words, we want to determine the

[1] Actually a complete, coordinate independent definition of asymptotic flatness would require the introduction of the concept of conformal infinity, which is beyond the scope of these lectures. However, for the derivation of the far field limit of isolated stationary objects the definition (2.1) is sufficient.

lowest order terms in $1/r$ in the expansion (2.2). We will show that the spacetime metric in the far field limit is

$$\begin{aligned} ds^2 &= -\left(1-\frac{2M}{r}\right)dt^2+\left(1+\frac{2M}{r}\right)dr^2 \\ &\quad +r^2(d\theta^2+\sin^2\theta d\phi^2)-\frac{4J}{r}\sin^2\theta dt d\phi \\ &\quad + \text{higher order terms in } 1/r\,. \end{aligned} \tag{2.3}$$

with M mass and J angular momentum of the central object.

Far away from the source the expansion (2.2) applies; it can also be written in the form

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu} \tag{2.4}$$

with $|h_{\mu\nu}|\ll 1$. The perturbation $h_{\mu\nu}$ is a solution of the equations of linearized gravity $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$,

$$\Box\bar{h}_{\mu\nu}=0 \tag{2.5}$$

$$\bar{h}^{\mu}{}_{\nu,\mu}=0 \tag{2.6}$$

where we have defined

$$\bar{h}_{\mu\nu}\equiv h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h^{\alpha}{}_{\alpha} \tag{2.7}$$

and we neglect terms $O(|h|^2)$ (which are also higher order terms in the $1/r$ expansion). We are assuming that the spacetime is stationary, therefore (2.5), (2.6) become

$$\nabla^2\bar{h}_{\mu\nu}=0 \tag{2.8}$$

$$\bar{h}^{i}{}_{\nu,i}=0\,. \tag{2.9}$$

We stress that (2.8), (2.9) hold only in the far field limit $r\gg R$; we are not making any assumption on the nature of the source.

Nevertheless, to have more insight on the physical meaning of (2.3), we will start deriving it in the particular case where the gravitational field on the source is weak. Subsequently, we will consider the general case.

2.1 The case of a weakly gravitating source

If the gravitational field on the source is weak, it is possible to linearize the metric on the source, and the linearized Einstein's equations give

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$$
$$\bar{h}^{\mu}{}_{\nu,\mu} = 0\,, \tag{2.10}$$

whose solution is

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = 4\int_V \frac{T_{\mu\nu}(t-|\mathbf{x}\text{-}\mathbf{x}'|,\mathbf{x}'\ )}{|\mathbf{x}\text{-}\mathbf{x}'\ |}\ d^3x' \tag{2.11}$$

where V is the three-volume of the source. We choose V such that its boundary falls outside the source, i.e. $T_{\mu\nu} = 0$ on the boundary ∂V.

In the standard derivation of the quadrupole formula for gravitational radiation, one considers the oscillating part of the integral (2.11), neglecting the terms constant in time. Here, on the contrary, we are considering a stationary source $T_{\mu\nu} = T_{\mu\nu}(\mathbf{x}')$; the integral (2.11) becomes

$$\bar{h}_{\mu\nu}(\mathbf{x}) = 4\int_V \frac{T_{\mu\nu}\,(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'\,. \tag{2.12}$$

We denote by Latin letters i,j the space indexes $1,2,3$; notice that, in linearized gravity, the indexes of $h_{\mu\nu}$ are raised by using the Minkowski metric, thus $h_{i\mu} = h^i{}_\mu$.

Let us consider the Taylor expansion of $1/|\mathbf{x}-\mathbf{x}'|$; $\mathbf{x}$ is the position vector of the point where we compute the gravitational perturbation $h_{\mu\nu}$ (with radial coordinate $r = |\mathbf{x}|$); $\mathbf{x}'$ is the position vector of a generic point inside the source; then, $|\mathbf{x}'| \leq R \ll r = |\mathbf{x}|$. Such an expansion is commonly named *multipolar expansion*; we have

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r} + \frac{\sum_i x^i x'^i}{r^3} + O\left(\frac{1}{r^3}\right)\,. \tag{2.13}$$

Substituting in (2.12) we find

$$\bar{h}_{\mu\nu}(\mathbf{x}) = \frac{4}{r}\int_V T_{\mu\nu} d^3x' + \frac{4\sum_i x^i}{r^3}\int_V T_{\mu\nu} x'^i d^3x'\,. \tag{2.14}$$

The first integral in (2.14) gives

$$\int_V T_{\mu\nu} d^3x'\,. \tag{2.15}$$

Being T_{00}, in the limit of weak gravitational field, the mass-energy density of the system, by definition

$$\int_V T_{00} d^3x' = M \,. \tag{2.16}$$

To compute the components μi of the integral (2.15), we first notice that the stress-energy tensor conservation, for a stationary source, reduces to

$$T^{\mu\nu}{}_{,\nu} = T^{\mu 0}{}_{,0} + T^{\mu i}{}_{,i} = T^{\mu i}{}_{,i} = 0 \,. \tag{2.17}$$

Using (2.17), and the property $\frac{\partial x^i}{\partial x^j} = \delta^i_j$, we find

$$\begin{aligned} \int_V T^{\mu i} d^3x' &= \int_V T^{\mu k} \delta^i_k d^3x' = \int_V T^{\mu k} \frac{\partial x'^i}{\partial x'^k} d^3x' \\ &= -\int_V \left(\frac{\partial T^{\mu k}}{\partial x'^k} \right) x'^i d^3x' = 0 \end{aligned} \tag{2.18}$$

where we have integrated by parts; the surface terms do not contribute, because, on the boundary of V, $T_{\mu\nu} = 0$. Therefore, the components μi of the integral (2.15) vanish.

Let us now compute the second integral in (2.14),

$$\int_V T_{\mu\nu} x'^i d^3x' \,. \tag{2.19}$$

The 00 component gives the position of the center of mass, which can be set to zero by a proper choice of the origin of the coordinates:

$$\int_V T_{00} x'^i d^3x' = M x'^i_{cdm} = 0 \,. \tag{2.20}$$

To compute the μi components of the integral (2.19), we start by proving that it is antisymmetric in the last two indexes:

$$\int_V T^{\mu i} x'^j d^3x' = -\int_V T^{\mu j} x'^i d^3x' \,. \tag{2.21}$$

Indeed,

$$\begin{aligned} &\int_V \left(T^{\mu i} x'^j + T^{\mu j} x'^i \right) d^3x' = \int_V T^{\mu k} \left(\frac{\partial x'^i}{\partial x'^k} x'^j + \frac{\partial x'^j}{\partial x'^k} x'^i \right) d^3x' \\ &= \int_V T^{\mu k} \frac{\partial}{\partial x'^k} \left(x'^i x'^j \right) d^3x' = -\int_V x'^i x'^j T^{\mu k}{}_{,k} d^3x' = 0 \end{aligned} \tag{2.22}$$

where we have integrated by parts, using the fact that, on the boundary of V, $T_{\mu\nu} = 0$.

Let us consider now the components $\mu\nu = jk$ of the integral (2.19)

$$\int_V T^{jk} x'^i d^3x' \,. \tag{2.23}$$

It is symmetric in the first two indexes, but, from (2.21), it follows that it is also antisymmetric in the last two indexes. Therefore, it is easy to show that it is the opposite of itself, and then it vanishes:

$$\begin{aligned} \int_V T^{ki} x'^j d^3x' &= -\int_V T^{kj} x'^i d^3x' = -\int_V T^{jk} x'^i d^3x' \\ &= \int_V T^{ji} x'^k d^3x' = \int_V T^{ij} x'^k d^3x' \qquad (2.24) \\ &= -\int_V T^{ik} x'^j d^3x' = 0 \,. \qquad (2.25) \end{aligned}$$

The last integral we have to compute is

$$\int_V T^{0i} x'^j d^3x' = -\int_V T^{0j} x'^i d^3x' \,. \tag{2.26}$$

To express it differently we introduce the tensor ϵ^{ijk}, which is antisymmetric in any exchange of its indexes, and has $\epsilon^{123} = 1$. Then, if we define

$$J^i \equiv -\int_V \epsilon^{ijk} T^{0j} x'^k d^3x' \,, \tag{2.27}$$

we can write

$$\int_V T^{0i} x'^j d^3x' = -\frac{1}{2}\epsilon^{ijk} J^k \tag{2.28}$$

It is easy to prove that (2.27) implies (2.28), using the very useful property[2]

$$\epsilon^{ijk}\epsilon^{klm} = \delta^{il}\delta^{jm} - \delta^{im}\delta^{jl} \,. \tag{2.29}$$

Then, given an antisymmetric tensor $B^{ij} = -B^{ji}$, we have

$$J^i = \epsilon^{ijk} B^{jk} \Longleftrightarrow B^{ij} = \frac{1}{2}\epsilon^{ijk} J^k \ : \tag{2.30}$$

[2]The prove of (2.29) is the following. $\epsilon^{ijk} \neq 0$ only if its three indexes are all different, thus $i \neq k$ and $j \neq k$; the same for ϵ^{lmk}. Therefore $\epsilon^{ijk}\epsilon^{lmk} \neq 0$ only if the indexes ij and lm are the same, with any ordering. If they have the same order, i.e. $ij = lm$, then $\epsilon^{ijk}\epsilon^{lmk} = 1$, while if they have the opposite order, i.e. $ij = ml$, then $\epsilon^{ijk}\epsilon^{lmk} = -1$. Thus (2.29).

$$\frac{1}{2}\epsilon^{ijk}J^k = \frac{1}{2}\epsilon^{ijk}\epsilon^{klm}B^{lm} = \frac{1}{2}\left(\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}\right)B^{lm} = B^{ij}\,. \quad (2.31)$$

Finally, replacing

$$B^{jk} = -\int_V T^{0j}x'^k d^3x'\,, \quad (2.32)$$

we find that (2.27) implies (2.28).

The physical meaning of J^k is simple. The tensor T^{0i} represents the momentum density of the source,

$$T^{0i} = \mathcal{P}^i \quad (2.33)$$

i.e. an element of the matter source in the volume d^3x' has momentum $\mathcal{P}d^3x'$. Writing (2.27) as a vector product,

$$\mathbf{J} = \int \mathbf{x}' \times \mathcal{P}d^3x' \quad (2.34)$$

since $(\mathbf{u}\times\mathbf{v})^i = \epsilon^{ijk}u^jv^k$.

We remind that the angular momentum of a point particle is the vector product between its momentum and its position vector. The angular momentum of an element of matter in the source, which has momentum $\mathcal{P}d^3x'$ and position $\mathbf{x}'$, is then $\mathbf{x}'\times\mathcal{P}d^3x'$. Therefore, $\mathbf{J}$ is the total angular momentum of the source.

Summarizing, the multipolar expansion (2.14) gives

$$\begin{aligned}\bar{h}_{00} &= \frac{4M}{r} + O\left(\frac{1}{r^3}\right)\\ \bar{h}_{0i} &= \frac{2}{r^3}\epsilon^{ijk}x^jJ^k + O\left(\frac{1}{r^3}\right)\\ \bar{h}_{ij} &= O\left(\frac{1}{r^3}\right)\end{aligned} \quad (2.35)$$

because

$$\begin{aligned}&\int_V T_{00}d^3x' = M\\ &\frac{4}{r^3}x^j\int_V T_{0i}x'^jd^3x' = -\frac{4}{r^3}x^j\int_V T^{0i}x'^jd^3x' = \frac{2}{r^3}\epsilon^{ijk}x^jJ^k\,.\end{aligned} \quad (2.36)$$

Notice that in flat space $\bar{h}_{00} = \bar{h}^{00}$, $\bar{h}_{0i} = -\bar{h}^{0i}$, $\bar{h}_{ij} = \bar{h}^{ij}$.

In terms of $h_{\mu\nu}$, given by [3]

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}^{\alpha}{}_{\alpha}\,, \tag{2.37}$$

we have[4]

$$\begin{aligned} h_{00} &= \frac{2M}{r} + O\left(\frac{1}{r^3}\right) \\ h_{0i} &= \frac{2}{r^3}\epsilon_{ijk}x^j J^k + O\left(\frac{1}{r^3}\right) \\ h_{ij} &= \frac{2M}{r}\delta_{ij} + O\left(\frac{1}{r^3}\right)\,. \end{aligned} \tag{2.38}$$

Let us transform the solution (2.38) in polar coordinates

$$\begin{aligned} x^1 &= r\sin\theta\cos\phi \\ x^2 &= r\sin\theta\sin\phi \\ x^3 &= r\cos\theta\,. \end{aligned} \tag{2.39}$$

We have

$$\sum_i (dx^i)^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\,, \tag{2.40}$$

thus

$$h_{ij}dx^i dx^j = \frac{2M}{r}\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right)\,. \tag{2.41}$$

The transformation of $h_{0i}dx^0dx^i$ is less trivial. If we choose the orientation of our polar coordinates such that

$$\mathbf{J} = (0,0,J)\,, \tag{2.42}$$

then

$$\begin{aligned} h_{0i}dx^0dx^i &= \left(\frac{2}{r^3}dx^0\right)\epsilon_{ijk}x^jJ^kdx^i = -\left(\frac{2}{r^3}dx^0\right)J(x^1dx^2 - x^2dx^1) \\ &= -\left(\frac{2}{r^3}dx^0\right)Jr^2\sin^2\theta d\phi = -\frac{2J}{r}\sin^2\theta dt d\phi \end{aligned} \tag{2.43}$$

[3]To invert (2.7) we take the trace of (2.7), finding $\bar{h}^{\lambda}{}_{\lambda} = -h^{\lambda}{}_{\lambda}$, and replace this relation into (2.7).

[4]Notice that $\frac{x^j}{r^3}$ is an $O\left(\frac{1}{r^2}\right)$ term, because in the far field limit $x^j \sim r$.

where the property $x^1 dx^2 - x^2 dx^1 = r^2 \sin^2\theta d\phi$ can be proven by differentiation of (2.39).

The corresponding line element is

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r} + O\left(\frac{1}{r^3}\right)\right) dt^2 \\ &+ \left(1 + \frac{2M}{r} + O\left(\frac{1}{r^3}\right)\right) \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \\ &+ \left(-\frac{4J}{r}\sin^2\theta + O\left(\frac{1}{r^2}\right)\right) dt d\phi \,. \end{aligned} \tag{2.44}$$

This is the solution of linearized Einstein's equations. If we consider the complete Einstein's equations

$$R_{\mu\nu} = 0 \tag{2.45}$$

we have terms $O(|h_{\mu\nu}|^2)$, which produce terms quadratic in M and J in the metric (and also higher order terms). Therefore, there are terms $\sim M^2/r^2$, $\sim J^2/r^2$ in the metric. Then, we cannot say, for instance, that

$$g_{00} = -1 + \frac{2M}{r} + O\left(\frac{1}{r^3}\right) \tag{2.46}$$

because there is also a term $\sim M^2/r^2$ arising from nonlinear terms in Einstein's equations. If we don't want to compute these terms, we truncate our expansion at $O(1/r^2)$:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\right) dt^2 \\ &+ \left(1 + \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\right) \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \\ &+ \left(-\frac{4J}{r} + O\left(\frac{1}{r^2}\right)\right) \sin^2\theta dt d\phi \,. \end{aligned} \tag{2.47}$$

Finally, we make the following redefinition of the radial coordinate:

$$r \to r - M \,. \tag{2.48}$$

Making this change in (2.47), and neglecting contributions $O(1/r^2)$, the only term which produces a change in the metric is

$$\left(1 + \frac{2M}{r}\right) r^2(d\theta^2 + \sin^2\theta d\phi^2) \to \left(1 + \frac{2M}{r}\right)(r - M)^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$= r^2 \left(1 + O\left(\frac{1}{r^2}\right)\right)(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{2.49}$$

With this coordinate redefinition, we find the expression (2.3):

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 + \frac{2M}{r}\right)dr^2 \\ &\quad + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{4J}{r}\sin^2\theta dt d\phi \\ &\quad + \text{higher order terms in } 1/r\,. \end{aligned} \tag{2.50}$$

Notice that in the case of a spherically symmetric spacetime (in which $J = 0$), the line element (2.50) corresponds to the linearization of the Schwarzschild metric in the standard coordinates (t, r, θ, ϕ). In the same limit, (2.38) corresponds to the linearization of the Schwarzschild metric in isotropic coordinates.

2.2 The case of a general source

Let us now consider a general source. The gravitational field can be strong near the source, whereas far away, where we want to solve Einstein's equations, it is weak. Thus, far away from the source, we can consider linearized gravity, neglecting terms $O(|h_{\mu\nu}|^2)$, and we can neglect terms with a large power of $1/r$, where r is the distance from the source.

Therefore, we expand the metric far away from the source as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.51}$$

with

$$\bar{h}_{\mu\nu} = \frac{a_{\mu\nu}(\theta, \phi)}{r} + \frac{b_{\mu\nu}(\theta, \phi)}{r^2} + O\left(\frac{1}{r^3}\right)\,. \tag{2.52}$$

The coefficients $a_{\mu\nu}$, $b_{\mu\nu}$ do not depend by r, but they depend by the angular variables θ, ϕ, so that they remain finite for $r \to \infty$. We can express the angular variables in an alternative way, using the director cosines n^i defined as

$$n^i = \frac{x^i}{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \qquad (n^i n^i = 1). \tag{2.53}$$

Thus, we can say that $a_{\mu\nu}, b_{\mu\nu}$ are functions of the n^i.

The metric perturbation, which we assume to be stationary, satisfies equation (2.8),

$$\nabla^2 \bar{h}_{\mu\nu} = 0\,. \tag{2.54}$$

The Laplace operator in spherical coordinates has the form[5]

$$\nabla^2 = \frac{1}{r^2}\partial_r r^2 \partial_r + \frac{\mathbb{L}}{r^2} \tag{2.55}$$

where $\mathbb{L}$ is an operator acting on the angular variables only:

$$\mathbb{L} \equiv \partial_\theta^2 + \cot\theta\partial_\theta + \sin^{-2}\theta\partial_\phi^2\,. \tag{2.56}$$

Substituting (2.52) in (2.55) we easily find

$$\mathbb{L}a_{\mu\nu}(\theta,\phi) = 0 \tag{2.57}$$

$$\mathbb{L}b_{\mu\nu}(\theta,\phi) = -2b_{\mu\nu}(\theta,\phi)\,. \tag{2.58}$$

The eigenfunctions of the operator $\mathbb{L}$ are the *spherical harmonics* $Y_{lm}(\theta,\phi)$, with $l = 0, 1, \ldots$ and $m = -l, -l+1, \ldots, l-1, l$. They are defined by the property

$$\mathbb{L}Y_{lm} = -l(l+1)Y_{lm}\,. \tag{2.59}$$

Equation (2.58) tells us that $b_{\mu\nu}$ is a linear combination of the spherical harmonics with $l = 1$, which are

$$\begin{aligned} Y_{11} &= -\sqrt{\frac{3}{8\pi}}\sin\theta e^{\mathrm{i}\phi} \\ Y_{10} &= \sqrt{\frac{3}{4\pi}}\cos\theta \\ Y_{1-1} &= \sqrt{\frac{3}{8\pi}}\sin\theta e^{-\mathrm{i}\phi}\,. \end{aligned} \tag{2.60}$$

This is equivalent to say that $b_{\mu\nu}$ is a linear combination of $n^i = x^i/r$, because

$$\begin{aligned} n^1 = \frac{x^1}{r} &= \sin\theta\cos\phi = \sqrt{\frac{8\pi}{3}}\frac{-Y_{11}+Y_{1-1}}{2} \\ n^2 = \frac{x^2}{r} &= \sin\theta\sin\phi = \sqrt{\frac{8\pi}{3}}\frac{-Y_{11}-Y_{1-1}}{2\mathrm{i}} \\ n^3 = \frac{x^3}{r} &= \cos\theta = \sqrt{\frac{4\pi}{3}}Y_{10}\,. \end{aligned} \tag{2.61}$$

[5]The theory of Laplace equation and the properties of spherical harmonics are extensively discussed in the literature. See for instance Jackson's book *Electromagnetism*, Chapter 3.

Therefore, $a^{\mu\nu}$ does not depend on the angular variables n^i, while $b^{\mu\nu}$, being a combination of the three harmonics Y_{1m}, is linear in n^i, and can be written as

$$b^{\mu\nu}(n^i) = b_i^{\mu\nu} n^i \,. \tag{2.62}$$

Expansion (2.52) can then be written as

$$\bar{h}^{\mu\nu} = \frac{a^{\mu\nu}}{r} + \frac{b_i^{\mu\nu} x^i}{r^3} + O\left(\frac{1}{r^3}\right) , \tag{2.63}$$

with $a_{\mu\nu}$, $b_{\mu\nu i}$ constant coefficients.

We now impose on (2.63) the gauge condition (2.6)

$$\bar{h}^{\mu\nu}_{\ \ ,\nu} = 0 \tag{2.64}$$

which in the case of stationary perturbations becomes

$$\bar{h}^{\mu i}_{\ \ ,i} = 0 \,. \tag{2.65}$$

We get (remember that in linearized gravity it is irrelevant if a space index i is high or low)

$$\bar{h}_{\mu j,j} = -\frac{a_{\mu j} x^j}{r^3} + \frac{b_{\mu j i}\left(\delta^{ij} r^2 - 3x^i x^j\right)}{r^5} = 0 \tag{2.66}$$

which has to be satisfied for all (large) values of r and for all values of $n^i = x^i/r$. Thus,

$$\begin{aligned} a_{\mu j} &= 0 \\ \left(\delta^{ij} - 3n^i n^j\right) b_{\mu i j} &= 0 \,. \end{aligned} \tag{2.67}$$

These equations do not involve a_{00}, b_{00i}, which are then in general nonvanishing free constants; to simplify the notation, we rewrite them as

$$\begin{aligned} a &\equiv a_{00} \\ b_i &\equiv b_{00i} \,. \end{aligned} \tag{2.68}$$

The first of the conditions (2.67) states that all the constants $a_{\mu\nu}$ different from a_{00} vanish. The second one is more complicate to analyze. It can be rewritten as

$$H^{ij} b_{0ij} = 0 \tag{2.69}$$

$$H^{ij} b_{kij} = 0 \tag{2.70}$$

where we have defined

$$H^{ij} \equiv \delta^{ij} - 3n^i n^j \,. \tag{2.71}$$

The general solution of (2.69), (2.70) is

$$\begin{aligned} b_{0ij} &= b\delta_{ij} + c_k \epsilon_{ijk} && (2.72) \\ b_{kij} &= d_k\delta_{ij} + d_i\delta_{kj} - d_j\delta_{ki} && (2.73) \end{aligned}$$

where b, c_k, d_k are constants. A rigorous proof of (2.72), (2.73) would require an use of the structures of Group Theory which goes beyond the scope of these lectures. We only give here an intuitive, non-rigorous proof of the first solution, (2.72).

Equation (2.69) must be satisfied for every value of n^i, i.e. for every value of the angular variables θ, ϕ. But, while H^{ij} depends on the angles, $b_{\mu ij}$ *cannot* depend on the angles. Thus, equation (2.69) can be satisfied only because of the symmetry properties of H^{ij}: it is symmetric and traceless

$$H^{ij} = H^{ji} \qquad \delta_{ij}H^{ij} = 0 \,. \tag{2.74}$$

All the quantities considered here are tensors in the euclidean three-dimensional space. The only constant tensors which vanish when contracted with H^{ij} are the Kronecker delta δ_{ij} and the completely antisymmetric tensor ϵ_{ijk}: the former vanishes because H^{ij} is traceless, the latter vanishes because H^{ij} is symmetric. The solution b_{0ij} must be a combination of them, thus we have (2.72).

Summarizing, by imposing the gauge condition (2.6) on the expansion (2.63) we get

$$\begin{aligned} \bar{h}_{00} &= \frac{a}{r} + \frac{b_i x^i}{r^3} + O\left(\frac{1}{r^3}\right) \\ \bar{h}_{0i} &= \frac{bx^i}{r^3} + \epsilon_{ijk}\frac{x^j c_k}{r^3} + O\left(\frac{1}{r^3}\right) \\ \bar{h}_{ij} &= \frac{1}{r^3}\left(-\delta_{ij}d_k x^k + d_i x^j + d_j x^i\right) + O\left(\frac{1}{r^3}\right) && (2.75) \end{aligned}$$

depending on the constants a, b_i, b, c_k, d_k.

The constants b, d_k can be eliminated by a (position dependent) infinitesimal diffeomorphism $x^\mu \to x^\mu + \xi^\mu$ with parameter

$$\xi^\mu = \left(-\frac{b}{r}, -\frac{d^i}{r}\right) \tag{2.76}$$

(it is infinitesimal in the sense that r is large and $\xi \sim 1/r$).

The change in the metric is

$$\begin{aligned} g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} &\rightarrow g_{\mu\nu} + g_{\mu\alpha}\xi^{\alpha}{}_{,\nu} + g_{\nu\alpha}\xi^{\alpha}{}_{,\mu} + g_{\mu\nu,\alpha}\xi^{\alpha} \\ &= \eta_{\mu\nu} + h_{\mu\nu} + g_{\mu\alpha}\xi^{\alpha}{}_{,\nu} + g_{\nu\alpha}\xi^{\alpha}{}_{,\mu} + g_{\mu\nu,\alpha}\xi^{\alpha} \end{aligned} \tag{2.77}$$

therefore the change in the perturbation $\delta h_{\mu\nu}$ is

$$\delta h_{\mu\nu} = g_{\mu\alpha}\xi^{\alpha}{}_{,\nu} + g_{\nu\alpha}\xi^{\alpha}{}_{,\mu} + g_{\mu\nu,\alpha}\xi^{\alpha} \tag{2.78}$$

and, being $\xi^{\mu} = O(|h_{\mu\nu}|)$, we have that, neglecting terms quadratic in $h_{\mu\nu}$,

$$\delta h_{\mu\nu} = \eta_{\mu\alpha}\xi^{\alpha}{}_{,\nu} + \eta_{\nu\alpha}\xi^{\alpha}{}_{,\mu} \,. \tag{2.79}$$

The change in the variable

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h_{\alpha\beta} \tag{2.80}$$

is

$$\begin{aligned} \delta\bar{h}_{\mu\nu} &= \delta h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\delta h_{\alpha\beta} \\ &= \eta_{\mu\alpha}\xi^{\alpha}{}_{,\nu} + \eta_{\nu\alpha}\xi^{\alpha}{}_{,\mu} - \eta_{\mu\nu}\xi^{\alpha}{}_{,\alpha} \,. \end{aligned} \tag{2.81}$$

We have

$$\begin{aligned} \xi^{\mu}{}_{,0} &= 0 \\ \xi^{0}{}_{,i} &= \frac{bx^i}{r^3} \\ \xi^{k}{}_{,i} &= \frac{d^k x^i}{r^3} \end{aligned} \tag{2.82}$$

then

$$\begin{aligned} \delta\bar{h}_{00} &= -\eta_{00}\xi^{k}{}_{,k} + O\left(\frac{1}{r^3}\right) = \frac{d^k x^k}{r^3} + O\left(\frac{1}{r^3}\right) \\ \delta\bar{h}_{0i} &= \eta_{00}\xi^{0}_{,i} + O\left(\frac{1}{r^3}\right) = -\frac{bx^i}{r^3} + O\left(\frac{1}{r^3}\right) \\ \delta\bar{h}_{ij} &= \eta_{ik}\xi^{k}{}_{,j} + \eta_{jk}\xi^{k}{}_{,i} - \eta_{ij}\xi^{k}{}_{,k} = \frac{1}{r^3}\left[d^i x^j + d^j x^j - \eta_{ij}d^k x^k\right] \end{aligned} \tag{2.83}$$

thus, after the diffeomorphism,

$$
\begin{aligned}
\bar{h}_{00} &= \frac{a}{r} + \frac{\tilde{b}_i x^i}{r^3} + O\left(\frac{1}{r^3}\right) \\
\bar{h}_{0i} &= \epsilon_{ijk} \frac{x^j c_k}{r^3} + O\left(\frac{1}{r^3}\right) \\
\bar{h}_{ij} &= O\left(\frac{1}{r^3}\right)
\end{aligned}
\tag{2.84}
$$

where we have defined

$$\tilde{b}_i \equiv b_i + d_i \,. \tag{2.85}$$

Furthermore, we can get rid of $\tilde{b}_i$ by making the (rigid) translation

$$x^i \to x^i + \frac{\tilde{b}_i}{a} \tag{2.86}$$

which produces the following change in the a/r term:

$$
\begin{aligned}
\frac{a}{r} = a\left((x^i)^2\right)^{-1/2} &\to a\left(\left(x^i + \frac{\tilde{b}_i}{a}\right)^2\right)^{-1/2} \\
&= a\left(r^2\left(1 + 2\frac{\tilde{b}_i x^i}{r^2 a}\right)\right)^{-1/2} + O\left(\frac{1}{r^3}\right) \\
&= \frac{a}{r}\left(1 - \frac{\tilde{b}_i x^i}{r^2 a}\right) + O\left(\frac{1}{r^3}\right) = \frac{a}{r} - \frac{\tilde{b}_i x^i}{r^3} + O\left(\frac{1}{r^3}\right) \,.
\end{aligned}
\tag{2.87}
$$

Therefore,

$$
\begin{aligned}
\bar{h}_{00} &= \frac{a}{r} + O\left(\frac{1}{r^3}\right) \\
\bar{h}_{0i} &= \epsilon_{ijk} \frac{x^j c_k}{r^3} + O\left(\frac{1}{r^3}\right) \\
\bar{h}_{ij} &= O\left(\frac{1}{r^3}\right) \,.
\end{aligned}
\tag{2.88}
$$

Finally, we compute

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\bar{h}_{\alpha\beta} \,. \tag{2.89}$$

We have

$$\frac{1}{2}\eta^{\alpha\beta}\bar{h}_{\alpha\beta} = -\frac{a}{2r} \tag{2.90}$$

therefore

$$\begin{aligned} h_{00} &= \frac{a}{2r} + O\left(\frac{1}{r^3}\right) \\ h_{0i} &= \epsilon_{ijk}\frac{x^j c_k}{r^3} + O\left(\frac{1}{r^3}\right) \\ h_{ij} &= \delta_{ij}\frac{a}{2r} + O\left(\frac{1}{r^3}\right) . \end{aligned} \tag{2.91}$$

With the identifications

$$a = 4M \qquad c^k = 2J^k \tag{2.92}$$

the solution (2.91) coincides with the solution (2.38), which we have derived in the case of a weak field source, and that we have already shown to be equivalent to the solution (2.3).

2.3 Mass and angular momentum of an isolated object

As we have seen, the metric

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 + \frac{2M}{r}\right)dr^2 \\ &\quad + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{4J}{r}\sin^2\theta dt d\phi \\ &\quad + \text{higher order terms in } 1/r\,. \end{aligned} \tag{2.93}$$

describes the far field limit of an isolated, stationary source. If the source is weakly gravitating, we can apply the definitions of Newtonian physics on the source, and in this case we have seen that M and J have a simple interpretation: they are the mass and the angular momentum of the source, respectively.

In the case of a source which is *not* weakly gravitating, M and J are simply integration constants of the general far field solution (2.93). We could ask: which is their physical interpretation in this case?

One answer which is often given to this question is the following. The motion of a test body in the metric (2.3), far away from a strongly gravitating source, cannot be distinguished from the motion it would have if the source were *weakly gravitating*, with mass M and angular momentum J. Thus, we can give an *operational definition* of the mass and angular momentum of the strongly gravitating source, by looking to the motion of test bodies far away from the source. The mass will be defined by looking to Kepler's third law, and the angular momentum by looking to the precession of gyroscopes orbiting around the source.

A different answer is based on the stress-energy pseudotensor $t^{\mu\nu}$, which describes the energy and momentum carried by the gravitational field, and satisfies, together with the stress-energy tensor $T_{\mu\nu}$, a conservation law:

$$[(-g)(T^{\mu\nu} + t^{\mu\nu})]_{,\nu} = 0\,. \tag{2.94}$$

It can be expressed as a divergence:

$$(-g)(T^{\mu\nu} + t^{\mu\nu}) = \frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha} \tag{2.95}$$

where

$$\zeta^{\mu\nu\alpha} = \frac{1}{16\pi}\frac{\partial}{\partial x^\beta}\left[(-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})\right]\,. \tag{2.96}$$

We can also write the quantity $\zeta^{\mu\nu\alpha}$ in a different way, which will be useful later:

$$\zeta^{\mu\nu\alpha} = \frac{\partial \lambda^{\mu\nu\alpha\beta}}{\partial x^\beta} \tag{2.97}$$

with

$$\lambda^{\mu\nu\alpha\beta} \equiv \frac{1}{16\pi}\left[(-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})\right]\,. \tag{2.98}$$

Furthermore, in a stationary spacetime, $g_{\mu\nu,0} = 0$ therefore

$$\zeta^{\mu\nu\alpha} = \frac{\partial \lambda^{\mu\nu\alpha i}}{\partial x^i}\,. \tag{2.99}$$

Let us now consider a spherical three-dimensional volume V centered on the source, extending up to a radius r far away from the source. Notice that it is different from the volume V considered in Section 2.1, which was only covering the source.

The total four-momentum enclosed in the volume V is due both to the source and to the gravitational field, and is

$$P^\mu = \int_V d^3x(-g)(T^{0\mu} + t^{0\mu}) \,. \tag{2.100}$$

As explained in Section 2.1, if the three-momentum in the volume element d^3x located at the space point with coordinates x^i is

$$\mathcal{P}^i d^3x \,, \tag{2.101}$$

then the angular momentum in the same volume element is

$$\epsilon^{ijk}\mathcal{P}^j x^k d^3x \,. \tag{2.102}$$

Therefore, the total angular momentum is

$$J^i = \epsilon^{ijk} \int_V d^3x(-g)(T^{0j} + t^{0j})x^k \,. \tag{2.103}$$

Substituting (2.95) in (2.100) and imposing the stationarity of the spacetime,

$$P^\mu = \int_V d^3x \frac{\partial\zeta^{0\mu\alpha}}{\partial x^\alpha} = \int_V d^3x \frac{\partial\zeta^{0\mu k}}{\partial x^k} \,. \tag{2.104}$$

Using Gauss' theorem we can express this integral as an integral on the spherical surface S, orthogonal, at each point, to the vector $\mathbf{n}$, which is the unit vector normal to the surface S:

$$P^\mu = \int_S dS \zeta^{0\mu k} n^k \,. \tag{2.105}$$

Substituting (2.95) in (2.103) we find that the total angular momentum enclosed in the volume V is

$$\begin{aligned} J^i &= \epsilon_{ijk} \int_V d^3x \frac{\partial\zeta^{0jl}}{\partial x^l} x^k = \epsilon_{ijk} \int_V d^3x \left[\frac{\partial(\zeta^{0jl}x^k)}{\partial x^l} - \zeta^{0jl}\frac{\partial x^k}{\partial x^l} \right] \\ &= \epsilon_{ijk} \int_V d^3x \left[\frac{\partial(\zeta^{0jl}x^k)}{\partial x^l} - \zeta^{0jk} \right] = \epsilon_{ijk} \int_V d^3x \frac{\partial}{\partial x^l} \left(\zeta^{0jl}x^k - \lambda^{0jkl} \right) \\ &= \epsilon_{ijk} \int_S dS \left(\zeta^{0jl}x^k - \lambda^{0jkl} \right) n^l \,. \end{aligned} \tag{2.106}$$

Substituting (2.96) and (2.98) in (2.105), (2.106) one finds

$$P^\mu = (M, 0, 0, 0) \,, \qquad J^i = (0, 0, J) \,. \tag{2.107}$$

We show this explicitly in the case of M. We have that

$$P^0 = \int_S dS \zeta^{00i} n^i \tag{2.108}$$

and $dS = r^2 d\Omega$. The metric perturbation is

$$\begin{aligned} h_{00} &= \frac{2M}{r} + O\left(\frac{1}{r^2}\right) \\ h_{0i} &= \frac{2}{r^3}\epsilon_{ijk} x^j J^k + O\left(\frac{1}{r^3}\right) \\ h_{ij} &= \frac{2M}{r}\delta_{ij} + O\left(\frac{1}{r^2}\right) . \end{aligned} \tag{2.109}$$

Being $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(|h_{\mu\nu}|^2)$, we have that the property $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$ implies

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(|h_{\mu\nu}|^2) \tag{2.110}$$

where the indexes of $h_{\mu\nu}$ have been raised with the Minkowski metric. Indeed,

$$(\eta^{\mu\nu} - h^{\mu\nu})(\eta_{\nu\rho} + h_{\nu\rho}) = \delta^\mu_\rho + O(|h_{\mu\nu}|^2) . \tag{2.111}$$

Therefore,

$$g^{00} = -1 - h^{00} + O(|h_{\mu\nu}|^2) = -1 - \frac{2M}{r} + O(|h_{\mu\nu}|^2) \tag{2.112}$$

$$g^{ij} = \delta^{ij} - h^{ij} + O(|h_{\mu\nu}|^2) = \left(1 - \frac{2M}{r}\right)\delta^{ij} + O(|h_{\mu\nu}|^2) . \tag{2.113}$$

Finally, the determinant of $g_{\mu\nu}$ is

$$g = (-1 + h_{00})(1 + h_{ii}) = -1 - \frac{4M}{r} + O(|h_{\mu\nu}|^2) . \tag{2.114}$$

Putting together (2.112), (2.113), (2.114), and neglecting the terms $O(|h_{\mu\nu}|^2)$ (like the terms $\sim M^2$, $\sim J^2$), we find

$$\begin{aligned} \zeta^{00i} n^i &= \frac{1}{16\pi} n^i \partial_j \left[(-g)\left(g^{00} g^{ij} - g^{0i} g^{0j}\right)\right] \\ &= \frac{1}{16\pi} n^i \partial_j \left[\left(1 + \frac{4M}{r}\right)\left(-1 - \frac{2M}{r}\right)\left(1 - \frac{2M}{r}\right)\delta_{ij}\right] + O(|h_{\mu\nu}|^2) \\ &= -\frac{1}{16\pi} n^i \partial_i \frac{4M}{r} + O(|h_{\mu\nu}|^2) = \frac{1}{4\pi}\frac{M}{r^2} n^i n^i = \frac{1}{4\pi}\frac{M}{r^2} \end{aligned} \tag{2.115}$$

thus

$$P^0 = \int_S \zeta^{00i} n^i r^2 d\Omega = M \, . \tag{2.116}$$

The case of angular momentum is conceptually similar, but the computation is more lengthy and we don't repeat it here.

We can conclude that the integration constants M and J appearing in the far field limit metric of an isolated source (2.3) can be correctly interpreted as the mass and the angular momentum, respectively, of the system. In the case of a weakly gravitating source, the system is the source alone, whereas if the source has a strong gravitational field, the field itself is part of the system and contributes to the total mass M and angular momentum J appearing in the line element (2.3).

We stress again that, since the source is isolated, the metric approaches the Minkowski metric far away from the source; this allows us to assume, for r sufficiently large, that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu}$ small. Furthermore, the $h_{\mu\nu}$ terms can be expanded in powers of $1/r$. The dominant contribution in this expansion tells us which are the total mass and angular momentum of the system.